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Uniqueness for Stochastic Scalar Conservation Laws on Riemannian Manifolds Revisited

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Abstract. We revise a uniqueness question for the scalar conservation law with stochastic forcing

$$du + \operatorname{div}_g \mathfrak{f}(\mathbf{x}, u)dt = \Phi(\mathbf{x}, u)dW_t, \quad \mathbf{x} \in M, \quad t \geq 0$$

on a smooth compact Riemannian manifold (M, g) where W_t is the Wiener process and $\mathbf{x} \mapsto \mathfrak{f}(\mathbf{x}, \xi)$ is a vector field on M for each $\xi \in \mathbb{R}$. We introduce admissibility conditions, derive the kinetic formulation and use it to prove uniqueness in a more straight-forward way than in the existing literature.

1. Introduction

The aim of the paper is to offer a simpler proof of uniqueness of admissible (i.e. kinetic) solution to the Cauchy problem for a stochastic scalar conservation law of the form

$$du + \operatorname{div}_g \mathfrak{f}(\mathbf{x}, u)dt = \Phi(\mathbf{x}, u)dW_t, \quad \mathbf{x} \in M, \quad t \geq 0 \tag{1}$$

$$u|_{t=0} = u_0(\mathbf{x}) \in L^\infty(M) \tag{2}$$

on a smooth, compact, d-dimensional (Hausdorff) Riemannian manifold (M, g). The object W is the Wiener process which can be finite or infinite dimensional which does not affect the essence of the proofs.

The proof of well-posedness has been recently presented in [14]. The authors considered the kinetic formulation of (1) and prove the uniqueness by finding a relation between the kinetic function and square of the kinetic function (see [14, (4.13)]). The procedure appeared to be quite complicated and we show here that it is possible to obtain the proof by considering the product of kinetic solution h and the function $(1-h)$.

More precisely, our idea of proof has the same starting point as in [14] since it is based on the appropriate kinetic reformulation of the problem (see (35) below). In [14], the authors then prove that the kinetic function h given by Definition 3.4 satisfies $h^2 = h$. However, unlike the method from [14] where the authors derive the equation for h^2 and then compare it with the equation for h in order to draw conclusions, we obtain

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an equation for $h(1-h)$ and use it to prove the uniqueness. Although the latter sounds the same, the regularization procedures in our situations are easier to follow (as we essentially closely follow the steps from the Euclidean case) and thus the method seems simpler than the one proposed in [14].

The basic reason for the simplification lies in the fact that the equations for h and $(1-h)$ are symmetric which is why we can fairly easily eliminate the terms appearing on the right hand-side of the latter equations and thus reach the Kato inequality (see (45)). Moreover, by regularizing the equation via the convolution with respect to $\mathbf{x}$ and ξ we obtain a function which is by assumption continuous with respect to time and we can directly use the Itô formula instead of using its generalized variant (see [14]).

We note that one of the ingredients of the proof is the classical method of the doubling of variables (see [22]). The method could be avoided since we regularized the equation (which means that we can use basic calculus for smooth functions). However, the analysis would then require various adaptations of the Friedrichs lemma (specially in a viewpoint that we have terms with the Wiener measure) and it appears that the proofs would not be easier.

Let us now introduce precise assumptions on the coefficients of the equation. First, we shall assume that we work with one-dimensional Wiener process defined on the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$. We will also assume that

- the flux $\mathfrak{f} \in C^1(M \times \mathbb{R}; \mathbb{R}^d)$ satisfies the geometry compatibility conditions and a decay property as follows respectively:

$$\operatorname{div}_g \mathfrak{f}(\mathbf{x}, \xi) = 0 \ \text{ for every } \xi \in \mathbb{R} \tag{3}$$

$$\|\mathfrak{f}(\cdot, \lambda)\|_{L^\infty(M)} \leq C(1 + |\lambda|); \tag{4}$$

- the function Φ is continuously differentiable and it decays to zero at infinity i.e. $\Phi \in C_0^1(M \times \mathbb{R})$, and

$$\sup_{\lambda \in \mathbb{R}} |\Phi(\cdot, \lambda)\lambda| \in L^1(M). \tag{5}$$

Nowadays, we are witnessing a rapid development of stochastic conservation laws and related equations. The rising interest to this field of research is motivated by concrete applications in biology, porous media, finances (see e.g. randomly chosen [1, 4, 32] and references therein) and, in general, any realistic situation in which we cannot determine parameters precisely (i.e. the coefficients of the equations governing the process).

Moreover, such equations have rich mathematical structure and therefore, they are very interesting and challenging from the mathematical point of view. We have numerous results in different directions beginning with the stochastic conservation laws [5, 6, 12, 13, 18, 19, 34], then velocity averaging results for stochastic transport equations [7, 25], stochastic degenerate parabolic equations [15, 36]. We remark that latter list of references is far from complete. As for the stochastic PDEs on manifolds, we mention [2] where the wave equation was considered.

Now we briefly recall the definition of the divergence on a manifold. We suppose that the map $(\mathbf{x}, \xi) \mapsto \mathfrak{f}(\mathbf{x}, \xi)$, $M \times \mathbb{R} \to TM$ is C^1 and that, for every $\xi \in \mathbb{R}$, $\mathbf{x} \mapsto \mathfrak{f}(\mathbf{x}, \xi) \in \mathfrak{X}(M)$ (the space of vector fields on M).

In local coordinates, we write

$$\mathfrak{f}(\mathbf{x}, \xi) = (f^1(\mathbf{x}, \xi), \dots, f^d(\mathbf{x}, \xi)).$$

The divergence operator appearing in the equation is to be formed with respect to the metric, so in local coordinates we have (cf. (10) below):

$$\operatorname{div}_g \mathfrak{f}(\mathbf{x}, u) = \operatorname{div}_g \left(\mathbf{x} \mapsto \mathfrak{f}(\mathbf{x}, u(t, \mathbf{x}))\right) = \frac{\partial}{\partial x_k}(f^k(\mathbf{x}, u(t, \mathbf{x})) + \Gamma^j_{kj}(\mathbf{x}) f^k(\mathbf{x}, u(t, \mathbf{x})) \tag{6}$$

where the Γ-terms are the Christoffel symbols of g and the Einstein summation convention is in effect.

As we can see, the divergence operator on manifolds is more involved than the one in Euclidean setting. Therefore, in order to prove uniqueness, we need to assume (3). Remark that (3) is the incompressibility condition from the fluid dynamics point of view, because, due to conservation of mass of an incompressible fluid, the density in a control volume changes according to the stochastic forcing

$$\frac{D\rho}{Dt} = \Phi(\mathbf{x}, \rho)\frac{dW_t}{dt} \tag{7}$$

where ρ is density of the control volume and $\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \frac{d\mathbf{x}}{dt}\cdot\nabla\rho$ is the material derivative for the flow velocity $\frac{d\mathbf{x}}{dt} = (\frac{dx_1}{dt}, \dots, \frac{dx_d}{dt})$. If we assume that the function ρ is smooth, we can rewrite equation (1) in the form

$$\frac{\partial\rho}{\partial t} + \partial_\xi\Big(\mathfrak{f}(\mathbf{x}, \xi)\Big)\Big|_{\xi=\rho}\cdot\nabla_g\rho + \operatorname{div}_g \mathfrak{f}(\mathbf{x}, \xi)\Big|_{\xi=\rho} = \Phi(\mathbf{x}, \rho)\frac{dW}{dt}. \tag{8}$$

Then, taking as usual $\frac{d\mathbf{x}}{dt} = \partial_\xi\Big(\mathfrak{f}(\mathbf{x}, \xi)\Big)\Big|_{\xi=\rho}$ and comparing (8) and (7), we arrive at

$$\operatorname{div}_g \mathfrak{f}(\mathbf{x}, \xi)\Big|_{\xi=\rho} = 0,$$

which immediately gives what is called the geometry compatibility condition.

Since the equation we consider is a nonlinear hyperbolic equation, its solution in general contains discontinuities and we need to pass to the weak solution concept. However, this induces uniqueness issues as one can in general construct several weak solutions satisfying the same initial data. Thus, in order to isolate the physically admissible one, we need to introduce entropy type admissibility conditions [22]. We will first derive them locally and then, using the geometry compatibility conditions, we shall show that the conditions hold globally as well.

Having the admissibility conditions, we can derive the kinetic formulation to (1) (see (33)). We will use it to prove the uniqueness to the considered Cauchy problem. The strategy of proof is adapted from [6]. We have tried to be as precise, self contained and intuitive as possible. We have therefore proven a simple corollary of the Itô lemma concerning the derivative of the product of two stochastic processes and derive the uniqueness proof first informally, and then also formally.

The paper is organized as follows. In Section 2 we introduce notions and notations from differential geometry and stochastic calculus. We then move on to derive the kinetic formulation of (1) and heuristically show how to get uniqueness to the solution. In Section 5, we formally prove the uniqueness result.

2. Preliminaries from Riemannian geometry and stochastic calculus

We shall split the section into two parts. In the first one, we will provide details from differential geometry, and in the second one, we recall necessary results from stochastic calculus.

2.1. Riemannian geometry

Our standard references for notions from Riemannian and distributional geometry are [17, 26, 27, 29]. As before, (M, g) will be a d-dimensional Riemannian manifold. If v is a distributional vector field on M then its gradient ∇v is the vector field metrically equivalent to the exterior derivative dv of v: $\langle\nabla v, X\rangle = dv(X) = X(v)$ for any $X \in \mathfrak{X}(M)$. In local coordinates,

$$\nabla v = g^{ij}\frac{\partial v}{\partial x^i}\partial_j, \tag{9}$$

with g^{ij} the inverse matrix to $g_{ij} = \langle\partial_{x^i}, \partial_{x^j}\rangle$.

As for the Laplace-Beltrami operator Δ_g on M, for a function $f \in C^2(M)$ in terms of local coordinates we have

$$\Delta_g f = \nabla_g^2 f = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right)$$

Finally, the divergence operator on M is locally defined via the Christofel symbols for a C^1 vector field on $X \in \mathcal{T}^1_0 = \mathfrak{X}(M)$ with local representation $X = X^i \frac{\partial}{\partial x^i}$:

$$\operatorname{div} X = \frac{\partial X^k}{\partial x^k} + \Gamma^j_{kj} X^k. \tag{10}$$

To proceed, we shall need basic notions from the Sobolev spaces on manifolds.

Since M is a compact manifold, we can define for a fixed $k \in \mathbb{N}$ (keeping in mind the Poincare inequality)

$$f \in H^k(M) \quad \Leftrightarrow \quad \|\nabla^k_g f\|_{L^2(M)} < \infty.$$

As for for the Sobolev spaces with negative indexes, we have

$$f \in H^{-k}(M) \;\text{ if }\; \exists F \in H^k(M) \;\text{ such that }\; \Delta^{2k} F = f$$

and we define

$$\|f\|_{H^{-k}(M)} = \|F\|_{H^k(M)}. \tag{11}$$

The spaces $H^k(M)$, $k \in \mathbb{Z}$, are Hilbert spaces and we denote by $\{e_k\}_{k\in\mathbb{N}}$ the orthogonal basis in $L^2(M)$ which is given as the set of eigenfunctions corresponding to the Laplace-Beltrami operator:

$$\Delta_g e_k(\mathbf{x}) = -\lambda_k e_k(\mathbf{x}).$$

At the same time, the set $\{e_k\}_{k\in\mathbb{N}}$ is the basis in $H^s(M)$, $s \in \mathbb{Z}$, according to the density arguments. We remark that it is usual to take the eigenvectors of the operator $(1 - \Delta_g)$ but since we are on the compact manifold, we can safely work with the simplified version.

Notice that if we have a function $g \in H^k(M)$ and we rewrite it in the basis $\{e_k/\|e_k\|_{H^k(M)}\}$:

$$g(\mathbf{x}) = \sum_{k=1}^{\infty} g_k e_k(\mathbf{x})/\|e_k\|_{H^k(M)} \tag{12}$$

then

$$g_k = \int_M g(\mathbf{x}) \frac{e_k(\mathbf{x})}{\|e_k\|_{H^k(M)}} d\mathbf{x} \tag{13}$$

which is easily obtained by multiplying (12) by $e_k/\|e_k\|_{H^k(M)}$, integrating the result over M and using the orthogonality of $\{e_k/\|e_k\|_{H^k(M)}\}$. Moreover,

$$\|g\|_{H^k(M)} = \sum_{k=1}^{\infty} g_k^2. \tag{14}$$

It is not difficult to notice that according to the definition of e_k and (11), we have

$$\|e_k\|_{L^2(M)} = \sqrt{\lambda_k}\|e_k\|_{H^{-1}(M)}. \tag{15}$$

Let us now recall basic notions from stochastic calculus.

2.2. *Stochastic calculus*

We start with the notion of predictability for the Hilbert-space valued stochastic processes.

Definition 2.1. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $\{\mathcal{F}_t\}_{t\in[0,T]}$, $T > 0$, be a filtration of the sigma algebra $\mathcal{F}$. Let V be a fixed Hilbert space with dual V^*.*

We say that the stochastic process $X : \Omega \times [0, T] \to V$ is adapted with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ if for every $\varphi \in V^*$ the stochastic process $\langle X(t), \varphi\rangle$ is measurable with respect to the σ–algebra $\mathcal{F}_t$ for any $t > 0$.

We note that in the latter definition we require the weak measurability of the mapping $X : \Omega \times [0, T] \to V$, but as we are going to deal with the Sobolev spaces $H^k(M)$, $k \in \mathbb{N} \cup \{0\}$ which are separable, the notions of weak and strong measurability coincide (see e.g. [23]). To this end, we use the following notations for $H^1(M)$ and $L^2(M)$-valued square integrable stochastic processes:

$$L^2_{\mathbf{P}}(\Omega; L^2((0, T); H^1(M))) = \{u : (0, T)\times M\times\Omega \to \mathbb{R} : \int_\Omega \int_0^T \|u(t, \cdot, \omega)\|^2_{H^1(M)} dt d\mathbf{P}(\omega) < \infty\}$$
$$L^2_{\mathbf{P}}(\Omega; L^2((0, T) \times M)) = \{u : (0, T)\times M\times\Omega \to \mathbb{R} : \int_\Omega \int_0^T \|u(t, \mathbf{x}, \omega)\|^2_{L^2(M)} dt d\mathbf{P}(\omega) < \infty\}$$

In both cases, the required measurability assumptions are tacitly assumed.

Let us now introduce the Itô lemma and some of its corollaries. To this end, let X_t be a stochastic process satisfying the following stochastic differential equation:

$$dX_t = \mu_1 dt + \sigma_1 dW_t. \tag{16}$$

We remark here that the latter equation is actually an informal way of expressing the integral equality

$$X_{t_0+s} - X_{t_0} = \int_{t_0}^{t_0+s} \mu_1 dt + \int_{t_0}^{t_0+s} \sigma_1 dW_t, \quad \forall t_0, s > 0. \tag{17}$$

By Itô's lemma, for each twice differentiable scalar function $f = f(t, z)$ the equation

$$df(X_t) = \left(\frac{\partial f}{\partial t} + \mu_1 \frac{\partial f}{\partial z} + \frac{\sigma_1^2}{2}\frac{\partial^2 f}{\partial z^2}\right) dt + \sigma_1 \frac{\partial f}{\partial z} dW_t \tag{18}$$

holds.

By taking $f(t, X_t) = X_t^2$, we get

$$dX_t^2 = 2\mu_1 X_t dt + \sigma_1^2 dt + 2\sigma_1 X_t dW_t. \tag{19}$$

Notice that $2\mu_1 X_t dt + 2\sigma_1 X_t dW_t = 2X_t(\mu_1 dt + \sigma_1 dW_t) = 2X_t dX_t$, so (19) becomes

$$dX_t^2 = 2X_t dX_t + \sigma_1^2 dt. \tag{20}$$

Similarly, if Y_t is a stochastic process satisfying the stochastic differential equation

$$dY_t = \mu_2 dt + \sigma_2 dW_t \tag{21}$$

then

$$dY_t^2 = 2Y_t dY_t + \sigma_2^2 dt, \tag{22}$$
$$d(X_t + Y_t)^2 = 2(X_t + Y_t)d(X_t + Y_t) + (\sigma_1 + \sigma_2)^2 dt. \tag{23}$$

The left-hand side of (23) is

$$\begin{aligned} d(X_t + Y_t)^2 &= d(X_t^2 + 2X_t Y_t + Y_t^2) = dX_t^2 + 2d(X_t Y_t) + dY_t^2 \\ &= 2X_t dX_t + \sigma_1^2 dt + 2d(X_t Y_t) + 2Y_t dY_t + \sigma_2^2 dt, \end{aligned} \tag{24}$$

and the right side is

$$2(X_t + Y_t)d(X_t + Y_t) + (\sigma_1 + \sigma_2)^2 dt = 2X_t dX_t + 2X_t dY_t + 2Y_t dX_t + 2Y_t dY_t + \sigma_1^2 dt + 2\sigma_1\sigma_2 dt + \sigma_2^2 dt. \quad (25)$$

By annuling the same terms on the left and right side respectively, and dividing the equation by 2, we get

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + \sigma_1\sigma_2 dt. \quad (26)$$

Let us finally recall the Itô isometry. The following equality holds

$$E\left[\left(\int_0^T X_t dW_t\right)^2\right] = E\left[\int_0^T X_t^2 dt\right].$$

3. Entropy admissibility and kinetic formulation

Let us first informally derive the admissibility conditions. As usual, we start with the parabolic approximation to (1)

$$du_\varepsilon + \operatorname{div}_g(\mathfrak{f}(\mathbf{x}, u_\varepsilon))dt = \Phi(\mathbf{x}, u_\varepsilon)dW_t + \varepsilon\Delta_g u_\varepsilon dt, \ \mathbf{x} \in M, \ t \in (0, T) \quad (27)$$

where, as before, $\mathfrak{f} = \mathfrak{f}(\mathbf{x}, \lambda) \in C^1(M \times \mathbb{R})$ and (M, g) is a d-dimensional Rimannian manifold with the metric g. We will assume that W_t is a Wiener process and $\Phi \in C_0^1(M \times \mathbb{R})$.

Let us recall the definition of the weak solution to (27), (2).

Definition 3.1. *We say that the measurable function $\Omega \ni \omega \mapsto u_\varepsilon(\cdot, \omega) \in L^2([0, T]; H^1(M))$ adapted with respect to the filtration $\{\mathcal{F}_t\}$ is the weak solution to* (27), (2) *if for a test function $\varphi \in C_0^2([0, T] \times M)$ it holds almost surely*

$$\int_0^T \int_M \left(u_\varepsilon \partial_t \varphi + \operatorname{div}_g(\mathfrak{f}(\mathbf{x}, u_\varepsilon))\nabla_g \varphi\right) d\mathbf{x}dt = \int_0^T \int_M \varphi\Phi(\mathbf{x}, u_\varepsilon)dW_t - \varepsilon \int_0^T \int_M u_\varepsilon \Delta_g \varphi d\mathbf{x}dt.$$

Existence of the solution to (27), (2) can be concluded from the general arguments given in [23]. One can also find a proof in [16].

Using the Itô formula, from (27) we get (here and in the sequel, we will set $\mathfrak{f}'(\mathbf{x}, \xi) = \partial_\xi \mathfrak{f}(\mathbf{x}, \xi)$):

$$\begin{aligned} d\theta(u_\varepsilon) = \Big(&- \theta'(u_\varepsilon)\mathfrak{f}'(\mathbf{x}, u_\varepsilon) \cdot \nabla_g u_\varepsilon + \theta'(u_\varepsilon) \operatorname{div}_g \mathfrak{f}(\mathbf{x}, \rho)\Big|_{\rho = u_\varepsilon} \\ &+ \varepsilon\Delta_g \theta(u_\varepsilon) - \varepsilon\theta''(u_\varepsilon)|\nabla_g u_\varepsilon|^2 + \frac{\Phi^2(\mathbf{x}, u_\varepsilon)}{2}\theta''(u_\varepsilon)\Big)dt + \Phi(\mathbf{x}, u_\varepsilon)\theta'(u_\varepsilon)dW_t \end{aligned} \quad (28)$$

for all twice differentiable scalar functions θ.

Using the standard approximation procedure and taking into account convexity of the function $\theta(u) = |u - \xi|_+ = \begin{cases} u - \xi, & u \geq \xi \\ 0, & else \end{cases}$, we know that we can safely plug it into (28). After letting $\varepsilon \to 0$ and assuming that $E(|u_\varepsilon(t, \mathbf{x}) - u(t, \mathbf{x})|) \to 0$ as $\varepsilon \to 0$, we get the following distributional inequality:

$$\begin{aligned} d|u - \xi|_+ \leq &-\mathfrak{f}'(\mathbf{x}, u)\nabla_g u \operatorname{sign}_+(u - \xi)dt + \theta'(u) \operatorname{div}_g \mathfrak{f}(\mathbf{x}, \rho)\Big|_{\rho = u} dt \\ &+ \frac{\Phi^2(\mathbf{x}, u)}{2}\delta(u - \xi)dt + \Phi(\mathbf{x}, u) \operatorname{sign}_+(u - \xi)dW_t. \end{aligned} \quad (29)$$

Taking into account the geometry compatibility condition (3), we have

$$\begin{aligned}&\mathfrak{f}'(\mathbf{x},u)\cdot(\nabla_g u)\operatorname{sign}_+(u-\xi)=\operatorname{div}_g\left(\operatorname{sign}_+(u-\xi)(\mathfrak{f}(\mathbf{x},u)-\mathfrak{f}(\mathbf{x},\xi))\right)\\&+\operatorname{sign}_+(u-\xi)\operatorname{div}_g\mathfrak{f}(\mathbf{x},\xi)=\operatorname{div}_g\left(\operatorname{sign}_+(u-\xi)(\mathfrak{f}(\mathbf{x},u)-\mathfrak{f}(\mathbf{x},\xi))\right),\end{aligned}\tag{30}$$

and using the Schwartz lemma on non-negative distributions, we conclude that there exists a non-negative stochastic kinetic measure m (to be precised later) such that the equation (29) can be written as

$$\begin{aligned}d|u-\xi|_+=&-\operatorname{div}_g(\operatorname{sign}_+(u-\xi)(\mathfrak{f}(\mathbf{x},u)-\mathfrak{f}(\mathbf{x},\xi)))dt+\frac{\Phi^2(\mathbf{x},u)}{2}\delta(u-\xi)dt\\&+\Phi(\mathbf{x},u)\operatorname{sign}_+(u-\xi)dW_t-dm(t,\mathbf{x},\xi)dt.\end{aligned}\tag{31}$$

Next, we find the partial derivative of the expression given in (31) with respect to ξ to get

$$\begin{aligned}d\partial_\xi|u-\xi|_+=&-\operatorname{div}_g(-\mathfrak{f}'(\mathbf{x},\xi)\operatorname{sign}_+(u-\xi))dt+\partial_\xi\left(\frac{\Phi^2(\mathbf{x},u)}{2}\delta(u-\xi)\right)dt\\&+\partial_\xi(\Phi(\mathbf{x},u)\operatorname{sign}_+(u-\xi)dW_t)-\partial_\xi dm.\end{aligned}\tag{32}$$

Introducing $h(t,x,\xi)=-\partial_\xi|u-\xi|_+=\operatorname{sign}_+(u-\xi)$ into (32) gives

$$dh+\operatorname{div}_g(\mathfrak{f}'(\mathbf{x},\xi)h)dt=-\partial_\xi\left(\frac{\Phi^2(\mathbf{x},u)}{2}\delta(u-\xi)\right)dt-\partial_\xi(\Phi(\mathbf{x},u)hdW_t)+\partial_\xi dm.\tag{33}$$

Notice that

$$\begin{aligned}&\partial_\xi(\Phi(\mathbf{x},u)hdW_t)=\partial_\xi(\Phi(\mathbf{x},u)\operatorname{sign}_+(u-\xi))dW_t=-\Phi(\mathbf{x},u)\delta(u-\xi)dW_t\\&=-\Phi(\mathbf{x},\xi)\delta(u-\xi)dW_t.\end{aligned}\tag{34}$$

Using $\frac{\Phi^2(\mathbf{x},u)}{2}\delta(u-\xi)=\frac{\Phi^2(\mathbf{x},\xi)}{2}\delta(u-\xi)$ and (34), and denoting the measure $-\partial_\xi h=\delta(u-\xi)$ by $\nu_{(t,\mathbf{x})}(\xi)$, we finally get the weak form of our equation:

$$dh+\operatorname{div}_g(\mathfrak{f}'(\mathbf{x},\xi)h)dt=-\partial_\xi\left(\frac{\Phi^2(\mathbf{x},\xi)}{2}\nu_{(t,\mathbf{x})}(\xi)\right)dt+\Phi(\mathbf{x},\xi)\nu_{(t,\mathbf{x})}(\xi)W_t+\partial_\xi dm\,.\tag{35}$$

We shall call the latter equation *the kinetic formulation* of (1).

It is important to notice that the function $\overline{h}=1-h$ satisfies

$$d\overline{h}+\operatorname{div}_g(\mathfrak{f}'(\mathbf{x},\xi)\overline{h})dt=\partial_\xi\left(\frac{\Phi^2(\mathbf{x},\xi)}{2}\nu_{(t,\mathbf{x})}(\xi)\right)dt-\Phi(\mathbf{x},\xi)\nu_{(t,\mathbf{x})}(\xi)dW_t-\partial_\xi dm.\tag{36}$$

We can now introduce a definition of an admissible solution. Let us first introduce what we meant under the stochastic measure here.

Definition 3.2. *We say that a mapping m from Ω into the space of Radon measures on $[0,T]\times M\times\mathbb{R}$ is a stochastic kinetic measure if:*

- *for every $\phi\in C_0([0,T]\times M\times\mathbb{R})$ the action $\langle m,\phi\rangle$ defines a $\mathbf{P}$-measurable function*

$$\langle m,\phi\rangle:\Omega\to R;$$

- *m vanishes for large ξ: if $B_R^c = \{\xi \in \mathbb{R} \mid |\xi| \geq R\}$, then*

$$\lim_{R\to\infty} Em(C_0([0,T]\times M\times B_R^c)) = 0$$

- *for every $\phi \in C_0(M\times\mathbb{R})$, the process*

$$t \mapsto \int_{[0,t]\times M\times\mathbb{R}} \phi(\mathbf{x},\xi)dm(s,\mathbf{x},\xi)$$

is predictable.

Definition 3.3. *The measurable function $u : [0,T]\times M\times\Omega \to \mathbb{R}$ almost surely continuous with respect to time in the sense that $u(\cdot,\cdot,\omega) \in C(\mathbb{R}^+; H^{-1}(M))$ for $\mathbf{P}$-almost every $\omega\in\Omega$, adapted with respect to the filtration $\{\mathcal{F}_t\}$, is an admissible stochastic solution to* (1), (2) *if*

- *there exists $C_2 > 0$ such that $E(\operatorname{esssup}_{t\in[0,T]} \|u(t)\|_{L^2(M)}) \leq C_2$;*
- *the kinetic function $h = \operatorname{sign}_+(u-\xi)$ adapted with respect to the filtration $\{\mathcal{F}_t\}$ satisfies* (31) *with the initial conditions $h(0,\mathbf{x},\xi) = \operatorname{sign}_+(u_0(\mathbf{x})-\xi)$ in the sense of weak traces and $\overline{h}$ satisfies* (36) *with the initial conditions $\overline{h}(0,\mathbf{x},\xi) = 1-\operatorname{sign}_+(u_0(\mathbf{x})-\xi)$ in the sense of weak traces.*

We shall also need a notion of the generalized stochastic kinetic solution.

Definition 3.4. *A measurable function $\omega \mapsto h(\cdot,\cdot,\cdot,\omega) \in L^2([0,T]\times M\times K)) \cap C_{LR}([0,T]; H^{-k}(M\times K))$ (with $C_{LR}(X)$ we denote the set of left and right continuous functions on X), for some $k\in\mathbb{N}$ and any $K \subset\subset \mathbb{R}$, adapted with respect to the filtration $\{\mathcal{F}_t\}$, bounded between zero and one and non-strictly decreasing with respect to $\xi\in\mathbb{R}$ such that $h = -\partial_\xi \nu_{(t,\mathbf{x})}$ is the generalized stochastic kinetic solution to* (1), (2) *if there exists a non-negative stochastic kinetic measure m such that h satisfies* (35) *and the initial conditions $h(0,\mathbf{x},\xi) = \operatorname{sign}_+(u_0(\mathbf{x})-\xi)$ in the sense of weak traces.*

Clearly, if we have the admissible solution to (1), (2) then we have the generalized stochastic kinetic solution as well. Interestingly, vice versa also holds which follows from the standard uniqueness arguments (see e.g. [6]). The concept of the generalized solution used here is essentially the same as the one from [14] except that we do not require boundedness of the p-moments, $p\in[1,\infty)$, of the measure $\nu_{t,\mathbf{x}}$ (see [14, Definition 3.3]). We note that the equation considered here is somewhat simpler than the one in [14] since we do not have cylindrical Wiener process and we require somewhat stricter conditions on the coefficients (compare in particular (4) and (5) here and [14, (2.1), (2.2), (2.3)]). Although insubstantial, the relaxation of the conditions seems sufficient to avoid additional requirements for the generalized stochastic kinetic solution from Definition 3.4.

4. Informal uniqueness proof – doubling of variables

In this section, we shall informally show how to get uniqueness. Formal proof does not essentially differ from the procedure given in this section but one needs to introduce several smoothing procedures which significantly complicates some steps of the proof. Therefore, for readers' convenience, in this section we essentially explain the basic ideas of the proof. We also remark that, in order to simplify the notation, we will denote by $d\mathbf{x}$ the measure on the manifold instead of usual $d\gamma(\mathbf{x})$.

Let $h^1(t,\mathbf{x},\xi)$ and $h^2(t,\mathbf{y},\zeta)$ be two different generalized kinetic solutions to (1), (2) (see Definition 3.4). Then

$$dh^1 + \operatorname{div}_g(\mathfrak{f}'(\mathbf{x},\xi)h^1)dt = -\partial_\xi\left(\frac{\Phi^2(\mathbf{x},\xi)}{2}\nu^1\right)dt + \Phi(\mathbf{x},\xi)\nu^1 dW_t + \partial_\xi dm_1, \tag{37}$$

$$d\overline{h^2} + \operatorname{div}_g(\mathfrak{f}'(\mathbf{y},\zeta)\overline{h^2})dt = \partial_\zeta\left(\frac{\Phi^2(\mathbf{y},\zeta)}{2}\nu^2\right)dt - \Phi(\mathbf{y},\zeta)\nu^2 dW_t - \partial_\zeta dm_2. \tag{38}$$

By (26), the following holds:

$$d(h^1\overline{h^2}) = h^1 d\overline{h^2} + \overline{h^2} dh^1 - \Phi(\mathbf{x},\xi)\Phi(\mathbf{y},\zeta)\nu^1 \otimes \nu^2 dt. \tag{39}$$

Multiplying (37) by $\overline{h^2} = \overline{h^2}(t,\mathbf{y},\zeta)$, (38) by $h^1 = h^1(t,\mathbf{x},\xi)$, adding them and using the geometry compatibility conditions (3), yields

$$\begin{aligned}
&\overline{h^2} dh^1 + h^1 d\overline{h^2} + \overline{h^2}\mathfrak{f}'(\mathbf{x},\xi)\cdot\nabla_{g,\mathbf{x}} h^1 dt + h^1 \mathfrak{f}'(\mathbf{y},\zeta)\cdot\nabla_{g,\mathbf{y}}\overline{h^2} dt\\
&= -\overline{h^2}\partial_\xi\left(\frac{\Phi^2(\mathbf{x},\xi)}{2}\nu^1\right)dt + h^1\partial_\zeta\left(\frac{\Phi^2(\mathbf{y},\zeta)}{2}\nu^2\right)dt + \overline{h^2}\Phi(\mathbf{x},\xi)\nu^1 dW_t - h^1\Phi(\mathbf{y},\zeta)\nu^2 dW_t\\
&+ \overline{h^2}\partial_\xi dm_1(t,\mathbf{x},\xi) - h^1\partial_\zeta dm_2(t,\mathbf{y},\zeta)dt.
\end{aligned} \tag{40}$$

Inserting (39) into (40), we get

$$\begin{aligned}
&d(h^1\overline{h^2}) + \Phi(\mathbf{x},\xi)\Phi(\mathbf{y},\zeta)\nu^1\otimes\nu^2 dt + \overline{h^2}\mathfrak{f}'(\mathbf{x},\xi)\cdot\nabla_{g,\mathbf{x}} h^1 dt + h^1 \mathfrak{f}'(\mathbf{y},\zeta)\cdot\nabla_{g,\mathbf{y}}\overline{h^2} dt\\
&= -\overline{h^2}\partial_\xi\left(\frac{\Phi^2(\mathbf{x},\xi)}{2}\nu^1\right)dt + h^1\partial_\zeta\left(\frac{\Phi^2(\mathbf{y},\zeta)}{2}\nu^2\right)dt + (\overline{h^2}\Phi(\mathbf{x},\xi)\nu^1 - h^1\Phi(\mathbf{y},\zeta)\nu^2)dW_t\\
&+ \overline{h^2}\partial_\xi dm_1(t,\mathbf{x},\xi)dt - h^1\partial_\zeta dm_2(t,\mathbf{y},\zeta)dt.
\end{aligned} \tag{41}$$

We now choose the non-negative test function $\varphi(t,\mathbf{x},\mathbf{y},\xi,\zeta) = \rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)$, where ρ and ψ are smooth non-negative functions defined on appropriate Euclidean spaces. Multiplying (41) with φ and integrating over $(0,T)\times M^2\times\mathbb{R}^2$ we get

$$\begin{aligned}
&\int_{M^2}\int_{\mathbb{R}^2} h^1(T,\mathbf{x},\xi)\overline{h^2}(T,\mathbf{y},\zeta)\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\zeta d\xi d\mathbf{y}d\mathbf{x}\\
&-\int_{M^2}\int_{\mathbb{R}^2} h^1_0\overline{h^2_0}\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\zeta d\xi d\mathbf{y}d\mathbf{x}\\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)\Phi(\mathbf{x},\xi)\Phi(\mathbf{y},\zeta)d\nu^2_{(t,\mathbf{y})}(\zeta)d\nu^1_{(t,\mathbf{x})}(\xi)d\mathbf{y}d\mathbf{x}dt\\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\mathfrak{f}'(\mathbf{x},\xi)\cdot\nabla_{g,\mathbf{x}}h^1(t,\mathbf{x},\xi)\overline{h^2}(t,\mathbf{y},\zeta)\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\zeta d\xi d\mathbf{y}d\mathbf{x}dt\\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\mathfrak{f}'(\mathbf{y},\zeta)\cdot\nabla_{g,\mathbf{y}}\overline{h^2}(t,\mathbf{y},\zeta)h^1(t,\mathbf{x},\xi)\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\zeta d\xi d\mathbf{y}d\mathbf{x}dt\\
&=\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\frac{\Phi^2(\mathbf{x},\xi)}{2}\overline{h^2}(t,\mathbf{y},\zeta)\rho(\mathbf{x}-\mathbf{y})\psi'(\xi-\zeta)d\nu^1_{(t,\mathbf{x})}(\xi)d\zeta d\mathbf{y}d\mathbf{x}dt\\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\frac{\Phi^2(\mathbf{y},\zeta)}{2}h^1(t,\mathbf{x},\xi)\rho(\mathbf{x}-\mathbf{y})\psi'(\xi-\zeta)d\nu^2_{(t,\mathbf{y})}(\zeta)d\xi d\mathbf{y}d\mathbf{x}dt\\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)\overline{h^2}(t,\mathbf{y},\zeta)\Phi(\mathbf{x},\xi)d\nu^1_{(t,\mathbf{x})}(\xi)d\zeta d\mathbf{y}d\mathbf{x}dW_t
\end{aligned} \tag{42}$$

$$
\begin{aligned}
&-\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)h^1(t,x,\xi)\Phi(\mathbf{y},\zeta)d\nu^2_{(t,\mathbf{y})}(\zeta)d\xi d\mathbf{y}d\mathbf{x}dW_t\\
&-\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\rho(\mathbf{x}-\mathbf{y})\psi'(\xi-\zeta)\overline{h^2}(t,\mathbf{y},\zeta)dm_1(t,\mathbf{x},\xi)d\zeta d\mathbf{y}\\
&-\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\rho(\mathbf{x}-\mathbf{y})\psi'(\xi-\zeta)h^1(t,\mathbf{x},\xi)dm_2(t,\mathbf{y},\zeta)d\xi d\mathbf{x}.
\end{aligned}
$$

By using integration by parts with respect to ζ and ξ in the first and second and in the last two terms on the right hand side in (42), and using $\partial_\xi h^1 = -\nu^1$ and $\partial_\zeta \overline{h^2} = \nu^2$, we obtain:

$$
\begin{aligned}
&\int_{M^2}\int_{\mathbb{R}^2}h^1(T,\mathbf{x},\xi)\overline{h^2}(T,\mathbf{y},\zeta)\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\zeta d\xi d\mathbf{y}d\mathbf{x} \qquad (43)\\
&-\int_{M^2}\int_{\mathbb{R}^2}h^1_0(\mathbf{x},\xi)\overline{h^2_0}(\mathbf{y},\zeta)\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\zeta d\xi d\mathbf{y}d\mathbf{x}\\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)\Phi(\mathbf{x},\xi)\Phi(\mathbf{y},\zeta)d\nu^2_{(t,\mathbf{y})}(\zeta)d\nu^1_{(t,\mathbf{x})}(\xi)d\mathbf{y}d\mathbf{x}dt\\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\mathfrak{f}'(\mathbf{x},\xi)\cdot\nabla_{g,\mathbf{x}}h^1(t,\mathbf{x},\xi)\overline{h^2}(t,\mathbf{y},\zeta)\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\zeta d\xi d\mathbf{y}d\mathbf{x}dt\\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\mathfrak{f}'(\mathbf{y},\zeta)\cdot\nabla_{g,\mathbf{y}}\overline{h^2}(t,\mathbf{y},\zeta)h^1(t,\mathbf{x},\xi)\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\zeta d\xi d\mathbf{y}d\mathbf{x}dt\\
&=\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\frac{\Phi^2(\mathbf{x},\xi)}{2}\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\nu^2_{(t,\mathbf{y})}(\zeta)d\nu^1_{(t,\mathbf{x})}(\xi)d\mathbf{y}d\mathbf{x}dt\\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\frac{\Phi^2(\mathbf{y},\zeta)}{2}\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\nu^2_{(t,\mathbf{y})}(\zeta)d\nu^1_{(t,\mathbf{x})}(\xi)d\mathbf{y}d\mathbf{x}dt\\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)\overline{h^2}(t,\mathbf{y},\zeta)\Phi(\mathbf{x},\xi)d\nu^1_{(t,\mathbf{x})}(\xi)d\zeta d\mathbf{y}d\mathbf{x}dW_t\\
&-\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)h^1(t,\mathbf{x},\xi)\Phi(\mathbf{y},\zeta)d\nu^2_{(t,\mathbf{y})}(\zeta)d\xi d\mathbf{y}d\mathbf{x}dW_t\\
&-\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)\nu^2_{(t,\mathbf{y})}(\zeta)dm_1(t,\mathbf{x},\xi)d\zeta d\mathbf{y}\\
&-\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)\nu^1_{(t,\mathbf{x})}(\xi)dm_2(t,\mathbf{y},\zeta)d\xi d\mathbf{x}.
\end{aligned}
$$

Finally, moving the third term on the left hand side in (43) to the right hand side and using non-negativity of the measures m_1 and m_2 yields

$$
\begin{aligned}
&\int_{M^2}\int_{\mathbb{R}^2} h^1(T,\mathbf{x},\xi)\overline{h^2}(T,\mathbf{y},\zeta)\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\zeta d\xi d\mathbf{y}d\mathbf{x} \\
&-\int_{M^2}\int_{\mathbb{R}^2} h^1_0(\mathbf{x},\xi)\overline{h^2_0}(\mathbf{y},\zeta)\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\zeta d\xi d\mathbf{y}d\mathbf{x} \\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2} \mathfrak{f}'(\mathbf{x},\xi)\cdot\nabla_{g,\mathbf{x}}h^1(t,\mathbf{x},\xi)\overline{h^2}(t,\mathbf{y},\zeta)\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\zeta d\xi d\mathbf{y}d\mathbf{x}dt \\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2} \mathfrak{f}'(\mathbf{y},\zeta)\cdot\nabla_{g,\mathbf{y}}\overline{h^2}(t,\mathbf{y},\zeta)h^1(t,\mathbf{x},\xi)\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\zeta d\xi d\mathbf{y}d\mathbf{x}dt \\
&\leq \frac{1}{2}\int_0^T\int_{M^2}\int_{\mathbb{R}^2} (\Phi(\mathbf{x},\xi)-\Phi(\mathbf{y},\zeta))^2\rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)d\nu^2_{(t,\mathbf{y})}(\zeta)d\nu^1_{(t,\mathbf{x})}(\xi)d\mathbf{y}d\mathbf{x}dt \\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2} \rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)\overline{h^2}(t,\mathbf{y},\zeta)\Phi(\mathbf{x},\xi)d\nu^1_{(t,\mathbf{x})}(\xi)d\zeta d\mathbf{y}d\mathbf{x}dW_t \\
&-\int_0^T\int_{M^2}\int_{\mathbb{R}^2} \rho(\mathbf{x}-\mathbf{y})\psi(\xi-\zeta)h^1(t,\mathbf{x},\xi)\Phi(\mathbf{y},\zeta)d\nu^2_{(t,\mathbf{y})}(\zeta)d\xi d\mathbf{y}d\mathbf{x}dW_t.
\end{aligned}
\tag{44}
$$

Setting $\psi(\xi)=\delta(\xi)$ and $\rho(\mathbf{x})=\delta(\mathbf{x})$ and rearranging it a bit, we obtain

$$
\begin{aligned}
&\int_M\int_{\mathbb{R}} h^1(T,\mathbf{x},\xi)\overline{h^2}(T,\mathbf{x},\xi)d\xi d\mathbf{x} \\
&\leq \int_M\int_{\mathbb{R}} h^1_0\overline{h^2_0}d\xi d\mathbf{x} - \int_0^T\int_M\int_{\mathbb{R}} \mathfrak{f}'(\mathbf{x},\xi)\cdot\nabla_{g,\mathbf{x}}(h^1(t,\mathbf{x},\xi)\overline{h^2}(t,\mathbf{x},\xi))d\xi d\mathbf{x}dt \\
&-\int_0^T\int_M\int_{\mathbb{R}} \Phi(\mathbf{x},\xi)\partial_\xi(h^1(t,\mathbf{x},\xi)\overline{h^2}(t,\mathbf{x},\xi))d\xi d\mathbf{x}dW_t.
\end{aligned}
\tag{45}
$$

Another integration by parts provides

$$
\begin{aligned}
&\int_M\int_{\mathbb{R}} h^1(T,\mathbf{x},\xi)\overline{h^2}(T,\mathbf{x},\xi)d\xi d\mathbf{x} \\
&\leq \int_M\int_{\mathbb{R}} h^1_0\overline{h^2_0}d\xi d\mathbf{x} + \int_0^T\int_M\int_{\mathbb{R}} \Phi'(\mathbf{x},\xi)h^1(t,\mathbf{x},\xi)\overline{h^2}(t,\mathbf{x},\xi)d\xi d\mathbf{x}dW(t)
\end{aligned}
\tag{46}
$$

where we used the geometry compatibility conditions to eliminate the flux term.

By using non-negativity of h^1 and $\overline{h^2}$, we have after finding expectation of square of (46) and taking into account the Itô isometry

$$E\left[\Big(\int_M\int_{\mathbb{R}} h^1(T,x,\xi)\overline{h^2}(T,\mathbf{x},\xi)d\xi d\mathbf{x}\Big)^2\right] \tag{47}$$
$$\lesssim E\left[\Big(\int_M\int_{\mathbb{R}} h_0^1\overline{h_0^2}d\xi d\mathbf{x}\Big)^2\right] + \|\Phi'\|_\infty^2 E\left[\int_0^T\Big(\int_M\int_{\mathbb{R}} h^1(t,\mathbf{x},\xi)\overline{h^2}(t,\mathbf{x},\xi)d\xi d\mathbf{x}\Big)^2 dt\right].$$

From here, using the Gronwall inequality, we get

$$E\left[\Big(\int_M\int_{\mathbb{R}} h^1(T,x,\xi)\overline{h^2}(T,\mathbf{x},\xi)d\xi d\mathbf{x}\Big)^2\right] \lesssim E\left[\Big(\int_M |u_{10}(\mathbf{x})-u_{20}(\mathbf{x})|d\mathbf{x}\Big)^2\right]. \tag{48}$$

From here, if assume that $u_{10}=u_{20}$, we get almost surely for almost every $(t,\mathbf{x},\xi)\in[0,\infty)\times M\times\mathbb{R}$:

$$h^1(t,\mathbf{x},\xi)\,(1-h^2(t,\mathbf{x},\xi))=0.$$

This implies that either $h^1(t,\mathbf{x},\xi)=0$ or $h^2(t,\mathbf{x},\xi)=1$. Since we can interchange the roles of h^1 and h^2, we conclude that 1 and 0 are actually the only values that h^1 or h^2 can attain and that $h^1=h^2=h$. Since h is also non-increasing with respect to ξ on $[0,\infty)$, we conclude (taking into account the initial data $h_0=\mathrm{sign}_+(u_0(\mathbf{x})-\xi)$) that there exists a function $u:[0,\infty)\times M\to\mathbb{R}$ such that

$$h(t,\mathbf{x},\xi)=\mathrm{sign}_+(u(t,\mathbf{x})-\xi). \tag{49}$$

We thus have the following corollary which is proven in the final section.

Corollary 4.1. *The generalized stochastic kinetic solution to* (1), (2) *has the form* (49). *If the function u satisfies the second item from Definition 3.3, then it is an admissible stochastic solution to* (1), (2).

5. Uniqueness – rigorous proof

In this section, we shall formalize the arguments from the previous section. To this end, it will be necessary to express (35) in local coordinates. So, assume we are given a generalized stochastic kinetic solution h. To prove uniqueness locally we take a chart (U,κ) for M and assume, without loss of generality, that $\kappa(U)=\mathbb{R}^d$. Define the local expression of h as the map (in order to avoid proliferation of symbols, we shall keep the same notations for global and local quantities but we shall write $\tilde{\mathbf{x}}$ to denote the local variable)

$$h:\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}\times\Omega\to\mathbb{R},\quad h(t,\tilde{\mathbf{x}},\xi,\omega)=h(t,\kappa^{-1}(\tilde{\mathbf{x}}),\xi,\omega)G(\tilde{\mathbf{x}}),$$

where $G(\tilde{\mathbf{x}})$ is the Gramian corresponding to the chart (U,κ). Similarly, for $\tilde{\mathbf{x}}\in\mathbb{R}^d$ we define

$$\begin{aligned}
\Phi(\tilde{\mathbf{x}},\xi)&=\Phi(\kappa^{-1}(\tilde{\mathbf{x}}),\xi),\\
\mathfrak{f}(\tilde{\mathbf{x}},\xi)&=\mathfrak{f}(\kappa^{-1}(\tilde{\mathbf{x}}),\xi),\ \ \mathfrak{f}'(\tilde{\mathbf{x}},\xi)=\mathfrak{f}'(\kappa^{-1}(\tilde{\mathbf{x}}),\xi)=a(\tilde{\mathbf{x}},\xi)\\
\nu_{(t,\tilde{\mathbf{x}})}(\lambda)&=\nu_{(t,\kappa^{-1}(\tilde{\mathbf{x}}))}(\lambda)G(\tilde{\mathbf{x}}),
\end{aligned} \tag{50}$$

and $m(t,\tilde{\mathbf{x}},\xi)$ will be the pushforward measure of m with respect to the mapping κ.

With such notations at hand, we now rewrite (35) locally in the chart (U,κ) into an equation in terms of $h_1(t,\tilde{\mathbf{x}},\xi)$ and $\overline{h_2}(t,\tilde{\mathbf{x}},\xi)$, which are two generalized kinetic solutions to Cauchy problems corresponding to (1) with the initial data u_{10} and u_{20}, respectively. Below, we use the Einstein summation convention and we remind that $a=(a_1,\dots,a_d)=\mathfrak{f}'=(f_1',\dots,f_d')$. Also, since the equations are to be understood in the weak sense, we need to add the Gramian in each of the terms below except in m_1 and m_2, since the corresponding part in these terms is implied there by the definition of the pushforward measure. This is why we introduce the conventions from (50).

$$
\begin{aligned}
&dh^1(t,\tilde{\mathbf{x}},\xi) + \operatorname{div}_{\tilde{\mathbf{x}}}(a(\tilde{\mathbf{x}},\xi)h^1)dt + h^1\Gamma^j_{kj}(\tilde{\mathbf{x}})a_k(t,\tilde{\mathbf{x}},\xi)dt \qquad (51)\\
&= -\partial_\xi\left(\frac{\Phi^2(\tilde{\mathbf{x}},\xi)}{2}\nu^1_{(t,\tilde{\mathbf{x}})}(\xi)\right)dt + \Phi(\tilde{\mathbf{x}},\xi)\nu^1_{(t,\tilde{\mathbf{x}})}(\xi)dW_t + \partial_\xi dm_1,\\
&d\overline{h^2}(t,\tilde{\mathbf{y}},\zeta) + \operatorname{div}_{\tilde{\mathbf{y}}}(a(\tilde{\mathbf{y}},\zeta)\overline{h^2})dt + \overline{h^2}\Gamma^j_{kj}(\tilde{\mathbf{y}})a_k(t,\tilde{\mathbf{y}},\zeta)dt \qquad (52)\\
&= \partial_\zeta\left(\frac{\Phi^2(\tilde{\mathbf{y}},\zeta)}{2}\nu^2_{(t,\tilde{\mathbf{y}})}(\zeta)\right)dt - \Phi(\tilde{\mathbf{y}},\zeta)\nu^2_{(t,\tilde{\mathbf{y}})}(\zeta)dW_t - \partial_\zeta dm_2
\end{aligned}
$$

We introduce two mollifying functions $\omega_1 \in \mathcal{D}(\mathbb{R}^d)$, $\omega_2 \in \mathcal{D}(\mathbb{R})$ where d is the dimension of the manifold $\mathbf{M}$, such that $\omega_i \geq 0$, $i = 1,2$ and $\int_{\mathbb{R}^d}\omega_1 = \int_{\mathbb{R}}\omega_2 = 1$. Taking $\omega_{\delta,r}(\tilde{\mathbf{x}},\xi,t) = \frac{1}{r\delta^d}\omega_1\left(\frac{\tilde{\mathbf{x}}}{\delta}\right)\omega_2\left(\frac{\xi}{r}\right)$, for some $\delta, r > 0$, and using convolution, (51) and (52) yield (below and in the sequel, subscripts δ and r denote convolution with respect to the corresponding variables):

$$
\begin{aligned}
&dh^1_{\delta,r} + \operatorname{div}_{\tilde{\mathbf{x}}}(a(\tilde{\mathbf{x}},\xi)h^1_{\delta,r})dt + g^1_{\delta,r}dt + \left(\Gamma^j_{kj}(\tilde{\mathbf{x}})a_k(t,\tilde{\mathbf{x}},\xi)h^1\right)_{\delta,r}dt \qquad (53)\\
&= -\partial_\xi\left(\frac{\Phi^2(\tilde{\mathbf{x}},\xi)}{2}\nu^1_{(t,\tilde{\mathbf{x}})}(\xi)dt\right)_{\delta,r} + (\Phi(\tilde{\mathbf{x}},\xi)\nu^1_{(t,\tilde{\mathbf{x}})}(\xi))_{\delta,r}dW_t + \partial_\xi dm_{1,\delta,r},\\
&d\overline{h^2_{\delta,r}} + \operatorname{div}_{\tilde{\mathbf{y}}}(a(\tilde{\mathbf{y}},\zeta)\overline{h^2_{\delta,r}})dt + g^2_{\delta,r}dt + \left(\Gamma^j_{kj}(\tilde{\mathbf{y}})a_k(t,\tilde{\mathbf{y}},\zeta)\overline{h^2}\right)_{\delta,r}dt \qquad (54)\\
&= \partial_\zeta\left(\frac{\Phi^2(\tilde{\mathbf{y}},\zeta)}{2}\nu^1_{(t,\tilde{\mathbf{y}})}(\zeta)dt\right)_{\delta,r} - (\Phi(\tilde{\mathbf{y}},\zeta)\nu^1_{(t,\tilde{\mathbf{y}})}(\zeta))_{\delta,r}dW_t - \partial_\zeta dm_{2,\delta,r}
\end{aligned}
$$

where

$$
\begin{aligned}
g^1_{\delta,r} &= \operatorname{div}_{\tilde{\mathbf{x}}}(a(\tilde{\mathbf{x}},\xi)h^1)_{\delta,r} - \operatorname{div}_{\tilde{\mathbf{x}}}(a(\tilde{\mathbf{x}},\xi)h^1_{\delta,r})\\
g^2_{\delta,r} &= \operatorname{div}_{\tilde{\mathbf{y}}}(a(\tilde{\mathbf{y}},\zeta)\overline{h^2})_{\delta,r} - \operatorname{div}_{\tilde{\mathbf{y}}}(a(\tilde{\mathbf{y}},\zeta)\overline{h^2_{\delta,r}}).
\end{aligned}
$$

These terms converge to zero as $\delta, r \to 0$ according to the Friedrichs lemma [30].

Now, multiplying (53) and (54) with $\overline{h^2_{\delta,r}} = \overline{h^2_{\delta,r}}(t,\mathbf{y},\zeta)$ and $h^1_{\delta,r} = h^1_{\delta,r}(t,\mathbf{x},\xi)$, respectively, and using (26), we obtain

$$
\begin{aligned}
&d(h^1_{\delta,r}\overline{h^2_{\delta,r}}) + (\Phi(\tilde{\mathbf{x}},\xi)\nu^1_{(t,\tilde{\mathbf{x}})}(\xi))_{\delta,r}(\Phi(\tilde{\mathbf{y}},\zeta)\nu^2_{(t,\mathbf{y})}(\zeta))_{\delta,r}dt \qquad (55)\\
&+ \overline{h^2_{\delta,r}}\operatorname{div}_{\tilde{\mathbf{x}}}(a(\tilde{\mathbf{x}},\xi)h^1_{\delta,r})dt + h^1_{\delta,r}\operatorname{div}_{\tilde{\mathbf{y}}}(a(\tilde{\mathbf{y}},\zeta)\overline{h^2_{\delta,r}})dt\\
&+ \left(\Gamma^j_{kj}(\tilde{\mathbf{x}})a_k(t,\tilde{\mathbf{x}},\xi)h^1\right)_{\delta,r}\overline{h^2_{\delta,r}}dt + \left(\Gamma^j_{kj}(\tilde{\mathbf{y}})a_k(t,\tilde{\mathbf{y}},\zeta)\overline{h^2}\right)_{\delta,r}h^1_{\delta,r}dt =\\
&- g^1_{\delta,r}\overline{h^2_{\delta,r}}dt - g^2_{\delta,r}h^1_{\delta,r}dt + \overline{h^2_{\delta,r}}(\Phi(\tilde{\mathbf{x}},\xi)\nu^1_{(t,\tilde{\mathbf{x}})}(\xi))_{\delta,r}dW_t - h^1_{\delta,r}(\Phi(\tilde{\mathbf{y}},\zeta)\nu^2_{(t,\tilde{\mathbf{y}})}(\zeta))_{\delta,r}dW_t\\
&+ h^1_{\delta,r}\partial_\zeta\left(\frac{\Phi^2(\tilde{\mathbf{y}},\zeta)}{2}\nu^2_{(t,\tilde{\mathbf{y}})}(\zeta)\right)_{\delta,r}dt - \overline{h^2_{\delta,r}}\partial_\xi\left(\frac{\Phi^2(\tilde{\mathbf{x}},\xi)}{2}\nu^1_{(t,\tilde{\mathbf{x}})}(\xi)\right)_{\delta,r}dt\\
&+ \overline{h^2_{\delta,r}}\partial_\xi dm_{1,\delta,r}(t,\tilde{\mathbf{x}},\xi)dt - h^1_{\delta,r}\partial_\zeta dm_{2,\delta,r}(t,\tilde{\mathbf{y}},\zeta)dt.
\end{aligned}
$$

Next, we choose non-negative functions $\rho \in \mathcal{D}(\mathbb{R}^d)$, $\psi, \varphi \in \mathcal{D}(\mathbb{R})$ such that $\int_{\mathbb{R}^d}\rho = \int_{\mathbb{R}}\psi = 1$. Using the test function $\rho_\varepsilon(\tilde{\mathbf{x}} - \tilde{\mathbf{y}})\psi_\varepsilon(\xi - \zeta)\varphi\left(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\right)$, with $\rho_\varepsilon(\tilde{\mathbf{x}}) = \frac{1}{\varepsilon^d}\rho\left(\frac{\tilde{\mathbf{x}}}{\varepsilon}\right)$, $\psi_\varepsilon(\xi) = \frac{1}{\varepsilon}\psi\left(\frac{\xi}{\varepsilon}\right)$, for some $\varepsilon > 0$, and integrating

(55) over $(0, T)$, the equation is rewritten in the variational formulation (recall that h^1 and h^2 are continuous with respect to $t \in \mathbb{R}^+$):

$$\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} h^1_{\delta,r}(T,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(T,\tilde{\mathbf{y}},\zeta)\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\right)d\xi d\zeta d\tilde{\mathbf{x}}d\tilde{\mathbf{y}} \tag{56}$$

$$-\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} h^1_{0,\delta,r}(\tilde{\mathbf{x}},\xi)\overline{h^2_{0,\delta,r}}(\tilde{\mathbf{y}},\zeta)\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\right)d\xi d\zeta d\tilde{\mathbf{x}}d\tilde{\mathbf{y}} \tag{57}$$

$$+\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\left(\overline{h^2_{\delta,r}}\operatorname{div}_{\tilde{\mathbf{x}}}(a(\tilde{\mathbf{x}},\xi)h^1_{\delta,r}) + h^1_{\delta,r}\operatorname{div}_{\tilde{\mathbf{y}}}(a(\tilde{\mathbf{y}},\zeta)\overline{h^2_{\delta,r}})\right)\times \tag{58}$$

$$\times\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\right)d\xi d\zeta d\tilde{\mathbf{x}}d\tilde{\mathbf{y}}dW_t$$

$$+\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\left(\left(\Gamma^j_{kj}(\tilde{\mathbf{x}})a_k(t,\tilde{\mathbf{x}},\xi)h^1\right)_{\delta,r}\overline{h^2_{\delta,r}} + \left(\Gamma^j_{kj}(\tilde{\mathbf{y}})a_k(t,\tilde{\mathbf{y}},\zeta)\overline{h^2}\right)_{\delta,r}h^1_{\delta,r}\right)\times \tag{59}$$

$$\times\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\right)d\xi d\zeta d\tilde{\mathbf{x}}d\tilde{\mathbf{y}}$$

$$=-\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\left(g^1_{\delta,r}\overline{h^2_{\delta,r}} + g^2_{\delta,r}h^1_{\delta,r} - \overline{h^2_{\delta,r}}(\Phi(\tilde{\mathbf{x}},\xi)\nu^1_{(t,\tilde{\mathbf{x}})}(\xi))_{\delta,r} + h^1_{\delta,r}(\Phi(\tilde{\mathbf{y}},\zeta)\nu^2_{(t,\tilde{\mathbf{y}})}(\zeta))_{\delta,r}\right)\times \tag{60}$$

$$\times\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\right)d\xi d\zeta d\tilde{\mathbf{x}}d\tilde{\mathbf{y}}dW_t$$

$$+\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\left(h^1_{\delta,r}\partial_\zeta\left(\frac{\Phi^2(\tilde{\mathbf{y}},\zeta)}{2}\nu^2_{(t,\tilde{\mathbf{y}})}(\zeta)\right)_{\delta,r} - \overline{h^2_{\delta,r}}\partial_\xi\left(\frac{\Phi^2(\tilde{\mathbf{x}},\xi)}{2}\nu^1_{(t,\tilde{\mathbf{x}})}(\xi)\right)_{\delta,r}\right. \tag{61}$$

$$\left.-(\Phi(\tilde{\mathbf{x}},\xi)\nu^1_{(t,\tilde{\mathbf{x}})}(\xi))_{\delta,r}(\Phi(\tilde{\mathbf{y}},\zeta)\nu^2_{(t,\mathbf{y})}(\zeta))_{\delta,r}\right)\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\right)d\xi d\zeta d\tilde{\mathbf{x}}d\tilde{\mathbf{y}}dt$$

$$+\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\left(\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\partial_\xi m_{1,\delta,r}(t,\tilde{\mathbf{x}},\xi) - h^1_{\delta,r}(t,\tilde{\mathbf{y}},\xi)\partial_\zeta m_{2,\delta,r}(t,\tilde{\mathbf{y}},\zeta)\right)\times \tag{62}$$

$$\times\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\right)d\xi d\zeta d\tilde{\mathbf{x}}d\tilde{\mathbf{y}}dt.$$

We shall analyze this equality term by term. We start with the terms from (56)–(58). We have:

$$\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} h^1_{\delta,r}(T,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(T,\tilde{\mathbf{y}},\zeta)\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\right)d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}} \tag{63}$$

$$-\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} h^1_{\delta,r}(0,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(0,\tilde{\mathbf{y}},\zeta)\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\right)d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}$$

$$-\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} a(\tilde{\mathbf{x}},\xi)h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\cdot\left[\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\right)\nabla\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\right.$$

$$\left.+\frac{1}{2}\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\nabla\varphi\left(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\right)\right]d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}dt$$

$$+\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} a(\tilde{\mathbf{y}},\zeta)h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\cdot\Big[\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)\nabla\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})$$

$$-\frac{1}{2}\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\nabla\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)\Big]d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}dt$$

$$=\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} h^1_{\delta,r}(T,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(T,\tilde{\mathbf{y}},\zeta)\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}-$$

$$-\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} h^1_{\delta,r}(0,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(0,\tilde{\mathbf{y}},\zeta)\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}$$

$$-\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}(a(\tilde{\mathbf{x}},\xi)-a(\tilde{\mathbf{y}},\zeta))\cdot\nabla\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}dt$$

$$-\frac{1}{2}\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}(a(\tilde{\mathbf{x}},\xi)+a(\tilde{\mathbf{y}},\zeta))\cdot\nabla\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}dt$$

The penultimate term in (63) can be rewritten as (below $dV = d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}dt$):

$$\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}(a(\tilde{\mathbf{x}},\xi)-a(\tilde{\mathbf{y}},\zeta))\cdot\nabla\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)dV= \tag{64}$$

$$\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}(a(\tilde{\mathbf{x}},\xi)-a(\tilde{\mathbf{y}},\zeta))\cdot\nabla\Big(\frac{1}{\varepsilon^d}\rho\Big(\frac{\tilde{\mathbf{x}}-\tilde{\mathbf{y}}}{\varepsilon}\Big)\Big)h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)dV=$$

$$\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}(a(\tilde{\mathbf{x}},\xi)-a(\tilde{\mathbf{y}},\zeta))\cdot\frac{1}{\varepsilon^d}\nabla\rho(\mathbf{z})\Big|_{\mathbf{z}=\frac{\tilde{\mathbf{x}}-\tilde{\mathbf{y}}}{\varepsilon}}h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)dV=$$

$$\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\frac{a(\tilde{\mathbf{x}},\xi)-a(\tilde{\mathbf{y}},\zeta)}{\varepsilon}\cdot\frac{1}{\varepsilon^d}\nabla\rho(\mathbf{z})\Big|_{\mathbf{z}=\frac{\tilde{\mathbf{x}}-\tilde{\mathbf{y}}}{\varepsilon}}h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)dV=$$

$$\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\frac{a_k(\varepsilon\mathbf{z}+\tilde{\mathbf{y}},\xi)-a_k(\tilde{\mathbf{y}},\zeta)}{\varepsilon z_k}z_k\partial_{z_k}\rho(\mathbf{z})h^1_{\delta,r}(t,\tilde{\mathbf{y}}+\varepsilon\mathbf{z},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\psi_\varepsilon(\xi-\zeta)\varphi\Big(\tilde{\mathbf{y}}+\frac{\varepsilon\mathbf{z}}{2}\Big)dV$$

where $\mathbf{z}=\frac{\tilde{\mathbf{x}}-\tilde{\mathbf{y}}}{\varepsilon}$. We notice that, as $r,\delta,\varepsilon\to 0$ (in any order), this term becomes

$$\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}}\partial_{\tilde{y}_k}a_k(\tilde{\mathbf{y}},\xi)h^1(t,\tilde{\mathbf{y}},\xi)\overline{h^2}(t,\tilde{\mathbf{y}},\xi)\varphi(\tilde{\mathbf{y}})\int_{\mathbb{R}^d}z_k\partial_{z_k}\rho(\mathbf{z})d\mathbf{z}\,d\xi d\tilde{\mathbf{y}}dt \tag{65}$$

$$=-\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}}\operatorname{div}_{\tilde{\mathbf{y}}}a(\tilde{\mathbf{y}},\xi)h^1(t,\tilde{\mathbf{y}},\xi)\overline{h^2}(t,\tilde{\mathbf{y}},\xi)\varphi(\tilde{\mathbf{y}})d\xi d\tilde{\mathbf{y}}dt$$

$$\overset{(3)}{=}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}}\Gamma^j_{kj}(\tilde{\mathbf{y}})a_k(t,\tilde{\mathbf{y}},\xi)h^1(t,\tilde{\mathbf{y}},\xi)\overline{h^2}(t,\tilde{\mathbf{y}},\xi)d\xi d\tilde{\mathbf{y}}dt. \tag{66}$$

due to properties of the mollifier ρ. Thus, from (65) and (63) we conclude that as $r, \delta, \varepsilon \to 0$ in any order

$$
\begin{aligned}
&(56) + (57) + (58) \underset{r,\delta,\varepsilon\to 0}{\longrightarrow} \\
&\int_{\mathbb{R}^d}\int_{\mathbb{R}} (h^1\overline{h^2})(T,\tilde{\mathbf{y}},\xi)\varphi(\tilde{\mathbf{x}})d\xi d\tilde{\mathbf{x}} - \int_{\mathbb{R}^d}\int_{\mathbb{R}} (h^1_0\overline{h^2_0})(\tilde{\mathbf{x}},\xi)\varphi(\tilde{\mathbf{x}})d\xi d\tilde{\mathbf{x}} \\
&- \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}} a(\tilde{\mathbf{x}},\xi)(h^1\overline{h^2})(t,\tilde{\mathbf{x}},\xi)\nabla\varphi(\tilde{\mathbf{x}})d\xi d\tilde{\mathbf{x}}dt - \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}} \Gamma^j_{kj}(\tilde{\mathbf{x}})a_k(t,\tilde{\mathbf{x}},\xi)h^1(t,\tilde{\mathbf{x}},\xi)\overline{h^2}(t,\tilde{\mathbf{x}},\xi)d\xi d\tilde{\mathbf{x}}dt.
\end{aligned} \tag{67}
$$

Term (59) is easy to handle. We simply let $r, \delta, \varepsilon \to 0$ to conclude

$$
(59) \underset{r,\delta,\varepsilon\to 0}{\longrightarrow} \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}} 2\Gamma^j_{kj}(\tilde{\mathbf{x}})a_k(t,\tilde{\mathbf{x}},\xi)h^1\overline{h^2}\varphi(\tilde{\mathbf{x}})d\xi d\tilde{\mathbf{x}}dt. \tag{68}
$$

In order to prepare handling (60) and (61), we use regularity of the function Φ (recall that $\Phi \in C^1_0(\mathbb{R}^d\times\mathbb{R})$). We have

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\Big[\Big(\frac{\Phi^2(\tilde{\mathbf{x}},\xi)}{2}\nu^1_{(t,\tilde{\mathbf{x}})}(\xi)\Big)_{\delta,r}\nu^2_{(t,\tilde{\mathbf{y}}),\delta,r}(\zeta) - \frac{\Phi^2(\tilde{\mathbf{x}},\xi)}{2}\nu^1_{(t,\tilde{\mathbf{x}}),\delta,r}(\xi)\nu^2_{(t,\tilde{\mathbf{y}}),\delta,r}(\zeta)\Big]\times \\
&\qquad\qquad \times\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}} \underset{r,\delta\to 0}{\longrightarrow} 0,
\end{aligned} \tag{69}
$$

and similarly

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\Big[\Big(\Phi(\tilde{\mathbf{x}},\xi)\nu^1_{(t,\tilde{\mathbf{x}})}(\xi)\Big)_{\delta,r}\Big(\Phi(\tilde{\mathbf{y}},\zeta)\nu^2_{(t,\tilde{\mathbf{y}})}(\zeta)\Big)_{\delta,r} - \Big(\Phi(\tilde{\mathbf{x}},\xi)\nu^1_{(t,\tilde{\mathbf{x}}),\delta,r}(\xi)\Big)\Big(\Phi(\tilde{\mathbf{y}},\zeta)\nu^2_{(t,\tilde{\mathbf{y}}),\delta,r}(\zeta)\Big)\Big]\times \\
&\qquad\qquad \times\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}dt \underset{r,\delta\to 0}{\longrightarrow} 0.
\end{aligned} \tag{70}
$$

In a similar fashion, we have

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\Big[\Big(\Phi(\tilde{\mathbf{x}},\xi)\nu^1_{(t,\tilde{\mathbf{x}})}(\xi)\Big)_{\delta,r}\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta) - \Big(\Phi(\tilde{\mathbf{y}},\zeta)\nu^2_{(t,\tilde{\mathbf{y}})}(\zeta)\Big)_{\delta,r}h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\Big]\times \\
&\qquad\qquad \times\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\right)d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}dW_t \\
&= \int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\Big[\Phi(\tilde{\mathbf{x}},\xi)\nu^1_{(t,\tilde{\mathbf{x}}),\delta,r}(\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta) - \Phi(\tilde{\mathbf{y}},\zeta)\nu^2_{(t,\tilde{\mathbf{y}}),\delta,r}(\zeta)h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\Big]\times \\
&\qquad\qquad \times\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\right)d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}dW_t + \int_0^T g_{3,\delta,r,\varepsilon}dW_t
\end{aligned}
$$

where

$$g_{3,\delta,r} = \int\limits_{\mathbb{R}^{2d}} \int\limits_{\mathbb{R}^2} \Big[\Big(\Big(\Phi(\tilde{\mathbf{x}},\xi) \nu^1_{(t,\tilde{\mathbf{x}})}(\xi) \Big)_{\delta,r} - \Phi(\tilde{\mathbf{x}},\xi) \Big(\nu^1_{(t,\tilde{\mathbf{x}})}(\xi) \Big)_{\delta,r} \Big) \overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta) - \Big(\Big(\Phi(\tilde{\mathbf{y}},\zeta) \nu^2_{(t,\tilde{\mathbf{y}})}(\zeta) \Big)_{\delta,r} - \Phi(\tilde{\mathbf{y}},\zeta) \Big(\nu^2_{(t,\tilde{\mathbf{y}})}(\zeta) \Big)_{\delta,r} \Big) h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi) \Big] \times \rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}}) \psi_\varepsilon(\xi-\zeta) \varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big) d\zeta d\xi d\tilde{\mathbf{y}} d\tilde{\mathbf{x}}$$

and $g_{3,\delta,r,\varepsilon} \to 0$ as $\delta, r \to 0$ almost surely. From here, using $\frac{dh^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)}{\partial\xi} = -\nu^1_{(t,\tilde{\mathbf{x}}),\delta,r}(\xi)$ and $\frac{d\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)}{\partial\zeta} = \nu^2_{(t,\tilde{\mathbf{y}}),\delta,r}(\zeta)$, integration by parts, we have the following conclusion for (60)

$$\int\limits_0^T \int\limits_{\mathbb{R}^{2d}} \int\limits_{\mathbb{R}^2} \Big[\Big(\Phi(\tilde{\mathbf{x}},\xi) \nu^1_{(t,\tilde{\mathbf{x}})}(\xi) \Big)_{\delta,r} \overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta) - \Big(\Phi(\tilde{\mathbf{y}},\zeta) \nu^2_{(t,\tilde{\mathbf{y}})}(\zeta) \Big)_{\delta,r} h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi) \Big] \times \rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}}) \psi_\varepsilon(\xi-\zeta) \varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big) d\zeta d\xi d\tilde{\mathbf{y}} d\tilde{\mathbf{x}} dW_t \underset{\varepsilon,r,\delta\to 0}{\longrightarrow} \int\limits_0^T \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}} \Phi'(\tilde{\mathbf{x}},\xi) h^1(t,\tilde{\mathbf{x}},\xi) \overline{h^2}(t,\tilde{\mathbf{x}},\xi) \varphi(\tilde{\mathbf{x}})\, d\xi d\tilde{\mathbf{x}} dW_t \tag{71}$$

where we used the procedure leading to (65).

Having in mind (69), (70), and (71), we conclude that (61) has the following asymptotics:

$$\begin{aligned}
&\int\limits_0^T \int\limits_{\mathbb{R}^{2d}} \int\limits_{\mathbb{R}^2} \Big(\frac{\Phi^2(\tilde{\mathbf{x}},\xi)}{2} \nu^1_{(t,\tilde{\mathbf{x}})}(\xi) \Big)_{\delta,r} \overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta) \rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}}) \psi'_\varepsilon(\xi-\zeta) \varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big) d\zeta d\xi d\tilde{\mathbf{y}} d\tilde{\mathbf{x}} dt + \\
&+ \int\limits_0^T \int\limits_{\mathbb{R}^{2d}} \int\limits_{\mathbb{R}^2} \Big(\frac{\Phi^2(\tilde{\mathbf{y}},\zeta)}{2} \nu^2_{(t,\tilde{\mathbf{y}})}(\zeta) \Big)_{\delta,r} h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi) \rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}}) \psi'_\varepsilon(\xi-\zeta) \varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big) d\zeta d\xi d\tilde{\mathbf{y}} d\tilde{\mathbf{x}} dt - \\
&- \int\limits_0^T \int\limits_{\mathbb{R}^{2d}} \int\limits_{\mathbb{R}^2} \Big(\Phi(\tilde{\mathbf{x}},\xi) \nu^1_{(t,\tilde{\mathbf{x}})}(\xi) \Big)_{\delta,r} \Big(\Phi(\tilde{\mathbf{y}},\zeta) \nu^2_{(t,\tilde{\mathbf{y}})}(\zeta) \Big)_{\delta,r} \times \\
&\qquad \times \rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}}) \psi_\varepsilon(\xi-\zeta) \varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big) d\zeta d\xi d\tilde{\mathbf{y}} d\tilde{\mathbf{x}} dt - \\
&- \int\limits_0^T \int\limits_{\mathbb{R}^{2d}} \int\limits_{\mathbb{R}^2} \Big(\Big(\Phi(\tilde{\mathbf{x}},\xi) \nu^1_{(t,\tilde{\mathbf{x}})}(\xi) \Big)_{\delta,r} \overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta) - \Big(\Phi(\tilde{\mathbf{y}},\zeta) \nu^2_{(t,\tilde{\mathbf{y}})}(\zeta) \Big)_{\delta,r} h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi) \Big) \times \\
&\qquad \times \rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}}) \psi_\varepsilon(\xi-\zeta) \varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big) d\zeta d\xi d\tilde{\mathbf{y}} d\tilde{\mathbf{x}} dW_t \underset{\varepsilon,r,\delta\to 0}{\longrightarrow} \\
&\lim_{\varepsilon\to 0} \frac{1}{2} \int\limits_0^T \int\limits_{\mathbb{R}^{2d}} \int\limits_{\mathbb{R}^2} (\Phi(\tilde{\mathbf{x}},\xi) - \Phi(\tilde{\mathbf{y}},\zeta))^2 \rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}}) \psi_\varepsilon(\xi-\zeta) \varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big) d\nu^2_{(t,\mathbf{y})}(\zeta) d\nu^1_{(t,\mathbf{x})}(\xi) d\mathbf{y} d\mathbf{x} dt \\
&\qquad - \int\limits_0^T \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}} \Phi'(\tilde{\mathbf{x}},\xi) \varphi(\tilde{\mathbf{x}}) h^1(t,\tilde{\mathbf{x}},\xi) \overline{h^2}(t,\tilde{\mathbf{x}},\xi) d\xi d\tilde{\mathbf{x}} dW_t
\end{aligned} \tag{72}$$

$$= -\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}} \Phi'(\tilde{\mathbf{x}},\xi)\varphi(\tilde{\mathbf{x}})h^1(t,\tilde{\mathbf{x}},\xi)\overline{h^2}(t,\tilde{\mathbf{x}},\xi)d\xi d\tilde{\mathbf{x}}dW_t. \tag{73}$$

Finally, we want to get rid of the entropy defect measures from (62). We use the fact that h^1 and h^2 are decreasing with respect to ξ (i.e. ζ) and that the measures m_1 and m_2 are non-negative. We have after two integration by parts (keep in mind that $\partial_\xi\psi(\xi-\zeta) = -\partial_\zeta\psi(\xi-\zeta)$)

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\Big(\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\partial_\xi m_{1,\delta,r}(t,\tilde{\mathbf{x}},\xi) - h^1_{\delta,r}(t,\tilde{\mathbf{y}},\xi)\partial_\zeta m_{2,\delta,r}(t,\tilde{\mathbf{y}},\zeta)\Big)\times \\
&\qquad\qquad \times\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)d\xi d\zeta d\tilde{\mathbf{x}}d\tilde{\mathbf{y}} \\
&= -\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\Big(\nu^2_{(t,\mathbf{y}),\varepsilon,\delta}(\zeta)m_{1,\delta,r}(t,\tilde{\mathbf{x}},\xi) + \nu^1_{(t,\mathbf{x}),\varepsilon,\delta}(\xi)m_{2,\delta,r}(t,\tilde{\mathbf{y}},\zeta)\Big)\times \\
&\qquad\qquad \times\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)d\xi d\zeta d\tilde{\mathbf{x}}d\tilde{\mathbf{y}} \leq 0.
\end{aligned} \tag{74}$$

Finally, from (67), (68), (71), (72), and (74), we conclude after letting $r,\delta,\varepsilon\to 0$ (first $r,\delta\to 0$ and then $\varepsilon\to 0$) that (56)–(62) becomes:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}} h^1(T,\tilde{\mathbf{x}},\xi)\overline{h^2}(T,\tilde{\mathbf{x}},\xi)\varphi(\tilde{\mathbf{x}})d\xi d\tilde{\mathbf{x}} + \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}}\Gamma^j_{kj}(\tilde{\mathbf{x}})a_k(t,\tilde{\mathbf{x}},\xi)h^1(t,\tilde{\mathbf{x}},\xi)\overline{h^2}(t,\tilde{\mathbf{x}},\xi)\varphi(\tilde{\mathbf{x}})d\xi d\tilde{\mathbf{x}}dt \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}} h^1_0\overline{h^2_0}\varphi(\tilde{\mathbf{x}})d\xi d\tilde{\mathbf{x}} + \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}} a(\tilde{\mathbf{x}},\xi)\cdot\nabla\varphi(\tilde{\mathbf{x}})(h^1\overline{h^2})(t,\tilde{\mathbf{x}},\xi)d\xi d\tilde{\mathbf{x}}dt \\
&+ \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}}\Phi'(\tilde{\mathbf{x}},\xi)h^1(t,\tilde{\mathbf{x}},\xi)\overline{h^2}(t,\tilde{\mathbf{x}},\xi)\varphi(\tilde{\mathbf{x}})d\xi d\tilde{\mathbf{x}}dW_t.
\end{aligned}$$

From here, using the definition of the integral over a manifold and recalling (50), we see that it holds

$$\begin{aligned}
&\int_M\int_{\mathbb{R}} h^1(T,\mathbf{x},\xi)\overline{h^2}(T,\mathbf{x},\xi)G(\kappa(\mathbf{x}))\varphi(\mathbf{x})d\xi d\mathbf{x} \\
&\leq \int_M\int_{\mathbb{R}} h^1_0(\mathbf{x},\xi)\overline{h^2_0}(\mathbf{x},\xi)G(\kappa(\mathbf{x}))\varphi(\mathbf{x})d\xi d\mathbf{x} - \int_0^T\int_M\int_{\mathbb{R}}(h^1\overline{h^2})(t,\mathbf{x},\xi)G(\kappa(\mathbf{x}))a(\mathbf{x},\xi)\cdot\nabla_g\varphi(\mathbf{x})d\xi d\mathbf{x}dt \\
&+ \int_0^T\int_M\int_{\mathbb{R}}\Phi'(\mathbf{x},\xi)(h^1\overline{h^2})(t,\mathbf{x},\xi)G(\kappa(\mathbf{x}))\varphi(\mathbf{x})d\xi d\mathbf{x}dW_t.
\end{aligned} \tag{75}$$

Since we are on the compact manifold, we can take $\varphi\equiv 1$ which yields:

$$\begin{aligned}
&\int\limits_M \int\limits_{\mathbb{R}} h^1(T,\mathbf{x},\xi)\overline{h^2}(T,\mathbf{x},\xi)G(\kappa(\mathbf{x}))d\xi d\mathbf{x} \qquad (76)\\
&\leq \int\limits_M \int\limits_{\mathbb{R}} h^1_0(\mathbf{x},\xi)\overline{h^2_0}(\mathbf{x},\xi)G(\kappa(\mathbf{x}))d\xi d\mathbf{x} - \int\limits_0^T \int\limits_M \int\limits_{\mathbb{R}} (h^1\overline{h^2})(t,\mathbf{x},\xi)G(\kappa(\mathbf{x}))a(\mathbf{x},\xi)\cdot\nabla_g 1\, d\xi d\mathbf{x} dt\\
&+ \int\limits_0^T \int\limits_M \int\limits_{\mathbb{R}} \Phi'(\mathbf{x},\xi)(h^1\overline{h^2})(t,\mathbf{x},\xi)G(\kappa(\mathbf{x}))d\xi d\mathbf{x} dW_t.
\end{aligned}$$

We arrived to (46) plus a term which does not affect using the Gronwall inequality and Itô isometry which give uniqueness as in (47). Remark that the Gramian has no influence on the procedure since it is a positive bounded function.

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