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On Kenmotsu Metric Spaces Satisfying Some Conditions on the W_1-Curvature Tensor

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Abstract. In this paper we present the curvature tensors of Kenmotsu manifold satisfying the conditions $W_1(X,Y)\cdot W_0 = 0$, $W_1(X,Y)\cdot W_1 = 0$, $W_1(X,Y)\cdot W_2 = 0$, $W_1(X,Y)\cdot W_3 = 0$, $W_1(X,Y)\cdot W_4 = 0$ and $W_1(X,Y)\cdot W_1^\star = 0$. According to these cases, Kenmotsu manifolds have been characterized. We consider that some interesting results on a Kenmotsu metric manifold are obtained.

1. Introduction

In 1972, K. Kenmotsu studied a class of contact Riemannian manifold and he called them as Kenmotsu manifold[8]. He proved that if a Kenmotsu manifold satisfies the condition R(X, Y).R = 0, then the manifold has negative curvature -1, where R is the Riemannian curvature tensor of type (1, 3) and R(X, Y) denotes the derivation of the tensor algebra at each point of the tangent space. The properties of Kenmotsu manifold have been studied by several authors such as A. Haseeb,[6], Y. Wang [20], [21], C. Özgür [10], M. M. Tripathi [18], R.N. Singh [15], D. G. Prakasha [13], U. C. De [4], K. De [3] and many others. Recently, some of these authors have worked as follows.

In 2013, On W2 -curvature tensor in a Kenmotsu manifold had been studied by R.N. Singh, S. K. Pandey and G. Pandey [15]. After, A. Haseeb examined the curvature tensor, the Ricci tensor and the scalar curvature in an $\epsilon-$Kenmotsu manifold with respect to the semi-symmetric metric connection. Also, he studied projectively flat and $\xi-$projectively flat $\epsilon-$Kenmotsu manifold with respect to the semi-symmetric metric connection. At the same time he investigated partially Ricci-pseudosymmetric $\epsilon-$Kenmotsu manifold with respect to the semi-symmetric metric connection and proved that such a manifold is an $\eta-$Einstein manifold[6].

Subsequently, D. G. Prakasha and B. S. Hadımanı researched a conharnanically flat Kenmotsu manifold with respect to the connection $\widetilde{\nabla}$. They studied locally $\phi-$conharmonically symmetric Kenmotsu manifold with respect to the connection $\widetilde{\nabla}$. Also, theiir study is devoted to Kenmotsu manifolds with respect to the

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connection $\widetilde{\nabla}$ satisfying the conditions $\widetilde{K}(\xi, X) \cdot R = 0$ and $\widetilde{P}(\xi, X) \cdot \widetilde{R} = 0$, respectively[13]. In addition, some authors study this topic in different manifolds.

The object of this paper is to study properties of some certain curvature tensors in a Kenmotsu metric manifold. Furthermore, we survey $W_1(X, Y) \cdot W_0 = 0$, $W_1(X, Y) \cdot W_1 = 0$, $W_1(X, Y) \cdot W_2 = 0$, $W_1(X, Y) \cdot W_3 = 0$, $W_1(X, Y) \cdot W_4 = 0$ and $W_1(X, Y) \cdot W_1^\star = 0$, where W_0, W_1, $W_1^\star$, W_2, W_3 and W_4 denote curvature tensors of manifold, respectively. Additionally, 3-dimensional Kenmotsu manifold example is given.

2. Preliminaries

Let M be a $(2n + 1)-$dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and the Riemanniann metric g on M satisfying the following conditions

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\phi X) = 0, \tag{1}$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi) = 0 \tag{2}$$

for all $X, Y \in \chi(M)$[9]. Let g be Riemannian metric compatible with (ϕ, ξ, η), that is

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y) \quad \text{and} \quad g(X, \xi) = \eta(X) \tag{4}$$

for all $X, Y \in \chi(M)$[2]. If in addition to above relations

$$(\nabla_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi, \tag{5}$$

and

$$\nabla_X \xi = X - \eta(X)\xi, \tag{6}$$

where ∇ denotes the Riemannian connection of g hold, then $M(\phi, \xi, \eta, g)$ is called an almost Kenmotsu manifold. An almost Kenmotsu manifold becomes a Kenmotsu manifold if

$$g(X, \phi Y) = d\eta(X, Y). \tag{7}$$

In a Kenmotsu manifold M, the following relations hold[4, 8]:

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{8}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{9}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{10}$$

$$S(X, \xi) = -(n - 1)\eta(X), \tag{11}$$

$$Q\xi = -(n - 1)\xi, \tag{12}$$

where R is the Riemannian curvature tensor and S is Ricci tensor defined by $S(X, Y) = g(QX, Y)$, where Q is Ricci operator. It yields to

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y). \tag{13}$$

Definition 2.1. *A Kenmotsu manifold M is said to be an $\eta-$Einstein manifold if its Ricci tensor S of the form*

$$S(X,Y) = \alpha g(X,Y) + \beta \eta(X)\eta(Y) \tag{14}$$

for arbitrary vector fields X, Y; where α and β are functions on (M^{2n+1}, g). If $\beta = 0$, then $\eta-$ Einstein manifold becomes Einstein manifold[8, 14].

Let M be a $(2n+1)-$dimensional Kenmotsu manifold. The curvature tensor $\widetilde{R}$ of M with respect to the connection $\widetilde{\nabla}$ is defined by

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z. \tag{15}$$

Then, in a Kenmotsu manifold, we have

$$\widetilde{R}(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y, \tag{16}$$

where $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, is the curvature tensor of M with respect to the connection ∇.

The Ricci tensor $\widetilde{S}$ and the scalar curvature $\widetilde{r}$ of the Kenmotsu manifold M with respect to the connection $\widetilde{\nabla}$ is given by

$$\widetilde{S}(X,Y) = \sum_{i=1}^{n} g(\widetilde{R}(e_i, X)Y, ei) = S(X,Y) + (n-1)g(X,Y) \tag{17}$$

and

$$\widetilde{r} = \sum_{i=1}^{n} \widetilde{S}(e_i, e_i) = r + n(n-1), \tag{18}$$

where $\widetilde{r}$ and r are the scalar curvatures of the connection $\widetilde{\nabla}$ and ∇, respectively[16, 17, 22].

The concept of W_0-curvature tensor was defined by [12]. W_0-curvature tensor, W_1-curvature tensor, $W_1^{\star}-$curvature tensor, W_2-curvature tensor, W_3 -curvature tensor and W_4-curvature tensor of a $(2n+1)$-dimensional Riemannian manifold are defined respectivelly as follows:

$$W_0(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - g(X,Z)QY], \tag{19}$$

$$W_1(X,Y)Z = R(X,Y)Z + \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{20}$$

$$W_1^{\star}(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{21}$$

$$W_2(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[g(Y,Z)QX - g(X,Z)QY, \tag{22}$$

$$W_3(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(X,Z)Y - g(Y,Z)QX], \tag{23}$$

$$W_4(X,Y)Z = R(X,Y)Z + \frac{1}{2n}[g(X,Z)QY - g(X,Y)QZ], \tag{24}$$

for all $X, Y, Z \in \chi(M)$[11, 12].

3. Some Curvature Results On Kenmotsu Metric Spaces

In this section, we will give the main results for this paper.

Let M be a $(2n+1)-$dimensional Kenmotsu metric manifold and we denote W_0 curvature tensor from (19), Therefore, we have

$$W_0(X,Y)\xi = \eta(X)Y - \frac{n+1}{2n}\eta(Y)X + \frac{1}{2n}\eta(X)QY. \tag{25}$$

Putting $X = \xi$ in (25), it is obtained

$$W_0(\xi,Y)\xi = Y - \frac{n+1}{2n}\eta(Y)\xi + \frac{1}{2n}QY. \tag{26}$$

In (20) choosing $Z = \xi$ and using (9), we obtain

$$W_1(X,Y)\xi = (\frac{3n-1}{2n})(\eta(X)Y - \eta(Y)X). \tag{27}$$

Setting $X = \xi$ in (20),

$$W_1(\xi,Y)Z = (\frac{3n-1}{2n})\eta(Z)Y - g(Y,Z)\xi + \frac{1}{2n}S(Y,Z)\xi \tag{28}$$

In (28), it follows

$$W_1(\xi,Y)\xi = \frac{3n-1}{2n}(Y - \eta(Y)\xi). \tag{29}$$

From (21) and (9), we acquire

$$W_1^{\star}(X,Y)\xi = \frac{n+1}{2n}(\eta(X)Y - \eta(Y)X). \tag{30}$$

and

$$W_1^{\star}(\xi,Y)\xi = \frac{n+1}{2n}(Y - \eta(Y)\xi). \tag{31}$$

In the same way, putting $Z = \xi$ in (22) and using (9), we have

$$W_2(X,Y)\xi = \eta(X)Y - \eta(Y)X - \frac{1}{2n}\{\eta(Y)QX - \eta(X)QY\}. \tag{32}$$

In (32), using $X = \xi$, we get

$$W_2(\xi,Y)\xi = Y - \frac{n+1}{2n}\eta(Y)\xi + \frac{1}{2n}QY. \tag{33}$$

Choosing $Z = \xi$, in (23), we obtain

$$W_3(X,Y)\xi = \frac{3n-1}{2n}\eta(X)Y - \eta(Y)X + \frac{1}{2n}\eta(Y)QX. \tag{34}$$

In (34) it follows

$$W_3(\xi,Y)\xi = \frac{3n-1}{2n}(Y - \eta(Y)\xi). \tag{35}$$

In (24), choosing $Z = \xi$ and using (9), we obtain

$$W_4(X,Y)\xi = \eta(X)Y - \eta(Y)X + \frac{1}{2n}\{\eta(X)QY + (n-1)g(X,Y)\xi\}. \tag{36}$$

Putting $X = \xi$ in (36),

$$W_4(\xi,Y)\xi = Y - \frac{n+1}{2n}\eta(Y)\xi + \frac{1}{2n}QY. \tag{37}$$

Theorem 3.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. Then, $W_1(X,Y)\cdot W_0 = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Suppose that $W_1(X,Y)\cdot W_0 = 0$. This implies that

$$\begin{aligned}(W_1(X,Y)W_0)(U,W)Z &= W_1(X,Y)W_0(U,W)Z - W_0(W_1(X,Y)U,W)Z\\ &\quad -W_0(U,W_1(X,Y)W)Z\\ &\quad -W_0(U,W)W_1(X,Y)Z = 0,\end{aligned} \tag{38}$$

for any $X,Y,U,W,Z \in \chi(M)$. Taking $X = Z = \xi$ in (38), making use of (25) and(28), for $A = -\frac{n+1}{2n}$, $B = \frac{1}{2n}$, $C = \frac{3n-1}{2n}$, we have

$$\begin{aligned}(W_1(\xi,Y)W_0)(U,W)\xi &= W_1(\xi,Y)(\eta(U)W + A\eta(W)U + B\eta(U)QW)\\ &\quad -W_0(C\eta(U)Y) - g(Y,U)\xi + BS(Y,U)\xi, W)\xi\\ &\quad -W_0(U, C\eta(W)Y - g(Y,W)\xi + BS(Y,W)\xi)\xi\\ &\quad -W_0(U,W)(CY - C\eta(Y)\xi) = 0.\end{aligned} \tag{39}$$

Taking into account (25), (26), (28) in (39), we obtain

$$\begin{aligned}&CW_0(U,W)Y + \eta(U)g(Y,W)\xi + BC(n-1)\eta(U)\eta(W)Y\\ &-B^2\eta(U)S(Y,QW)\xi - g(Y,U)W - Bg(Y,U)QW\\ &+BS(Y,U)W + B^2S(Y,U)QW + BC\eta(U)\eta(W)QY\\ &+g(Y,W)U + A\eta(U)g(Y,W)\xi + Bg(Y,W)QU\\ &-BS(Y,W)U - AB\eta(U)S(Y,W)\xi - B^2S(Y,W)QU = 0.\end{aligned} \tag{40}$$

Putting (19), (4), choosing $W = \xi$ and inner product both sides of (40) by $\xi \in \chi(M)$, we arrive

$$\begin{aligned}&-C\eta(Y)\eta(U) + Cg(U,Y) + BCS(U,Y) + 2\eta(Y)\eta(U)\\ &+BC(n-1)\eta(Y)\eta(U) + B^2(n-1)\eta(Y)\eta(U) - g(Y,U)\\ &+B(n-1)g(U,Y) + BS(Y,U) - B^2(n-1)S(Y,U)\\ &+A\eta(Y)\eta(U) + AB(n-1)\eta(Y)\eta(U) - B^2(n-1)^2\eta(Y)\eta(U) = 0.\end{aligned} \tag{41}$$

From (11), we conclude

$$S(U,Y) = 2(1-n)g(U,Y) + \frac{(1-n)}{2}\eta(U)\eta(Y).$$

So, M is an $\eta-$Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an $\eta-$Einstein manifold i.e. $S(U,Y) = 2(1-n)g(U,Y) + \frac{(1-n)}{2}\eta(U)\eta(Y)$, then from (41), (40), (39) and (38), we have $W_1(X,Y)\cdot W_0 = 0$. □

Theorem 3.2. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. Then, $W_1(X,Y)\cdot W_1 = 0$ if and only if M is an Einstein manifold.*

Proof. Let $W_1(X,Y)\cdot W_1 = 0$. This yields to

$$\begin{aligned}(W_1(X,Y)W_1)(U,W)Z &= W_1(X,Y)W_1(U,W)Z - W_1(W_1(X,Y)U,W)Z\\ &-W_1(U,W_1(X,Y)W)Z\\ &-W_1(U,W)W_1(X,Y)Z = 0,\end{aligned} \tag{42}$$

for any $X,Y,U,W,Z \in \chi(M)$. Taking $X = Z = \xi$ in (42) and using (27), (28), (29), for $A = \frac{3n-1}{2n}$, $B = \frac{1}{2n}$, we obtain

$$\begin{aligned}(W_1(\xi,Y)W_1)(U,W)\xi &= W_1(\xi,Y)(A\eta(U)W - A\eta(W)U) - W_1(A\eta(U)Y\\ &-g(Y,U)\xi + BS(Y,U)\xi, W)\xi - W_1(U,\eta(W)Y\\ &-g(Y,W)\xi + BS(Y,W)\xi)\xi - W_1(U,W)(AY\\ &-A\eta(Y)\xi) = 0\end{aligned} \tag{43}$$

and we arrive

$$\begin{aligned}&A\eta(U)W_1(\xi,Y)W - A\eta(W)W_1(\xi,Y)U - A\eta(U)W_1(Y,W)\xi\\ &+g(Y,U)W_1(\xi,W)\xi - BS(Y,U)W_1(\xi,W)\xi - A\eta(W)W_1(U,Y)\xi\\ &+g(Y,W)W_1(U,\xi)\xi - AW_1(U,W)Y + A\eta(Y)W_1(U,W)\xi = 0.\end{aligned} \tag{44}$$

Taking into account that (28), (27), (29) and inner product both sides of (44) by $Z \in \chi(M)$, we get

$$\begin{aligned}&g(AW_1(U,W)Y,Z) - A\eta(W)\eta(Z)g(Y,U) - Ag(Y,U)g(W,Z)\\ &+A\eta(W)\eta(Z)g(Y,U) + ABS(Y,U)g(W,Z) + Ag(Y,W)g(U,Z)\\ &-ABS(Y,W)g(U,Z) = 0.\end{aligned} \tag{45}$$

Making use of (11), (12) and choosing $U = Z = e_i$, ξ $1 \le i \le n$, for orthonormal basis of $\chi(M)$ in (45), we conclude

$$S(Y,W) = -2ng(Y,W).$$

Thus, M is an Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an Einstein manifold i.e. $S(Y,W) = -2ng(Y,W)$, then from equations (45), (44), (43) and (42), we have $W_1(X,Y)\cdot W_1 = 0$. Thus, this completes the proof. □

Theorem 3.3. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. Then, $W_1(X,Y)\cdot W_1^{\star} = 0$ if and only if M is an Einstein manifold.*

Proof. Suppose that $W_1(X,Y)\cdot W_1^{\star} = 0$. This yields to

$$\begin{aligned}(W_1(X,Y)W_1^{\star})(U,W)Z &= W_1(X,Y)W_1^{\star}(U,W)Z - W_1^{\star}(W_1(X,Y)U,W)Z\\ &-W_1^{\star}(U,W_1(X,Y)W)Z\\ &-W_1^{\star}(U,W)W_1(X,Y)Z = 0,\end{aligned} \tag{46}$$

for any $X,Y,U,W,Z \in \chi(M)$. Taking $X = Z = \xi$ in (46) and using (28), (30), (31), for $A = \frac{n+1}{2n}$, $B = -\frac{1}{2n}$, $C = \frac{3n-1}{2n}$, we obtain

$$\begin{aligned}(W_1(\xi,Y)W_1^{\star})(U,W)\xi &= W_1(\xi,Y)(A\eta(U)W - A\eta(W)U) - W_1^{\star}(C\eta(U)Y\\ &-g(Y,U)\xi - BS(Y,U)\xi, W)\xi - W_1^{\star}(U,\eta(W)Y\\ &-g(Y,W)\xi - BS(Y,W)\xi)\xi - W_1^{\star}(U,W)(CY\\ &-C\eta(Y)\xi) = 0\end{aligned} \tag{47}$$

and from (47), we arrive

$$A\eta(U)W_1(\xi,Y)W - A\eta(W)W_1(\xi,Y)U - C\eta(U)W_1^\star(Y,W)\xi$$
$$+g(Y,U)W_1^\star(\xi,W)\xi + BS(Y,U)W_1^\star(\xi,W)\xi - C\eta(W)W_1^\star(U,Y)\xi$$
$$+g(Y,W)W_1^\star(U,\xi)\xi - CW_1^\star(U,W)Y + C\eta(Y)W_1^\star(U,W)\xi = 0. \quad (48)$$

Taking into account that (21), in (48), we get

$$CW_1^\star(U,W)Y - Ag(Y,U)W - ABS(Y,U)W$$
$$+Ag(Y,W)U + ABS(Y,W)U = 0. \quad (49)$$

Setting $U = \xi$, using (11) and inner product both sides of (49) by $\xi \in \chi(M)$, we have

$$S(Y,W) = (2n+1)g(Y,W).$$

Thus, M is an Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an Einstein manifold, i.e. $S(Y,W) = (2n+1)g(Y,W)$, then from (49), (48), (47) and (46), we obtain $W_1(X,Y)\cdot W_1^\star = 0$. □

Theorem 3.4. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. Then, $W_1(X,Y)\cdot W_2 = 0$ if and only if M is an Einstein manifold.*

Proof. Let $W_1(X,Y)\cdot W_2 = 0$. This implies that

$$(W_1(X,Y)W_2)(U,W)Z = W_1(X,Y)W_2(U,W)Z - W_2(W_1(X,Y)U,W)Z$$
$$-W_2(U,W_1(X,Y)W)Z$$
$$-W_2(U,W)W_1(X,Y)Z = 0, \quad (50)$$

for any $X,Y,U,W,Z \in \chi(M)$. Setting $X = Z = \xi$ in (50) and making use of (32), (29), (28), for $A = -\frac{n+1}{2n}$, $B = -\frac{1}{2n}$, $C = \frac{3n-1}{2n}$, we obtain

$$(W_1(\xi,Y)W_2)(U,W)\xi = W_1(\xi,Y)(\eta(U)W - \eta(W)U) + B\eta(W)QU$$
$$-B\eta(U)QW) - W_2(C\eta(U)Y - g(Y,U)\xi$$
$$-BS(Y,U)\xi, W) - W_2(U,\eta(W)Y - g(Y,W)\xi$$
$$-BS(Y,W)\xi)\xi - W_2(U,W)(CY - C\eta(Y)\xi) = 0. \quad (51)$$

Using of (28), (29), (32)and (51), we get

$$CW_2(U,W)Y + \eta(U)g(Y,W)\xi - \eta(W)g(Y,U)\xi + B^2\eta(W)S(Y,QU)\xi$$
$$-B^2\eta(U)S(Y,QW)\xi - g(Y,U)W - A\eta(W)g(Y,U)\xi + Bg(Y,U)QW$$
$$-BS(Y,U)W - AB\eta(W)S(Y,U)\xi + B^2S(Y,U)QW + g(Y,W)U$$
$$+A\eta(U)g(Y,W)\xi - Bg(Y,W)QU + BS(Y,W)U$$
$$+A\eta(U)S(Y,W)\xi - B^2S(Y,W)QU = 0. \quad (52)$$

Inner product both sides of (52) by $\xi \in \chi(M)$ and using $U = \xi$, putting (22), we have

$$C\eta(W)\eta(Y) + CAg(W,Y) + BC(n-1)\eta(W)\eta(Y)$$
$$-\eta(W)\eta(Y) - B^2(n-1)\eta(W)\eta(Y) - B^2S(Y,QW)$$
$$-\eta(W)\eta(Y) - A\eta(W)\eta(Y) + AB(n-1)\eta(W)\eta(Y)$$
$$+B^2(n-1)^2\eta(W)\eta(Y) + 2g(Y,W) + Ag(W,Y)$$
$$+B(n-1)g(W,Y) + BS(Y,W) + AS(Y,W)$$
$$+B^2(n-1)S(Y,W) = 0. \quad (53)$$

From (53) and by using (11), for the sake of brevity, we set

$$\begin{aligned} E &= 2n(3n-1), \\ F &= n+1, \\ D &= 0 \end{aligned}$$

and we arrive

$$FS(Y,W) = Eg(Y,W) + D\eta(W)\eta(Y),$$

Thus, M is an Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an Einstein manifold, i.e. $FS(Y,W) = Eg(Y,W) + D\eta(W)\eta(Y)$, $(F \neq 0)$, then from equations (53), (52), (51) and (50), we obtain $W_1(X,Y)\cdot W_2 = 0$, which verifies our assertion. □

Theorem 3.5. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. Then, $W_1(X,Y)\cdot W_3 = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Let $W_1(X,Y)\cdot W_3 = 0$. This means that

$$\begin{aligned} (W_1(X,Y)W_3)(U,W,Z) &= W_1(X,Y)W_3(U,W)Z - W_3(W_1(X,Y)U,W)Z \\ &\quad -W_3(U,W_1(X,Y)W)Z \\ &\quad -W_3(U,W)W_1(X,Y)Z = 0, \end{aligned} \tag{54}$$

for any $X,Y,U,W,Z \in \chi(M)$. Setting $X = Z = \xi$ in (54) and making use of (34), (28), for $A = \frac{3n-1}{2n}$, $B = \frac{1}{2n}$, we obtain

$$\begin{aligned} (W_1(\xi,Y)W_3)(U,W)\xi &= W_1(\xi,Y)(A\eta(U)W - \eta(W)U + B\eta(W)QU) \\ &\quad -W_3(A\eta(U)Y - g(Y,U)\xi + BS(Y,U)\xi, W)\xi \\ &\quad -W_3(U,\eta(W)Y - g(Y,W)\xi + BS(Y,W)\xi)\xi \\ &\quad -W_3(U,W)(AY - A\eta(Y)\xi) = 0. \end{aligned} \tag{55}$$

Using (34), (35), (28) in (55), we get

$$\begin{aligned} &AW_3(U,W)Y - \eta(W)g(Y,U)\xi + B\eta(W)S(Y,U)\xi \\ &+AB(n-1)\eta(W)\eta(U)Y + B\eta(W)S(Y,U)\xi \\ &+AB\eta(U)\eta(W)QY - Ag(Y,U)W - B^2\eta(W)S(Y,QU)\xi \\ &+ABS(Y,U)W - AB\eta(W)S(Y,U)\xi + Ag(Y,W)U \\ &+A\eta(W)g(U,Y)\xi - ABS(Y,W)U = 0. \end{aligned} \tag{56}$$

Making use of (23), choosing $U = \xi$, and inner product both sides of (56) by $\xi \in \chi(M)$, we have

$$\begin{aligned} &A^2\eta(Y)\eta(W) - A^2g(Y,W) - \eta(Y)\eta(W) - 2B(n-1)\eta(Y)\eta(W) \\ &+B^2(n-1)\eta(Y)\eta(W) - ABS(Y,W) = 0. \end{aligned} \tag{57}$$

From (57) and by using (11), we conclude

$$S(Y,W) = (\frac{4n-3n^2-1}{3n-1})g(Y,W) + (\frac{n^2-2n}{3n-1})\eta(Y)\eta(W).$$

This tell us, M is an η-Einstein manifold. Conversely, let M be an η-Einstein manifold, i.e. $S(Y,W) = (\frac{4n-3n^2-1}{3n-1})g(Y,W) + (\frac{n^2-2n}{3n-1})\eta(Y)\eta(W)$, then from (54), (55), (56) and (57), we have $W_1(X,Y)\cdot W_3 = 0$. This completes of the proof. □

Theorem 3.6. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. Then, $W_1(X,Y)\cdot W_4 = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Suppose that $W_1(X,Y)\cdot W_4 = 0$. This implies that

$$\begin{aligned}(W_1(X,Y)W_4)(U,W,Z) &= W_1(X,Y)W_4(U,W)Z - W_4(W_1(X,Y)U,W)Z \\ &-W_4(U,W_1(X,Y)W)Z \\ &-W_4(U,W)W_1(X,Y)Z = 0, \end{aligned} \tag{58}$$

for any $X,Y,U,W,Z \in \chi(M)$. Setting $X = Z = \xi$ in (58) and making use of (36), (28), (29), for $A = \frac{n-1}{2n}$, $B = \frac{n-1}{2n}$, $C = \frac{3n-1}{2n}$, $D = \frac{1}{2n}$, we obtain

$$\begin{aligned}(W_1(\xi,Y)W_4)(U,W)\xi &= W_1(\xi,Y)(\eta(U)W - \eta(W)U + D\eta(U)QW \\ &+Ag(U,W)\xi) - W_4(C\eta(U)Y - g(Y,U)\xi \\ &+DS(Y,U)\xi,W)\xi - W_4(U,C\eta(W)Y \\ &-g(Y,W)\xi + DS(Y,W)\xi)\xi \\ &-W_4(U,W)(CY - C\eta(Y)\xi) = 0. \end{aligned} \tag{59}$$

Using (36) and (37) in (59), we get

$$\begin{aligned}&CW_4(U,W)Y - \eta(U)g(Y,W)\xi + D\eta(U)S(Y,W)\xi \\ &+\eta(W)g(Y,U)\xi - D\eta(W)S(Y,U)\xi - DC(n-1)\eta(U)\eta(W)Y \\ &-D\eta(U)S(Y,W)\xi + D^2\eta(U)S(Y,QW)\xi + ACg(U,W)Y \\ &-AC\eta(U)g(Y,W)\xi + g(U,Y)W - B\eta(W)g(Y,U)\xi \\ &+Dg(Y,U)QW - DS(Y,U)W + BD\eta(W)S(Y,U)\xi \\ &-D^2S(Y,U)QW - CD\eta(W)\eta(U)QY - AC\eta(W)g(U,Y)\xi \\ &-g(Y,W)U + B\eta(U)g(Y,W)\xi - Dg(Y,W)QU \\ &+DS(Y,W)U - BD\eta(U)S(Y,W)\xi + D^2S(Y,W)QU = 0. \end{aligned} \tag{60}$$

Making use of (24) and choosing $W = \xi$ and inner product both sides of in (60) by $\xi \in \chi(M)$, we have

$$\begin{aligned}&C\eta(Y)\eta(U) - Cg(Y,U) - 2\eta(Y)\eta(U) + 2g(Y,U) \\ &-DS(Y,U) - D^2(n-1)\eta(Y)\eta(U) - Bg(Y,U) \\ &-D(n-1)g(Y,U) - DS(Y,U) + BDS(Y,U) \\ &+D^2(n-1)S(Y,U) - ACg(U,Y) + B\eta(Y)\eta(U) \\ &+BD(n-1)\eta(Y)\eta(U) + D^2(n-1)^2\eta(Y)\eta(U) = 0. \end{aligned} \tag{61}$$

From (61) and (11), we obtain

$$S(Y,U) = (\frac{6n-5n^2-1}{2n})g(Y,U) + (\frac{1-2n^2-3n}{2n})\eta(Y)\eta(U).$$

Thus, M is an $\eta-$Einstein manifold. Conversely, let M be an $\eta-$Einstein manifold, i.e. $S(Y,U) = (\frac{6n-5n^2-1}{2n})g(Y,U) + (\frac{1-2n^2-3n}{2n})\eta(Y)\eta(U)$, then from (61), (60), (59) and (58), we obtain $W_1(X,Y)\cdot W_4 = 0$. So, this completes of the proof. □

Example 3.7. *We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3,\ z \neq 0\}$, where (x,y,z) are standart coordinates of $\mathbb{R}^3$. The vector fields*

$$e_1 = e^z\frac{\partial}{\partial x}, \quad e_2 = e^z\frac{\partial}{\partial y}, \quad e_3 = -\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1,e_2) &= g(e_1,e_3) = g(e_2,e_3) = 0, \\ g(e_1,e_1) &= g(e_2,e_2) = g(e_3,e_3) = 1. \end{aligned}$$

Let η be the 1-form defined by $\eta(X) = g(X,e_3)$ for any $X \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by

$$\phi(\frac{\partial}{\partial x}) = -\frac{\partial}{\partial y}, \qquad \phi(\frac{\partial}{\partial y}) = \frac{\partial}{\partial x}, \qquad \phi(\frac{\partial}{\partial z}) = 0$$

Thus,

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0,$$

for any vector field $X = \lambda_1\frac{\partial}{\partial x} + \lambda_2\frac{\partial}{\partial y} + \lambda_3\frac{\partial}{\partial z} \in \chi(T\mathbb{R}^3)$ than we have

$$g(X,X) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad g(\phi X,\phi X) = \lambda_1^2 + \lambda_2^2$$

and

$$\begin{aligned} \phi^2 X &= -\lambda_1\frac{\partial}{\partial x} - \lambda_2\frac{\partial}{\partial y} = -X + \eta(X)e_3 \\ \eta(e_3) &= 1, \\ g(\phi X,\phi X) &= g(X,X) - \eta(X)\eta(X) \end{aligned}$$

for any $X \in \chi(M)$.

Then, for $e_3 = \xi$, the structure (ϕ,ξ,η,g) defines an almost contact metric structure on M. Let ∇ be the Levi-Civita connection with respect to the metric tensor g. Then, therefore, we get

$$[e_3,e_1] = -e_1, \quad [e_1,e_2] = 0, \quad [e_2,e_3] = e_2.$$

The Levi-Civita connection ∇ of the metric g is given by the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y,Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

By using the above formula, we obtain

$$\begin{aligned} \nabla_{e_1}e_1 &= -e_1, \quad \nabla_{e_2}e_1 = 0, \quad \nabla_{e_3}e_1 = 0, \\ \nabla_{e_1}e_2 &= 0, \quad \nabla_{e_2}e_2 = -e_3, \quad \nabla_{e_3}e_2 = 0, \\ \nabla_{e_1}e_3 &= e_1, \quad \nabla_{e_2}e_3 = e_2, \quad \nabla_{e_3}e_3 = 0. \end{aligned}$$

From the above properties the manifold satisfies $\nabla_X\xi = X - \eta(X)\xi$, for $\xi = e_3$. Hence, the manifold is a Kenmotsu manifold. Using the formula $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, we calculate the following expressions:

$$\begin{aligned} R(e_1,e_2)e_3 &= 0, \quad R(e_2,e_3)e_3 = -e_2, \quad R(e_1,e_3)e_3 = -e_1, \\ R(e_1,e_2)e_2 &= -e_1, \quad R(e_2,e_3)e_2 = -e_3, \quad R(e_1,e_3)e_2 = 0, \\ R(e_1,e_2)e_1 &= 0, \quad R(e_2,e_3)e_1 = 0, \quad R(e_1,e_3)e_1 = e_1. \end{aligned}$$

From the above expressions of the curvature tensor R, we obtain that

$$\begin{aligned} S(e_1,e_1) &= g(R(e_1,e_2)e_2,e_1) + g(R(e_1,e_3)e_3,e_1) \\ &= -2. \end{aligned}$$

Similarly, we have

$$S(e_2, e_2) = S(e_3, e_3) = -2.$$

So,

$$r = \sum_{i=1}^{3} S(e_i, e_i) = -6.$$

We note that, here , r is a constant. Therefore, it can be easily verified that the manifold is an Einstein manifold with respect to Levi-Civita connection.

References

[1] M. Atçeken and P. Uygun, *Characterizations for totally geodesic submanifolds of (k,μ)-paracontact metric manifolds*, Korean J. Math. 28(2020), 555-571.

[2] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Note in Mathematics, 509, Springer-Verlag, Berlin-New York, (1976).

[3] K. De and U. C. De, *Conharmonic curvature tensor on Kenmotsu manifolds*, In Bulletin of the Transilvania University of Brasov, Vol. 6(55), (2013), No. 1.

[4] U. C. De and G. Pathak, *On 3-dimensional Kenmotsu manifolds*, Indian J. pure Appl. Math., 35, (2004), 159-165.

[5] S. Dirik, M. Atçeken and Ü. Yıldırım, *On contact CR-Submanifolds of a Kenmotsu manifold*, Turk.J.Math. Comput. Sci. 10 (2018),153-159.

[6] A. Haseeb, *Some results on projective on projective curvature tensor in an $\epsilon-$Kenmotsu manifold*, Plestine Journal of Math. 6 (2017), 196-203.

[7] S. Ianuş and D. Smaranda, *Some remarkable structures on the products of an almost contact metric manifold with the real line*, Papers from the National Coll. on Geometry and Topology, Univ. Timişoara, (1997), 107-110.

[8] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J., 24 (1972), 93-103.

[9] R. S. Mishra, *Structure on a Differentiable manifold and their applications*, Chandrama Prakashan, 50-A Bairampur House Allahabad (1984).

[10] C. Özgür and U.C. De, *On the quasi conformal curvature tensor of a Kenmotsu manifold*, Mathematica Pannonica, 17 (2) (2006), 221-228.

[11] G.P. Pokhariyal and R.S.Mishra, *Curvature tensors and their relativistic significance*, Yokohama Math. J., 18 (1970), 105-108.

[12] G.P. Pokhariyal and R.S.Mishra, *Curvature tensors and their relativistic significance II*, Yokohama Math. J., 19(2) (1971), 97-103.

[13] D. G. Prakasha and B. S. Hadımanı, *On the conharmonic curvature tensor of Kenmotsu manifolds with generalized Tanaka-Webster connection*, Miskole Mathematical Notes, Vol. 19(1) (2018), 491-503.

[14] S. Sasaki, *Almost contact manifolds*, 1,2,3, A Lecture note, Tohoku University (1965, 1967, 1968).

[15] R.N. Singh, S. K. Pandey, and G. Pandey. *On W2 -curvature tensor in a Kenmotsu manifold*. Tamsui Oxf. J. Inf. Math. Sci, 29(2) (2013),129-141.

[16] N. Tanaka, *On non-degenerate real hypersurfaces, graded Lie algebra and Cartan connections*, Japanese journal of mathematics. New series, 2(1), (1976), 131-190,.

[17] S. Tanno, *Variational problems on contact Riemannian manifolds*, Transactions of the American Mathematical Society, 314(1),(1989),349-379 .

[18] M. M. Tripathi and P. Gupta, *$T-$curvature tensor on a semi-Riemannian manifold*, J. Adv. Math. Studies, 4(1) (2011),117-129.

[19] P. Uygun and M. Atçeken, *On (k,μ)-paracontact metric spaces satisfying some conditions on the $W_0^\star-$curvature tensor*, NTMSCI 9,(2) (2021) , 26-37.

[20] Y. Wang and X. Liu, *Second Order Parallel Tensors on Almost Kenmotsu Manifolds Satisfying The Nullity Distributions*, Filomat, 28(4) (2014), 839-847.

[21] Y. Wang and W. Wang, *An Einstein-like Metric on Almost Kenmotsu Manifolds*, Filomat, 31(15) (2017), 4695-4702.

[22] S. M. Webster, *Pseudo-Hermitian structures on a real hypersurface*, Journal of Differential Geometry, vol. 13(1) (1978) , 25-41.