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Warped-Twisted Product Semi-Slant Submanifolds

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Abstract. We introduce the notion of warped-twisted product semi-slant submanifolds of the form ${}_{f_2}M^T \times_{f_1} M^\theta$ with warping function f_2 on M^θ and twisting function f_1, where M^T is a holomorphic and M^θ is a slant submanifold of a globally conformal Kaehler manifold. We prove that a warped-twisted product semi-slant submanifold of a globally conformal Kaehler manifold is a locally doubly warped product. Then we establish a general inequality for doubly warped product semi-slant submanifolds and get some results for such submanifolds by using the equality sign of the general inequality.

1. Introduction

Şahin [15] proved the non-existence of non-trivial warped product semi-slant submanifolds in Kaehlerian manifolds. More precisely, there do not exist warped product semi-slant submanifolds in Kaehlerian manifolds of the forms $M^\theta \times_f M^T$ and $M^T \times_f M^\theta$, where M^T is a holomorphic and M^θ is a slant submanifold of a Kaehlerian manifold (see, Theorems 3.1 and 3.2 of [15]). Also, Şahin [16] showed that there exists no non-trivial warped product hemi-slant submanifolds in Kaehlerian manifolds of the form $M^\perp \times_f M^\theta$, where $M^\perp$ is a totally real submanifold of a Kaehlerian manifold (see, Theorem 4.2 of [16]). We are inspired by the results of Şahin [15, 16] and deduce that Kaehlerian structures do not admit non-trivial doubly warped product semi-slant or hemi-slant submanifolds. Recently, Matsumoto studied warped product semi-slant submanifolds in locally conformal Kaehler manifolds of the forms $M^\theta \times_f M^T$ and $M^T \times_f M^\theta$ in [9, 10].

In [18], we defined two classes of doubly twisted products under the names of *nearly doubly twisted products of type 1* and *type 2*. In this article, we rename the nearly doubly twisted products of type 1 as *warped-twisted products*.

Motivated by the above papers, we consider and study warped-twisted product semi-slant submanifolds in globally conformal Kaehler manifolds in this paper.

2. Preliminaries

In this section, we recall the fundamental definitions and notions needed for the further study. Actually, in subsection 2.1, we will recall the definition of the warped-twisted product manifolds. The definitions of

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locally and globally conformal Kaehler manifolds will be presented in subsection 2.2. In subsection 2.3, we will give the basic background for submanifolds of Riemannian manifolds.

2.1. *Warped-twisted products*

Let M_1 and M_2 be Riemannian manifolds endowed with metric tensors g_1 and g_2, respectively and let f_1 and f_2 are positive smooth functions defined on $M_1 \times M_2$. Then the *doubly twisted product manifold* [14] ${}_{f_2}M_1 \times_{f_1} M_2$ is the product manifold $\bar{M} = M_1 \times M_2$ equipped with metric g given by

$$g = {f_2}^2 \pi_1^* g_1 + {f_1}^2 \pi_2^* g_2,$$

where $\pi_i : M_1 \times M_2 \to M_i$ is the canonical projections for $i = 1, 2$. Each function f_i is called a *twisting function* of the doubly twisted product ${}_{f_2}M_1 \times_{f_1} M_2$. If the twisting functions f_1 and f_2 depend only on the points of M_1 and M_2 respectively, then ${}_{f_2}M_1 \times_{f_1} M_2$ becomes a *doubly warped product manifold* [7] and each function f_i is called a *warping function* of the doubly warped product manifold. In this case, if $f_1 \equiv 1$ or $f_2 \equiv 1$, then we get a *warped product* [1].

Let ${}_{f_2}M_1 \times_{f_1} M_2$ be doubly twisted product manifold. If $f_1 \equiv 1$ or $f_2 \equiv 1$, then we get a *twisted product* [4] with the twisting function f_1 or a twisted product with the twisting function f_2. In a warped or twisted product case, the notation ${}_{f_2}M_1 \times_{f_1} M_2$ is simplified to ${}_{f_2}M_1 \times M_2$ or $M_1 \times_{f_1} M_2$. In addition, if both f_1 and f_2 are constant, then we get a *usual* or *direct product manifold* [3].

Let us recall the definition of a warped-twisted product manifold. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and let $f_2 : M_2 \to (0, \infty)$ and $f_1 : M_1 \times M_2 \to (0, \infty)$ be smooth functions. The *warped-twisted product* ${}_{f_2}M_1 \times_{f_1} M_2$ [18] is the product manifold $M_1 \times M_2$ equipped with the metric tensor g defined by

$$g = (f_2 \circ \pi_2)^2 \pi_1^*(g_1) + f_1^2 \pi_2^*(g_2). \tag{1}$$

The function $f_2 \in C^\infty(M_2)$ is called a *warping function* and the function $f_1 \in C^\infty(M_1 \times M_2)$ is called a *twisting function* of ${}_{f_2}M_1 \times_{f_1} M_2$. In this case, if the function f_1 depends only on the points of M_2, then the warped-twisted product ${}_{f_2}M_1 \times_{f_1} M_2$ becomes a *base conformal warped product* [5]. We say that a warped-twisted product is *non-trivial* if it is neither doubly warped product nor warped product or base conformal warped product.

Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a warped-twisted product manifold with the Levi-Civita connection $\bar{\nabla}$ of g, given in (1). Also we denote by ∇^i the Levi-Civita connection of g_i for $i \in \{1, 2\}$, respectively. By usual convenience, we denote the set of lifts of vector fields on M_i by $\mathcal{L}(M_i)$ and we use the same notation for a vector field and for its lift. On the other hand, each π_i is a positive homothety, so it preserves the Levi-Civita connection. Thus, there is no confusion using the same notation for a connection ∇^i on M_i and for its pullback via π_i. Then, the covariant derivative formulas of the warped-twisted product manifold ${}_{f_2}M_1 \times_{f_1} M_2$ with the warping function $f_2 \in C^\infty(M_2)$ and twisting function f_1 are given by

$$\bar{\nabla}_X Y = \nabla^1_X Y - g(X, Y)\bar{\nabla} \ln(f_2 \circ \pi_2), \tag{2}$$

$$\bar{\nabla}_V X = \bar{\nabla}_X V = V(\ln(f_2 \circ \pi_2))X + X(\ln f_1)V, \tag{3}$$

$$\bar{\nabla}_U V = \nabla^2_U V + U(\ln f_1)V + V(\ln f_1)U - g(U, V)\bar{\nabla} \ln f_1, \tag{4}$$

for any $X, Y \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$. These formulas immediately come from Lemma 2.1 of [8] with $X(\ln(f_2 \circ \pi_2)) = Y(\ln(f_2 \circ \pi_2)) = 0$.

Remark 2.1. *Until the section 5, we will use the same symbol for the warping function f_2 and its pullback $f_2 \circ \pi_2$, i.e., we will put $f_2 = f_2 \circ \pi_2$.*

2.2. Locally and globally conformal Kaehler manifolds

Let $(\bar{M}, J, g)$ be a Hermitian manifold of dimension $2m$. Then it is called a *locally conformal Kaehler manifold* (briefly *l.c.K. manifold*) [6], if each point of $p \in \bar{M}$ has an open neighborhood $\mathcal{U}$ with smooth function $\sigma : \mathcal{U} \to R$ such that $\tilde{g} = e^{-\sigma} g|_{\mathcal{U}}$ is a Kaehler metric on $\mathcal{U}$. If one choose $\mathcal{U} = \bar{M}$, then $(\bar{M}, J, g)$ is called a *globally conformal Kaehler manifold* (briefly *g.c.K. manifold*).

Theorem 2.2. *[6] Let $(\bar{M}, J, g)$ be a Hermitian manifold and let Ω be a $2-$ form defined by $\Omega(\bar{X}, \bar{Y}) = g(\bar{X}, J\bar{Y})$ for all vector fields in $\bar{M}$. Then $(\bar{M}, J, g)$ is a l.c.K. manifold if and only if there exists a globally defined $1-$ form ω such that*

$$d\Omega = \omega \wedge \Omega \qquad \textit{and} \qquad d\omega = 0. \tag{5}$$

The closed $1-$ form ω is called the *Lee form* of the l.c.K. manifold $(\bar{M}, J, g)$. In addition, the manifold $(\bar{M}, J, g)$ is g.c.K., if its Lee form ω is also exact. In this case, we have $\omega = d\sigma$ [20]. The *Lee vector field B* is defined by

$$\omega(\bar{X}) = g(B, \bar{X}), \tag{6}$$

for any vector fields $\bar{X}$ on $\bar{M}$. One can see that, the globally conformal Kaehler case is a special case of the locally conformal Kaehler case. We denote by $\tilde{\nabla}$ (resp. $\bar{\nabla}$) the Levi-Civita connection on $\bar{M}$ with respect to $\tilde{g} = e^{-\sigma} g$ (resp. g). Then we have [6]

$$\tilde{\nabla}_{\bar{X}} \bar{Y} = \bar{\nabla}_{\bar{X}} \bar{Y} - \frac{1}{2}\Big\{\omega(\bar{X})\bar{Y} + \omega(\bar{Y})\bar{X} - g(\bar{X}, \bar{Y})B\Big\}, \tag{7}$$

for any vector fields $\bar{X}$ and $\bar{Y}$ on $\bar{M}$. The connection $\tilde{\nabla}$ is a torsionless linear connection on $\bar{M}$ which is called the *Weyl connection* of g. It is easy to see that the Weyl connection $\tilde{\nabla}$ satisfies the condition

$$\tilde{\nabla} J = 0. \tag{8}$$

Remark 2.3. *Throughout this paper, we denote by $(\bar{M}, J, \omega, g)$ the g.c.K. manifold with the Lee form ω.*

2.3. Submanifolds of Riemannian manifolds

Let M be an isometrically immersed submanifold in a Riemannian manifold $(\bar{M}, g)$. Let $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$ with respect to the metric g and let ∇ and $\nabla^{\perp}$ be the induced, and induced normal connection on M, respectively. Then, for all $X, Y \in TM$ and $Z \in T^{\perp}M$, the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{9}$$

$$\bar{\nabla}_X Z = -A_Z X + \nabla^{\perp}_X Z, \tag{10}$$

where TM is the tangent bundle and $T^{\perp}M$ is the normal bundle of M in $\bar{M}$. Additionally, h is the *second fundamental form* of M and A_Z is the Weingarten endomorphism associated with Z. The second fundamental form h and the *shape operator A* are related by

$$g(h(X, Y), Z) = g(A_Z X, Y). \tag{11}$$

The *mean curvature vector field H* of M is given by $H = \frac{1}{m}(trace\, h)$, where $dim(M) = m$. We say that the submanifold M is *totally geodesic* in $\bar{M}$ if $h = 0$, and *minimal* if $H = 0$. The submanifold M is called *totally umbilical* if $h(X, Y) = g(X, Y)H$ for all $X, Y \in TM$.

Let M be any submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then the Gauss and Weingarten formulas with respect to $\tilde{\nabla}$ are given by

$$\tilde{\nabla}_X Y = \hat{\nabla}_X Y + \tilde{h}(X, Y), \tag{12}$$

$$\tilde{\nabla}_X Z = -\tilde{A}_Z X + \tilde{\nabla}^{\perp}_X Z, \tag{13}$$

for $X, Y \in TM$ and $Z \in T^{\perp}M$. Thus, using (9), (10) and (13), we have

$$\hat{\nabla}_X Y = \nabla_X Y - \frac{1}{2}\Big\{\omega(X)Y + \omega(Y)X - g(X,Y)B^M\Big\}, \tag{14}$$

$$\tilde{A}_Z X = A_Z X + \frac{1}{2}\omega(Z)X, \tag{15}$$

$$\tilde{h}(X,Y) = h(X,Y) + \frac{1}{2}g(X,Y)B^N, \tag{16}$$

from (7), where B^M and B^N are respectively the tangential and the normal part of B.

3. Semi-slant submanifolds of a g.c.K. manifold

In this section, we recall the definition of a semi-slant submanifold and give some auxiliary results related to the semi-slant submanifolds of a g.c.K. manifold to prove our main theorems.

Let $(\bar{M}, J, g)$ be an almost Hermitian manifold and let M be a Riemannian manifold isometrically immersed in $\bar{M}$. A distribution $\mathcal{D}$ on M is called a *slant distribution* if for $U \in \mathcal{D}_p$, the angle θ between JU and $\mathcal{D}_p$ is constant, i.e., independent of $p \in M$ and $U \in \mathcal{D}_p$. The constant angle θ is called the *slant angle* of the slant distribution $\mathcal{D}$. We know that holomorphic and totally real distributions on M are slant distributions with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant distribution is called *proper* if it is neither holomorphic nor totally real. A submanifold M of $\bar{M}$ is said to be a *slant submanifold* [2] if the tangent bundle TM of M is slant. For examples and more details, see [2].

A *semi-slant submanifold* M [13] of a g.c.K. manifold $(\bar{M}, J, g)$ is a submanifold such that its tangent bundle TM admits two orthogonal complementary holomorphic distribution $\mathcal{D}^T$ and slant distribution $\mathcal{D}^\theta$, i.e., we have

$$TM = \mathcal{D}^T \oplus \mathcal{D}^\theta. \tag{17}$$

We say that the semi-slant submanifold M is *proper* if $dim(\mathcal{D}^T) \neq 0$ and $\theta \neq 0, \frac{\pi}{2}$.
For any $Y \in TM$ we write

$$JY = PY + FY, \tag{18}$$

where PY is the tangential part of JY, and FY is the normal part of JY. Then the normal bundle $T^{\perp}M$ of M is decomposed as

$$T^{\perp}M = F\mathcal{D}^\theta \oplus \overline{\mathcal{D}}, \tag{19}$$

where $\overline{\mathcal{D}}$ is the orthogonal complementary distribution of $F\mathcal{D}^\theta$ in $T^{\perp}M$ and it is an invariant subbundle of $T^{\perp}M$ with respect to J. For a semi-slant submanifold, we have [15]

$$P^2 U = -\cos^2\theta U, \tag{20}$$

$$g(PU, PV) = \cos^2\theta g(U,V) \qquad \text{and} \qquad g(FU, FV) = \sin^2\theta g(U,V) \tag{21}$$

for $U, V \in \Gamma(\mathcal{D}^\theta)$.

Lemma 3.1. *Let M be a semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then we have*

$$\begin{aligned} g(\nabla_X Y, U) = \ & \csc^2\theta\Big\{g\Big(A_{FU}JY - A_{FPU}Y, X\Big) + \tfrac{1}{2}\omega(FU)g(JY, X) \\ & -\tfrac{1}{2}\omega(FPU)g(X, Y)\Big\} - \tfrac{1}{2}\omega(U)g(X, Y), \end{aligned} \tag{22}$$

for $X, Y \in \Gamma(\mathcal{D}^T)$ and $U \in \Gamma(\mathcal{D}^\theta)$.

Proof. Let $X, Y \in \Gamma(\mathcal{D}^T)$ and $U \in \Gamma(\mathcal{D}^\theta)$. Since $(\bar{M}, J, \omega, \tilde{g} = e^{-\sigma}g)$ is a Kaehler manifold, by using (8), (13), (18) and (20), we have

$$\begin{aligned} \tilde{g}(\hat{\nabla}_X Y, U) = \ & \tilde{g}(\tilde{\nabla}_X Y, U) = \tilde{g}(\tilde{\nabla}_X JY, JU) \\ = \ & \tilde{g}(\tilde{\nabla}_X JY, PU) + \tilde{g}(\tilde{\nabla}_X JY, FU) \\ = \ & -\tilde{g}(\tilde{\nabla}_X Y, JPU) + \tilde{g}(\tilde{A}_{FU}X, JY) \\ = \ & -\tilde{g}(\tilde{\nabla}_X Y, P^2U) - \tilde{g}(\tilde{\nabla}_X Y, FPU) + \tilde{g}(\tilde{A}_{FU}X, JY) \\ = \ & \cos^2\theta\tilde{g}(\hat{\nabla}_X Y, U) + \tilde{g}(\tilde{A}_{FU}JY, X) - \tilde{g}(\tilde{A}_{FPU}Y, X). \end{aligned}$$

Hence, it follows that

$$\tilde{g}(\hat{\nabla}_X Y, U) = \csc^2\theta\tilde{g}(\tilde{A}_{FU}JY, X) - \tilde{g}(\tilde{A}_{FPU}Y, X).$$

Now, by using (6), (14) and (15), we derive the conclusion. □

Theorem 3.2. *Let M be a proper semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then the holomorphic distribution $\mathcal{D}^T$ is integrable if and only if*

$$g(A_{FU}JY, X) - g(A_{FU}JX, Y) = \omega(FU)g(JX, Y), \tag{23}$$

for $X, Y \in \Gamma(\mathcal{D}^T)$ and $U \in \Gamma(\mathcal{D}^\theta)$.

Proof. Let M be a proper semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then the holomorphic distribution D^T is integrable if and only if $g([X, Y], U) = 0$ for all $X, Y \in \Gamma(\mathcal{D}^T)$ and $U \in \Gamma(\mathcal{D}^\theta)$. Thus, the assertion (23) comes from (22). □

Lemma 3.3. *Let M be a semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then we have*

$$g(\nabla_U V, X) = -\csc^2\theta g\Big(A_{FV}JX - A_{FPV}X,\ U\Big) - \tfrac{1}{2}\omega(X)g(U, V), \tag{24}$$

for $X \in \Gamma(\mathcal{D}^T)$ and $U, V \in \Gamma(\mathcal{D}^\theta)$.

Proof. Let $X \in \Gamma(\mathcal{D}^T)$ and $U, V \in \Gamma(\mathcal{D}^\theta)$. Since $(\bar{M}, J, \omega, \tilde{g} = e^{-\sigma}g)$ is a Kaehler manifold, using (8), (13), (18) and (20), we have

$$\begin{aligned} \tilde{g}(\hat{\nabla}_U V, X) = \ & \tilde{g}(\tilde{\nabla}_U V, X) = \tilde{g}(\tilde{\nabla}_U JV, JX) \\ = \ & \tilde{g}(\tilde{\nabla}_U PV, JX) + \tilde{g}(\tilde{\nabla}_U FV, JX) \\ = \ & -\tilde{g}(\tilde{\nabla}_U JPV, X) - \tilde{g}(\tilde{A}_{FV}JX, U) \\ = \ & -\tilde{g}(\tilde{\nabla}_U P^2V, X) - \tilde{g}(\tilde{\nabla}_U FPV, X) - \tilde{g}(\tilde{A}_{FV}JX, U) \\ = \ & \cos^2\theta\tilde{g}(\hat{\nabla}_U V, X) + \tilde{g}(\tilde{A}_{FPV}X, U) - \tilde{g}(\tilde{A}_{FV}JX, U). \end{aligned}$$

Hence, it follows that

$$\tilde{g}(\hat{\nabla}_U V, X) = -\csc^2\theta\Big\{\tilde{g}(\tilde{A}_{FV}JX - \tilde{A}_{FPV}X, U)\Big\}.$$

Now, by using (6), (14) and (15), we derive the conclusion. □

Theorem 3.4. *Let M be a proper semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then the slant distribution $\mathcal{D}^\theta$ is integrable if and only if*

$$g(A_{FV}JX - A_{FPV}X, U) = g(A_{FU}JX - A_{FPU}X, V), \tag{25}$$

for $X \in \Gamma(\mathcal{D}^T)$ and $U, V \in \Gamma(\mathcal{D}^\theta)$.

Proof. Let M be a proper semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then the slant distribution D^θ is integrable if and only if $g([U, V], X) = 0$ for all $X \in \Gamma(\mathcal{D}^T)$ and $U, V \in \Gamma(\mathcal{D}^\theta)$. Thus, the assertion (25) follows from (24). □

Remark 3.5. *Throughout this paper, for a semi-slant submanifold M of a g.c.K. manifold $(\bar{M}, J, \omega, g)$, we write $B^M = B^T + B^\theta$, where B^T(resp. B^θ) is tangential part of B^M to $\mathcal{D}^T$(resp. $\mathcal{D}^\theta$).*

For some properties of semi-slant submanifolds of a g.c.K. manifold, we refer to the paper [17].

4. Warped-twisted product semi-slant submanifolds of a g.c.K. manifold

In this section, we consider warped-twisted product semi-slant submanifolds in the form ${}_{f_2}M^T \times_{f_1} M^\theta$, where M^T is a holomorphic and M^θ is a slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. We give necessary and sufficient conditions for such manifolds to be twisted product, base-conformal warped product and direct product. Then we give a characterization for these kind of submanifolds in a main theorem. We first give an (non-trivial) example of such a submanifold.

Example 4.1. *Let $(z_1, ..., z_6)$ be natural coordinates of the six-dimensional Euclidean space R^6 and let $\bar{R}^6 = \{(z_1, ..., z_6) \in R^6 : z_1 \neq 0$ and $z_3 + z_4 \neq 0\}$. Then $(\bar{R}^6, J, g_0)$ is a Kaehler manifold with usual Kaehler structure (J, g_0). Now, we consider the Riemannian metric $g = e^\sigma g_0$ conformal to Kaehler metric g_0 on $\bar{R}^6$, where $e^\sigma = z_1^2 \dfrac{(z_3 + z_4)^2}{4}$. Then $(\bar{R}^6, J, g)$ is clearly a g.c.K. manifold. Let M be a submanifold given by*

$$z_1 = x\,,\ z_2 = y\,,\ z_3 = u + v\,,\ z_4 = -u + v\,,\ z_5 = u\,,\ z_6 = 0\,,$$

where $x, y, u, v \neq 0$. Then, the local frame field of the tangent bundle TM of M is given by

$$X = \partial_1\,,\quad Y = \partial_2\,,\quad U = \frac{1}{\sqrt{3}}\Big\{\partial_3 - \partial_4 + \partial_5\Big\},\quad V = \frac{1}{\sqrt{2}}\Big\{\partial_3 + \partial_4\Big\},$$

where $\partial_i = \frac{\partial}{\partial z_i}$ for $i \in \{1, 2, ..., 6\}$. Then $\mathcal{D}^T = span\{X, Y\}$ is a holomorphic and $\mathcal{D}^\theta = span\{U, V\}$ is a (proper) slant distribution with the slant angle $\theta = \cos^{-1}(\frac{2}{\sqrt{6}})$. Thus, M is a proper semi-slant submanifold of $(\bar{R}^6, J, g)$. One can see that both $\mathcal{D}^T$ and $\mathcal{D}^\theta$ are integrable. Let us denote the integral submanifolds of $\mathcal{D}^T$ and $\mathcal{D}^\theta$ by M^T and M^θ, respectively. Let g_T and g_θ be the induced metrics from the Kaehler metric g_0 on M^T and M^θ, respectively. We choose the conformal Riemann metric $\bar{g}_T = x^2 g_T$ on M^T. Since $x = z_1$ and $v = \dfrac{z_3 + z_4}{2}$ on M, the induced metric of M from the conformal Kaehler metric g is

$$\begin{aligned} ds^2 &= x^2v^2(dx^2 + dy^2) + x^2v^2(du^2 + dv^2) \\ &= v^2x^2 g_T + x^2v^2 g_\theta \\ &= v^2\bar{g}_T + (xv)^2 g_\theta\,. \end{aligned}$$

Thus, M is a warped-twisted product of $(M^T, \bar{g}_T)$ and (M^θ, g_θ). So, ${}_{f_2}M^T \times_{f_1} M^\theta$ is a (non-trivial) warped-twisted product proper semi-slant submanifold of the g.c.K. manifold $(\bar{R}^6, J, g)$ with warping function $f_2 = v$ and twisting function $f_1 = xv$. Moreover, the Lee form $(\bar{R}^6, J, g)$ is

$$\omega = 2\Big(\frac{1}{x}dx + \frac{1}{v}dv\Big).$$

Consequently, the Lee vector field is

$$B = \frac{2}{x^2v^2}\Big(\frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{v}\frac{\partial}{\partial v}\Big)$$

which is tangent to M.

Lemma 4.2. *Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a warped-twisted product semi-slant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then, for all $X \in \mathcal{L}(M^T)$, we have*

$$\omega(X) = \tfrac{2}{3}X(\ln f_1). \tag{26}$$

Proof. Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a warped-twisted product semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. For $U, V \in \mathcal{L}(M^\theta)$ and $X \in \mathcal{L}(M^T)$, using the exterior differentiation formula (see, [21], p.17), we have

$$\begin{aligned} 3d\Omega(X,U,V) = {} & X\Omega(U,V) + U\Omega(V,X) + V\Omega(X,U) \\ & -\Omega([X,U],V) - \Omega([U,V],X) - \Omega([V,X],U) \\ = {} & Xg(U,PV), \end{aligned}$$

since $[X, V] = [X, U] = 0$ and $[U, V] \in \mathcal{L}(M^\theta)$. Hence,

$$\begin{aligned} 3d\Omega(X,U,V) = {} & Xg(U,PV) \\ = {} & g(\nabla_X U, PV) + g(U, \nabla_X PV). \end{aligned}$$

Using (3), we obtain

$$3d\Omega(X,U,V) = 2X(\ln f_1)g(U,PV). \tag{27}$$

On the other hand, using (5) and (18), we have

$$\begin{aligned} d\Omega(X,U,V) = {} & \omega \wedge \Omega(X,U,V) \\ = {} & \omega(X)\Omega(U,V) + \omega(U)\Omega(V,X) + \omega(V)\Omega(X,U) \\ = {} & \omega(X)g(U,PV) \end{aligned}$$

from (5). Namely,

$$d\Omega(X,U,V) = \omega(X)g(U,PV). \tag{28}$$

Thus, the assertion comes from (27) and (28). □

By Lemma 4.2, we immediately have the following result.

Theorem 4.3. *Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a warped-twisted product semi-slant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then M is a base conformal warped product submanifold in the form ${}_{f_2}M^T \times_{f_1} M^\theta$ if and only if the Lee vector field B is normal to M^T.*

Proof. Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a warped-twisted product semi-slant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. If M is a base conformal warped product submanifold in the form ${}_{f_2}M^\perp \times_{f_1} M^\theta$, then for any $X \in \mathcal{L}(M^T)$, $X(\ln f_1)$=0, since f_1 depends only on the points of M^θ. From (26), we find $g(B, X) = 0$. So, the Lee vector field B is normal to M^T.

Conversely, if the Lee vector field B is normal to M^T, we have $g(B, X) = 0$. Then, we get $X(\ln f_1) = 0$ for any $X \in \mathcal{L}(M^T)$ from (26). So f_1 depends only on the points of M^θ. Then the induced metric tensor g_M of M has the form $g_M = {f_2}^2 g_T \oplus \tilde{g}_\theta$, where f_2 is warping function and $\tilde{g}_\theta = {f_1}^2 g_\theta$. Thus, $M = {}_{f_2}M^T \times_{f_1} M^\theta$ is a base conformal warped product. □

Lemma 4.4. *Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a warped-twisted product semi-slant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then, for all $V \in \mathcal{L}(M^\theta)$, we have*

$$\omega(V) = \tfrac{2}{3}V(\ln f_2). \tag{29}$$

Proof. Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a warped-twisted product semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then using the exterior differentiation formula, we have

$$\begin{aligned} 3d\Omega(V, X, Y) &= V\Omega(X, Y) + X\Omega(Y, V) + Y\Omega(V, X) \\ &\quad -\Omega([V, X], Y) - \Omega([X, Y], V) - \Omega([Y, V], X) \\ &= Vg(X, JY) - Xg(JY, V) + Yg(V, JX) \\ &\quad -g([V, X], JY) + g(J[X, Y], V) - g([Y, V], JX). \end{aligned}$$

Here, we know $g(JY, V) = g(V, JX) = 0$, since M is a semi-slant submanifold. Also, by (3), we have $[V, X] = [Y, V] = 0$ and by (2), we have $[X, Y] = \nabla^1_X Y - \nabla^1_Y X$. So $J[X, Y] \in \Gamma(TM^T)$. Thus, we obtain

$$\begin{aligned} 3d\Omega(V, X, Y) &= Vg(X, JY) \\ &= g(\nabla_V X, JY) + g(X, \nabla_V JY). \end{aligned}$$

Again, using (3), we find

$$3d\Omega(V, X, Y) = g(X(\ln f_1)V + V(\ln f_2)X, JY) + g(X, JY(\ln f_1)V + V(\ln f_2)JY).$$

So, we obtain

$$3d\Omega(V, X, Y) = 2V(\ln f_2)g(X, JY). \tag{30}$$

On the other hand, using (5) and (18), we have

$$\begin{aligned} d\Omega(V, X, Y) &= \omega \wedge \Omega(V, X, Y) \\ &= \omega(V)\Omega(X, Y) + \omega(X)\Omega(Y, V) + \omega(Y)\Omega(V, X) \\ &= \omega(V)g(X, JY). \end{aligned} \tag{31}$$

Thus, the assertion comes from (30) and (31). □

By Lemma 4.4, we immediately have the following result.

Theorem 4.5. *Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a warped-twisted product semi-slant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then M is a twisted product submanifold in the form $M^T \times_{f_1} M^\theta$ if and only if the Lee vector field B is normal to M^θ.*

Proof. Let M is a twisted product submanifold in the form $M^T \times_{f_1} M^\theta$, where f_1 is a twisting function. Then, for any $V \in \mathcal{L}(M^\theta)$, $V(\ln f_2)$=0, since f_2 is a constant. From (29), we find $g(B, V) = 0$, for any $V \in \mathcal{L}(M^\theta)$. So, the Lee vector field B is normal to M^θ.

Conversely, if the Lee vector field B is normal to M^θ, we have $g(B, V) = 0$, for any $V \in \mathcal{L}(M^\theta)$. Then, we get $V(\ln f_2) = 0$ from (29). So, f_2 is a constant, say $f_2 = c$. Then the induced metric tensor g_M of M has the form $g_M = c^2 g_T \oplus {f_1}^2 g_\theta$, where c is constant and f_1 is the twisting function. Thus, $M = M^T \times_{f_1} M^\theta$ is a twisted product. □

We conclude from Theorems 4.3 and 4.5 that:

Theorem 4.6. *Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a warped-twisted product semi-slant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then M is a locally direct product manifold if and only if the Lee vector field B is normal to M.*

Proof. Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a warped-twisted product semi-slant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. If M is a locally direct product, then the functions f_1 and f_2 are constants. In that case, for any $X \in \mathcal{L}(M^T)$ and $V \in \mathcal{L}(M^\theta)$, we have $g(B, X) = g(B, V) = 0$ from (26) and (29), respectively. It follows that B is normal to M.

Conversely, let B is normal to M. Then, for any $X \in \mathcal{L}(M^T)$ and $V \in \mathcal{L}(M^\theta)$, we have $X(\ln f_1) = V(\ln f_2) = 0$. It follows that f_2 is a constant, say $f_2 = c$ and f_1 depends only on the points of M^θ. Then the induced metric tensor g_M of M has the form $g_M = c^2 g_T \oplus {f_1}^2 g_\theta$. Hence, we conclude that M is a locally direct product of $(M^\perp, \tilde{g_T})$ and $(M^\theta, \tilde{g_\theta})$, where $\tilde{g_T} = c^2 g_T$ and $\tilde{g_\theta} = {f_1}^2 g_\theta$. □

By using (22) and (26), we deduce the following result.

Lemma 4.7. *Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a warped-twisted product semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then we have*

$$g(A_{FU}JX - A_{FPU}X, Y) = \frac{1}{2}\Big(\omega(FPU)g(X, Y) - \omega(FU)g(JX, Y)\Big) - \sin^2\theta\omega(U)g(X, Y) \tag{32}$$

for $X, Y \in \mathcal{L}(M^T)$ and $U \in \mathcal{L}(M^\theta)$.

By using (24) and (29), we deduce the following result.

Lemma 4.8. *Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a warped-twisted product semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then we have*

$$g(A_{FV}JX - A_{FPV}X, U) = \sin^2\theta\,\omega(X)g(V, U) \tag{33}$$

for $X \in \mathcal{L}(M^T)$ and $U, V \in \mathcal{L}(M^\theta)$.

Now, we recall the following two facts to prove the main theorem.

Lemma 4.9. *(Proposition 3-a [14]) Let g be a pseudo-Riemannian metric on the manifold $M = M_1 \times M_2$ and $(\mathcal{D}_1, \mathcal{D}_2)$ the canonical foliations. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ intersect perpendicularly everywhere. Then (M, g) is a doubly twisted product ${}_{f_2}M_1 \times_{f_1} M_2$ if and only if $\mathcal{D}_1$ and $\mathcal{D}_2$ are totally umbilic foliations.*

Lemma 4.10. *(Lemma 3.1.1 [11]) Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a doubly twisted product. It is a doubly warped product if and only if the mean curvature vector fields of canonical foliations are closed.*

Motivated by Lemma 4.9 and Lemma 4.10, we can obtain the following result.

Lemma 4.11. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a doubly twisted product. It is a warped-twisted product with warping function $f_2 \in C^\infty(M_2)$ and twisting function f_1 if and only if the mean curvature vector field of canonical foliation $\mathcal{D}_1$ is closed.*

Proof. The proof is very similar to the proof of Lemma 2.3 [8], so we omit it. □

We now are ready to prove the main theorem.

Theorem 4.12. *Let M be a semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then M is a locally warped-twisted product submanifold if and only if its shape operator A satisfies the following equation*

$$A_{FU}JX - A_{FPU}X = \frac{1}{2}\Big\{\omega(FPU)X - \omega(FU)JX\Big\} + \sin^2\theta\Big\{\omega(X)U - \omega(U)X\Big\} \tag{34}$$

for $X \in \Gamma(\mathcal{D}^T)$ and $U \in \Gamma(\mathcal{D}^\theta)$. Moreover, M is also a locally doubly warped product submanifold.

Proof. Let M be a warped-twisted product submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ of type ${}_{f_2}M^T \times_{f_1} M^\theta$. For any $X \in \mathcal{L}(M^T)$ and $V \in \mathcal{L}(M^\theta)$, we write

$$A_{FU}JX - A_{FPU}X = \Big(A_{FU}JX - A_{FPU}X\Big)^T + \Big(A_{FU}JX - A_{FPU}X\Big)^\theta, \tag{35}$$

where $\Big(A_{FU}JX - A_{FPU}X\Big)^T$ is the tangent part of $A_{FU}JX - A_{FPU}X$ to M^T and $\Big(A_{FU}JX - A_{FPU}X\Big)^\theta$ is the tangent part of $A_{FU}JX - A_{FPU}X$ to M^θ. Hence, for any $Y \in \mathcal{L}(M^T)$, using (32), we have

$$g(A_{FU}JX - A_{FPU}X,\, Y) = g\Big(\frac{1}{2}\omega(FPU)X - \frac{1}{2}\omega(FU)JX - \sin^2\theta\omega(U)X,\, Y\Big).$$

Since $Y \in \mathcal{L}(M^T)$ is arbitrary and the metric g is Riemannian, it follows that

$$\Big(A_{FU}JX - A_{FPU}X\Big)^T = \tfrac{1}{2}\omega(FPU)X - \tfrac{1}{2}\omega(FU)JX - \sin^2\theta\omega(U)X. \tag{36}$$

Similarly, for any $V \in \mathcal{L}(M^\theta)$, using (33), we have

$$g(A_{FU}JX - A_{FPU}X,\, V) = g\Big(\sin^2\theta\,\omega(X)U,\, V\Big).$$

Since $V \in \mathcal{L}(M^\theta)$ is arbitrary and the metric g is Riemannian, it follows that

$$\Big(A_{FU}JX - A_{FPU}X\Big)^\theta = \sin^2\theta\,\omega(X)U. \tag{37}$$

Thus, by (35)~(37), we get (34).

Conversely, suppose that M is a semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ such that (34) holds. Then, for any $X \in \Gamma(\mathcal{D}^T)$ and $U, V \in \Gamma(\mathcal{D}^\theta)$, using (34), we deduce (23). Thus, by Theorem 3.2, the holomorphic distribution $\mathcal{D}^T$ is integrable. On the other hand, again using (34), we obtain (25). Thus, by Theorem 3.4, the slant distribution $\mathcal{D}^\theta$ is integrable. Let M^T and M^θ be the integral manifolds of $\mathcal{D}^T$ and $\mathcal{D}^\theta$, respectively and let denote by h^T and h^θ the second fundamental forms of M^T and M^θ in M, respectively. Then, for any $X, Y \in \Gamma(\mathcal{D}^T)$ and $U \in \Gamma(\mathcal{D}^\theta)$, using (9), we have

$$g(h^T(X, Y), U) = g(\nabla_X Y, U).$$

Here, if we use (22) and (34), we find

$$g(h^T(X, Y), V) = -\tfrac{3}{2}\omega(U)g(X, Y).$$

After some calculation, we obtain

$$g(h^T(X, Y), U) = g(-g(X, Y)\tfrac{3}{2}B^\theta, U).$$

Hence, we conclude that

$$h^T(X, Y) = -g(X, Y)\tfrac{3}{2}B^\theta.$$

This equation says that M^T is totally umbilic with the mean curvature vector field $-\frac{3}{2}B^\theta$. On the other hand, for any $X \in \Gamma(\mathcal{D}^T)$ and $U, V \in \Gamma(\mathcal{D}^\theta)$, using (9), we have

$$g(h^\theta(U, V), X) = g(\nabla_U V, X).$$

Here, if we use (24) and (34), we find

$$g(h^\theta(U,V),X) = -\tfrac{3}{2}\omega(X)g(U,V).$$

After some calculation, we obtain

$$g(h^\theta(U,V),X) = g(-g(U,V)\tfrac{3}{2}B^T, X).$$

Hence, we conclude that

$$h^\theta(U,V) = -g(U,V)\tfrac{3}{2}B^T.$$

It means that M^θ is totally umbilic in M with the mean curvature vector field $-\frac{3}{2}B^T$.

Next, we prove B^T and B^θ are closed. Let denote by ω^T (resp. ω^θ) the dual 1-form of B^T (resp. B^θ). For any $X \in \Gamma(\mathcal{D}^T)$, we have $\omega^T(X) = \omega(X)$. Thus, for $X, Y \in \Gamma(\mathcal{D}^T)$, we obtain

$$d\omega^T(X,Y) = X\omega^T(Y) - Y\omega^T(X) - \omega^T([X,Y]) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) = d\omega(X,Y).$$

It follows that $d\omega^T = 0$, since $d\omega = 0$. Namely, ω^T is closed. Hence, B^T is closed, since its dual 1-form is closed. Thus, by Lemma 4.11, M is a locally warped-twisted product submanifold. Moreover, we can prove that B^θ is closed in a similar way. Thereby, by Lemma 4.10, M is also a locally doubly warped product submanifold. □

Remark 4.13. *We have just proved that a warped-twisted product semi-slant submanifold of a g.c.K. manifold* $(\bar{M}, J, \omega, g)$ *is also a doubly warped product submanifold in Theorem 4.12. Therefore, from now on we will focus on doubly warped product submanifolds of a g.c.K. manifold.*

5. An inequality for doubly warped product proper semi-slant submanifolds

In this section, we shall establish an inequality for the squared norm of the second fundamental form of a doubly warped product proper semi-slant submanifold in the form ${}_{f_2}M^T \times_{f_1} M^\theta$, where M^T is a holomorphic and M^θ is a slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Note that a general inequality for any doubly warped product submanifold in arbitrary Riemannian manifolds was established in Theorem 3 of [12].

Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a doubly warped product manifold equipped with the metric g defined by

$$g = (f_2 \circ \pi_2)^2 \pi_1^*(g_1) + (f_1 \circ \pi_1)^2 \pi_2^*(g_2). \tag{38}$$

Then the covariant derivative formulas (2)~(5) become

$$\bar{\nabla}_X Y = \nabla^1_X Y - g(X,Y)\bar{\nabla}(\ln f_2 \circ \pi_2), \tag{39}$$

$$\bar{\nabla}_V X = \bar{\nabla}_X V = V(\ln f_2 \circ \pi_2)X + X(\ln f_1 \circ \pi_1)V, \tag{40}$$

$$\bar{\nabla}_U V = \nabla^2_U V - g(U,V)\bar{\nabla}(\ln f_1 \circ \pi_1), \tag{41}$$

for $X, Y \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$. It follows that $M_1 \times \{p_2\}$ and $\{p_1\} \times M_2$ are totally umbilical submanifolds with closed mean curvature vector fields in ${}_{f_2}M_1 \times_{f_1} M_2$ [11], where $p_1 \in M_1$ and $p_2 \in M_2$. We say that a doubly warped product is non-trivial if it is neither warped nor a direct product.

Remark 5.1. *[7] For a doubly warped product manifold* ${}_{f_2}M_1 \times_{f_1} M_2$*, we have*

$$\begin{aligned} \bar{\nabla}(\ln f_1 \circ \pi_1) &= \frac{1}{(f_2 \circ \pi_2)^2}\nabla^1(\ln f_1 \circ \pi_1) \\ \bar{\nabla}(\ln f_2 \circ \pi_2) &= \frac{1}{(f_1 \circ \pi_1)^2}\nabla^2(\ln f_2 \circ \pi_2). \end{aligned} \tag{42}$$

In view of the above convenience together with (38) and (42), the covariant derivative formulas (39) and (41) become

$$\bar{\nabla}_X Y = \nabla^1_X Y - \frac{(f_2 \circ \pi_2)^2}{(f_1 \circ \pi_1)^2} g_1(X, Y)\nabla^2(\ln f_2 \circ \pi_2), \tag{43}$$

$$\bar{\nabla}_U V = \nabla^2_U V - \frac{(f_1 \circ \pi_1)^2}{(f_2 \circ \pi_2)^2} g_2(U, V)\nabla^1(\ln f_1 \circ \pi_1), \tag{44}$$

for $X, Y \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$.

For more details on doubly warped products, we refer to the papers [7], [8], [11] and [19].

Remark 5.2. *From now on, we will use the same symbol for a warping function f_i and its pullback $f_i \circ \pi_i$ for $i = 1, 2.$, i.e. we will put $f_i = f_i \circ \pi_i$.*

Lemma 5.3. *Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a doubly warped product semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ and h be the second fundamental form of M in $\bar{M}$. Then we have*

$$g(h(X, Y), FU) = -\Big(\tfrac{1}{2}\omega(FU) - \omega(PU)\Big)g(X, Y) + \omega(U)g(X, JY), \tag{45}$$

$$g(h(X, U), FV) = -\omega(JX)g(U, V) - \omega(X)g(U, PV), \tag{46}$$

where $X, Y \in \mathcal{L}(M^T)$ and $U, V \in \mathcal{L}(M^\theta)$.

Proof. Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a doubly warped product semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ and let $X, Y \in \mathcal{L}(M^T)$ and $U \in \mathcal{L}(M^\theta)$. Since $(\bar{M}, J, \omega, \tilde{g} = e^{-\sigma}g)$ is a Kaehler manifold, using (12), (18) and (8), we have

$$\begin{aligned} \tilde{g}(\tilde{h}(X, Y), FU) =&\ \tilde{g}(\tilde{\nabla}_X Y, FU) \\ =&\ \tilde{g}(\tilde{\nabla}_X Y, JU) - \tilde{g}(\tilde{\nabla}_X Y, PU) \\ =&\ -\tilde{g}(\tilde{\nabla}_X JY, U) - \tilde{g}(\hat{\nabla}_X Y, PU) \\ =&\ -\tilde{g}(\hat{\nabla}_X JY, U) - \tilde{g}(\hat{\nabla}_X Y, PU) . \end{aligned}$$

Now, using (2), (14), (15) and (29), we get (45). Next, let $X, Y \in \mathcal{L}(M^T)$ and $V \in \mathcal{L}(M^\theta)$, since $(\bar{M}, J, \omega, \tilde{g} = e^{-\sigma}g)$ is a Kaehler manifold, using (12), (18) and (8), we have

$$\begin{aligned} \tilde{g}(\tilde{h}(X, U), FV) =&\ \tilde{g}(\tilde{\nabla}_U X, FV) \\ =&\ \tilde{g}(\tilde{\nabla}_U X, JV) - \tilde{g}(\tilde{\nabla}_U X, PV) \\ =&\ -\tilde{g}(\tilde{\nabla}_U JX, V) - \tilde{g}(\tilde{\nabla}_U X, PV) \\ =&\ -\tilde{g}(\hat{\nabla}_U JX, V) - \tilde{g}(\hat{\nabla}_U X, PV) . \end{aligned}$$

Now, using (3), (14), (15) and (26), we get (46). □

Remark 5.4. *We say that a semi-slant submanifold M is* mixed geodesic, *if $h(X, U) = 0$ for $X \in \Gamma(\mathcal{D}^T)$ and $U \in \Gamma(\mathcal{D}^\theta)$.*

Theorem 5.5. *Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a doubly warped product proper semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. If M is mixed geodesic, then M is a warped product of the form ${}_{f_2}M^T \times M^\theta$.*

Proof. Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a doubly warped product proper semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. If M is mixed geodesic, then we have

$$\omega(JX)g(U, V) = -\omega(X)g(U, PV) \tag{47}$$

from (46), where $X \in \mathcal{L}(M^T)$ and $U, V \in \mathcal{L}(M^\theta)$. Replacing X by JX in (47), we obtain

$$\omega(X)g(U,V) = \omega(JX)g(U,PV). \tag{48}$$

Now, replacing V by PV in (48), we get

$$\omega(X)g(U,PV) = \omega(JX)g(U,P^2V). \tag{49}$$

By using (20) in (49), we arrive to

$$-\cos^2\theta\omega(JX)g(U,V) = \omega(X)g(U,PV). \tag{50}$$

By summing (47) and (50), we find

$$\sin^2\theta\omega(JX)g(U,V) = 0.$$

Since $\sin^2\theta \neq 0$ in proper case and g is non-degenerate, it follows that

$$\omega(JX) = 0.$$

But, (26) implies $JX(\ln f_1) = 0$. Which says us that the warping function f_1 is constant. Thus, M is a warped product of the form $M = {}_{f_2}M^T \times M^\theta$. □

By using (19), (26) and (46), we can prove the following result.

Theorem 5.6. *Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a doubly warped product proper semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ such that the invariant subnormal bundle $\overline{\mathcal{D}} = \{0\}$. Then M is mixed geodesic if and only if it is a warped product of the form $M = {}_{f_2}M^T \times M^\theta$.*

Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a $(m_1 + m_2)$-dimensional doubly warped product proper semi-slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. We choose a canonical orthonormal basis $\{e_1, ..., e_{n_1}, e_{n_1+1} = Je_1, ...e_{2n_1} = Je_{n_1}, \bar{e}_1, ..., \bar{e}_{2n_2}, e_1^*, ..., e_{2n_2}^*, \hat{e}_1, ..., \hat{e}_l\}$ of $\bar{M}$ such that $\{e_1, ..., e_{n_1}, e_{n_1+1} = Je_1, ..., e_{2n_1} = Je_{n_1}\}$ is an orthonormal basis of $\mathcal{D}^T$, $\{\bar{e}_1, ..., \bar{e}_{2n_2}\}$ is an orthonormal basis of $\mathcal{D}^\theta$, $\{e_1^*, ..., e_{2n_2}^*\}$ is an orthonormal basis of $F\mathcal{D}^\theta$ and $\{\hat{e}_1, ..., \hat{e}_l\}$ is an orthonormal basis of $\overline{\mathcal{D}}$. Here, $2n_1 = dim(\mathcal{D}^T)$, $2n_2 = dim(\mathcal{D}^\theta)$ and $l = dim(\overline{\mathcal{D}})$.

Remark 5.7. *Since $\mathcal{D}^T$ is a holomorphic distribution, $\{Je_1, ..., Je_{m_1}\}$ is also an orthonormal basis of $\mathcal{D}^T$, where $m_1 = 2n_1 = dim(M^T)$. Moreover, by (21), we observe that $\{\bar{a}_1 = \sec\theta P\bar{e}_2, \bar{a}_2 = -\sec\theta P\bar{e}_1, ..., \bar{a}_{2n_2-1} = \sec\theta P\bar{e}_{2n_2}, \bar{a}_{2n_2} = -\sec\theta P\bar{e}_{2n_2-1}\}$ is also an orthonormal basis of $\mathcal{D}^\theta$ and $\{\csc\theta F\bar{e}_1, ..., \csc\theta F\bar{e}_{m_2}\}$ is also an orthonormal basis of $F\mathcal{D}^\theta$, where θ is the slant angle of $\mathcal{D}^\theta$ and $m_2 = 2n_2 = dim(M^\theta)$.*

Theorem 5.8. *Let $M = {}_{f_2}M^T \times_{f_1} M^\theta$ be a doubly warped product proper semi-slant submanifold a g.c.K. manifold $(\bar{M}, J, \omega, g)$ such that the Lee vector field B is tangent to M. Then*
(i) *the squared norm of the second fundamental form h of M satisfies*

$$\|h\|^2 \geq m_1\Big(\csc^2\theta + \cot^2\theta\Big)\|B^\theta\|^2 + m_2\Big(\csc^2\theta + (m_2 - 1)\cot^2\theta\Big)\|B^T\|^2, \tag{51}$$

where $m_1 = 2n_1 = dim(M^T)$, $m_2 = 2n_2 = dim(M^\theta)$.
(ii) *If the equality sign of (51) holds identically, then M^θ is also totally umbilical in the ambient manifold $\bar{M}$.*

Proof. The squared norm of the second fundamental form h can be written as

$$\|h\|^2 = \|h(\mathcal{D}^T, \mathcal{D}^T)\|^2 + \|h(\mathcal{D}^T, \mathcal{D}^\theta)\|^2 + \|h(\mathcal{D}^\theta, \mathcal{D}^\theta)\|^2.$$

In view of decomposition (17), which can be explicitly written as follows:

$$\begin{aligned}\|h\|^2 = &\sum_{r,s=1}^{m_1}\sum_{i=1}^{m_2} g(h(e_r,e_s),e_i^*)^2 + \sum_{i,j=1}^{m_2}\sum_{r=1}^{m_1} g(h(e_r,\bar{e}_i),e_j^*)^2 + \sum_{r,s=1}^{m_1}\sum_{t=1}^{l} g(h(e_r,e_s),\hat{e}_t)^2 \\ &+ \sum_{r=1}^{m_1}\sum_{i=1}^{m_2}\sum_{t=1}^{l} g(h(e_r,\bar{e}_i),\hat{e}_t)^2 + \|h(\mathcal{D}^\theta,\mathcal{D}^\theta)\|^2,\end{aligned} \tag{52}$$

where $l = dim(\overline{\mathcal{D}})$. Hence, we have

$$\|h\|^2 \geq \sum_{r,s=1}^{m_1}\sum_{i=1}^{m_2} g(h(e_r,e_s),e_i^*)^2 + \sum_{i,j=1}^{m_2}\sum_{r=1}^{m_1} g(h(\bar{e}_i,e_r),e_j^*)^2.$$

By Remark 5.7, we write

$$\|h\|^2 \geq \sum_{r,s=1}^{m_1}\sum_{i=1}^{m_2} g(h(e_r,e_s),\csc\theta F\bar{e}_j)^2 + \sum_{i,j=1}^{m_2}\sum_{r=1}^{m_1} g(h(\bar{e}_i,e_r),\csc\theta F\bar{e}_j)^2.$$

Using (45) and (46), we obtain

$$\|h\|^2 \geq \quad \csc^2\theta \textstyle\sum_{r,s=1}^{m_1} \displaystyle\sum_{i=1}^{m_2}\Big(\omega(P\bar{e}_i)g(e_r,e_s) + \omega(\bar{e}_i)g(e_r,Je_s)\Big)^2 + \csc^2\theta \sum_{i,j=1}^{m_2}\sum_{r=1}^{m_1}\Big(\omega(Je_r)g(\bar{e}_i,\bar{e}_j) + \omega(e_r)g(\bar{e}_i,P\bar{e}_j)\Big)^2,$$

since $\omega(F\bar{e}_i) = 0$ in the case of the Lee vector field B is tangent to M. By a direct calculation, we get

$$\begin{aligned}\|h\|^2 \geq \quad &\csc^2\theta \textstyle\sum_{r,s=1}^{m_1} \displaystyle\sum_{i=1}^{m_2}\Big\{\omega^2(P\bar{e}_i)g^2(e_r,e_s) + \omega^2(\bar{e}_i)g^2(e_r,Je_s) + 2\omega(P\bar{e}_i)\omega(\bar{e}_i)g(e_r,e_s)g(e_r,Je_s)\Big\} \\ &+ \csc^2\theta\sum_{i,j=1}^{m_2}\sum_{r=1}^{m_1}\Big\{\omega^2(Je_r)g^2(\bar{e}_i,\bar{e}_j) + \omega^2(e_r)g^2(\bar{e}_i,P\bar{e}_j) + 2\omega(Je_r)\omega(e_r)g(\bar{e}_i,\bar{e}_j)g(\bar{e}_i,P\bar{e}_j)\Big\}.\end{aligned}$$

Here, by using (6)

$$\begin{aligned}&\sum_{i,j=1}^{m_2}\sum_{r=1}^{m_1}\omega(Je_r)\omega(e_r)g(\bar{e}_i,\bar{e}_j)g(\bar{e}_i,P\bar{e}_j) \\ &= \sum_{i,j=1}^{m_2}\sum_{r=1}^{m_1} g(B,Je_r)g(\bar{e}_i,\bar{e}_j)g(B,e_r)g(\bar{e}_i,P\bar{e}_j) \\ &= \sum_{i,j=1}^{m_2}\sum_{r=1}^{m_1} g(B,Je_r)g(B,e_r)g(\bar{e}_i,\bar{e}_j)g(\bar{e}_i,P\bar{e}_j) \\ &= -\sum_{i,j=1}^{m_2}\sum_{r=1}^{m_1} g(JB,e_r)g(B,e_r)g(\bar{e}_i,\bar{e}_j)g(\bar{e}_i,P\bar{e}_j) \\ &= -g(JB^T,B^T)\sum_{i,j=1}^{m_2} g(\bar{e}_i,\bar{e}_j)g(\bar{e}_i,P\bar{e}_j) = 0\,.\end{aligned}$$

In a similar way, we can conclude that

$$\sum_{r,s=1}^{m_1}\sum_{i=1}^{m_2}\omega(P\bar{e}_i)\omega(\bar{e}_i)g(e_r,e_s)g(e_r,Je_s) = 0\,.$$

Thus, we arrive at

$$\begin{aligned}\|h\|^2 \geq \quad & \csc^2\theta \textstyle\sum_{r,s=1}^{m_1} \displaystyle\sum_{i=1}^{m_2} \Big\{\omega^2(P\bar{e}_i)g^2(e_r, e_s) + \omega^2(\bar{e}_i)g^2(e_r, Je_s)\Big\} \\ & + \csc^2\theta \sum_{i,j=1}^{m_2} \sum_{r=1}^{m_1} \Big\{\omega^2(Je_r)g^2(\bar{e}_i, \bar{e}_j) + \omega^2(e_r)g^2(\bar{e}_i, P\bar{e}_j)\Big\}.\end{aligned}$$

Again by Remark 5.7, we find

$$\begin{aligned}\|h\|^2 \geq \quad & \csc^2\theta\Big\{\cos^2\theta \textstyle\sum_{r,s=1}^{m_1} \displaystyle\sum_{k=1}^{m_2} \omega^2(\bar{a}_k)g^2(e_r, e_s) + \sum_{r,s=1}^{m_1}\sum_{i=1}^{m_2} \omega^2(\bar{e}_i)g^2(e_r, Je_s)\Big\} \\ & + \csc^2\theta\Big\{\sum_{i,j=1}^{m_2} \sum_{r=1}^{m_1} \omega^2(Je_r)g^2(\bar{e}_i, \bar{e}_j) + \sum_{i,j=1}^{m_2} \sum_{r=1}^{m_1} \omega^2(e_r)g^2(\bar{e}_i, P\bar{e}_j)\Big\}.\end{aligned}$$

On the other hand, for $i, j \in \{1, 2, ..., m_2\}$, we have

$$g(\bar{e}_i, P\bar{e}_j) = \begin{cases} \cos\theta & , if \quad i \neq j, \\ 0 & , if \quad i = j, \end{cases}$$

since $\mathcal{D}^\theta$ is a slant distribution with slant angle θ.

Consequently, $\displaystyle\sum_{i,j=1}^{m_2} g^2(\bar{e}_i, P\bar{e}_j) = m_2(m_2-1)\cos^2\theta$. Upon a straightforward calculation, we obtain the following inequality:

$$\|h\|^2 \geq m_1 \cot^2\theta\|B^\theta\|^2 + m_1 \csc^2\theta\|B^\theta\|^2 + m_2 \csc^2\theta\|B^T\|^2 + m_2(m_2 - 1)\cot^2\theta\|B^T\|^2.$$

Rearranging the last inequality, we get the inequality (51). If the equality sign of (51) holds identically, then we have $h(\mathcal{D}^\theta, \mathcal{D}^\theta) = 0$ from (52). Namely, h vanishes on $\mathcal{D}^\theta$. Since $\mathcal{D}^\theta$ is a totally umbilical distribution on M, it follows that M^θ is totally umbilical in $\bar{M}$. $\square$

Remark 5.9. *Whether the Lee form ω is exact or not does not change all the results in this paper. Thus, these results also hold for locally conformal Kaehler case.*

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