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$\mathcal{A}^{\mathcal{I}}$-Statistical Limit Points and $\mathcal{A}^{\mathcal{I}}$-Statistical Cluster Points

Prasanta Malik[a], Samiran Das[a]

[a]*Department of Mathematics, The University of Burdwan, Golapbag, Burdwan-713104, West Bengal, India.*

Abstract. In this paper using a non-negative regular summability matrix $\mathcal{A}$ and a non trivial admissible ideal $\mathcal{I}$ of subsets of $\mathbb{N}$ we have introduced the notion of $\mathcal{A}^{\mathcal{I}}$-statistical limit point as a generalization of $\mathcal{A}$-statistical limit point of sequences of real numbers. We have also studied some basic properties of the sets of all $\mathcal{A}^{\mathcal{I}}$-statistical limit points and $\mathcal{A}^{\mathcal{I}}$-statistical cluster points of real sequences including their interrelationship. Also introducing additive property of $\mathcal{A}^{\mathcal{I}}$-density zero sets we have established $\mathcal{A}^{\mathcal{I}}$-statistical analogue of some completeness theorems of $\mathbb{R}$.

1. Introduction and background:

The notion of statistical convergence of real sequences was introduced by Fast [12] (also independently by Schoenberg [33]) as a generalization of the usual notion of convergence, using the notion of natural density of subsets of $\mathbb{N}$, the set of all natural numbers. A set $\mathcal{B} \subset \mathbb{N}$ is said to have natural density $d(\mathcal{B})$ if

$$d(\mathcal{B}) = \lim_{n\to\infty} \frac{|\mathcal{B}(n)|}{n},$$

where $\mathcal{B}(n) = \{m \leq n : m \in \mathcal{B}\}$ and $|\mathcal{B}(n)|$ denotes the number of elements in $\mathcal{B}(n)$. Note that one can write $d(\mathcal{B}) = \lim_{n\to\infty}(C_1\chi_{\mathcal{B}})_n$, where $C_1 = (C, 1)$ is the Cesaro matrix of order 1 and $\chi_{\mathcal{B}}$ is the characteristic function of $\mathcal{B}$.

A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $\xi \in \mathbb{R}$, if for every $\epsilon > 0$, $d(\mathcal{B}(\epsilon)) = 0$, where $\mathcal{B}(\epsilon) = \{k \in \mathbb{N} : |x_k - \xi| \geq \epsilon\}$. Study in this line turned out to be one of the active research area in summability theory after the works of Salat [29] and Fridy [14]. Applying this notion of statistical convergence, the concepts of statistical limit point and statistical cluster point of real sequences were introduced by Fridy [15].

If $\{x_{k_j}\}_{j\in\mathbb{N}}$ is a subsequence of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$ and $Q = \{k_j : j \in \mathbb{N}\}$, then we use the notation $\{x\}_Q$ to denote the subsequence $\{x_{k_j}\}_{j\in\mathbb{N}}$. In case $d(Q) = 0$, $\{x\}_Q$ is called a thin subsequence of x. On the other hand $\{x\}_Q$ is called a nonthin subsequence of x if $d(Q) \neq 0$, where $d(Q) \neq 0$ means that either $d(Q)$ is a positive number or Q fails to have natural density.

A real number p is called a statistical limit point of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if there exists a nonthin subsequence of x that converges to p.

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Corresponding author: Prasanta Malik
Email addresses: pmjupm@yahoo.co.in (Prasanta Malik), das91samiran@gmail.com (Samiran Das)

A real number q is called a statistical cluster point of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if for every $\epsilon > 0$ the set $\{k \in \mathbb{N} : \left|x_k - q\right| < \epsilon\}$ does not have natural density zero.

More primary work on this convergence can be found in [1–3, 16, 28, 34], where many more references are mentioned.

In 1981, Freedman and Sember [13] generalized the concept of natural density to the notion of $\mathcal{A}$-density by replacing the Cesaro matrix C_1 with an arbitrary non-negative regular summability matrix $\mathcal{A}$. An $\mathbb{N}\times\mathbb{N}$ matrix $\mathcal{A} = (a_{nk}), a_{nk} \in \mathbb{R}$ is said to be a regular summability matrix if for any convergent sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers with limit ξ, $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{\infty} a_{nk}x_k = \xi$, and $\mathcal{A}$ is called non-negative if $a_{nk} \geq 0, \forall n, k$. The well-known Silvermann- Toepliz's theorem asserts that an $\mathbb{N}\times\mathbb{N}$ matrix $\mathcal{A} = (a_{nk}), a_{nk} \in \mathbb{R}$ is regular if and only if the following three conditions are satisfied:

(i) $\|\mathcal{A}\| = \sup\limits_n \sum\limits_k |a_{nk}| < \infty$,

(ii) $\lim\limits_{n\to\infty} a_{nk} = 0$ for each k,

(iii) $\lim\limits_{n\to\infty} \sum\limits_k a_{nk} = 1$.

Throughout the paper we take $\mathcal{A} = (a_{nk})$ as an $\mathbb{N}\times\mathbb{N}$ non negative regular summability matrix.

For a non negative regular summability matrix $\mathcal{A} = (a_{nk})$, a set $\mathcal{B} \subset \mathbb{N}$ is said to have $\mathcal{A}$-density $\delta_{\mathcal{A}}(\mathcal{B})$, if

$$\delta_{\mathcal{A}}(\mathcal{B}) = \lim_{n\to\infty}\sum_{k\in\mathcal{B}} a_{nk}.$$

Using this notion of $\mathcal{A}$-density, the notion of statistical convergence was extended to the notion of $\mathcal{A}$-statistical convergence by Kolk [19], which included the ideas of statistical convergence [12, 33], λ-statistical convergence [25] or lacunary statistical convergence [17] as special cases.

A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be $\mathcal{A}$-statistically convergent to ξ if for every $\epsilon > 0$, $\delta_{\mathcal{A}}(B(\epsilon)) = 0$, where $B(\epsilon) = \{k \in \mathbb{N} : |x_k - \xi| \geq \epsilon\}$.

Using this notion of $\mathcal{A}$-statistical convergence, the concepts of statistical limit point and statistical cluster point of real sequences were extended to the notions of $\mathcal{A}$-statistical limit point and $\mathcal{A}$-statistical cluster point by Connor et al. [4].

If $\{x\}_Q$ is a subsequence of a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ and $\delta_{\mathcal{A}}(Q) = 0$, then $\{x\}_Q$ is called an $\mathcal{A}$-thin subsequence of x. On the other hand $\{x\}_Q$ is called an $\mathcal{A}$-nonthin subsequence of x if $\delta_{\mathcal{A}}(Q) \neq 0$, where $\delta_{\mathcal{A}}(Q) \neq 0$ means that either $\delta_{\mathcal{A}}(Q)$ is a positive number or Q fails to have $\mathcal{A}$-density.

A real number p is called an $\mathcal{A}$-statistical limit point of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if there exists an $\mathcal{A}$-nonthin subsequence of x that converges to p.

A real number q is called an $\mathcal{A}$-statistical cluster point of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if for every $\epsilon > 0$ the set $\{k \in \mathbb{N} : \left|x_k - q\right| < \epsilon\}$ does not have $\mathcal{A}$-density zero.

If $\Lambda_x^{\mathcal{A}}$, $\Gamma_x^{\mathcal{A}}$ and L_x denote the set of all $\mathcal{A}$-statistical limit points, the set of all $\mathcal{A}$-statistical cluster points and the set of all ordinary limit points of x, then clearly $\Lambda_x^{\mathcal{A}} \subset \Gamma_x^{\mathcal{A}} \subset L_x$

More primary works on this convergence can be found in [8, 9, 18, 25], where many more references are mentioned.

The concept of statistical convergence was generalized to $\mathcal{I}$-convergence by Kostyrko et al. [20] based on the notion of an ideal $\mathcal{I}$ of subsets of $\mathbb{N}$.

A non-empty family $\mathcal{I}$ of subsets of a non empty set S is called an ideal in S if $\mathcal{I}$ is hereditary (i.e. $A \in \mathcal{I}, B \subset A \Rightarrow B \in \mathcal{I}$) and additive (i.e. $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$).

An ideal $\mathcal{I}$ in a non-empty set S is called non-trivial if $S \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$.

A non-trivial ideal $\mathcal{I}$ in $S(\neq \emptyset)$ is called admissible if $\{z\} \in \mathcal{I}$ for each $z \in S$.

Throughout the paper we take $\mathcal{I}$ as a non-trivial admissible ideal in $\mathbb{N}$ unless otherwise mentioned.

A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent to ξ if for any $\epsilon > 0, \{k \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\} \in \mathcal{I}$. In this case we write $\mathcal{I}\text{-}\lim\limits_{k\to\infty} x_k = \xi$.

More works in this line can be seen in [7, 21–23, 26, 27] and many others.

Recently in 2012 using the notion of $\mathcal{I}$-convergence, the concept of $\mathcal{A}$-statistical convergence was extended to $\mathcal{A}^{\mathcal{I}}$-statistical convergence by Savas et al. [31], which included the ideas of $\mathcal{I}$-statistical convergence [5], $\mathcal{I}_\lambda$-statistical convergence [30] or $\mathcal{I}$-lacunary statistical convergence [5] as special cases. More recent works in this line can be seen in [10, 11, 32] where many references are mentioned.

If $\mathcal{A} = (a_{nk})$ is a $\mathbb{N} \times \mathbb{N}$ non negative regular summability matrix, then a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be $\mathcal{A}^{\mathcal{I}}$-statistically convergent to ξ if for any $\epsilon > 0$ and $\delta > 0$, $\{n \in \mathbb{N} : \sum\limits_{k\in B(\epsilon)} a_{nk} \geq \delta\} \in \mathcal{I}$, where $B(\epsilon) = \{k \in \mathbb{N} : |x_k - \xi| \geq \epsilon\}$. In this case we write $\mathcal{I}\text{-}st_{\mathcal{A}}\text{-}\lim\limits_{k\to\infty} x_k = \xi$ or simply $x_k \xrightarrow{\mathcal{A}^{\mathcal{I}}\text{-}st} \xi$. Note that if $\mathcal{I} = \mathcal{I}_{fin} = \{\mathcal{K} \subset \mathbb{N} : |\mathcal{K}| < \infty\}$, then the notions of $\mathcal{A}^{\mathcal{I}}$-statistical convergence coincide with the notion of $\mathcal{A}$-statistical convergence [19].

Also in [18], the notion of $\mathcal{A}^{\mathcal{I}}$-statistical cluster point was introduced as a generalization of $\mathcal{A}$-statistical cluster point, via the notion of $\mathcal{A}^{\mathcal{I}}$-density. A subset $\mathcal{M}$ of $\mathbb{N}$ is said to have $\mathcal{A}^{\mathcal{I}}$-density $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M})$, if

$$\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}) = \mathcal{I} - \lim_{n\to\infty} \sum_{k\in\mathcal{M}} a_{nk}.$$

Using the notion of $\mathcal{A}^{\mathcal{I}}$-density, the definition of $\mathcal{A}^{\mathcal{I}}$-statistical convergence can be restated as follows: A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be $\mathcal{A}^{\mathcal{I}}$-statistically convergent to ξ if for any $\epsilon > 0$, $\delta_{\mathcal{A}^{\mathcal{I}}}(B(\epsilon)) = 0$, where $B(\epsilon) = \{k \in \mathbb{N} : |x_k - \xi| \geq \epsilon\}$.

A real number p is said to be an $\mathcal{A}^{\mathcal{I}}$-statistical cluster point of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if for each $\epsilon > 0$, $\delta_{\mathcal{A}^{\mathcal{I}}}(B(\epsilon)) \neq 0$, where $B(\epsilon) = \{k \in \mathbb{N} : |x_k - p| < \epsilon\}$. Note that $\delta_{\mathcal{A}^{\mathcal{I}}}(B(\epsilon)) \neq 0$ means, either $\delta_{\mathcal{A}^{\mathcal{I}}}(B(\epsilon)) > 0$ or $\mathcal{A}^{\mathcal{I}}$-density of $B(\epsilon)$ does not exist.

In this paper using the concept of $\mathcal{A}^{\mathcal{I}}$-density we have extended the concept of $\mathcal{A}$-statistical limit point of sequences of real numbers to $\mathcal{A}^{\mathcal{I}}$-statistical limit point. We have established relationship among $\mathcal{A}^{\mathcal{I}}$-statistical limit points, $\mathcal{A}^{\mathcal{I}}$-statistical cluster points and $\mathcal{A}$-statistical cluster points of a sequence of real numbers. We also have studied some basic properties of the sets of all $\mathcal{A}^{\mathcal{I}}$-statistical limit points and $\mathcal{A}^{\mathcal{I}}$-statistical cluster points of sequences of real numbers not done earlier. Also we have introduced additive property of $\mathcal{A}^{\mathcal{I}}$-density zero sets and established $\mathcal{A}^{\mathcal{I}}$-statistical analogue of some completeness theorems of $\mathbb{R}$.

2. $\mathcal{A}^{\mathcal{I}}$-statistical limit points and $\mathcal{A}^{\mathcal{I}}$-statistical cluster points

In this section we introduce the notion of $\mathcal{A}^{\mathcal{I}}$-statistical limit point and discuss some basic properties of the set of all $\mathcal{A}^{\mathcal{I}}$-statistical limit points and the set of all $\mathcal{A}^{\mathcal{I}}$-statistical cluster point of real sequences. For this we first study some properties of $\mathcal{A}^{\mathcal{I}}$-density not done earlier.

Throughout the paper $\mathbb{N}$, $\mathbb{R}$ denote the set of all natural numbers, the set of all real numbers respectively, x denotes a real sequence $\{x_k\}_{k\in\mathbb{N}}$ and L_x denotes the set of all ordinary limit points of the sequence x. Also $\mathcal{I}$ denotes a non-trivial admissible ideal in $\mathbb{N}$ and $\mathcal{A} = (a_{nk})$ denotes an $\mathbb{N} \times \mathbb{N}$ non negative regular summability matrix unless otherwise mentioned.

Following the line of Freedman et al. [13] and Kostyrko et al. [20] we first introduce the concepts of lower $\mathcal{A}^{\mathcal{I}}$-density and upper $\mathcal{A}^{\mathcal{I}}$-density associated with a lower $\mathcal{A}^{\mathcal{I}}$-density of a set $\mathcal{M} \subset \mathbb{N}$.

Definition 2.1. *A set $\mathcal{M} \subset \mathbb{N}$ is said to have lower $\mathcal{A}^{\mathcal{I}}$-density $\underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M})$ if*

$$\underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}) = \mathcal{I}\text{-}\liminf_{n\to\infty} (\mathcal{A}\chi_{\mathcal{M}})_n = \mathcal{I}\text{-}\liminf_{n\to\infty} \sum_{m\in\mathcal{M}} a_{nm}.$$

Definition 2.2. *The upper $\mathcal{A}^{\mathcal{I}}$-density $\bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(M)$ associated with a lower $\mathcal{A}^{\mathcal{I}}$-density $\underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M})$ of a set $M \subset \mathbb{N}$ is defined by*

$$\bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(M) = 1 - \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathbb{N} \setminus \mathcal{M}).$$

Lemma 2.1. $\bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(M) = \mathcal{I}\text{-}\limsup\limits_{n\to\infty} (\mathcal{A}\chi_{\mathcal{M}})_n = \mathcal{I}\text{-}\limsup\limits_{n\to\infty} \sum\limits_{m\in\mathcal{M}} a_{nm}.$

Proof. Let $\bar{1} = (1, 1, 1, ...)$ and $J = (i_{nk})$ be an $\mathbb{N} \times \mathbb{N}$ matrix such that $i_{nk} = 1$ for $n = k$ and 0 otherwise. Since $\chi_{\mathbb{N}\setminus\mathcal{M}} = 1 - \chi_{\mathcal{M}}$ and $\mathcal{I}\text{-}\lim_{n\to\infty}\{(J.\bar{1})_n - (\mathcal{A}.\bar{1})_n\} = 0$ so

$$\begin{aligned}
\bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(M) &= 1 - \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathbb{N}\setminus\mathcal{M}) = 1 - \mathcal{I}\text{-}\liminf_{n\to\infty}(\mathcal{A}\chi_{\mathbb{N}\setminus\mathcal{M}})_n \\
&= 1 - \mathcal{I}\text{-}\liminf_{n\to\infty}\Big(\sum_{k\in\mathbb{N}\setminus\mathcal{M}} a_{nk}\Big) = 1 - \mathcal{I}\text{-}\liminf_{n\to\infty}\Big(\sum_{k\in\mathbb{N}} a_{nk} - \sum_{k\in\mathcal{M}} a_{nk}\Big) \\
&= \mathcal{I}\text{-}\limsup_{n\to\infty}\Big(1 - \sum_{k\in\mathbb{N}} a_{nk} + \sum_{k\in\mathcal{M}} a_{nk}\Big) \\
&= \mathcal{I}\text{-}\limsup_{n\to\infty}\{(J.\bar{1})_n - (\mathcal{A}.\bar{1})_n + (\mathcal{A}\chi_{\mathcal{M}})_n\} \\
&= \mathcal{I}\text{-}\limsup_{n\to\infty}(\mathcal{A}\chi_{\mathcal{M}})_n = \mathcal{I}\text{-}\limsup_{n\to\infty}\sum_{m\in\mathcal{M}} a_{nm}.
\end{aligned}$$

□

Lemma 2.2. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two subsets of $\mathbb{N}$. Then*

(i) $\underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\emptyset) = 0$, $\underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathbb{N}) = 1$ *and* $0 \le \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i) \le 1$, *for* $i = 1, 2$,
(ii) $|\mathcal{M}_1 \Delta \mathcal{M}_2| < \infty \Rightarrow \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) = \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2)$,
(iii) $\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset \Rightarrow \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) + \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2) \le \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2)$,
(iv) $\underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) + \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2) \le 1 + \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cap \mathcal{M}_2)$,
(v) $\mathcal{M}_1 \subset \mathcal{M}_2 \Rightarrow \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) \le \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2)$.

Proof. (*i*) It is clear that $\underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\emptyset) = 0$ and $\underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathbb{N}) = 1$. Now for $i = 1, 2$

$$\begin{aligned}
& 0 \le \sum_{k\in\mathcal{M}_i} a_{nk} \le \sum_{k\in\mathbb{N}} a_{nk} \Rightarrow \ 0 \le \liminf_{n\to\infty}\sum_{k\in\mathcal{M}_i} a_{nk} \le \liminf_{n\to\infty}\sum_{k\in\mathbb{N}} a_{nk} \\
\Rightarrow &\quad 0 \le \mathcal{I}\text{-}\liminf_{n\to\infty}\sum_{k\in\mathcal{M}_i} a_{nk} \le \mathcal{I}\text{-}\liminf_{n\to\infty}\sum_{k\in\mathbb{N}} a_{nk} \\
\Rightarrow &\quad 0 \le \mathcal{I}\text{-}\liminf_{n\to\infty}\sum_{k\in\mathcal{M}_i} a_{nk} \le 1 \Rightarrow \ 0 \le \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i) \le 1.
\end{aligned}$$

(*ii*) Let $|\mathcal{M}_1 \Delta \mathcal{M}_2| < \infty$. Then $\exists$ an $N_0 \in \mathbb{N}$ such that $\chi_{\mathcal{M}_1}(m) = \chi_{\mathcal{M}_2}(m)$ except $m = 1, 2, ..., N_0$. So

$$\begin{aligned}
\left|(\mathcal{A}\chi_{\mathcal{M}_1})_n - (\mathcal{A}\chi_{\mathcal{M}_2})_n\right| &= \left|\sum_m a_{nm}\chi_{\mathcal{M}_1}(m) - \sum_m a_{nm}\chi_{\mathcal{M}_2}(m)\right| \\
\le \quad \sum_{m=1}^{N_0} a_{nm}\left|\chi_{\mathcal{M}_1}(m) - \chi_{\mathcal{M}_2}(m)\right| &\le \sum_{m=1}^{N_0} a_{nm} \longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

This gives $\liminf_{n\to\infty}(\mathcal{A}\chi_{\mathcal{M}_1})_n = \liminf_{n\to\infty}(\mathcal{A}\chi_{\mathcal{M}_2})_n$ and since $\mathcal{I}$ is a nontrivial admissible ideal so $\mathcal{I}\text{-}\liminf_{n\to\infty}(\mathcal{A}\chi_{\mathcal{M}_1})_n = \mathcal{I}\text{-}\liminf_{n\to\infty}(\mathcal{A}\chi_{\mathcal{M}_2})_n$. Therefore $\underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) = \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2)$.

(*iii*) Let $\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset$. Then $\chi_{\mathcal{M}_1\cup\mathcal{M}_2} = \chi_{\mathcal{M}_1} + \chi_{\mathcal{M}_2}$ and so

$$\begin{aligned}
\underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2) &= \mathcal{I}\text{-}\liminf_{n\to\infty}(\mathcal{A}\chi_{\mathcal{M}_1\cup\mathcal{M}_2})_n = \mathcal{I}\text{-}\liminf_{n\to\infty}(\mathcal{A}\chi_{\mathcal{M}_1} + \mathcal{A}\chi_{\mathcal{M}_2})_n \\
&\ge \mathcal{I}\text{-}\liminf_{n\to\infty}(\mathcal{A}\chi_{\mathcal{M}_1})_n + \mathcal{I}\text{-}\liminf_{n\to\infty}(\mathcal{A}\chi_{\mathcal{M}_2})_n \\
&= \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) + \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2).
\end{aligned}$$

(*iv*) Since $\chi_{\mathcal{M}_1\cup\mathcal{M}_2} = \chi_{\mathcal{M}_1} + \chi_{\mathcal{M}_2} - \chi_{\mathcal{M}_1\cap\mathcal{M}_2}$, so

$$\begin{aligned} & 1 + \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cap \mathcal{M}_2) = 1 + \mathcal{I}\text{-}\liminf_{n\to\infty}(\mathcal{A}\chi_{\mathcal{M}_1\cap\mathcal{M}_2})_n \\ \geq\ & 1 + \mathcal{I}\text{-}\liminf_{n\to\infty}(\mathcal{A}\chi_{\mathcal{M}_1})_n + \mathcal{I}\text{-}\liminf_{n\to\infty}(\mathcal{A}\chi_{\mathcal{M}_2})_n - \mathcal{I}\text{-}\limsup_{n\to\infty}(\mathcal{A}\chi_{\mathcal{M}_1\cup\mathcal{M}_2})_n \\ =\ & 1 + \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) + \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2) - \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2) \\ =\ & \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) + \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2) + \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathbb{N} \setminus (\mathcal{M}_1 \cup \mathcal{M}_2)) \\ \geq\ & \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) + \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2). \end{aligned}$$

(*v*)

$$\begin{aligned} & \mathcal{M}_1 \subset \mathcal{M}_2 \Rightarrow \sum_{m\in\mathcal{M}_1} a_{nm} \leq \sum_{m\in\mathcal{M}_2} a_{nm} \\ \Rightarrow\ & \liminf_{n\to\infty} \sum_{m\in\mathcal{M}_1} a_{nm} \leq \liminf_{n\to\infty} \sum_{m\in\mathcal{M}_2} a_{nm} \\ \Rightarrow\ & \mathcal{I}\text{-}\liminf_{n\to\infty} \sum_{m\in\mathcal{M}_1} a_{nm} \leq \mathcal{I}\text{-}\liminf_{n\to\infty} \sum_{m\in\mathcal{M}_2} a_{nm} \\ \Rightarrow\ & \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) \leq \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2). \end{aligned}$$

□

Lemma 2.3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two subsets of $\mathbb{N}$. Then*

(i) $\bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\emptyset) = 0$, $\bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathbb{N}) = 1$ *and* $0 \leq \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i) \leq 1$, *for* $i = 1, 2$,
(ii) $|\mathcal{M}_1 \Delta \mathcal{M}_2| < \infty \Rightarrow \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) = \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2)$,
(iii) $\bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) + \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2) \geq \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2)$,
(iv) $\mathcal{M}_1 \subset \mathcal{M}_2 \Rightarrow \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) \leq \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2)$,
(v) $\underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i) \leq \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i)$, *for* $i = 1, 2$.

Proof. Proof is similar to that of Lemma 2.2 and so is omitted. □

Note 2.1. *If $\mathcal{M} \subset \mathbb{N}$ and $\underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M})$, $\bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M})$ both exist and equal then their common value is called $\mathcal{A}^{\mathcal{I}}$-density of $\mathcal{M}$ and is denoted by $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M})$ (see [18]) i.e. $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}) = \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}) = \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M})$. In this case we say $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M})$ exists.*

If a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ satisfies a property $\mathfrak{P}$ for each k except for a set of $\mathcal{A}^{\mathcal{I}}$-density zero, then we say that the sequence x satisfies the property $\mathfrak{P}$ for "almost all $k(\mathcal{A}^{\mathcal{I}})$" or in short "$a.a.k(\mathcal{A}^{\mathcal{I}})$".

It is clear that, if $\delta_{\mathcal{A}}(\mathcal{M}) = u$ for $\mathcal{M} \subset \mathbb{N}$, then $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}) = u$.

Lemma 2.4. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two subsets of $\mathbb{N}$ such that $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1)$ and $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M})_2$ exist. Then*

(i) $\delta_{\mathcal{A}^{\mathcal{I}}}(\emptyset) = 1$, $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathbb{N}) = 1$ *and* $0 \leq \delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i) \leq 1$ *for* $i = 1, 2$,
(ii) $|\mathcal{M}_1 \Delta \mathcal{M}_2| < \infty \Rightarrow \delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) = \delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2)$,
(iii) $\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset \Rightarrow \delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) + \delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2) = \delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2)$,
(iv) $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i^c) = 1 - \delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i)$ *for* $i = 1, 2$,
(v) $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i) = 0$ *for* $i = 1, 2 \Rightarrow \delta_{\mathcal{A}^{\mathcal{I}}}\left(\bigcup_{i=1}^{2} \mathcal{M}_i\right) = 0$,
(vi) $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i) = 1$ *for* $i = 1, 2 \Rightarrow \delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cap \mathcal{M}_2) = 1, \delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2) = 1$.

Proof. (i), (ii), and (iii) directly follow from Lemma 2.2 and Lemma 2.3.

(*iv*) For $i = 1, 2$, $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i)$ exists implies $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i^c)$ exists. Now $\bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i) = 1 - \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathbb{N} \setminus \mathcal{M}_i) \Rightarrow \delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i^c) = 1 - \delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i)$.

(*v*) Let $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i) = 0$ for $i = 1, 2$. Then by Lemma 2.3 (iii), we have $\bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2) \leq \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) + \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2) = \delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) + \delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_2) = 0$. Again by Lemma 2.2 (i) and Lemma 2.3 (v) we have $0 \leq \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2) \leq \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2) \leq 0 \Rightarrow \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2) = \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2) = 0 \Rightarrow \delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2) = 0$.

(*vi*) Let $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i) = 1$ for $i = 1, 2$. Then by (iv) we have $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_i^c) = 0$, for $i = 1, 2$. Now by (v) $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1^c \cup \mathcal{M}_2^c) = 0$ and so $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cap \mathcal{M}_2) = 1$.

Again using Lemma 2.2 (v), Lemma 2.3 (i) and (v) we have, $1 = \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1) \leq \underline{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2) \leq \bar{\delta}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2) \leq 1$. Therefore $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}_1 \cup \mathcal{M}_2) = 1$. □

Now following Fridy [15], Kostyrko et al. [20], Connor et al. [4] and Gürdal et al. [18] we introduce the notion of $\mathcal{A}^{\mathcal{I}}$-statistical limit point.

If $\{x\}_{\mathcal{M}}$ is a subsequence of a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ and $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}) = 0$, then $\{x\}_{\mathcal{M}}$ is said to be a subsequence of $\mathcal{A}^{\mathcal{I}}$ density zero or an $\mathcal{A}^{\mathcal{I}}$-thin subsequence of x. On the other hand if $\mathcal{M}$ does not have $\mathcal{A}^{\mathcal{I}}$ density zero i.e., if either $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M})$ is a positive number or $\mathcal{M}$ fails to have $\mathcal{A}^{\mathcal{I}}$ density then $\{x\}_{\mathcal{M}}$ is called an $\mathcal{A}^{\mathcal{I}}$-nonthin subsequence of x.

Definition 2.3. *A real number L is an $\mathcal{A}^{\mathcal{I}}$-statistical limit point of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if there exists an $\mathcal{A}^{\mathcal{I}}$-nonthin subsequence of x that converges to L.*

Note 2.2. *If $\mathcal{I} = \mathcal{I}_{fin} = \{\mathcal{K} \subset \mathbb{N} : |\mathcal{K}| < \infty\}$, then the notions of $\mathcal{A}^{\mathcal{I}}$-statistical limit point and $\mathcal{A}^{\mathcal{I}}$-statistical cluster point [18] coincide with the notions of $\mathcal{A}$-statistical limit point [4] and $\mathcal{A}$-statistical cluster point [4] respectively.*

The set of all $\mathcal{A}^{\mathcal{I}}$-statistical limit points and $\mathcal{A}^{\mathcal{I}}$-statistical cluster points of a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ are denoted by $\Lambda_x^{\mathcal{A}}(\mathcal{I})$ and $\Gamma_x^{\mathcal{A}}(\mathcal{I})$ respectively.

Theorem 2.1. *Let $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers. Then $\Lambda_x^{\mathcal{A}}(\mathcal{I}) \subset \Gamma_x^{\mathcal{A}}(\mathcal{I}) \subset \Gamma_x^{\mathcal{A}}$.*

Proof. Let $\xi \in \Lambda_x^{\mathcal{A}}(\mathcal{I})$. So we get a subsequence $\{x_{k_q}\}_{q\in\mathbb{N}}$ of x with $\lim_{q\to\infty} x_{k_q} = \xi$ and $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}) \neq 0$, where $\mathcal{M} = \{k_q : q \in \mathbb{N}\}$. Let $\varepsilon > 0$ be given. Since $\lim_{q\to\infty} x_{k_q} = \xi$, so $\mathcal{H} = \{k_q : \left|x_{k_q} - \xi\right| \geq \varepsilon\}$ is a finite set. Hence

$$\{k \in \mathbb{N} : |x_k - \xi| < \varepsilon\} \supset \{k_q : q \in \mathbb{N}\} \setminus \mathcal{H}$$

$$\Rightarrow \mathcal{M} = \{k_q : q \in \mathbb{N}\} \subset \{k \in \mathbb{N} : |x_k - \xi| < \varepsilon\} \cup \mathcal{H}.$$

Now if $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : |x_k - \xi| < \varepsilon\}) = 0$, then we have $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{M}) = 0$, which is a contradiction. Thus ξ is an $\mathcal{A}^{\mathcal{I}}$-statistical cluster point of x. Since $\xi \in \Lambda_x^{\mathcal{A}}(\mathcal{I})$ is arbitrary, $\Lambda_x^{\mathcal{A}}(\mathcal{I}) \subset \Gamma_x^{\mathcal{A}}(\mathcal{I})$.

Now let $\eta \in \Gamma_x^{\mathcal{A}}(\mathcal{I})$. Then for any $\varepsilon > 0$,

$$\begin{aligned}
& \delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : \left|x_k - \eta\right| < \varepsilon\} \neq 0 \\
\Rightarrow\quad & \mathcal{I}\text{-}\lim_{n\to\infty} \sum_{\left|x_k-\eta\right|<\varepsilon} a_{nk} \neq 0 \\
\Rightarrow\quad & \lim_{n\to\infty} \sum_{\left|x_k-\eta\right|<\varepsilon} a_{nk} \neq 0 \text{ [since } \mathcal{I} \text{ is an admissible ideal]} \\
\Rightarrow\quad & \delta_{\mathcal{A}}(\{k \in \mathbb{N} : \left|x_k - \eta\right| < \varepsilon\}) \neq 0 \\
\Rightarrow\quad & \eta \in \Gamma_x^{\mathcal{A}}.
\end{aligned}$$

Therefore, $\Lambda_x^{\mathcal{A}}(\mathcal{I}) \subset \Gamma_x^{\mathcal{A}}(\mathcal{I}) \subset \Gamma_x^{\mathcal{A}}$. □

Theorem 2.2. *If $x = \{x_k\}_{k\in\mathbb{N}}$ and $y = \{y_k\}_{k\in\mathbb{N}}$ are two sequences of real numbers such that $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$, then $\Lambda_x^{\mathcal{A}}(\mathcal{I}) = \Lambda_y^{\mathcal{A}}(\mathcal{I})$ and $\Gamma_x^{\mathcal{A}}(\mathcal{I}) = \Gamma_y^{\mathcal{A}}(\mathcal{I})$.*

Proof. Let $\zeta \in \Gamma_x^{\mathcal{A}}(\mathcal{I})$ and $\varepsilon > 0$ be given. Then $\{k \in \mathbb{N} : |x_k - \zeta| < \varepsilon\}$ does not have $\mathcal{A}^{\mathcal{I}}$-density zero. Let $\mathcal{H} = \{k \in \mathbb{N} : x_k = y_k\}$. As $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H}) = 1$ so $\{k \in \mathbb{N} : |x_k - \zeta| < \varepsilon\} \cap \mathcal{H}$ does not have $\mathcal{A}^{\mathcal{I}}$-density zero. Thus $\zeta \in \Gamma_y^{\mathcal{A}}(\mathcal{I})$. Since $\zeta \in \Gamma_x^{\mathcal{A}}(\mathcal{I})$ is arbitrary, so $\Gamma_x^{\mathcal{A}}(\mathcal{I}) \subset \Gamma_y^{\mathcal{A}}(\mathcal{I})$. By symmetry we have $\Gamma_y^{\mathcal{A}}(\mathcal{I}) \subset \Gamma_x^{\mathcal{A}}(\mathcal{I})$. Hence $\Gamma_x^{\mathcal{A}}(\mathcal{I}) = \Gamma_y^{\mathcal{A}}(\mathcal{I})$.

Also let $\eta \in \Lambda_x^{\mathcal{A}}(\mathcal{I})$. Then x has an $\mathcal{A}^{\mathcal{I}}$-nonthin subsequence $\{x_{k_q}\}_{q\in\mathbb{N}}$ that converges to η. Let $Q = \{k_q \in \mathbb{N} : q \in \mathbb{N}\}$. Since $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k_q \in \mathbb{N} : x_{k_q} \neq y_{k_q}\}) = 0$, we have $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k_q \in \mathbb{N} : x_{k_q} = y_{k_q}\}) \neq 0$. Therefore from the latter set we have an $\mathcal{A}^{\mathcal{I}}$-nonthin subsequence $\{y\}_{Q'}$ of $\{y\}_Q$ that converges to η. Thus $\eta \in \Lambda_y^{\mathcal{A}}(\mathcal{I})$. As $\eta \in \Lambda_x^{\mathcal{A}}(\mathcal{I})$ is arbitrary, so $\Lambda_x^{\mathcal{A}}(\mathcal{I}) \subset \Lambda_y^{\mathcal{A}}(\mathcal{I})$. By similar way we get $\Lambda_x^{\mathcal{A}}(\mathcal{I}) \supset \Lambda_y^{\mathcal{A}}(\mathcal{I})$. Hence $\Lambda_x^{\mathcal{A}}(\mathcal{I}) = \Lambda_y^{\mathcal{A}}(\mathcal{I})$. □

We now investigate some topological properties of the set $\Gamma_x^{\mathcal{A}}(\mathcal{I})$ of all $\mathcal{A}^{\mathcal{I}}$-statistical cluster points of x.

Theorem 2.3. *Let $C \subset \mathbb{R}$ be a compact set and $C \cap \Gamma_x^{\mathcal{A}}(\mathcal{I}) = \emptyset$. Then the set $\{k \in \mathbb{N} : x_k \in C\}$ has $\mathcal{A}^{\mathcal{I}}$-density zero.*

Proof. Since $C \cap \Gamma_x^{\mathcal{A}}(\mathcal{I}) = \emptyset$, so for every $\alpha \in C$ there exists a positive real number $\gamma = \gamma(\alpha)$ such that

$$\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : |x_k - \alpha| < \gamma(\alpha)\}) = 0.$$

Let $B(\alpha;\gamma(\alpha)) = \{z \in \mathbb{R} : |z - \alpha| < \gamma(\alpha)\}$. Then the family of open sets $\{B(\alpha;\gamma(\alpha)) : \alpha \in C\}$ form an open cover of C. As C is a compact subset of $\mathbb{R}$ so there exists a finite subcover of the open cover $\{B(\alpha;\gamma(\alpha)) : \alpha \in C\}$ for C, say $\{B(\alpha_j;\gamma(\alpha_j)) : j = 1,2,...,r\}$. Then $C \subset \bigcup_{j=1}^{r} B(\alpha_j;\gamma(\alpha_j))$ and also

$$\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : |x_k - \alpha_j| < \gamma(\alpha_j)\}) = 0 \text{ for } j = 1,2,...,r.$$

Now since $\mathcal{A}$ is a non-negative regular summability matrix so there exists an $N_0 \in \mathbb{N}$ such that for each $n \geq N_0$, we get

$$\sum_{x_k \in C} a_{nk} \leq \sum_{j=1}^{r} \sum_{x_k \in B(\alpha_j;\gamma(\alpha_j))} a_{nk},$$

and by the property of $\mathcal{I}$-convergence,

$$\mathcal{I}\text{-}\lim_{n\to\infty} \sum_{x_k \in C} a_{nk} \leq \sum_{j=1}^{r} \mathcal{I}\text{-}\lim_{n\to\infty} \sum_{x_k \in B(\alpha_j;\gamma(\alpha_j))} a_{nk} = 0.$$

This gives $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k \in C\}) = 0$. □

Theorem 2.4. *Let $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If x has a bounded $\mathcal{A}^{\mathcal{I}}$-nonthin subsequence, then the set $\Gamma_x^{\mathcal{A}}(\mathcal{I})$ is a nonempty closed set.*

Proof. Let $\{x_{k_m}\}_{m\in\mathbb{N}}$ be a bounded $\mathcal{A}^{\mathcal{I}}$-nonthin subsequence of x and C be a compact set such that $x_{k_m} \in C$ for each $m \in \mathbb{N}$. Let $Q = \{k_m : m \in \mathbb{N}\}$. Then $\delta_{\mathcal{A}^{\mathcal{I}}}(Q) \neq 0$. Now if $\Gamma_x^{\mathcal{A}}(\mathcal{I}) = \emptyset$, then $C \cap \Gamma_x^{\mathcal{A}}(\mathcal{I}) = \emptyset$ and then by Theorem 2.3 we get

$$\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k \in C\}) = 0.$$

Now since $\mathcal{A}$ is a non-negative regular summability matrix so there exists an $N_0 \in \mathbb{N}$ such that for each $n \geq N_0$, we get

$$\sum_{k\in Q} a_{nk} \leq \sum_{x_k \in C} a_{nk}$$

so $\delta_{\mathcal{A}^{\mathcal{I}}}(Q) = 0$, which is a contradiction. Therefore $\Gamma_x^{\mathcal{A}}(\mathcal{I}) \neq \emptyset$.

Now to prove $\Gamma_x^{\mathcal{A}}(\mathcal{I})$ is a closed set in $\mathbb{R}$, let ζ be a limit point of $\Gamma_x^{\mathcal{A}}(\mathcal{I})$. Then for all $\varepsilon > 0$, $B(\zeta;\varepsilon) \cap (\Gamma_x^{\mathcal{A}}(\mathcal{I}) \setminus \{\zeta\}) \neq \emptyset$. Let $\eta \in B(\zeta;\varepsilon) \cap (\Gamma_x^{\mathcal{A}}(\mathcal{I}) \setminus \{\zeta\})$. Now we can choose $\varepsilon' > 0$ so that $B(\eta;\varepsilon') \subset B(\zeta;\varepsilon)$. Since $\eta \in \Gamma_x^{\mathcal{A}}(\mathcal{I})$ so

$$\begin{aligned} & \delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : |x_k - \eta| < \varepsilon'\}) \neq 0 \\ \Rightarrow\ & \delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : |x_k - \zeta| < \varepsilon\}) \neq 0. \end{aligned}$$

Therefore $\zeta \in \Gamma_x^{\mathcal{A}}(\mathcal{I})$. □

Definition 2.4. *(a) A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be $\mathcal{A}^{\mathcal{I}}$-statistically bounded above if, there exists $L_1 \in \mathbb{R}$ such that $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k > L_1\}) = 0$.*
(b) A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be $\mathcal{A}^{\mathcal{I}}$-statistically bounded below if, there exists $L_2 \in \mathbb{R}$ such that $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k < L_2\}) = 0$.

Definition 2.5. *[18] A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be $\mathcal{A}^{\mathcal{I}}$-statistically bounded if, there exists $\mathcal{L} > 0$ such that for all $\beta > 0$, the set*

$$\{n \in \mathbb{N} : \sum_{|x_k|>\mathcal{L}} a_{nk} \geq \beta\} \in \mathcal{I}$$
$$\text{i.e.,} \quad \delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : |x_k| > \mathcal{L}\}) = 0.$$

Note 2.3. (i) *Definition 2.5 can be restated as follows: A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is said to be $\mathcal{A}^{\mathcal{I}}$-statistically bounded if there exists a compact set C in $\mathbb{R}$ such that for all $\beta > 0$, the set $\{n \in \mathbb{N} : \sum_{x_k \notin C} a_{nk} \geq \beta\} \in \mathcal{I}$ i.e., $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k \notin C\}) = 0$.*

(ii) *If $\mathcal{I} = \mathcal{I}_{fin} = \{\mathcal{M} \subset \mathbb{N} : |\mathcal{M}| < \infty\}$, then the notion of $\mathcal{A}^{\mathcal{I}}$-statistical boundedness coincide with the notion of $\mathcal{A}$-statistical boundedness.*

Corollary 2.1. *If $x = \{x_k\}_{k\in\mathbb{N}}$ is $\mathcal{A}^{\mathcal{I}}$-statistically bounded, then the set $\Gamma_x^{\mathcal{A}}(\mathcal{I})$ is nonempty and compact.*

Proof. Let C be a compact set in $\mathbb{R}$ such that $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k \notin C\}) = 0$. Then $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k \in C\}) = 1$ and this implies that C contains a bounded $\mathcal{A}^{\mathcal{I}}$-nonthin subsequence of x. So by Theorem 2.4, $\Gamma_x^{\mathcal{A}}(\mathcal{I})$ is a nonempty closed set.

Now to show $\Gamma_x^{\mathcal{A}}(\mathcal{I})$ is compact, it is sufficient to prove that $\Gamma_x^{\mathcal{A}}(\mathcal{I}) \subset C$. If possible let us assume that $\zeta \in \Gamma_x^{\mathcal{A}}(\mathcal{I})$ but $\zeta \notin C$. Since C is compact, so there exists $\varepsilon > 0$ such that $B(\zeta;\varepsilon) \cap C = \emptyset$. Now since $\mathcal{A}$ is a non-negative regular summability matrix so there exists an $N_0 \in \mathbb{N}$ such that for each $n \geq N_0$, we get

$$\sum_{x_k \in B(\zeta;\varepsilon)} a_{nk} \leq \sum_{x_k \notin C} a_{nk}.$$

Therefore $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : |x_k - \zeta| < \varepsilon\}) = 0$, which is a contradicts that $\zeta \in \Gamma_x^{\mathcal{A}}(\mathcal{I})$. Hence $\Gamma_x^{\mathcal{A}}(\mathcal{I}) \subset C$. Therefore the set $\Gamma_x^{\mathcal{A}}(\mathcal{I})$ is nonempty and compact. □

Theorem 2.5. *Let $x = \{x_k\}_{k\in\mathbb{N}}$ be an $\mathcal{A}^{\mathcal{I}}$-statistically bounded sequence. Then for any $\varepsilon > 0$ the set*

$$\left\{k \in \mathbb{N} : d(\Gamma_x^{\mathcal{A}}(\mathcal{I}), x_k) \geq \varepsilon\right\}$$

has $\mathcal{A}^{\mathcal{I}}$-density zero, where $d(\Gamma_x^{\mathcal{A}}(\mathcal{I}), x_k) = \inf_{z\in\Gamma_x^{\mathcal{A}}(\mathcal{I})} |z - x_k|$-the distance from x_k to the set $\Gamma_x^{\mathcal{A}}(\mathcal{I})$.

Proof. Let C be a compact set such that $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k \notin C\}) = 0$. Then by Corollary 2.1, we get $\Gamma_x^{\mathcal{A}}(\mathcal{I})$ is nonempty and $\Gamma_x^{\mathcal{A}}(\mathcal{I}) \subset C$.

If possible, let $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : d(\Gamma_x^{\mathcal{A}}(\mathcal{I}), x_k) \geq \varepsilon'\}) \neq 0$ for some $\varepsilon' > 0$. We define $B(\Gamma_x^{\mathcal{A}}(\mathcal{I});\varepsilon') = \{z \in \mathbb{R} : d(\Gamma_x^{\mathcal{A}}(\mathcal{I}), z) < \varepsilon'\}$ and let $\mathcal{H} = C \setminus B(\Gamma_x^{\mathcal{A}}(\mathcal{I});\varepsilon')$. Then $\mathcal{H}$ is a compact set which contains an $\mathcal{A}^{\mathcal{I}}$-nonthin subsequence of x. Then by Theorem 2.3, $\mathcal{H} \cap \Gamma_x^{\mathcal{A}}(\mathcal{I}) \neq \emptyset$, which is absurd, since $\Gamma_x^{\mathcal{A}}(\mathcal{I}) \subset B(\Gamma_x^{\mathcal{A}}(\mathcal{I});\varepsilon')$. Therefore, $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : d(\Gamma_x^{\mathcal{A}}(\mathcal{I}), x_k) \geq \varepsilon\}) = 0$ for every $\varepsilon > 0$. □

3. Condition AP$\mathcal{A}^{\mathcal{I}}$O

The additive property for sets of zero natural density (APO) was introduced by Freedman et al. [13] and they further extended it for sets of zero $\mathcal{A}$-density. Here we introduce the additive property for sets of zero $\mathcal{A}^{\mathcal{I}}$ density (AP$\mathcal{A}^{\mathcal{I}}$O).

Definition 3.1. ***(Additive property for $\mathcal{A}^{\mathcal{I}}$-density zero sets).*** *The $\mathcal{A}^{\mathcal{I}}$-density $\delta_{\mathcal{A}^{\mathcal{I}}}$ is said to satisfy the condition AP$\mathcal{A}^{\mathcal{I}}$O if given any countable collection of mutually disjoint sets $\{\mathcal{G}_m\}_{m\in\mathbb{N}}$ in $\mathbb{N}$ with $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{G}_m) = 0$ for all $m \in \mathbb{N}$, there exists a collection of sets $\{\mathcal{H}_m\}_{m\in\mathbb{N}}$ in $\mathbb{N}$ such that $\left|\mathcal{G}_m \Delta \mathcal{H}_m\right| < \infty$ for each $m \in \mathbb{N}$ and $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H} = \bigcup\limits_{m=1}^{\infty} \mathcal{H}_m) = 0$.*

Theorem 3.1. *A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real number is $\mathcal{A}^{\mathcal{I}}$-statistically convergent to $\mathcal{L}$ implies there exists a subset $\mathcal{W}$ of $\mathbb{N}$ with $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{W}) = 1$ and $\lim\limits_{\substack{k\in\mathcal{W}\\ k\to\infty}} x_k = \mathcal{L}$ if and only if $\delta_{\mathcal{A}^{\mathcal{I}}}$ has the property AP$\mathcal{A}^{\mathcal{I}}$O.*

Proof. Suppose any sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is $\mathcal{A}^{\mathcal{I}}$-statistically convergent to $\mathcal{L}$ implies there exists a subset $\mathcal{W}$ of $\mathbb{N}$ with $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{W}) = 1$ and $\lim\limits_{\substack{k\in\mathcal{W}\\ k\to\infty}} x_k = \mathcal{L}$. We have to show $\delta_{\mathcal{A}^{\mathcal{I}}}$ has the property AP$\mathcal{A}^{\mathcal{I}}$O.

Let $\{\mathcal{G}_m\}_{m\in\mathbb{N}}$ be a countable collection of mutually disjoint sets in $\mathbb{N}$ with $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{G}_m) = 0$, for every $m \in \mathbb{N}$. Let us construct a sequence $\{x_k\}_{k\in\mathbb{N}}$ as follows

$$x_k = \begin{cases} \frac{1}{m} & \text{if } k \in \mathcal{G}_m, \\ 0 & \text{if } k \notin \bigcup\limits_{m=1}^{\infty} \mathcal{G}_m. \end{cases}$$

Let $\varepsilon > 0$ be given. Then there exists $j \in \mathbb{N}$ such that $\frac{1}{j+1} < \varepsilon$. Then we have

$$\{k \in \mathbb{N} : x_k \geq \varepsilon\} \subset \mathcal{G}_1 \cup \mathcal{G}_2 \cup ... \cup \mathcal{G}_j.$$

Since $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{G}_m) = 0$, for $m = 1, 2, ..., j$, we get $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k \geq \varepsilon\}) = 0$. So $\{x_k\}_{k\in\mathbb{N}}$ is $\mathcal{A}^{\mathcal{I}}$-statistically convergent to 0. Then by the assumption there exists a set $\mathcal{H} \subset \mathbb{N}$, where $\mathcal{H} = \mathbb{N} \setminus \mathcal{W}$, $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H}) = 0$ such that $\lim\limits_{\substack{k\in\mathbb{N}\setminus\mathcal{H}\\ k\to\infty}} x_k = 0$. Therefore for each $m = 1, 2, ...$ we have $n_m \in \mathbb{N}$ such that $n_{m+1} > n_m$ and $x_k < \frac{1}{m}$ for all $k \geq n_m$, $k \in \mathcal{W}$. Thus if $x_k \geq \frac{1}{m}$ and $k \geq n_m$ then $k \in \mathcal{H}$.

Set $\mathcal{H}_m = \{k \in \mathbb{N} : k \in \mathcal{G}_m,\ k \geq n_{m+1}\} \cup \{k \in \mathbb{N} : k \in \mathcal{H},\ n_m \leq k < n_{m+1}\}$, $m \in \mathbb{N}$. Clearly for all $m \in \mathbb{N}$ we have $\left|\mathcal{G}_m \Delta \mathcal{H}_m\right| < \infty$. We now show that $\mathcal{H} = \bigcup\limits_{m=1}^{\infty} \mathcal{H}_m$. Fix $m \in \mathbb{N}$ and let $k \in \mathcal{H}_m$. If $k \in \{j \in \mathbb{N} : j \in \mathcal{H}, n_m \leq j < n_{m+1}\}$, then we are done. If $k \geq n_{m+1}$ and $k \in \mathcal{G}_m$ we have $x_k = \frac{1}{m}$ and so $k \in \mathcal{H}$. Therefore $\mathcal{H}_m \subset \mathcal{H}$ for all $m \in \mathbb{N}$.

Again let $k \in \mathcal{H}$. Then there exists $u \in \mathbb{N}$ such that $n_u \leq k < n_{u+1}$, which implies $k \in \mathcal{H}_u$. Therefore $\mathcal{H} \subset \bigcup\limits_{m=1}^{\infty} \mathcal{H}_m$. Thus $\mathcal{H} = \bigcup\limits_{m=1}^{\infty} \mathcal{H}_m$ and $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H} = \bigcup\limits_{m=1}^{\infty} \mathcal{H}_m) = 0$. This proves that $\delta_{\mathcal{A}^{\mathcal{I}}}$ has the property AP$\mathcal{A}^{\mathcal{I}}$O.

Conversely suppose that $\delta_{\mathcal{A}^{\mathcal{I}}}$ has the property AP$\mathcal{A}^{\mathcal{I}}$O. Let $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence such that x is $\mathcal{A}^{\mathcal{I}}$-statistically convergent to $\mathcal{L}$. Then for each $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - \mathcal{L}| \geq \varepsilon\}$ has $\mathcal{A}^{\mathcal{I}}$-density zero. Let $\mathcal{G}_1 = \{k \in \mathbb{N} : |x_k - \mathcal{L}| \geq 1\}$, $\mathcal{G}_m = \{k \in \mathbb{N} : \frac{1}{m-1} > |x_k - \mathcal{L}| \geq \frac{1}{m}\}$ for $m \geq 2$, $m \in \mathbb{N}$. Then $\{\mathcal{G}_m\}_{m\in\mathbb{N}}$ is a sequence of mutually disjoint sets with $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{G}_m) = 0$ for every $m \in \mathbb{N}$. Then by the assumption there exists a sequence of sets $\{\mathcal{H}_m\}_{m\in\mathbb{N}}$ with $\left|\mathcal{G}_m \Delta \mathcal{H}_m\right| < \infty$ and $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H} = \bigcup\limits_{m=1}^{\infty} \mathcal{H}_m) = 0$. We claim that $\lim\limits_{\substack{k\in\mathbb{N}\setminus\mathcal{H}\\ k\to\infty}} x_k = \mathcal{L}$. To establish our claim, let $\beta > 0$ be given. Then there exists a positive integer j such that $\frac{1}{j+1} < \beta$. Then $\{k \in \mathbb{N} : |x_k - \mathcal{L}| \geq \beta\} \subset \bigcup\limits_{m=1}^{j+1} \mathcal{G}_m$. Now since $\left|\mathcal{G}_m \Delta \mathcal{H}_m\right| < \infty$, for each $m = 1, 2, ..., j+1$, there exists $n' \in \mathbb{N}$ such that $\bigcup\limits_{m=1}^{j+1} \mathcal{G}_m \cap (n', \infty) = \bigcup\limits_{m=1}^{j+1} \mathcal{H}_m \cap (n', \infty)$. Now if $k \notin \mathcal{H}$, $k > n'$, then $k \notin \bigcup\limits_{m=1}^{j+1} \mathcal{H}_m$ and consequently $k \notin \bigcup\limits_{m=1}^{j+1} \mathcal{G}_m$, which implies $|x_k - \mathcal{L}| < \beta$. This completes the proof. □

Theorem 3.2. *If $\delta_{\mathcal{A}^I}$ has the property AP$\mathcal{A}^I$O, then for any sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers there exists a sequence $y = \{y_k\}_{k\in\mathbb{N}}$ such that $L_y = \Gamma_x^{\mathcal{A}}(\mathcal{I})$ and the set $\{k \in \mathbb{N} : x_k \neq y_k\}$ has $\mathcal{A}^I$-density zero.*

Proof. We first prove that $\Gamma_x^{\mathcal{A}}(\mathcal{I}) \subset L_x$. Let $\alpha \in \Gamma_x^{\mathcal{A}}(\mathcal{I})$. So from Theorem 2.1 we have $\alpha \in \Gamma_x^{\mathcal{A}}$. Then $\mathcal{A}$-density of the set

$$\{k \in \mathbb{N} : |x_k - \alpha| < \varepsilon\}$$

is not zero, for every $\varepsilon > 0$. So there exists a subsequence $\{x\}_{\mathcal{K}}$ of x that converges to α. So, $\alpha \in L_x$. Hence $\Gamma_x^{\mathcal{A}}(\mathcal{I}) \subset L_x$.

If $\Gamma_x^{\mathcal{A}}(\mathcal{I}) = L_x$ then the proof is trivial, we take $y = \{y_k\}_{k\in\mathbb{N}} = \{x_k\}_{k\in\mathbb{N}} = x$. Now suppose that $\Gamma_x^{\mathcal{A}}(\mathcal{I})$ is a proper subset of L_x. Let $\zeta \in L_x \setminus \Gamma_x^{\mathcal{A}}(\mathcal{I})$. Choose an open interval J_ζ with center at ζ such that $\delta_{\mathcal{A}^I}(\{k \in \mathbb{N} : x_k \in J_\zeta\}) = 0$. Then the collection of all such J_ζ's is an open cover of $L_x \setminus \Gamma_x^{\mathcal{A}}(\mathcal{I})$ and by the Lindelöf covering lemma there exists a countable subcover, say $\{J_{\zeta_m}\}_{m\in\mathbb{N}}$ of $\{J_\zeta : \zeta \in L_x \setminus \Gamma_x^{\mathcal{A}}(\mathcal{I})\}$ for $L_x \setminus \Gamma_x^{\mathcal{A}}(\mathcal{I})$. Since each ζ_m is a limit point of x, consequently each J_{ζ_m} contains an $\mathcal{A}^I$-thin subsequence of x. Let $J_1 = \{k \in \mathbb{N} : x_k \in J_{\zeta_1}\}, J_m = \{k \in \mathbb{N} : x_k \in J_{\zeta_m}\} \setminus (J_1 \cup J_2 ... \cup J_{m-1}), \forall m \geq 2, m \in \mathbb{N}$. Then $\{J_m\}_{m\in\mathbb{N}}$ is a sequence of mutually disjoint sets with $\delta_{\mathcal{A}^I}(J_m) = 0, \forall m \in \mathbb{N}$. Since $\delta_{\mathcal{A}^I}$ has the property AP$\mathcal{A}^I$O, so there exists a sequence of sets $\{\mathcal{H}_m\}_{m\in\mathbb{N}}$ such that $|J_m \Delta \mathcal{H}_m| < \infty$ for each $m \in \mathbb{N}$ and $\delta_{\mathcal{A}^I}(\mathcal{H} = \bigcup_{m=1}^{\infty} \mathcal{H}_m) = 0$. Then $J_m \setminus \mathcal{H}$ is a finite set and so $\{k \in \mathbb{N} : x_k \in J_{\zeta_m}\} \setminus \mathcal{H}$ is a finite set for each $m \in \mathbb{N}$. Let $\mathbb{N} \setminus \mathcal{H} = \{m_1 < m_2 < ...\}$ and we define a sequence $y = \{y_k\}_{k\in\mathbb{N}}$ as follows

$$y_k = \begin{cases} x_{m_k} & \text{if } k \in \mathcal{H}, \\ x_k & \text{if } k \in \mathbb{N} \setminus \mathcal{H}. \end{cases}$$

Obviously the set $\{k \in \mathbb{N} : x_k \neq y_k\} (\subset \mathcal{H})$ has $\mathcal{A}^I$-density zero and by Theorem 2.2 we have $\Gamma_x^{\mathcal{A}}(\mathcal{I}) = \Gamma_y^{\mathcal{A}}(\mathcal{I})$.

Now we show that $L_y = \Gamma_y^{\mathcal{A}}(\mathcal{I})$. If possible, let $\Gamma_y^{\mathcal{A}}(\mathcal{I}) \subsetneq L_y$ and $\eta \in L_y \setminus \Gamma_y^{\mathcal{A}}(\mathcal{I})$. Then there exists an $\mathcal{A}^I$-thin subsequence of y converging to η.

Now we claim that $\{y\}_{\mathcal{H}}$ has no limit point which is not an $\mathcal{A}^I$-statistical cluster point of y.

Since $\{y_k : k \in \mathcal{H}\} \subset \{y_k : k \in \mathbb{N} \setminus \mathcal{H}\} \Rightarrow \{x_{m_k} : k \in \mathcal{H}\} \subset \{x_k : k \in \mathbb{N} \setminus \mathcal{H}\}$. Now there does not exist any limit point of $\{x\}_{\mathbb{N}\setminus\mathcal{H}}$ which is not an $\mathcal{A}^I$-statistical cluster point of x. For this let γ be a limit point of $\{x\}_{\mathbb{N}\setminus\mathcal{H}}$ which is not an $\mathcal{A}^I$-statistical cluster point of x. So there is an $\mathcal{A}^I$-thin subsequence $\{x\}_{\mathcal{K}}$ of $\{x\}_{\mathbb{N}\setminus\mathcal{H}}$ converging to γ. Now $\{J_{\zeta_m}\}_{m\in\mathbb{N}}$ covers $L_x \setminus \Gamma_x^{\mathcal{A}}(\mathcal{I})$ so it covers $L_{\{x\}_{\mathbb{N}\setminus\mathcal{H}}} \setminus \Gamma_x^{\mathcal{A}}(\mathcal{I})$. Then $\mathcal{K} \setminus \mathcal{M} \subset \{k \in \mathbb{N} : x_k \in J_{\zeta_q}\} \setminus \mathcal{H}$, where $\mathcal{M}$ is a finite subset of $\mathbb{N}$, for some $\zeta_q \in L_x \setminus \Gamma_x^{\mathcal{A}}(\mathcal{I})$, a contradiction.

So there does not exist any limit point of $\{x\}_{\mathbb{N}\setminus\mathcal{H}}$ which is not an $\mathcal{A}^I$-statistical cluster point of x and so there does not exist any limit point of $\{y\}_{\mathbb{N}\setminus\mathcal{H}}$ which is not an $\mathcal{A}^I$-statistical cluster point of y and this gives $\{y\}_{\mathcal{H}}$ has no limit point which is not an $\mathcal{A}^I$-statistical cluster point of y. Therefore no such η can exist. Hence $L_y = \Gamma_y^{\mathcal{A}}(\mathcal{I})$. Consequently $L_y = \Gamma_x^{\mathcal{A}}(\mathcal{I})$. □

Theorem 3.3. *Suppose $x = \{x_k\}_{k\in\mathbb{N}}$ is a sequence of real numbers and $\delta_{\mathcal{A}^I}$ satisfies the property AP$\mathcal{A}^I$O. Then $x_k \xrightarrow{\mathcal{A}^I\text{-}st} \mathcal{L}$ if and only if there exists a sequence $\{g_k\}_{k\in\mathbb{N}}$ so that $x_k = g_k$ for a.a.k($\mathcal{A}^I$) and $g_k \to \mathcal{L}$.*

Proof. Let $x_k \xrightarrow{\mathcal{A}^I\text{-}st} \mathcal{L}$. So by Theorem 3.1, there is a set $\mathcal{W} = \{q_1 < q_2 < ... < q_n < ...\} \subset \mathbb{N}$ such that $\delta_{\mathcal{A}^I}(\mathcal{W}) = 1$ and $\lim_{n\to\infty} x_{q_n} = \mathcal{L}$.

Now we define a sequence $\{g_k\}_{k\in\mathbb{N}}$ as follows:

$$g_k = \begin{cases} x_k, & \text{if } k \in \mathcal{W} \\ \mathcal{L}, & \text{if } k \notin \mathcal{W}. \end{cases}$$

Then clearly, $g_k \to \mathcal{L}$ and also $\delta_{\mathcal{A}^I}(\{k \in \mathbb{N} : x_k \neq g_k\}) = 0$ i.e., $x_k = g_k$ for *a.a.k*($\mathcal{A}^I$).

Conversely, let there exist a sequence $\{g_k\}_{k\in\mathbb{N}}$ such that $x_k = g_k$ for $a.a.k(\mathcal{A}^{\mathcal{I}})$ and $g_k \to \mathcal{L}$. Let $\varepsilon > 0$ be given. Since $\mathcal{A}$ is non-negative regular summability matrix so there exists an $N_0 \in \mathbb{N}$ such that for each $n \geq N_0$, we have

$$\sum_{|x_k-\mathcal{L}|\geq\varepsilon} a_{nk} \leq \sum_{x_k\neq g_k} a_{nk} + \sum_{\left|g_k-\mathcal{L}\right|\geq\varepsilon} a_{nk}.$$

As $\{g_k\}_{k\in\mathbb{N}}$ is convergent to $\mathcal{L}$, so the set $\{k \in \mathbb{N} : \left|g_k - \mathcal{L}\right| \geq \varepsilon\}$ is finite and hence $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : \left|g_k - \mathcal{L}\right| \geq \varepsilon\}) = 0$.
Thus,

$$\begin{aligned}&\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : |x_k - \mathcal{L}| \geq \varepsilon\})\\ \leq\ &\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k \neq g_k\}) + \delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : \left|g_k - \mathcal{L}\right| \geq \varepsilon\}) = 0.\end{aligned}$$

Therefore, $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : |x_k - \mathcal{L}| \geq \varepsilon\}) = 0$. Hence the sequence x is $\mathcal{A}^{\mathcal{I}}$-statistically convergent to $\mathcal{L}$. □

Theorem 3.4. *Suppose $x = \{x_k\}_{k\in\mathbb{N}}$ is a sequence of real numbers and $\delta_{\mathcal{A}^{\mathcal{I}}}$ satisfies the property $AP\mathcal{A}^{\mathcal{I}}O$. If $\mathcal{I}\text{-}st_{\mathcal{A}}\text{-}\lim_{k\to\infty} x_k = \zeta$, then $\Lambda_x^{\mathcal{A}}(\mathcal{I}) = \Gamma_x^{\mathcal{A}}(\mathcal{I}) = \{\zeta\}$.*

Proof. Let $\mathcal{I}\text{-}st_{\mathcal{A}}\text{-}\lim_{k\to\infty} x_k = \zeta$. So for every $\varepsilon > 0$, $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : |x_k - \zeta| < \varepsilon\}) = 1$. Therefore, $\zeta \in \Gamma_x^{\mathcal{A}}(\mathcal{I})$. If possible, let there exist $\eta \in \Gamma_x^{\mathcal{A}}(\mathcal{I})$ such that $\zeta \neq \eta$. Let $\left|\zeta - \eta\right| = \sigma$. Then $\sigma > 0$. Since $\zeta, \eta \in \Gamma_x^{\mathcal{A}}(\mathcal{I})$, so $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{G}) \neq 0$ and $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H}) \neq 0$, where $\mathcal{G} = \{k \in \mathbb{N} : |x_k - \zeta| < \frac{\sigma}{2}\}$ and $\mathcal{H} = \{k \in \mathbb{N} : \left|x_k - \eta\right| < \frac{\sigma}{2}\}$. Since $\zeta \neq \eta$, so $\mathcal{G} \cap \mathcal{H} = \emptyset$ and so $\mathcal{H} \subset \mathcal{G}^c$. Since $\mathcal{I}\text{-}st_{\mathcal{A}}\text{-}\lim_{k\to\infty} x_k = \zeta$, so $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{G}^c) = 0$. Hence $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H}) = 0$, a contradiction. Therefore, $\Gamma_x^{\mathcal{A}}(\mathcal{I}) = \{\zeta\}$.

As $\mathcal{I}\text{-}st_{\mathcal{A}}\text{-}\lim_{k\to\infty} x_k = \zeta$, so by Theorem 3.3, we have $\zeta \in \Lambda_x^{\mathcal{A}}(\mathcal{I})$. Then by Theorem 2.1, we get $\Lambda_x^{\mathcal{A}}(\mathcal{I}) = \Gamma_x^{\mathcal{A}}(\mathcal{I}) = \{\zeta\}$. □

4. $\mathcal{A}^{\mathcal{I}}$-statistical analogous of Completeness Theorems

In this section, following Fridy [15] and Malik et al. [24] we formulate $\mathcal{A}^{\mathcal{I}}$-statistical analogue of the theorems concerning sequences that are equivalent to the completeness of $\mathbb{R}$.

We first consider a sequential version of the least upper bound axiom (in $\mathbb{R}$), namely, Monotone sequence Theorem: every monotone increasing sequence of real numbers which is bounded above, is convergent. The following result is an $\mathcal{A}^{\mathcal{I}}$-statistical analogue of that Theorem.

Theorem 4.1. *Let $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers and $\mathcal{Q} = \{k \in \mathbb{N} : x_k \leq x_{k+1}\}$. If $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{Q}) = 1$ and x is bounded above on $\mathcal{Q}$, then x is $\mathcal{A}^{\mathcal{I}}$-statistically convergent.*

Proof. Since x is bounded above on $\mathcal{Q}$, so let $\mathcal{L}$ be the least upper bound of the range of $\{x_k\}_{k\in\mathcal{Q}}$. Then we have
(i) $x_k \leq \mathcal{L}$, $\forall k \in \mathcal{Q}$
(ii) for a pre-assigned $\varepsilon > 0$, there exists a natural number $k_0 \in \mathcal{Q}$ such that $x_{k_0} > \mathcal{L} - \varepsilon$.
Now let $k \in \mathcal{Q}$ and $k > k_0$. Then $\mathcal{L} - \varepsilon < x_{k_0} \leq x_k < \mathcal{L} + \varepsilon$. Thus $\mathcal{Q} \cap \{k \in \mathbb{N} : k > k_0\} \subset \{k \in \mathbb{N} : \mathcal{L} - \varepsilon < x_k < \mathcal{L} + \varepsilon\}$. Since the set on the left hand side of the inclusion is of $\mathcal{A}^{\mathcal{I}}$-density 1, we have $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : \mathcal{L} - \varepsilon < x_k < \mathcal{L} + \varepsilon\}) = 1$ i.e., $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : |x_k - \mathcal{L}| \geq \varepsilon\}) = 0$. Hence x is $\mathcal{A}^{\mathcal{I}}$-statistically convergent to $\mathcal{L}$. □

Theorem 4.2. *Let $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers and $\mathcal{Q} = \{k \in \mathbb{N} : x_k \geq x_{k+1}\}$. If $\delta_{\mathcal{A}^{\mathcal{I}}}(\mathcal{Q}) = 1$ and x is bounded below on $\mathcal{Q}$, then x is $\mathcal{A}^{\mathcal{I}}$-statistically convergent.*

Proof. The proof is similar to that of Theorem 4.1 and so is omitted. □

Note 4.1. *(a) In the Theorem 4.1 if we replace the criteria that 'x is bounded above on Q' by 'x is $\mathcal{A}^{\mathcal{I}}$-statistically bounded above on Q' then the result still holds. Indeed if x is $\mathcal{A}^{\mathcal{I}}$-statistically bounded above on Q, then there exists $L \in \mathbb{R}$ such that $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in Q : x_k > L\}) = 0$ i.e., $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in Q : x_k \leq L\}) = 1$. Let $\mathcal{S} = \{k \in Q : x_k \leq L\}$ and $L^{'} = sup\{x_k : k \in \mathcal{S}\}$. Then*

(i) $x_k \leq L^{'}$ for all $k \in \mathcal{S}$

(ii) for any $\varepsilon > 0$, there exists a natural number $k_0 \in \mathcal{S}$ such that $x_{k_0} > L^{'} - \varepsilon$. Then proceeding in a similar way as in Theorem 4.1 we get the result.

(b) Similarly, In the Theorem 4.2 if we replace the criteria that 'x is bounded below on Q' by 'x is $\mathcal{A}^{\mathcal{I}}$-statistically bounded below on Q' then the result still holds.

Another completeness result for $\mathbb{R}$ is the Bolzano-Weierstrass Theorem, which tells us that, every bounded sequence of real numbers has a cluster point. The following result is an $\mathcal{A}^{\mathcal{I}}$-statistical analogue of that result.

Theorem 4.3. *Suppose $x = \{x_k\}_{k\in\mathbb{N}}$ is a sequence of real numbers and $\delta_{\mathcal{A}^{\mathcal{I}}}$ has the property $AP\mathcal{A}^{\mathcal{I}}O$. If x has a bounded $\mathcal{A}^{\mathcal{I}}$-nonthin subsequence, then x has an $\mathcal{A}^{\mathcal{I}}$-statistical cluster point.*

Proof. Using Theorem 3.2, we have a sequence $y = \{y_k\}_{k\in\mathbb{N}}$ such that $L_y = \Gamma_x^{\mathcal{A}}(\mathcal{I})$ and $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k = y_k\}) = 1$. Let $\{x\}_Q$ be the bounded $\mathcal{A}^{\mathcal{I}}$-nonthin subsequence of x. Then $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k = y_k\} \cap Q) \neq 0$. Thus y has a bounded $\mathcal{A}^{\mathcal{I}}$-nonthin subsequence and hence by Bolzano-Weierstrass Theorem, $L_y \neq \emptyset$. Thus $\Gamma_x^{\mathcal{A}}(\mathcal{I}) \neq \emptyset$. □

Corollary 4.1. *Suppose $\delta_{\mathcal{A}^{\mathcal{I}}}$ has the property $AP\mathcal{A}^{\mathcal{I}}O$. If x is a bounded sequence of real numbers, then x has an $\mathcal{A}^{\mathcal{I}}$-statistical cluster point.*

The next result is an $\mathcal{A}^{\mathcal{I}}$-statistical analogue of the Heine-Börel Covering Theorem.

Theorem 4.4. *Suppose $\delta_{\mathcal{A}^{\mathcal{I}}}$ has the property $AP\mathcal{A}^{\mathcal{I}}O$. If $x = \{x_k\}_{k\in\mathbb{N}}$ is a bounded sequence of real numbers, then it has an $\mathcal{A}^{\mathcal{I}}$-thin subsequence $\{x\}_Q$ such that $\{x_k : k \in \mathbb{N} \setminus Q\} \cup \Gamma_x^{\mathcal{A}}(\mathcal{I})$ is a compact set.*

Proof. Using Theorem 3.2, we have a sequence $y = \{y_k\}_{k\in\mathbb{N}}$ such that $L_y = \Gamma_x^{\mathcal{A}}(\mathcal{I})$ and $\delta_{\mathcal{A}^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k = y_k\}) = 1$. Let $Q = \{k \in \mathbb{N} : x_k \neq y_k\}$. Then $\delta_{\mathcal{A}^{\mathcal{I}}}(Q) = 0$. Therefore $\{x\}_Q$ is an $\mathcal{A}^{\mathcal{I}}$-thin subsequence of x and $\{x_k : k \in \mathbb{N} \setminus Q\} \cup \Gamma_x^{\mathcal{A}}(\mathcal{I}) = \{y_k : k \in \mathbb{N}\} \cup L_y$. Since the set on the right hand side is compact, so the set on the left hand side is also compact. This completes the proof. □

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