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Modified Mann-type Inertial Subgradient Extragradient Methods for Solving Variational Inequalities in Real Hilbert Spaces

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Abstract. This paper is to investigate the monotone variational inequalities (VIPs) in real Hilbert spaces. We constructed two iterative algorithms based on subgradient extragradient algorithms and Tseng's algorithms for solving VIPs. Convergence analysis of the suggested methods are proved. Several numerical examples to illustrate the efficiency of the methods are given.

1. Introduction

This paper is to consider the following VIPs ([18]) of finding a point $x^* \in C$ such that

$$\langle \Phi x^*, x - x^* \rangle \geq 0, \ \forall x \in C, \tag{1}$$

where $\mathcal{H}$ is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and the norm $\| \cdot \|$, C is a nonempty closed convex subset of $\mathcal{H}$, and $\Phi : C \to \mathcal{H}$ is an operator. We use $VI(\Phi, C)$ to denote the solution set of problem (1).

VIPs is an important research field in fundamental mathematics, which has been expanded in application mathematics and computational mathematics, especially in optimization theory and methods, fixed point theory and methods, equilibrium problems and split problems, see for example, [2, 4, 5, 9, 10], [25]-[38] and so on. It is known that the point $x^* \in VI(\Phi, C)$ iff $x^* \in FixP_C(Id - \rho\Phi)$, i.e.,

$$x^* = P_C(x^* - \rho\Phi x^*), \rho > 0. \tag{2}$$

Recently, many scholars devoted to study VIPs' numerical solutions ([13, 19]) and semivariational inequlity problems ([12]). The basic algorithm for solving VIPs is defined by

$$\begin{cases} x_0 \in \mathcal{H}, \\ x_{m+1} = P_C(x_m - \rho\Phi x_m), \ \forall m \geq 0, \end{cases} \tag{3}$$

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where $\rho > 0$ is a constant.

This algorithm is so-called *the projection gradient algorithm* ([11, 14]). However, the projection gradient method (3) requires that Φ is inverse-strongly monotone ([16, 17]. In order to weaken this assumption, Tseng considered to relax extragradient technique in ([22]). The specific structure is as follows:

$$\begin{cases} x_0 \in \mathcal{H}, \\ y_m = P_C(x_m - \rho\Phi x_m), \\ x_{m+1} = y_m - \rho(\Phi y_m - \Phi x_m),\ m \geq 0, \end{cases} \tag{4}$$

where $\rho \in (0, \frac{1}{S})$ and S is the Lipschitz constant of Φ.

On the basis of (4), in ([20]), Thong and Hieu used the inertial term $\omega_m = x_m + \eta_m(x_m - x_{m-1})$ and constructed an iterative algorithm which generates a sequence $\{x_m\}$ via the following algorithm

$$\begin{cases} x_0, x_1 \in \mathcal{H}, \\ \omega_m = x_m + \eta_m(x_m - x_{m-1}), \\ y_m = P_C(\omega_m - \rho\Phi\omega_m), \\ x_{m+1} = y_m - \rho(\Phi y_m - \Phi\omega_m), m \geq 0. \end{cases} \tag{5}$$

Censor, Gibali and Reich ([6–8]) presented several subgradient extragradient algorithms for solving VIPs. In ([9]), Verlan studied a modified extragradient algorithm with non-Lipschitz operator for solving VIPs, and strong convergence analysis of this method is proved.

Very recently, Thong et. al. ([21]) constructed *two acceleration methods* by adding half-space projection in (5) and obtained the following procedures

$$\begin{cases} x_0, x_1 \in \mathcal{H}, \\ \omega_m = x_m + \eta_m(x_m - x_{m-1}), \\ y_m = P_C(\omega_m - \rho\Phi\omega_m), \\ \mathcal{J}_m := \{z \in \mathcal{H} : \langle \omega_m - \rho\Phi\omega_m - y_m, z - y_m \leq 0\rangle\}, \\ z_m = P_{\mathcal{J}_m}(\omega_m - \rho\Phi y_m), \\ x_{m+1} = (1 - \beta_m)z_m + \beta_m e(x_m), \end{cases} \text{and} \begin{cases} x_0, x_1 \in \mathcal{H}, \\ \omega_m = x_m + \eta_m(x_m - x_{m-1}), \\ y_m = P_C(\omega_m - \rho\Phi\omega_m), \\ \mathcal{J}_m := \{z \in H : \langle \omega_m - \rho\Phi\omega_m - y_m, z - y_m \leq 0\rangle\}, \\ z_m = P_{\mathcal{J}_m}(\omega_m - \rho\Phi y_m), \\ x_{m+1} = (1 - \nu_m - \beta_m)x_m + \nu_m z_m. \end{cases} \tag{6}$$

where $\rho \in (0, \frac{1}{S})$, $\{\eta_m\} \subset [0, \eta)$, $\{\nu_m\} \subset (a, b) \subset (0, 1 - \beta_m)$ and $\{\beta_m\} \subset (0, 1)$ satisfying $\lim_{m\to\infty} \beta_m = 0$ and $\sum_{m=1}^{\infty} \beta_m = \infty$. Here, $e : \mathcal{H} \to \mathcal{H}$ is a contraction mapping with coefficient $\kappa \in [0, 1)$.

Inspired and motivated by the above work, our main purpose of this paper is to construct two iterative algorithms for solving VIPs. Our algorithms are based on subgradient extragradient methods, Tseng's method (4) and iterative algorithms (6). Convergence analysis of these algorithms are proved. Several numerical examples to illustrate the efficiency of the methods are given.

2. Preliminaries

This section contains some definitions and basic lemmas, which will be used in section 3. First, in real Hilbert space $\mathcal{H}$, the following results hold: $\forall x, y \in \mathcal{H}$ and $\eta \in \mathbb{R}$,

$$\begin{cases} \|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \\ \|\eta x + (1 - \eta)y\|^2 = \eta\|x\|^2 + (1 - \eta)\|y\|^2 - \eta(1 - \eta)\|x - y\|^2, \\ \|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \end{cases}$$

The following expressions will be used in the sequel.

- $x_m \hookrightarrow x$ denotes that the sequence $\{x_m\}$ weak convergence to x;

- $x_m \to x$ denotes that the sequence $\{x_m\}$ strong convergence to x;
- $Fix\Phi$ denotes that the set solution of fixed points of Φ.

Definition 2.1. $\forall x, y \in \mathcal{H}$,

- *if* $\|\Phi x - \Phi y\| \leq \kappa \| x - y \|$ $(\kappa \geq 0)$, *then* Φ *is called* κ*-Lipschitz continuous.*
  If $\kappa \in [0, 1)$, *then* Φ *called contractive mapping.*
- *if* $\langle \Phi x - \Phi y, x - y\rangle \geq 0$, *then* Φ *is called monotone.*

Lemma 2.2 ([15]). *Suppose that* $\{a_m\} \subset \mathbb{R}^+$ *and there exists a subsequence* $\{a_{m_j}\}$ *of* $\{a_m\}$ *such that* $a_{m_j} < a_{m_j+1}$, $\forall j \in \mathbb{N}$. *Then, there exists a nondecreasing sequence* $\{sz_k\} \subset \mathbb{N}$ *satisfying* $\lim_{k\to\infty} sz_k = \infty$ *and for all* $k \in \mathbb{N}$,

- $a_{sz_k} \leq a_{sz_k+1}$,
- $a_k \leq a_{sz_k+1}$.

More precisely sz_k *is the largest number* m *in the set* $\{1, 2, ..., k\}$ *such that* $a_m \leq a_{m+1}$.

Lemma 2.3 ([23]). *Suppose that* $\{a_m\}$ *and* $\{\eta_m\}$ *are nonnegative real sequences with* $\eta_m \in (0, 1)$ *and* $\sum_{m=0}^{\infty}\eta_m = \infty$. *If there exists a sequence* $\{b_m\}$ *with* $\limsup_{m\to\infty} b_m \leq 0$, *and* $0 < N \in \mathbb{N}$ *such that* $a_{m+1} \leq (1 - \eta_m)a_m + \eta_m b_m$ *for all* $m \geq N$, *then* $\lim_{m\to\infty} a_m = 0$.

Lemma 2.4 ([1]). *Suppose that* $\{x_m\}, \{\delta_m\}$ *and* $\{\eta_m\}$ *are sequences in* $[0, +\infty)$, *such that*

$$x_{m+1} \leq x_m + \eta_m(x_m - x_{m-1}) \ \forall m \geq 1, \sum_{m=1}^{+\infty} \delta_m < +\infty.$$

And $\exists \eta \in \mathbb{R}$, *for any* $m \in \mathbb{N}$, $0 \leq \eta_m \leq \eta < 1$. *Then*

1. $\sum_{m=1}^{+\infty}[x_m - x_{m-1}]_+ < +\infty$, *where* $[x_m - x_{m-1}]_+ = \max\{x_m - x_{m-1}, 0\}$.
2. *There is a point* $\theta \in [0, +\infty)$, *such that* $\lim_{m\to+\infty} x_m = \theta$.

Lemma 2.5 ([3]). *Let* $\rho > 0$. *Then* $x \in VI(C, \Phi)$ *iff* $x \in FixP_C(Id - \rho\Phi)$.

Lemma 2.6 ([21]). *Let* $\{x_m\}$ *be a sequence defined by* (6). *Let* $\{\eta_m\}$ *be a non-increasing real number sequence such that*

$$\lim_{m\to\infty} \frac{\eta_m}{\beta_m}\|x_m - x_{m-1}\| = 0. \tag{7}$$

Then, the sequence $\{x_m\}$ *strong convergence to* $\theta \in VI(C, \Phi)$.

Lemma 2.7 ([3]). *Suppose that* C *is a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$. *Given* $x \in \mathcal{H}$ *and* $x^* \in C$. *Then,* $x^* = P_C x \Leftrightarrow \langle x - x^*, x^* - y\rangle \geq 0$, $\forall y \in C$.

Lemma 2.8 ([3]). *Let* $f : \mathcal{H} \to \mathbb{R}$ *be a convex and subdifferentiable function with* $S(f, 0) \neq \emptyset$. *Then the subgradient projection* $P_{f,0}$ *is a cutter and satisfies*

$$\langle y - x, z - x\rangle \leq 0,$$

here $y = P_{f,0}x$ *and* $z \in S(f, 0)$.

Lemma 2.9. *Let $\Phi : \mathcal{H} \to \mathcal{H}$ be a monotone and S-Lipschitz mapping on C and ρ be a positive number. Suppose that $VI(C, \Phi)$ is nonempty. Let $f : \mathcal{H} \to \mathbb{R}$ be a continuously subdifferentiable and convex function, $g_f \in \partial f$ and $\partial f(z) := \{x \in \mathcal{H} : f(y) \geq f(z) + \langle x, y - z\rangle, \forall y \in \mathcal{H}\}$. Let $x \in \mathcal{H}$. Set*

$$y = P_C(x - \rho\Phi x),\ t = x - \rho\Phi y,\ z = P_{f,0}(x - \rho\Phi y),$$

where

$$P_{f,0}(t) := \begin{cases} t - \dfrac{[f(t)]_+}{\|g_f(t)\|^2} g_f(t), & \text{when } g_f(t) \neq 0, \\ t, & g_f(t) = 0. \end{cases}$$

Then, $\forall \theta \in VI(C, \Phi)$, we have

$$\|z - \theta\|^2 \leq \|x - \theta\|^2 - (1 - \rho S)\|y - x\|^2 - (1 - \rho S)\|z - y\|^2.$$

Proof. In fact, we have

$$\begin{aligned}
&\langle (x-\rho\Phi y) - z, z - \theta\rangle \geq 0 \\
&\quad\Leftrightarrow \|(x - \rho\Phi y) - z\|^2 + \langle (x - \rho\Phi y) - z, z - \theta\rangle \geq \|(x - \rho\Phi y) - z\|^2 \\
&\quad\Leftrightarrow \langle (x - \rho\Phi y) - z, (x - \rho\Phi y) - \theta\rangle \geq \|(x - \rho\Phi y) - z\|^2 \\
&\quad\Leftrightarrow 2\langle (x - \rho\Phi y) - z, (x - \rho\Phi y) - \theta\rangle \geq 2\|(x - \rho\Phi y) - z\|^2 \\
&\quad\Leftrightarrow 2\langle z - (x - \rho\Phi y), (x - \rho\Phi y) - \theta\rangle \leq -2\|(x - \rho\Phi y) - z\|^2 \\
&\quad\Leftrightarrow \|z - (x - \rho\Phi y)\|^2 + 2\langle z - (x - \rho\Phi y), (x - \rho\Phi y) - \theta\rangle \leq -\|(x - \rho\Phi y) - z\|^2 \\
&\quad\Leftrightarrow \|z - (x - \rho\Phi y)\|^2 + \|(x - \rho\Phi y) - \theta\|^2 + 2\langle z - (x - \rho\Phi y), (x - \rho\Phi y) - \theta\rangle \\
&\quad\leq \|(x - \rho\Phi y) - \theta\|^2 - \|(x - \rho\Phi y) - z\|^2 \\
&\quad\Leftrightarrow \|z - \theta\|^2 \leq \|(x - \rho\Phi y) - \theta\|^2 - \|(x - \rho\Phi y) - z\|^2.
\end{aligned} \tag{8}$$

According to (8), we have

$$\begin{aligned}
\|z - \theta\|^2 &\leq \|(x - \rho\Phi y) - \theta\|^2 - \|(x - \rho\Phi y) - z\|^2 \\
&= \|x - \theta\|^2 + 2\rho\langle \theta - z, \Phi y\rangle - \|x - z\|^2 \\
&= \|x - \theta\|^2 + 2\rho\langle \theta - y, \Phi y - \Phi\theta\rangle + 2\rho\langle \theta - y, \Phi\theta\rangle + 2\rho\langle y - z, \Phi y\rangle - \|x - z\|^2 \\
&\leq \|x - \theta\|^2 + 2\rho\langle y - z, \Phi y\rangle - \|x - z\|^2 \\
&\leq \|x - \theta\|^2 + 2\rho\langle y - z, \Phi y\rangle - \|x - y\|^2 - 2\rho\langle x - y, y - z\rangle - \|y - z\|^2 \\
&= \|x - \theta\|^2 - \|x - y\|^2 - \|y - z\|^2 - 2\langle x - \rho\Phi y - y, y - z\rangle.
\end{aligned}$$

Now we estimate

$$\begin{aligned}
\langle x - \rho\Phi y - y, y - z\rangle &= \langle x - \rho\Phi x - y, y - z\rangle + \langle \rho\Phi x - \rho\Phi y, y - z\rangle \\
&\leq \langle \rho\Phi x - \rho\Phi y, y - z\rangle \\
&\leq \rho S\|x - y\|\|y - z\|.
\end{aligned}$$

So,

$$\begin{aligned}
\|z - \theta\|^2 &= \|x - \theta\|^2 - \|x - y\|^2 - \|y - z\|^2 + 2\rho S\|x - y\|\|y - z\| \\
&= \|x - \theta\|^2 - (1 - \rho S)\|x - y\|^2 - (1 - \rho S)\|y - z\|^2 - \rho S(\|x - y\| - \|z - y\|)^2 \\
&\leq \|x - \theta\|^2 - (1 - \rho S)\|y - x\|^2 - (1 - \rho S)\|z - y\|^2.
\end{aligned}$$

□

3. Main results

In this section, we present our main results.

Let Φ be an $S-$Lipschitz continuous and monotone operator on $\mathcal{H}$ with $VI(C,\Phi) \neq \emptyset$. Let $f : \mathcal{H} \to \mathbb{R}$ be a continuously subdifferentiable and convex function, $g_f \in \partial f$ and $\partial f(z) := \{x \in \mathcal{H} : f(y) \geq f(z)+\langle x, y-z\rangle, \forall y \in \mathcal{H}\}$.

Next, we introduce our iterative algorithms.

Take $\rho \in (0, \frac{1}{S})$, $\{\eta_m\} \subset [0, \eta)$ for some $\eta > 0$, $\{\nu_m\} \subset (a, b) \subset (0, 1 - \beta_m)$ and $\{\beta_m\} \subset (0, 1)$ satisfying the following conditions:

$$\lim_{m\to\infty} \beta_m = 0 \text{ and } \sum_{m=1}^{\infty} \beta_m = \infty.$$

Algorithm 3.1. ***Initialization:*** *Let $x_0, x_1 \in C$. Set $m = 1$.*

S1. Compute

$$\omega_m = x_m + \eta_m(x_m - x_{m-1}),$$
$$y_m = P_C(\omega_m - \rho\Phi\omega_m).$$

If $y_m = \omega_m$, then stop and y_m is a solution to the VIP. Else, go to S2.

S2. Construct the subgradient projection by

$$\begin{cases} P_{f,0}(t_m) = \begin{cases} t_m - \dfrac{[f(t_m)]_+}{\|g_f(t_m)\|^2} g_f(t_m), & \text{when } g_f(t_m) \neq 0, \\ t_m, & g_f(t_m) = 0, \end{cases} \\ t_m = \omega_m - \rho\Phi y_m. \end{cases}$$

and compute

$$z_m = P_{f,0}(\omega_m - \rho\Phi y_m).$$

S3. Calculate

$$x_{m+1} = (1 - \nu_m - \beta_m)x_m + \nu_m z_m.$$

S4. Set $m := m + 1$ and return to S1.

Theorem 3.2. *Assume that the $\{\eta_m\}$ is a non-increasing sequence such that*

$$\lim_{m\to\infty} \frac{\eta_m}{\beta_m}\|x_m - x_{m-1}\| = 0. \tag{9}$$

Then $\{x_m\}$ strongly converges to a point $\theta \in VI(C,\Phi)$, here $\|\theta\| = min\{\|z\| : z \in VI(C,\Phi)\}$.

Proof. We divide the proof into four steps.
Step 1. We show that $\{x_m\}$, $\{z_m\}$, $\{\omega_m\}$ are bounded. According to Lemma 2.9, we get

$$\|z_m - \theta\|^2 \leq \|\omega_m - \theta\|^2 - (1 - \rho S)\|y_m - \omega_m\|^2 - (1 - \rho S)\|z_m - y_m\|^2. \tag{10}$$

From the definition of ω_m, we get

$$\begin{aligned} \|\omega_m - \theta\| &= \|x_m + \eta_m(x_m - x_{m-1}) - \theta\| \\ &\leq \|x_m - \theta\| + \eta_m\|x_m - x_{m-1}\| \\ &\leq \|x_m - \theta\| + \beta_m \cdot \frac{\eta_m}{\beta_m}\|x_m - x_{m-1}\|. \end{aligned} \tag{11}$$

Since $\frac{\eta_m}{\beta_m}\|x_m - x_{m-1}\| \to 0$, $\exists N_1 \geq 0$ such that

$$\|z_m - \theta\| \leq \|\omega_m - \theta\| \leq \|x_m - \theta\| + \beta_m N_1. \tag{12}$$

Therefore, we have

$$\begin{aligned}\|x_{m+1} - \theta\| &= \|(1 - \nu_m - \beta_m)x_m + \nu_m z_m - \theta\| \\ &= \|(1 - \nu_m - \beta_m)(x_m - \theta) + \nu_m(z_m - \theta) - \beta_m\theta\| \\ &\leq \|(1 - \nu_m - \beta_m)(x_m - \theta) + \nu_m(z_m - \theta)\| + \beta_m\|\theta\|.\end{aligned} \tag{13}$$

Note that

$$\begin{aligned}&\|(1 - \nu_m - \beta_m)(x_m - \theta) + \nu_m(z_m - \theta)\| \\ &\quad= (1 - \nu_m - \beta_m)^2\|x_m - \theta\|^2 + \nu_m^2\|z_m - \theta\|^2 + 2(1 - \nu_m - \beta_m)\nu_m\langle x_m - \theta, z_m - \theta\rangle \\ &\quad\leq (1 - \nu_m - \beta_m)^2\|x_m - \theta\|^2 + \nu_m^2\|z_m - \theta\|^2 + 2(1 - \nu_m - \beta_m)\nu_m\|x_m - \theta\|\|z_m - \theta\| \\ &\quad\leq (1 - \nu_m - \beta_m)^2\|x_m - \theta\|^2 + \nu_m^2\|z_m - \theta\|^2 + (1 - \nu_m - \beta_m)\nu_m(\|x_m - \theta\|^2 + \|z_m - \theta\|^2). \\ &\quad= (1 - \nu_m - \beta_m)(1 - \beta_m)\|x_m - \theta\|^2 + (1 - \beta_m)\nu_m\|z_m - \theta\|^2.\end{aligned} \tag{14}$$

According to (12) and (14), we get

$$\begin{aligned}&\|(1 - \nu_m - \beta_m)(x_m - \theta) + \nu_m(z_m - \theta)\| \\ &\quad\leq (1 - \nu_m - \beta_m)(1 - \beta_m)\|x_m - \theta\|^2 + (1 - \beta_m)\nu_m(\|x_m - \theta\| + \beta_m N_1)^2 \\ &\quad= (1 - \nu_m - \beta_m)(1 - \beta_m)\|x_m - \theta\|^2 + (1 - \beta_m)\nu_m(\|x_m - \theta\|^2 + \beta_m^2 N_1^2 + 2\beta_m N_1\|x_m - \theta\|) \\ &\quad\leq (1 - \beta_m)^2\|x_m - \theta\|^2 + 2(1 - \beta_m)\beta_m\|x_m - \theta\|N_1 + \beta_m^2 N_1^2 \\ &\quad= [(1 - \beta_m)\|x_m - \theta\| + \beta_m N_1]^2.\end{aligned} \tag{15}$$

Therefore, according to (13) and (15), we have

$$\begin{aligned}\|x_{m+1} - \theta\| &\leq (1 - \beta_m)\|x_m - \theta\| + \beta_m N_1 + \beta_m\|\theta\| \\ &= (1 - \beta_m)\|x_m - \theta\| + \beta_m(N_1 + \|\theta\|) \\ &\leq \max\{\|x_m - \theta\|, N_1 + \|\theta\|\} \\ &\leq \cdots \\ &\leq \max\{\|x_0 - \theta\|, N_1 + \|\theta\|\}.\end{aligned}$$

So $\{x_m\}$ is bounded and $\{z_m\}$, $\{\omega_m\}$ are bounded.
Step 2. We show that $\exists N_4 > 0$, such that

$$(1 - \rho S)\nu_m\|y_m - \omega_m\|^2 + (1 - \rho S)\nu_m\|z_m - y_m\|^2 \leq \|x_m - \theta\|^2 - \|x_{m+1} - \theta\|^2 + \beta_m N_4.$$

In fact,

$$\begin{aligned}\|x_{m+1} - \theta\|^2 &= \|(1 - \nu_m - \beta_m)x_m + \nu_m z_m - \theta\|^2 \\ &= \|(1 - \nu_m - \beta_m)(x_m - \theta) + \nu_m(z_m - \theta) - \beta_m\theta\|^2 \\ &= \|(1 - \nu_m - \beta_m)(x_m - \theta) + \nu_m(z_m - \theta)\|^2 \\ &\quad- 2\beta_m\langle(1 - \nu_m - \beta_m)(x_m - \theta) + \nu_m(z_m - \theta), \theta\rangle + \beta_m^2\|\theta\|^2 \\ &\leq \|(1 - \nu_m - \beta_m)(x_m - \theta) + \nu_m(z_m - \theta)\|^2 + \beta_m N_2.\end{aligned} \tag{16}$$

According to (14) and (16), we get

$$\|x_{m+1}-\theta\|^2 \le (1-\nu_m-\beta_m)(1-\beta_m)\|x_m-\theta\|^2 + (1-\beta_m)\nu_m\|z_m-\theta\|^2 + \beta_m N_2. \tag{17}$$

By (10) and (17), we obtain

$$\begin{aligned}\|x_{m+1}-\theta\|^2 \le{}& (1-\nu_m-\beta_m)(1-\beta_m)\|x_m-\theta\|^2 + (1-\beta_m)\nu_m\|\omega_m-\theta\|^2 + \beta_m N_2.\\ &- (1-\beta_m)\nu_m(1-\rho S)\|y_m-\omega_m\|^2 - (1-\beta_m)\nu_m(1-\rho S)\|z_m-y_m\|^2.\end{aligned} \tag{18}$$

Since $\{x_m\}$ is bounded and $\|\omega_m-\theta\| \le \|x_m-\theta\| + \beta_m N_1$, we have

$$\|\omega_m-\theta\|^2 \le \|x_m-\theta\|^2 + \beta_m N_3. \tag{19}$$

From (18) and (19), we get

$$\begin{aligned}\|x_{m+1}-\theta\|^2 \le{}& (1-\nu_m-\beta_m)(1-\beta_m)\|x_m-\theta\|^2 + (1-\beta_m)\nu_m\|x_m-\theta\|^2 + (1-\beta_m)\nu_m\beta_m N_3\\ &- (1-\beta_m)\nu_m(1-\rho S)\|y_m-\omega_m\|^2 - (1-\beta_m)\nu_m(1-\rho S)\|z_m-y_m\|^2 + \beta_m N_2\\ \le{}& (1-\beta_m)^2\|x_m-\theta\|^2 - (1-\beta_m)\nu_m(1-\rho S)\|y_m-\omega_m\|^2\\ &- (1-\beta_m)\nu_m(1-\rho S)\|z_m-y_m\|^2 + \beta_m[(1-\beta_m)\nu_m N_3 + N_2]\\ \le{}& \|x_m-\theta\|^2 - (1-\beta_m)\nu_m(1-\rho S)\|y_m-\omega_m\|^2\\ &- (1-\beta_m)\nu_m(1-\rho S)\|z_m-y_m\|^2 + \beta_m N_4,\end{aligned}$$

and

$$(1-\rho S)\nu_m\|y_m-\omega_m\|^2 + (1-\rho S)\nu_m\|z_m-y_m\|^2 \le \|x_m-\theta\|^2 - \|x_{m+1}-\theta\|^2 + \beta_m N_4.$$

Step 3. We show that

$$\begin{aligned}\|x_{m+1}-\theta\|^2 \le{}& (1-\beta_m)\|x_m-\theta\|^2 + \beta_m[\frac{\eta_m}{\beta_m}\|x_m-x_{m-1}\|(1-\beta_m)N_5\\ &+ 2\nu_m\|x_m-z_m\|\|x_m-\theta\| + 2\langle\theta,\theta-x_{m+1}\rangle].\end{aligned}$$

In fact, we have

$$x_{m+1} = (1-\nu_m)x_m + \nu_m z_m - \beta_m x_m.$$

Set $s_m = (1-\nu_m)x_m + \nu_m z_m$. Then, we get

$$\begin{aligned}\|s_m-\theta\|^2 &= \|(1-\nu_m)x_m + \nu_m z_m - \theta\|^2\\ &= \|(1-\nu_m)(x_m-\theta) + \nu_m(z_m-\theta)\|^2\\ &= (1-\nu_m)\|x_m-\theta\|^2 + \nu_m\|z_m-\theta\|^2 + 2\nu_m(1-\nu_m)\langle x_m-\theta, z_m-\theta\rangle\\ &\le (1-\nu_m)\|x_m-\theta\|^2 + \nu_m\|z_m-\theta\|^2 + 2\nu_m(1-\nu_m)\|x_m-\theta\|\|z_m-\theta\|\\ &\le (1-\nu_m)\|x_m-\theta\|^2 + \nu_m\|z_m-\theta\|^2 + \nu_m(1-\nu_m)(\|x_m-\theta\|^2 + \|z_m-\theta\|^2)\\ &= (1-\nu_m)\|x_m-\theta\|^2 + \nu_m\|z_m-\theta\|^2\\ &\le (1-\nu_m)\|x_m-\theta\|^2 + \nu_m\|\omega_m-\theta\|^2.\end{aligned} \tag{20}$$

Oh the other hand, we have

$$\begin{aligned}\|\omega_m-\theta\|^2 &= \|x_m + \eta_m(x_m-x_{m-1}) - \theta\|^2\\ &= \|(x_m-\theta) + \eta_m(x_m-x_{m-1})\|^2\\ &= \|x_m-\theta\|^2 + \eta_m\|x_m-x_{m-1}\|^2 + 2\eta_m\langle x_m-\theta, x_m-x_{m-1}\rangle\\ &\le \|x_m-\theta\|^2 + \eta_m\|x_m-x_{m-1}\|^2 + 2\eta_m\|x_m-\theta\|\|x_m-x_{m-1}\|\\ &\le \|x_m-\theta\|^2 + \eta_m\|x_m-x_{m-1}\|[\eta_m\|x_m-x_{m-1}\| + 2\|x_m-\theta\|]\\ &\le \|x_m-\theta\|^2 + \eta_m\|x_m-x_{m-1}\|N_5.\end{aligned} \tag{21}$$

Taking into account (20) and (21), we get

$$\begin{aligned}\|s_m-\theta\|^2 &\le (1-\nu_m)\|x_m-\theta\|^2+\nu_m(\|x_m-\theta\|^2+\eta_m\|x_m-x_{m-1}\|N_5)\\ &\le \|x_m-\theta\|^2+\eta_m\|x_m-x_{m-1}\|N_5.\end{aligned} \tag{22}$$

By $s_m=(1-\nu_m)x_m+\nu_m z_m$, we get $x_m-s_m=\nu_m(x_m-z_m)$ and

$$x_{m+1}=s_m-\beta_m x_m=(1-\beta_m)s_m-\beta_m(x_m-s_m)=(1-\beta_m)s_m-\beta_m\nu_m(x_m-z_m).$$

It means that

$$\begin{aligned}&\|x_{m+1}-\theta\|^2\\ &\quad=\|(1-\beta_m)s_m-\beta_m\nu_m(x_m-z_m)-\theta\|^2\\ &\quad=\|(1-\beta_m)(s_m-\theta)-(\beta_m\nu_m(x_m-z_m)+\beta_m\theta)\|^2\\ &\quad\le(1-\beta_m)\|s_m-\theta\|-2\langle\beta_m\nu_m(x_m-z_m)+\beta_m\theta,x_{m+1}-\theta\rangle\\ &\quad\le(1-\beta_m)\|s_m-\theta\|+2\langle\beta_m\nu_m(x_m-z_m),\theta-x_{m+1}\rangle+2\beta_m\langle\theta,\theta-x_{m+1}\rangle\\ &\quad\le(1-\beta_m)\|s_m-\theta\|+2\beta_m\nu_m\|x_m-z_m\|\|\theta-x_m\|+2\beta_m\langle\theta,\theta-x_{m+1}\rangle.\end{aligned} \tag{23}$$

Combining (22) and (23), we get

$$\begin{aligned}\|x_{m+1}-\theta\|^2\le&(1-\beta_m)\|x_m-\theta\|^2+(1-\beta)\eta_m\|x_m-x_{m-1}\|N_5\\ &+2\beta_m\nu_m\|x_m-z_m\|\|\theta-x_{m+1}\|+2\beta_m\langle\theta,\theta-x_{m+1}\rangle\\ \le&(1-\beta_m)\|x_m-\theta\|^2+\beta_m[\frac{\eta_m}{\beta_m}\|x_m-x_{m-1}\|(1-\beta_m)N_5\\ &+2\nu_m\|x_m-z_m\|\|x_m-\theta\|+2\langle\theta,\theta-x_{m+1}\rangle].\end{aligned}$$

Step 4. Now, we proof that $\{\|x_m-\theta\|^2\to 0\}$ by considering two possible cases.
Case 1. $\exists M\in\mathbb{N},\forall m\ge M$ such that $\|x_{m+1}-\theta\|^2\le\|x_m-\theta\|^2$. It means that $\lim_{m\to\infty}\|x_m-\theta\|^2$ exists. By Step 2, we have

$$\lim_{m\to\infty}\|y_m-\omega_m\|=0,\ \lim_{m\to\infty}\|y_m-z_m\|=0,$$

which implies that

$$\|z_m-\omega_m\|\le\|z_m-y_m\|+\|y_m-\omega_m\|\to 0.$$

Similarly, we have

$$\|\omega_m-x_m\|=\eta_m\|x_{m+1}-x_m\|=\frac{\eta_m}{\beta_m}\|x_{m+1}-x_m\|\cdot\beta_m\to 0,$$

and

$$\|x_m-z_m\|\le\|x_m-\omega_m\|+\|\omega_m-z_m\|\to 0.$$

Thus,

$$\begin{aligned}\|x_{m+1}-x_m\|&=\|(1-\nu_m-\beta_m)x_m+\nu_m z_m-x_m\|\\ &=\|\nu_m(z_m-x_m)-\beta_m x_m\|\\ &\le\nu_m\|z_m-x_m\|+\beta_m\|x_m\|\to 0.\end{aligned}$$

Since $\{x_m\}$ is bounded, there exists a subsequence $\{x_{m_j}\}$ of $\{x_m\}$ such that $\{x_{m_j}\}\rightharpoonup q$ and

$$\limsup_{m\to\infty}\langle\theta,\theta-x_m\rangle=\limsup_{j\to\infty}\langle\theta,\theta-x_{m_j}\rangle=\langle\theta,\theta-q\rangle.$$

From $x_{m_j} \rightharpoonup q$ and $\|x_m - \omega_m\| \to 0$, we have $\omega_m \to q$, and

$$\|\omega_m - y_m\| = \|\omega_m - P_C(\omega_m - \rho\Phi\omega_m)\| \to 0.$$

According to Lemma 2.5, it follows that $q \in VI(C, \Phi)$. By $\theta = P_{VI(C,\Phi)}\theta$, we get

$$\limsup_{m\to\infty} \langle \theta, \theta - x_m \rangle = \langle \theta, \theta - q \rangle \le 0.$$

Since $\|x_{m+1} - x_m\| \to 0$, we have

$$\limsup_{m\to\infty} \langle \theta, \theta - x_{m+1} \rangle \le 0.$$

By Step 3 the condition $\lim_{m\to\infty} \frac{\eta_m}{\beta_m}\|x_{m+1} - x_m\| = 0$ and Lemma 2.3, we get $\lim_{m\to\infty} \|x_{m+1} - \theta\| = 0$, that is, $x_m \to p$.

Case 2. There is a subsequence $\{\|x_{m_j} - \theta\|^2\}$ of $\{\|x_m - \theta\|^2\}$ such that

$$\|x_{m_j} - \theta\|^2 < \|x_{m_j+1} - \theta\|^2, \forall j \in \mathbb{N}.$$

In this case, according to Lemma 2.2, we get that there is a non-decreasing sequence $\{sz_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} sz_k = \infty$ and the following inequality holds: $\forall k \in \mathbb{N}$, we have

$$\|x_{sz_k} - \theta\|^2 \le \|x_{sz_k+1} - \theta\|^2, \|x_k - \theta\|^2 \le \|x_{sz_k+1} - \theta\|^2.$$

By Step 2, we have

$$\begin{aligned}
&(1 - \beta_{sz_k})\theta_{sz_k}(1 - \rho S)\|y_{sz_k} - \omega_{sz_k}\|^2 + (1 - \beta_{sz_k})\theta_{sz_k}(1 - \rho S)\|z_{sz_k} - y_{sz_k}\|^2 \\
&\le \|x_{sz_k} - \theta\|^2 - \|x_{sz_k+1} - \theta\|^2 + \beta_{sz_k}N_4 \le \beta_{sz_k}N_4.
\end{aligned}$$

It follows that

$$\lim_{k\to\infty} \|y_{sz_k} - \omega_{sz_k}\| = 0, \lim_{k\to\infty} \|z_{sz_k} - y_{sz_k}\| = 0.$$

Using the same arguments as in the proof of **Case 1**, we get

$$\lim_{k\to\infty} \|x_{sz_k} - z_{sz_k}\| = 0 \text{ and } \lim_{k\to\infty} \|x_{sz_k+1} - x_{sz_k}\| = 0,$$

and

$$\limsup_{k\to\infty} \langle \theta, \theta - x_{sz_k+1} \rangle \le 0.$$

According to Step 3, we get

$$\begin{aligned}
\|x_{sz_k+1} - \theta\|^2 \le (1 - \beta_{sz_k})\|x_{sz_k} - \theta\|^2 + \beta_{sz_k}[&\frac{\eta_{sz_k}}{\beta_{sz_k}}\|x_{sz_k} - x_{sz_k-1}\|(1 - \beta_{sz_k})N_5 \\
&+ 2\theta_{sz_k}\|x_{sz_k} - z_{sz_k}\|\|x_{sz_k} - \theta\| + 2\langle \theta, \theta - x_{sz_k+1} \rangle].
\end{aligned}$$

It yields

$$\begin{aligned}
\|x_k - \theta\|^2 &\le \|x_{sz_k+1} - \theta\|^2 \\
&\le (1 - \beta_{sz_k})\|x_{sz_k} - p\|^2 + \beta_{sz_k}[\frac{\eta_{sz_k}}{\beta_{sz_k}}\|x_{sz_k} - x_{sz_k-1}\|(1 - \beta_{sz_k})N_5 \\
&+ 2\theta_{sz_k}\|x_{sz_k} - z_{sz_k}\|\|x_{sz_k}\theta\| + 2\langle \theta, \theta - x_{sz_k+1} \rangle].
\end{aligned}$$

Therefore, $\limsup_{k\to\infty} \|x_k - \theta\| \le 0$, that is, $x_k \to \theta$. □

Next, we construct iterative Algorithm 3.3 based on ([23]). It is worth noting that in the algorithm 3.3 the nonempty closed convex subset C is defined as $C := \{\psi \in \mathcal{H} : f(\psi) \leq 0\}$ ([10]).

Let $\rho_0 > 0$ and $\mu \in (\frac{1}{\sqrt{2}}, 1)$ be two constants.

Algorithm 3.3. ***Initialization:*** *Let $x_0, x_1 \in \mathcal{H}$. Set $m = 1$.*

S1. Compute

$$\omega_m = x_m + \eta_m(x_m - x_{m-1}).$$

S2. Construct the half-space

$$\mathcal{J}_m := \{\omega_m \in \mathcal{H} : f(\omega_m) + \langle g_f(\omega_m), x - \omega_m\rangle \leq 0\}, x \in \mathcal{H},$$

and compute

$$y_m = P_{\mathcal{J}_m}(\omega_m - \rho_m \Phi\omega_m).$$

If $y_m = \omega_m$, then stop and y_m is a solution to the VIP. Else, go to S3.

S3. Calculate

$$x_{m+1} = y_m - \rho_m(\Phi y_m - \Phi\omega_m),$$

and

$$\rho_{m+1} = \begin{cases} \min\{\frac{\mu\|y_m - \omega_m\|}{\|\Phi y_m - \Phi\omega_m\|}, \rho_m\}, & \text{when } \Phi y_m - \Phi\omega_m \neq 0, \\ \rho_m, & \text{others.} \end{cases}$$

S4. Set $m := m + 1$ and repeat steps S1–S3.

Lemma 3.4 ([24]). *Suppose that sequence $\{\rho_m\}$ is defined by Algorithm 3.3, then we have*

1. $\underline{\lim}_{m\to\infty}\rho_m \geq \min\{\frac{\mu}{S}, \rho_0\}$,
2. $0 \leq \rho_{m+1} \leq \rho_m$.

Theorem 3.5. *Suppose that sequence $\{x_m\}$ is defined by Algorithm 3.3. Then $\forall\theta \in VI(C, \Phi)$, we have*

$$\|x_{m+1} - \theta\| \leq \|\omega_m - \theta\|. \tag{24}$$

Proof.

$$\begin{aligned}
\|x_{m+1} - \theta\|^2 &= \|y_m - \rho_m(\Phi y_m - \Phi\omega_m) - \theta\|^2 \\
&= \|y_m - \theta\|^2 + {\rho_m}^2\|\Phi y_m - \Phi\omega_m\|^2 - 2\rho_m\langle y_m - \theta, \Phi y_m - \Phi\omega_m\rangle \\
&= \|\omega_m - \theta\|^2 + \|\omega_m - y_m\|^2 + 2\langle y_m - \theta, \Phi y_m - \Phi\omega_m\rangle \\
&+ {\rho_m}^2\|\Phi y_m - \Phi\omega_m\|^2 - 2\rho_m\langle y_m - \theta, \Phi y_m - \Phi\omega_m\rangle \\
&= \|\omega_m - \theta\|^2 + \|\omega_m - y_m\|^2 - 2\langle y_m - \omega_m, y_m - \omega_m\rangle \\
&+ 2\langle y_m - \omega_m, y_m - \theta\rangle + {\rho_m}^2\|\Phi y_m - \Phi\omega_m\|^2 \\
&- 2\rho_m\langle y_m - \theta, \Phi y_m - \Phi\omega_m\rangle \\
&= \|\omega_m - \theta\|^2 - \|\omega_m - y_m\|^2 + 2\langle y_m - \omega_m, y_m - \theta\rangle \\
&+ {\rho_m}^2\|\Phi y_m - \Phi\omega_m\|^2 - 2\rho_m\langle y_m - \theta, \Phi y_m - \Phi\omega_m\rangle
\end{aligned} \tag{25}$$

Since $y_m = P_{\mathcal{J}_m}(\omega_m - \rho_m \Phi\omega_m)$, we have

$$\langle y_m - \omega_m + \rho_m \Phi\omega_m, y_m - \theta\rangle \le 0$$

or

$$\langle y_m - \omega_m, y_m - \theta\rangle \le -\rho_m \langle \Phi\omega_m, y_m - \theta\rangle. \tag{26}$$

It follows from (25) and (26) that

$$\begin{aligned}
\|x_{m+1} - \theta\|^2 &\le \|\omega_m - \theta\|^2 - \|\omega_m - y_m\|^2 - 2\rho_m\langle \Phi\omega_m, y_m - \theta\rangle \\
&+ {\rho_m}^2 \|\Phi y_m - \Phi\omega_m\|^2 - 2\rho_m \langle y_m - \theta, \Phi y_m - \Phi\omega_m\rangle \\
&= \|\omega_m - \theta\|^2 - \|\omega_m - y_m\|^2 + {\rho_m}^2\|\Phi y_m - \Phi\omega_m\|^2 - 2\rho_m\langle y_m - \theta, \Phi y_m\rangle \\
&\le \|\omega_m - \theta\|^2 - \|\omega_m - y_m\|^2 + {\rho_m}^2 \cdot \frac{\mu^2}{\rho_{m+1}^2}\|y_m - \omega_m\|^2 \\
&- 2\rho_m\langle y_m - \theta, \Phi y_m - \Phi\theta\rangle - 2\rho_m\langle y_m - \theta, \Phi\theta\rangle \\
&\le \|\omega_m - \theta\|^2 - (1 - {\rho_m}^2 \cdot \frac{\mu^2}{\rho_{i+1}^2})\|y_m - \omega_m\|^2.
\end{aligned}$$

So, we have $\|x_{m+1} - \theta\| \le \|\omega_m - \theta\|$. □

Theorem 3.6. *Let the sequence $\{x_m\}$ be defined by Algorithm 3.3. Suppose that $\{\eta_m\}$ is a non-increasing sequence such that*

$$0 \le \eta_m \le \eta < \frac{\sqrt{1+8\Re} - 1 - 2\Re}{2(1-2\Re)}, \tag{27}$$

where $\Re := 1 - \mu^2$. Then $\{x_m\}$ is bounded in $\mathcal{H}$.

Proof. To prove this theorem, we divide it into 2 steps.

Step 1. We show that $\eta < \frac{\sqrt{1+8\Re_m}-1-2\Re_m}{2(1-2\Re_m)} < \frac{\sqrt{1+8\Re}-1-2\Re}{2(1-2\Re)}$, where $\Re = 1 - \mu^2$. Let $\Re_m := \frac{1-{\rho_m}^2\cdot\frac{\mu^2}{\rho_{m+1}^2}}{1+{\rho_m}^2\cdot\frac{\mu^2}{\rho_{i+1}^2}} \le 1 - {\rho_m}^2 \cdot \frac{\mu^2}{\rho_{m+1}^2} \le 1 - \mu^2 = \Re$. Since $\frac{\sqrt{1+8\Re_m}-1-2\Re_m}{2(1-2\Re_m)}$ is monotonically increasing in $(0, \frac{1}{2})$ and $\{\rho_m\}$ is monotonically decreasing, so $\frac{\rho_m^2}{{\rho_{m+1}}^2} \ge 1$. Then, we have $\eta < \frac{\sqrt{1+8\Re_m}-1-2\Re_m}{2(1-2\Re_m)} < \frac{\sqrt{1+8\Re}-1-2\Re}{2(1-2\Re)}$.

Step 2. First, we proof that $\{x_m\}$ is bounded. According to Theorem 3.5, we can get

$$\|x_{m+1} - \theta\| \le \|\omega_m - \theta\|. \tag{28}$$

By the construction of ω_m, we can get

$$\begin{aligned}
\|\omega_m - \theta\|^2 &= \|x_m + \eta_m(x_m - x_{m-1}) - \theta\| \\
&= \|(1+\eta_m)(x_m - \theta) - \eta_m(x_{m-1} - \theta)\| \\
&\le (1+\eta_m)\|x_m - \theta\|^2 - \eta_m\|x_{m-1} - \theta\|^2 + \eta_m(1+\eta_m)\|x_m - x_{m-1}\|^2.
\end{aligned} \tag{29}$$

From (28) and (29), we can get

$$\begin{aligned}
\|x_{m+1} - \theta\|^2 &\le (1+\eta_m)\|x_m - \theta\|^2 - \eta_m\|x_{m-1} - \theta\|^2 + \eta_m(1+\eta_m)\|x_m - x_{m-1}\|^2 \\
&\le (1+\eta_m)\|x_m - \theta\|^2 - \eta_m\|x_{m-1} - \theta\|^2 + 2\eta\|x_m - x_{m-1}\|^2.
\end{aligned} \tag{30}$$

On the one hand,

$$\begin{aligned}\|x_{m+1}-\omega_m\|^2 &= \|x_{m+1}-x_m-\eta_m(x_m-x_{m-1})\|^2\\ &= \|x_{m+1}-x_m\|^2+\eta_m{}^2\|x_m-x_{m-1}\|^2-2\eta_m\langle x_{m+1}-x_m, x_m-x_{m-1}\rangle\\ &\geq \|x_{m+1}-x_m\|^2+\eta_m{}^2\|x_m-x_{m-1}\|^2-2\eta_m\|x_{m+1}-x_m\|\|x_m-x_{m-1}\|\\ &\geq (1-\eta_m)\|x_{m+1}-x_m\|^2+(\eta_m{}^2-\eta_m)\|x_m-x_{m-1}\|^2.\end{aligned} \tag{31}$$

According to (28), (29) and (31) we get

$$\begin{aligned}\|x_{m+1}-\theta\|^2 &\leq (1+\eta_m)\|x_m-\theta\|^2-\eta_m\|x_{m-1}-\theta\|^2+\eta_m(1+\eta_m)\|x_m-x_{m-1}\|^2\\ &-\mathfrak{K}(1-\eta_m)\|x_{m+1}-x_m\|^2-\mathfrak{K}(\eta_m{}^2-\eta_m)\|x_m-x_{m-1}\|^2\\ &=(1+\eta_m)\|x_m-\theta\|^2-\eta_m\|x_{m-1}-\theta\|^2-\mathfrak{K}(1-\eta_m)\|x_{m+1}-x_m\|^2\\ &+[\eta_m(1+\eta_m)-\mathfrak{K}(\eta_m{}^2-\eta_m)]\|x_m-x_{m-1}\|^2\\ &=(1+\eta_m)\|x_m-\theta\|^2-\eta_m\|x_{m-1}-\theta\|^2-\gamma_m\|x_{m+1}-x_m\|^2+\mu_m\|x_m-x_{m-1}\|^2,\end{aligned} \tag{32}$$

where $\gamma_m := \mathfrak{K}(1-\eta_m)$ and $\mu_m := \eta_m(1+\eta_m)-\mathfrak{K}(\eta_m{}^2-\eta_m) \geq 0$.

Set $\Gamma_m := \|x_m-\theta\|^2-\eta_m\|x_{m-1}-\theta\|^2+\mu_m\|x_m-x_{m-1}\|^2$. From (32), we obtain

$$\begin{aligned}\Gamma_{m+1}-\Gamma_m &= \|x_{m+1}-\theta\|^2-(1+\eta_{m+1})\|x_m-\theta\|^2+\eta_m\|x_{m-1}-\theta\|^2\\ &+\mu_{m+1}\|x_{m+1}-x_m\|^2-\mu_m\|x_m-x_{m-1}\|^2\\ &\leq \|x_{m+1}-\theta\|^2-(1+\eta_m)\|x_m-\theta\|^2+\eta_m\|x_{m-1}-\theta)\|^2\\ &+\mu_{m+1}\|x_{m+1}-x_m\|^2-\mu_m\|x_m-x_{m-1}\|^2\\ &\leq -(\gamma_m-\mu_{m+1})\|x_{m+1}-x_m\|^2.\end{aligned} \tag{33}$$

Because $0 \leq \eta_m \leq \eta_{m+1} \leq \eta$, we have

$$\begin{aligned}\gamma_m-\mu_{m+1} &= \mathfrak{K}(1-\eta_m)-\eta_{m+1}(1+\eta_{m+1})+\mathfrak{K}(\eta_{m+1}{}^2-\eta_{m+1})\\ &\geq \mathfrak{K}(1-\eta_{m+1})-\eta_{m+1}(1+\eta_{m+1})+\mathfrak{K}(\eta_{m+1}^2-\eta_{m+1})\\ &\geq \mathfrak{K}(1-\eta)-\eta(1+\eta)+\mathfrak{K}(\eta^2-\eta)\\ &\geq -(1-\mathfrak{K})\eta^2-(1+2\mathfrak{K})\eta+\mathfrak{K}.\end{aligned} \tag{34}$$

Combining (33) and (34), we get

$$\Gamma_{m+1}-\Gamma_m \leq -\delta\|x_{m+1}-x_m\|^2 \tag{35}$$

where, $\delta := -(1-\mathfrak{K})\eta^2-(1+2\mathfrak{K})\eta+\mathfrak{K}$. According to (27) we can get $\delta > 0$.

So, we have

$$\Gamma_{m+1}-\Gamma_m \leq 0. \tag{36}$$

Therefore, the sequence $\{\Gamma_m\}$ is nonincreasing.

On the other hand, due to $\mu_m \geq 0$, we get

$$\begin{aligned}\Gamma_m &= \|x_m-\theta\|^2-\eta_m\|x_{m-1}-\theta\|^2+\mu_m\|x_m-x_{m-1}\|^2\\ &\geq \|x_m-\theta\|^2-\eta_m\|x_{m-1}-\theta\|^2.\end{aligned}$$

This means

$$\begin{aligned}\|x_m-\theta\|^2 &\le \eta_m\|x_{m-1}-\theta\|^2+\Gamma_m\\ &\le \eta\|x_{m-1}-\theta\|^2+\Gamma_1\\ &\le \cdots \le \eta^m\|x_0-\theta\|^2+\Gamma_1(1+\ldots+\eta^{m-1})\\ &\le \eta^m\|x_0-\theta\|^2+\frac{\Gamma_1}{1-\eta}.\end{aligned}\tag{37}$$

Similarly,

$$\begin{aligned}\Gamma_{m+1} &= \|x_{m+1}-\theta\|^2-\eta_{m+1}\|x_m-\theta\|^2+\mu_{m+1}\|x_{m+1}-x_m\|^2\\ &\ge \|x_{m+1}-\theta\|^2-\eta_{m+1}\|x_m-\theta\|^2\\ &\ge -\eta_{m+1}\|x_m-\theta\|^2.\end{aligned}\tag{38}$$

From (37) and (38), we get

$$-\Gamma_{m+1}\le \eta_{m+1}\|x_m-\theta\|^2\le \eta\|x_m-\theta\|^2\le \eta^{m+1}\|x_0-\theta\|^2+\frac{\eta\Gamma_1}{1-\eta}.$$

It follows from (35) that

$$\begin{aligned}\delta\sum_{m=1}^{k}\|x_{m+1}-x_m\|^2 \le \Gamma_1-\Gamma_{k+1} &\le \eta^{k+1}\|x_0-\theta\|^2+\frac{\Gamma_1}{1-\eta}\\ &\le \|x_0-\theta\|^2+\frac{\Gamma_1}{1-\eta}.\end{aligned}$$

So we get $\sum_{m=1}^{+\infty}\|x_{m+1}-x_m\|^2<\infty$ and $\|x_{m+1}-x_m\|\to 0$. Note that

$$\|x_{m+1}-\omega_m\|^2=\|x_{m+1}-x_m\|^2+\eta_m{}^2\|x_m-x_{m-1}\|^2-2\eta_m\langle x_{m+1}-x_m,x_m-x_{m-1}\rangle.$$

So, $\|x_{m+1}-\omega_m\|\to 0$. By (30) and Lemma 2.4, we have

$$\lim_{m\to+\infty}\|x_m-\theta\|^2=l.$$

And from (29), we can get

$$\|\omega_m-\theta\|^2=\|x_m-\theta\|^2+\eta_m(\|x_m-\theta\|^2-\|x_{m-1}-\theta\|^2)+\eta_m(1+\eta_m)\|x_m-x_{m-1}\|^2.$$

We know the sequence $\{\eta_m\}$ is bounded, and

$$\lim_{m\to+\infty}\|\omega_m-\theta\|^2=l.$$

Thus, $\{x_m\}$, $\{\omega_m\}$ and $\{z_m\}$ are all bounded. □

Theorem 3.7. *Suppose that the sequence $\{x_m\}$ is defined by Algorithm 3.3. Then the sequence $\{x_m\}$ weakly converges to $\theta\in VI(C,\Phi)$.*

Proof. Because $\{x_m\}$ is bounded, there exists a subsequence of $\{x_m\}$, which weakly converges to $\theta\in\mathcal{H}$. Without loss of generality, we use $\{x_m\}$ to represent the subsequence, that is $x_m\hookrightarrow\theta$. Since $\|x_m-\omega_m\|\to 0$, $\omega_m\hookrightarrow\theta$. Since $y_m=P_{\mathcal{T}_m}(\omega_m-\rho_m\Phi\omega_m)$, $\forall x\in C$, we have

$$\begin{aligned}0 &\le \langle y_m-\omega_m+\rho_m\Phi\omega_m,x-y_m\rangle\\ &= \langle y_m-\omega_m,x-y_m\rangle+\rho_m\langle\Phi\omega_m,x-y_m\rangle\\ &= \langle y_m-\omega_m,x-y_m\rangle+\rho_m\langle\Phi\omega_m,x-\omega_m\rangle+\rho_m\langle\Phi x_m,x_m-y_m\rangle\\ &\le \langle y_m-\omega_m,x-y_m\rangle+\rho_m\langle\Phi\omega_m,x-\omega_m\rangle+\rho_m\langle\Phi\omega_m,\omega_m-y_m\rangle.\end{aligned}$$

Let $m\to+\infty$, then, $\forall x\in C$, we have $\langle\Phi\theta,x-\theta\rangle\ge 0$ and $\theta\in VI(C,\Phi)$. This completes the proof. □

4. Numerical illustrations

In section 4, we give some concrete examples to illustrate the efficiency of the suggested algorithms. All the projection over C are computed effectively by math, numpy and matplotlib.pyplot in Python 3.8. All the programs are performed on a PC Desktop Intel(R) Core(TM) i5-5200U @ 2.20Ghz 2.20Ghz, RAM 4.00GB. We apply Algorithm 3.1 (shortly, Alg.2) and Algorithm 3.3 (shortly, Alg.3) for numerical calculations to solve VIPs and some other algorithms are used for comparison. We will write the results of the numerical calculations in the table below, among them, 'Iter.' and 'Msec.' respectively represent the iteration steps and total running time of the algorithm (in milliseconds). In addition, we rename the following algorithm.

1. Algorithm 3.1 (Alg.2),
2. Algorithm 3.3 (Alg.3),
3. Migorski's algorithm[5] (Algorithm M).

Example 4.1. *We consider the space of $\mathcal{H} = \mathbb{R}^4$, operator $\Phi := T =$*

$$\begin{pmatrix} 0.5 & 0.1 & 0 & 0 \\ -0.1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.4 & 0.1 \\ 0 & 0 & 0.11 & 0.5 \end{pmatrix}$$

$C : \sum_{m=1}^{4} x_m = 0.$

Now we use Algorithm M and Alg.2 to solve this problem

1. Set step size $\rho = 0.5$ and $\mu = 0.5$,
2. Error ≤ 0.001,
3. $\beta_m = \frac{1}{m+1}, \nu_m = \frac{1}{2(m+1)}$,

Case 1. x0=(6,6,1,2), x1=(0,7,3,2),
Case 2. x0=(0,0,1,2), x1=(0,0,3,2),
Case 3. x0=(1,1,1,2), x1=(2,2,2,4).

Table 1: Comparison of Algorithm M and Algorithm 1

		Case.1		Case.2		Case.3	
		Iter.	Msec.	Iter.	Msec.	Iter.	Msec.
1.	Alg.2	335	87005	335	78604	335	90604
2.	Algorithm M	959	125006	953	105008	1045	135010

Example 4.2. *Let $\mathcal{H} = \mathbb{R}^2$. Define the operator $\Phi := T =$*

$$\begin{pmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{pmatrix}$$

and $C : \sum_{m=1}^{2} x_m = 0.$

Now we use Alg.2 and Alg.3 to solve this problem.

1. Set step size $\rho = 0.5$ and $\mu = 0.5$,
2. Error ≤ 0.001,
3. $\beta_m = \frac{1}{m+1}, \nu_m = \frac{1}{2(m+1)}$.

Case 1. x0=(0,1), x1=(2,2),
Case 2. x0=(3,1), x1=(4,2),
Case 3. x0=(13,11), x1=(14,12).

Table 2: Comparison of Algorithm 1, Algorithm 2 and Algorithm 3

		Case.1		Case.2		Case.3	
		Iter.	Msec.	Iter.	Msec.	Iter.	Msec.
1.	Alg.2	37	2000	41	13000	24	4000
2.	Alg.3	20	3000	24	3000	81	12000

It can be seen from Table 1 that the number of iteration steps of Algorithm 3.1 is less than Migorski's Algorithm, and the running time is also faster than Migorski's algorithm; from Table 2 we can see that Algorithm 3.2 is stronger than Algorithm 3.1 in terms of iterative steps and running time.

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