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Generalized Bivariate Baskakov Durrmeyer Operators and Associated GBS Operators

Mamta Rani[a], Nadeem Rao[a], Pradeep Malik[a]

[a]*Department of Mathematics, Shree Guru Gobind Singh Tricentenary University, Gurugram-122505, Haryana, India*

Abstract. In the present research article, we construct a new sequence of Generalized Bivariate Baskakov Durrmeyer Operators. We investigate rate of convergence and the order of approximation with the aid of modulus of continuity in terms of well known Peetre's K-functional, Voronovskaja type theorems and Lipschitz maximal functions. Further, graphical analysis is discussed. Moreover, we study the approximation properties of the operators in Bögel-spaces with the aid of mixed-modulus of continuity.

1. Introduction

In the last decade of nineteenth century (1885), Weierstrass [25] proposed a prominent and historical theorem termed as Weierstrass approximation theorem. This theorem plays a vital role in the field of approximation theorem defined as: "Corresponding to each continuous function defined on closed interval $[a, b]$ there corresponds a polynomial function of degree n such that for any $\epsilon > 0$, we have $|f(x) - P_n(x)| < \epsilon$ for the large value of n." The proof of this celebrated theorem was very difficult and lengthy to understand. In order to give a simple and easy proof of this theorem, Bernstein (1912) [5] proposed the Bernstein polynomials as follows:

$$B_n(f;x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) f\left(\frac{\nu}{n}\right),\ n \in \mathbb{N}, \tag{1}$$

where $p_{n,\nu}(x) = \binom{n}{k} x^{\nu}(1-x)^{\nu-n}$. He proved that $B_n(f;x) \rightrightarrows f$ for each $f \in C[0,1]$ where $\rightrightarrows$ stands for uniform convergence. Several researchers, *e.g.*, Mursaleen et al. ([16], [17],), Nasiruzamman et al. [18, 20–22], Acar et al. ([1], [2]), Mohiuddine et al. [15], Ana et al. [3], İçöz et al. ([11]), [12]), Kajla et al. ([13], [14]), Nasiruzamman et al.([19]), Rao et al. ([23], [24]) constructed new sequences of linear positive operators to investigate the rapidity of convergence and order of approximation in different functional spaces in terms of several generating functions.

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Corresponding author: Nadeem Rao
Email addresses: mamtakalra84@gmail.com (Mamta Rani), nadeemrao1990@gmail.com (Nadeem Rao), pradeepmaths@gmail.com (Pradeep Malik)

In the recent past, for $g \in C[0,1], m \in \mathbb{N}$ and $\alpha \in [-1,1]$, Chen et al. [8] constructed a sequnce of new linear positive operators as:

$$T_{m,\alpha}(g;y) = \sum_{i=0}^{m} g\left(\frac{i}{m}\right) p_{m,i}^{\alpha}(y) \qquad (y \in [0,1]), \tag{2}$$

where $p_{1,0}^{(\alpha)} = 1-y, \quad p_{1,1}^{(\alpha)} = y$ and

$$p_{m,i}^{\alpha}(y) = \left[(1-\alpha)y\binom{m-2}{i} + (1-\alpha)(1-y)\binom{m-2}{i-2} + \alpha y(1-y)\binom{m}{i}\right] y^{i-1}(1-y)^{m-i-1} \quad (m \geq 2). \tag{3}$$

The operators defined in (2) are named as $\alpha-$Berstein operator of order m.

Remark 1.1. *One can not that for $\alpha = 1$, the relation (2) is reduced to classical Bernstein operators [5].*

The bivariate version of $\alpha-$ Bernstein-Durrmeyer operators were developed and investigated by Micláu s and Kajla [13] where they studied GBS operator of $\alpha-$Bernstein-Durrmeyer operators. Further, Kajla and Acar [14] proposed the classical case of these linear positive operators. While the Kantorovich variant of $\alpha-$Bernstein operators developed by Mohiuddine et al. [15]. Later, Aral and Erbay [4] introduced a parametric extension of Baskakov operators as: for every $f \in C_B[0,\infty)$(space of bounded and continuous functions)

$$L_{n,\alpha}(f;x) = \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right), \tag{4}$$

where $n \geq 1, x \in [0,\infty)$ and

$$Q_{n,k}^{(\alpha)}(x) = \frac{x^{k-1}}{(1+x)^{n+k-1}}\left\{\frac{\alpha x}{1+x}\binom{n+k-1}{k} - (1-\alpha)(1+x)\binom{n+k-3}{k-2} + (1-\alpha)x\binom{n+k-1}{k}\right\}, \tag{5}$$

with $\binom{n-3}{-2} = \binom{n-2}{-1} = 0$. The operators defined in (4) are restricted for the space of continuous functions only.

2. Construction of bivariate Szász-Durrmeyer-Operators $H^*_{n_1,n_2}(.;.)$ and their Basic Estimates

Take $\mathcal{I}^2 = \{(y_1,y_2) : 0 \leq y_1 < \infty,\ 0 \leq y_2 < \infty\}$ and $C\left(\mathcal{I}^2\right)$ is the class of all continuous functions on $\mathcal{I}^2$ equipped with the norm $\|g\|_{C(\mathcal{I}^2)} = \sup_{(y_1,y_2)\in\mathcal{I}^2} |g(y_1,y_2)|$. Then for all $h \in C\left(\mathcal{I}^2\right)$ and $n_1, n_2 \in \mathbb{N}$, we construct sequence of bivariate generalized baskakov operators as follows:

$$H^*_{n_1,n_2}(f;y_1,y_2) = \sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} Q^{*,\alpha}(n_1,y_1)Q^{*,\alpha}(n_2,y_2) f\left(\frac{\nu_1}{n_1},\frac{\nu_2}{n_2}\right), \tag{6}$$

where

$$Q^{*,\alpha}(n_i,y_i) = \frac{y_i^{k-1}}{(1+y_i)^{n+k-1}}\left\{\frac{\alpha y_i}{1+y_i}\binom{n+k-1}{k} - (1-\alpha)(1+y_i)\binom{n+k-3}{k-2} + (1-\alpha)y_i\binom{n+k-1}{k}\right\}, \quad i \in \{1,2\}. \tag{7}$$

Lemma 2.1. *[4] Let $e_i(t) = t^i$, $i = 0, 1, 2$ be the test functions. Then, for the operators $\mathcal{B}^*_{n,\alpha}(, ; ,)$, we have*

$$\begin{aligned} L_{n,\alpha}(e_0; x) &= 1, \\ L_{n,\alpha}(e_1; x) &= \left(1 + \frac{2}{n}(\alpha - 1)\right)x + \frac{\lambda + 1}{n}, \\ L_{n,\alpha}(e_2; x) &= x^2\left(1 + \frac{4\alpha - 3}{n}\right) + x\left(\frac{2\lambda + 3}{n} + \frac{4\alpha - 4 + (2\lambda + 3)(\alpha - 1)}{n^2}\right) + \frac{\lambda^2 + 3\lambda + 2}{n^2}. \end{aligned}$$

Lemma 2.2. *[4] Let $\psi^i_x(t) = (t - x)^i$, $i = 0, 1, 2$ be the central moments. Then, for the operators defined by (4), we have*

$$\begin{aligned} \mathcal{B}^*_{n,\alpha}(\psi^0_x; x) &= 1, \\ \mathcal{B}^*_{n,\alpha}(\psi^1_x; x) &= \frac{2}{n}(\alpha - 1)x + \frac{\lambda + 1}{n}, \\ \mathcal{B}^*_{n,\alpha}(\psi^2_x; x) &= \frac{4}{n}(1 - \alpha)x^2 + x\left(\frac{1}{n} + \frac{4\alpha - 4 + (2\lambda + 3)(\alpha - 1)}{n^2}\right)x + \frac{\lambda^2 + 3\lambda + 2}{n^2}. \end{aligned}$$

Lemma 2.3. *Let $e_{i,}(y_1, y_2) = y_1^i y_2^j$. Then, for the operator (4), we have*

$$\begin{aligned} H^*_{n_1,n_2}(e_{0,0}; y_1, y_2) &= 1, \\ H^*_{n_1,n_2}(e_{1,0}; y_1, y_2) &= \left(1 + \frac{2}{n_1}(\alpha - 1)\right)y_1 + \frac{\lambda + 1}{n_1}, \\ H^*_{n_1,n_2}(e_{0,1}; y_1, y_2) &= \left(1 + \frac{2}{n_2}(\alpha - 1)\right)y_2 + \frac{\lambda + 1}{n_2}, \\ H^*_{n_1,n_2}(e_{1,1}; y_1, y_2) &= \left(\left(1 + \frac{2}{n_1}(\alpha - 1)\right)y_1 + \frac{\lambda + 1}{n}\right)\left(\left(1 + \frac{2}{n_2}(\alpha - 1)\right)y_2 + \frac{\lambda + 1}{n_1}\right) \\ H^*_{n_1,n_2}(e_{2,0}; y_1, y_2) &= \frac{4}{n_1}(1 - \alpha)y_1^2 + \left(\frac{1}{n_1} + \frac{4\alpha - 4 + (2\lambda + 3)(\alpha - 1)}{n^2}\right)y_1, + \frac{\lambda^2 + 3\lambda + 2}{n_1^2}, \\ H^*_{n_1,n_2}(e_{0,2}; y_1, y_2) &= \frac{4}{n_2}(1 - \alpha)y_2^2 + \left(\frac{1}{n_2} + \frac{4\alpha - 4 + (2\lambda + 3)(\alpha - 1)}{n_2^2}\right)y_2 + \frac{\lambda^2 + 3\lambda + 2}{n_2^2}. \end{aligned}$$

Proof. In the light of Lemma (2.1) and linearity property, we have

$$\begin{aligned} H^*_{n_1,n_2}(e_{0,0}; y_1, y_2) &= H^*_{n_1,n_2}(e_0; y_1, y_2)H^*_{n_1,n_2}(e_0; y_1, y_2), \\ H^*_{n_1,n_2}(e_{1,0}; y_1, y_2) &= H^*_{n_1,n_2}(e_1; y_1, y_2)H^*_{n_1,n_2}(e_0; y_1, y_2), \\ H^*_{n_1,n_2}(e_{0,1}; y_1, y_2) &= H^*_{n_1,n_2}(e_0; y_1, y_2)H^*_{n_1,n_2}(e_1; y_1, y_2), \\ H^*_{n_1,n_2}(e_{1,1}; y_1, y_2) &= H^*_{n_1,n_2}(e_1; y_1, y_2)H^*_{n_1,n_2}(e_1; y_1, y_2), \\ H^*_{n_1,n_2}(e_{2,0}; y_1, y_2) &= H^*_{n_1,n_2}(e_2; y_1, y_2)H^*_{n_1,n_2}(e_0; y_1, y_2), \\ H^*_{n_1,n_2}(e_{0,2}; y_1, y_2) &= H^*_{n_1,n_2}(e_0; y_1, y_2)H^*_{n_1,n_2}(e_2; y_1, y_2), \end{aligned}$$

which proves Lemma (2.3). □

Lemma 2.4. *Let $\Psi^{i,j}_{y_1,y_2}(t, s) = \eta_{i,j}(t, s) = (t - y_1)^i(s - y_2)^j$, $\quad i, j \in \{0, 1, 2\}$ be the central moments. Then, from the* *operators $H^*_{n_1,n_2}(.;.)$ defined by* (6) *satisfies the following identities*

$$\begin{aligned}
H^*_{n_1,n_2}(\eta_{0,0};y_1,y_2) &= 1\\
H^*_{n_1,n_2}(\eta_{1,0};y_1,y_2) &= \frac{2}{n_1}(\alpha-1)y_1+\frac{\lambda+1}{n_1},\\
H^*_{n_1,n_2}(\eta_{0,1};y_1,y_2) &= \frac{2}{n_1}(\alpha-1)y_1+\frac{\lambda+1}{n_1},\\
H^*_{n_1,n_2}(\eta_{1,1};y_1,y_2) &= \Big(\frac{2}{n_1}(\alpha-1)y_1+\frac{\lambda+1}{n_1}\Big)\Big(\frac{2}{n_2}(\alpha-1)y_2+\frac{\lambda+1}{n_2}\Big),\\
H^*_{n_1,n_2}(\eta_{2,0};y_1,y_2) &= y_1^2\Big(1+\frac{4\alpha-3}{n_1}\Big)+y_1\Big(\frac{2\lambda+3}{n_1}+\frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n_1^2}\Big)+\frac{\lambda^2+3\lambda+2}{n_2^2},\\
H^*_{n_1,n_2}(\eta_{0,2};y_1,y_2) &= y_2^2\Big(1+\frac{4\alpha-3}{n_2}\Big)+y_2\Big(\frac{2\lambda+3}{n_2}+\frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n_2^2}\Big)+\frac{\lambda^2+3\lambda+2}{n_2^2}.
\end{aligned}$$

Proof. In the light of Lemma (2.2) and linearity property, we have

$$\begin{aligned}
H^*_{n_1,n_2}(\eta_{0,0};y_1,y_2) &= H^*_{n_1,n_2}(\eta_0;y_1,y_2)H^*_{n_1,n_2}(\eta_0;y_1,y_2),\\
H^*_{n_1,n_2}(\eta_{1,0};y_1,y_2) &= H^*_{n_1,n_2}(\eta_1;y_1,y_2)H^*_{n_1,n_2}(\eta_0;y_1,y_2),\\
H^*_{n_1,n_2}(\eta_{0,1};y_1,y_2) &= H^*_{n_1,n_2}(\eta_0;y_1,y_2)H^*_{n_1,n_2}(\eta_1;y_1,y_2),\\
H^*_{n_1,n_2}(\eta_{1,1};y_1,y_2) &= H^*_{n_1,n_2}(\eta_1;y_1,y_2)H^*_{n_1,n_2}(\eta_1;y_1,y_2),\\
H^*_{n_1,n_2}(\eta_{2,0};y_1,y_2) &= H^*_{n_1,n_2}(\eta_2;y_1,y_2)H^*_{n_1,n_2}(\eta_0;y_1,y_2),\\
H^*_{n_1,n_2}(\eta_{0,2};y_1,y_2) &= H^*_{n_1,n_2}(\eta_0;y_1,y_2)H^*_{n_1,n_2}(\eta_2;y_1,y_2),
\end{aligned}$$

which proves Lemma (2.3). □

Lemma 2.5. *For all $(y_1,y_2)\in\mathcal{I}^2$ and sufficiently large $n_1,n_2\in\mathbb{N}$ the operators $H^*_{n_1,n_2}(.\,;\,.)$ given by* (4) *satisfy*

(1) $H^*_{n_1,n_2}(\Psi^{2,0}_{y_1,y_2};y_1,y_2) = o\left(\frac{1}{n_1}\right)(y_1+1)^2\le C_1(y_1+1)^2$ *as* $n_1,n_2\to\infty$;

(2) $H^*_{n_1,n_2}(\Psi^{0,2}_{y_1,y_2};y_1,y_2) = o\left(\frac{1}{n_2}\right)(y_2+1)^2\le C_2(y_2+1)^2$ *as* $n_1,n_2\to\infty$;

(3) $H^*_{n_1,n_2}(\Psi^{4,0}_{y_1,y_2};y_1,y_2) = o\left(\frac{1}{n_1^2}\right)(y_1+1)^4\le C_3(y_1+1)^4$ *as* $n_1,n_2\to\infty$;

(4) $H^*_{n_1,n_2}(\Psi^{0,4}_{y_1,y_2};y_1,y_2) = o\left(\frac{1}{n_2^2}\right)(y_2+1)^4\le C_4(y_2+1)^4$ *as* $n_1,n_2\to\infty$.

3. Some Approximation Results in Weighted Space and Their Degree of Convergence

Let φ be weight function such that $\varphi(y_1,y_2)=1+y_1^2+y_2^2$ and satisfying $B_\varphi\left(\mathcal{I}^2\right)=\{g:|\ g(y_1,y_2)\ |\le C_g\varphi(y_1,y_2),\quad C_g>0\}$, where $B_\varphi\left(\mathcal{I}^2\right)$ is the set of all bounded function on $\mathcal{I}^2=[0,\infty)\times[0,\infty)$. Suppose $C^{(m)}\left(\mathcal{I}^2\right)$ be the m-times continuously differentiable functions defined on $\mathcal{I}^2=\{(y_1,y_2)\in\mathcal{I}^2:y_1,y_2\in[0,\infty)\}$. The equipped norm on B_φ defined by $\parallel g\parallel_\varphi=\sup_{y_1,y_2\in\mathcal{I}^2}\frac{|g(y_1,y_2)|}{\varphi(y_1,y_2)}$. Moreover we have classified here some classes of function as follows:

$$C^m_\varphi\left(\mathcal{I}^2\right)=\{g:g\in C_\varphi\left(\mathcal{I}^2\right);\quad \text{such that}\lim_{(y_1,y_2)\to\infty}\frac{g(y_1,y_2)}{\varphi(y_1,y_2)}=k_g<\infty\};$$

$$C^0_\varphi\left(\mathcal{I}^2\right)=\{f:f\in C^m_\varphi\left(\mathcal{I}^2\right);\quad \text{such that}\lim_{(y_1,y_2)\to\infty}\frac{g(y_1,y_2)}{\varphi(y_1,y_2)}=0\}.$$

$$C_\varphi\left(\mathcal{I}^2\right)=\{g: g\in B_\varphi\cap C_\varphi\left(\mathcal{I}^2\right)\}.$$

Suppose $\omega_\varphi(g;\delta_1,\delta_2)$ is the weighted modulus of continuity for all $g\in C_\varphi^0\left(\mathcal{I}^2\right)$ and $\delta_1,\delta_2>0$, defined by

$$\omega_\varphi(g;\delta_1,\delta_2)=\sup_{(y_1,y_2)\in[0,\infty)}\ \sup_{0\le|\theta_1|\le\delta_1,\ 0\le|\theta_2|\le\delta_2}\frac{\mid g(y_1+\theta_1,y_2+\theta_2)-g(y_1,y_2)\mid}{\varphi(y_1,y_2)\varphi(\theta_1,\theta_2)}. \tag{8}$$

For any $\eta_1,\eta_2>0$ one has

$$\omega_\varphi(g;\eta_1\delta_1,\eta_2\delta_2)\le 4(1+\eta_1)(1+\eta_2)(1+\delta_1^2)(1+\delta_2^2)\omega_\varphi(g;\delta_1,\delta_2),$$

$$\begin{aligned}\mid g(t,s)-g(y_1,y_2)\mid &\le \varphi(y_1,y_2)\varphi\left(\mid t-y_1\mid,\mid s-y_2\mid\right)\omega_\varphi\left(g;\mid t-y_1\mid,\mid s-y_2\mid\right)\\ &\le (1+y_1^2+y_2^2)(1+(t-y_1)^2)(1+(s-y_2)^2)\omega_\varphi\left(g;\mid t-y_1\mid,\mid s-y_2\mid\right).\end{aligned}$$

Theorem 3.1. *Let $g\in C_\varphi^0\left(\mathcal{I}^2\right)$, then for sufficiently large $n_1,n_2\in\mathbb{N}$ operator $H^*_{n_1,n_2}$ satisfying the inequality*

$$\frac{\mid H^*_{n_1,n_2}(g;y_1,y_2)-g(y_1,y_2)\mid}{(1+y_1^2+y_2^2)}\ \le\ \Psi_{y_1,y_2}\left(1+o\left(n_1^{-1}\right)\right)\left(1+o\left(n_2^{-1}\right)\right)\omega_\varphi\left(g;o\left(n_1^{-\frac{1}{2}}\right),\left(n_2^{-\frac{1}{2}}\right)\right),$$

where $\Psi_{y_1,y_2}=\left(1+(y_1+1)+C_1(y_1+1)^2+\sqrt{C_3}(y_1+1)^3\right)\left(1+(y_2+1)+C_2(y_2+1)^2+\sqrt{C_4}(y_2+1)^3\right)$ *and* $C_1,\ C_2,\ C_3,\ C_4>0$.

Proof. For all $\delta_{n_1},\delta_{n_2}>0$ we have

$$\begin{aligned}\mid g(t,s)-g(y_1,y_2)\mid &\le 4(1+y_1^2+y_2^2)\left(1+(t-y_1)^2\right)\left(1+(s-y_2)^2\right)\\ &\times \left(1+\frac{\mid t-y_1\mid}{\delta_{n_1}}\right)\left(1+\frac{\mid s-y_2\mid}{\delta_{n_2}}\right)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\varphi\left(g;\delta_{n_1},\delta_{n_2}\right)\\ &= 4(1+y_1^2+y_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\\ &\times \left(1+\frac{\mid t-y_1\mid}{\delta_{n_1}}+(t-y_1)^2+\frac{1}{\delta_{n_1}}\mid t-y_1\mid(t-y_1)^2\right)\\ &\times \left(1+\frac{\mid s-y_2\mid}{\delta_{n_2}}+(s-y_2)^2+\frac{\mid s-y_2\mid}{\delta_{n_2}}(s-y_2)^2\right)\omega_\varphi\left(g;\delta_{n_1},\delta_{n_2}\right).\end{aligned}$$

Applying operator $H^*_{n_1,n_2}(,;,)$ in the above equation both the sides and using Cauchy-Schwarz inequality,

$$\begin{aligned}\mid H^*_{n_1,n_2}(g;y_1,y_2)-g(y_1,y_2)\mid &\le H^*_{n_1,n_2}\left(\mid g(.,.)-g(y_1,y_2)\mid;y_1,y_2\right)4(1+y_1^2+y_2^2)\\ &\times H^*_{n_1,n_2}\left(1+\frac{\mid t-y_1\mid}{\delta_{n_1}}+(t-y_1)^2+\frac{1}{\delta_{n_1}}\mid t-y_1\mid(t-y_1)^2;y_1,y_2\right)\\ &\times H^*_{n_1,n_2}\left(1+\frac{\mid s-y_2\mid}{\delta_{n_2}}+(s-y_2)^2+\frac{\mid s-y_2\mid}{\delta_{n_2}}(s-y_2)^2;y_1,y_2\right)\\ &\times (1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\varphi\left(g;\delta_{n_1},\delta_{n_2}\right)\\ &= 4(1+y_1^2+y_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\varphi\left(g;\delta_{n_1},\delta_{n_2}\right)\\ &\times \left(1+\frac{1}{\delta_{n_1}}H^*_{n_1,n_2}(\mid t-y_1\mid;y_1,y_2)+H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)\right.\\ &+ \left.\frac{1}{\delta_{n_1}}H^*_{n_1,n_2}(\mid t-y_1\mid(t-y_1)^2;y_1,y_2\right)\\ &\times \left(1+\frac{1}{\delta_{n_2}}H^*_{n_1,n_2}(\mid s-y_2\mid;y_1,y_2)+H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2)\right.\\ &+ \left.\frac{1}{\delta_{n_2}}H^*_{n_1,n_2}(\mid s-y_2\mid(s-y_2)^2;y_1,y_2\right);\end{aligned}$$

$$
\begin{aligned}
| H^*_{n_1,n_2}(g;y_1,y_2)-g(y_1,y_2) | \;\leq\;& 4(1+y_1^2+y_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}\,(g;\delta_{n_1},\delta_{n_2})\\
\times\;& \Bigg[1+\frac{1}{\delta_{n_1}}\sqrt{H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)}+H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)\\
+\;& \frac{1}{\delta_{n_1}}\sqrt{H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)}\sqrt{H^*_{n_1,n_2}((t-y_1)^4;y_1,y_2)}\Bigg]\\
\times\;& \Bigg[1+\frac{1}{\delta_{n_2}}\sqrt{H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2)}+H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2)\\
+\;& \frac{1}{\delta_{n_2}}\sqrt{H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2)}\sqrt{H^*_{n_1,n_2}((s-y_2)^4;y_1,y_2)}\Bigg].
\end{aligned}
$$

In view of Lemma 2.5 and choose $\delta_{n_1}=o\left(n_1^{-\frac{1}{2}}\right)$ and $\delta_{n_2}=o\left(n_2^{-\frac{1}{2}}\right)$, then

$$
\begin{aligned}
| H^*_{n_1,n_2}(g;y_1,y_2)-g(y_1,y_2) | \;\leq\;& 4(1+y_1^2+y_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}\,(g;\delta_{n_1},\delta_{n_2})\\
\times\;& \Bigg[1+\frac{1}{\delta_{n_1}}\sqrt{o\left(\frac{1}{n_1}\right)(y_1+1)^2}+o\left(\frac{1}{n_1}\right)(y_1+1)^2\\
+\;& \frac{1}{\delta_{n_1}}\sqrt{o\left(\frac{1}{n_1}\right)(y_1+1)^2}\sqrt{o\left(\frac{1}{n_1}\right)(y_1+1)^4}\Bigg]\\
\times\;& \Bigg[1+\frac{1}{\delta_{n_2}}\sqrt{o\left(\frac{1}{n_2}\right)(y_2+1)^2}+o\left(\frac{1}{n_2}\right)(y_2+1)^2\\
+\;& \frac{1}{\delta_{n_2}}\sqrt{o\left(\frac{1}{n_2}\right)(y_2+1)^2}\sqrt{o\left(\frac{1}{n_2}\right)(y_2+1)^4}\Bigg]\\
\leq\;& 4(1+y_1^2+y_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}\,(g;\delta_{n_1},\delta_{n_2})\\
\times\;& \Big[1+(y_1+1)+C_1(y_1+1)^2+\sqrt{C_2}(y_1+1)^3\Big]\Big[1+(y_2+1)\\
+\;& C_3(y_2+1)^2+\sqrt{C_4}(y_2+1)^3\Big].
\end{aligned}
$$

Which completes the proof. □

Lemma 3.2 ([9, 10]). *For the positive sequence of operators $\{L_{n_1,n_2}\}_{n_1,n_2\geq 1}$, which acting $C_{\varphi}\to B_{\varphi}$ defined earlier then there exists some positive K such that*

$$\| L_{n_1,n_2}(\varphi;y_1,y_2)\|_{\varphi}\leq K.$$

Theorem 3.3 ([9, 10]). *or the positive sequence of operators $\{L_{n_1,n_2}\}_{n_1,n_2\geq 1}$ acting $C_{\varphi}\to B_{\varphi}$ defined earlier satisfying the following conditions*

$$
\begin{aligned}
(1)\quad & \lim_{n_1,n_2\to\infty}\| L_{n_1,n_2}(1;y_1,y_2)-1\|_{\varphi} &=& 0;\\
(2)\quad & \lim_{n_1,n_2\to\infty}\| L_{n_1,n_2}(t;y_1,y_2)-y_1\|_{\varphi} &=& 0;\\
(3)\quad & \lim_{n_1,n_2\to\infty}\| L_{n_1,n_2}(s;y_1,y_2)-y_2\|_{\varphi} &=& 0;\\
(4)\quad & \lim_{n_1,n_2\to\infty}\| L_{n_1,n_2}((t^2+s^2);y_1,y_2)-(y_1^2+y_2^2)\|_{\varphi} &=& 0.
\end{aligned}
$$

Then for all $g\in C^0_{\varphi}$,

$$\lim_{n_1,n_2\to\infty}\| L_{n_1,n_2}g-g\|_{\varphi}=0$$

and there exists another function $f \in C_\varphi \setminus C^0_\varphi$*, such that*

$$\lim_{n_1,n_2\to\infty} \| L_{n_1,n_2} f - f \|_\varphi \geq 1.$$

Theorem 3.4. *If* $g \in C^0_\varphi\left(\mathcal{I}^2\right)$*, then we have*

$$\lim_{n_1,n_2\to\infty} \| H^*_{n_1,n_2}(g) - g \|_\varphi = 0.$$

Proof.

$$\begin{aligned}
\| H^*_{n_1,n_2}(\varphi; y_1, y_2) \|_\varphi &= \sup_{(y_1,y_2)\in\mathcal{I}^2} \frac{| H^*_{n_1,n_2}(1+y_1^2+y_2^2; y_1, y_2) |}{1+y_1^2+y_2^2} \\
&= 1 + \sup_{(y_1,y_2)\in\mathcal{I}^2} \left[\frac{1}{1+y_1^2+y_2^2} \left| \left(1 + H^*_{n_1,n_2}(y_1^2; y_1, y_2) + H^*_{n_1,n_2}(y_2^2; y_1, y_2) \right) \right| \right] \\
&= 1 + \left| \frac{4}{n_1}(1-\alpha) \right| \sup_{(y_1,y_2)\in\mathcal{I}^2} \frac{y_1^2}{1+y_1^2+y_2^2} \\
&+ \left| + \left(\frac{1}{n_1} + \frac{4\alpha - 4 + (2\lambda+3)(\alpha-1)}{n_1^2} \right) \right| \sup_{(y_1,y_2)\in\mathcal{I}^2} \frac{y_1}{1+y_1^2+y_2^2} \\
&+ \left| \frac{\lambda^2+3\lambda+2}{n_1^2} \right| \sup_{(y_1,y_2)\in\mathcal{I}^2} \frac{1}{1+y_1^2+y_2^2} \\
&+ 1 + \left| \frac{4}{n_2}(1-\alpha) \right| \sup_{(y_1,y_2)\in\mathcal{I}^2} \frac{y_2^2}{1+y_1^2+y_2^2} \\
&+ \left| + \left(\frac{1}{n_2} + \frac{4\alpha - 4 + (2\lambda+3)(\alpha-1)}{n_2^2} \right) \right| \sup_{(y_1,y_2)\in\mathcal{I}^2} \frac{y_2}{1+y_1^2+y_2^2} \\
&+ \left| \frac{\lambda^2+3\lambda+2}{n_2^2} \right| \sup_{(y_1,y_2)\in\mathcal{I}^2} \frac{1}{1+y_1^2+y_2^2} \\
&\leq 2 + \left| \frac{4}{n_1}(1-\alpha) \right| + \left| + \left(\frac{1}{n_1} + \frac{4\alpha - 4 + (2\lambda+3)(\alpha-1)}{n_1^2} \right) \right| \\
&+ \left| \frac{4}{n_2}(1-\alpha) \right| + \left| + \left(\frac{1}{n_2} + \frac{4\alpha - 4 + (2\lambda+3)(\alpha-1)}{n_2^2} \right) \right| \\
&+ \left| \frac{\lambda^2+3\lambda+2}{n_1^2} \right| + \left| \frac{\lambda^2+3\lambda+2}{n_2^2} \right|.
\end{aligned}$$

Now for all $n_1, n_2 \in \mathbb{N} \setminus \{1, 2\}$, there exists a positive constant K such that

$$\| H^*_{n_1,n_2}(\varphi; y_1, y_2) \|_\varphi \leq K.$$

Therefore, in order to prove Theorem 3.4 it is sufficient from the Lemma 3.2 and Theorem 3.3. Thus we led to prove of Theorem 3.4. □

For any $g \in C(\mathcal{I}^2)$ and $\delta > 0$ modulus of continuity of order second is given by

$$\omega\left(g; \delta_{n_1}, \delta_{n_2}\right) = \sup\{| g(t,s) - g(y_1, y_2) |: (t,s), (y_1, y_2) \in \mathcal{I}^2\}$$

with $| t - y_1 | \leq \delta_{n_1}$, $| s - y_2 | \leq \delta_{n_2}$ with the partial modulus of continuity defined as:

$$\omega_1(g; \delta) = \sup_{0\leq y_2\leq\infty} \sup_{|x_1-x_2|\leq\delta} \{| g(x_1, y_2) - g(x_2, y_2) |\},$$

$$\omega_2(g;\delta) = \sup_{0\le y_1\le\infty}\ \sup_{|y_1-y_2|\le\delta} \{|\, g(y_1,y_1) - g(y_1,y_2)\,|\}.$$

Theorem 3.5. *For any $g \in C(\mathcal{I}^2)$ we have*

$$|\, H^*_{n_1,n_2}(g;y_1,y_2) - g(y_1,y_2)\,| \le 2\Big(\omega_1(g;\delta_{y_1,n_1}) + \omega_2(g;\delta_{n_2,y_2})\Big).$$

Proof. In order to give the prove of Theorem 3.5, in general we use well-known Cauchy-Schwarz inequality. Thus we see that

$$\begin{aligned}
|\, H^*_{n_1,n_2}(g;y_1,y_2) - g(y_1,y_2)\,| &\le H^*_{n_1,n_2}(|\, g(t,s) - g(y_1,y_2)\,|; y_1,y_2)\\
&\le H^*_{n_1,n_2}(|\, g(t,s) - g(y_1,s)\,|; y_1,y_2)\\
&+ H^*_{n_1,n_2}(|\, g(y_1,s) - g(y_1,y_2)\,|; y_1,y_2)\\
&\le H^*_{n_1,n_2}(\omega_1(g;|\,t-y_1\,|); y_1,y_2) + H^*_{n_1,n_2}(\omega_2(g;|\,s-y_2\,|); y_1,y_2)\\
&\le \omega_1(g;\delta_{n_1})\left(1+\delta_{n_1}^{-1}H^*_{n_1,n_2}(|\,t-y_1\,|; y_1,y_2)\right)\\
&+ \omega_2(g;\delta_{n_2})\left(1+\delta_{n_2}^{-1}H^*_{n_1,n_2}(|\,s-y_2\,|; y_1,y_2)\right)\\
&\le \omega_1(g;\delta_{n_1})\left(1+\frac{1}{\delta_{n_1}}\sqrt{H^*_{n_1,n_2}((t-y_1)^2; y_1,y_2)}\right)\\
&+ \omega_2(g;\delta_{n_2})\left(1+\frac{1}{\delta_{n_2}}\sqrt{H^*_{n_1,n_2}((s-y_2)^2; y_1,y_2)}\right).
\end{aligned}$$

If we choose $\delta^2_{n_1} = \delta^2_{n_1,y_1} = H^*_{n_1,n_2}((t-y_1)^2; y_1,y_2)$ and $\delta^2_{n_2} = \delta^2_{n_2,y_2} = H^*_{n_1,n_2}((s-y_2)^2; y_1,y_2)$, then we easily to reach our desired results. □

Example 3.6. *For a function $f(y_1,y_2) = y_1^3y_2^2 + y_1^2y_2^3$ and the values of $n_1 = n_2 = 20$. The convergence of the operator $H^*_{n_1,n_2}(f;y_1,y_2)$ to the mentioned function $f(y_1,y_2)$ gives as the values of α decreases $0.8, 0.6, 0.4$ on $\mathbb{R}^{[0,1]\times[0,1]}$. Which is shown in the given figure 1.*

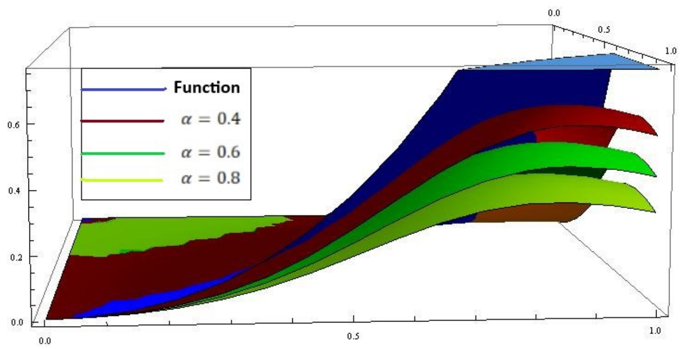

Figure 1: Approximation by $H^*_{20,20}(f;y_1,y_2)$ for different values of $\alpha = 0.8, 0.6, 0.4$

Example 3.7. *The operator $H^*_{n_1,n_2}(f;y_1,y_2)$ to the same function $f(y_1,y_2) = y_1^3y_2^2 + y_1^2y_2^3$ shows better convergence (compare with example 3.6) for same the values of $\alpha = 0.8, 0.6, 0.4$ as increases $n_1 = n_2 = 40$ on $\mathbb{R}^{[0,1]\times[0,1]}$. Which is illustrated in below figure 2.*

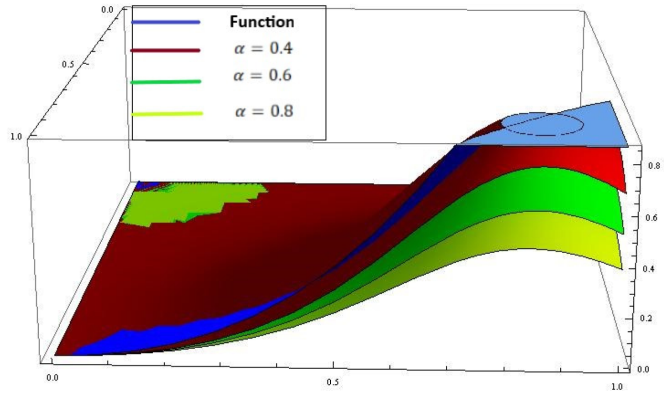

Figure 2: Approximation by $H^*_{40,40}(f;y_1,y_2)$ for different values of $\alpha = 0.8, 0.6, 0.4$

The both examples 3.6 and 3.7 are demonstrated our analytical results.

Here, we find convergence in terms of the Lipschitz class for bivariate function. For $M > 0$ and $\rho_1, \rho_2 \in [0,\infty)$, Lipschitz maximal function space on $E \times E \subset \mathcal{I}^2$ defined by

$$\begin{aligned} \mathcal{L}_{\rho_1,\rho_2}(E\times E) &= \Big\{g : \sup(1+t)^{\rho_1}(1+s)^{\rho_2}\left(g_{\rho_1,\rho_2}(t,s) - g_{\rho_1,\rho_2}(y_1,y_2)\right) \\ &\leq M\frac{1}{(1+y_1)^{\rho_1}}\frac{1}{(1+y_2)^{\rho_2}}\Big\}, \end{aligned}$$

where g is continuous and bounded on $\mathcal{I}^2$, and

$$g_{\rho_1,\rho_2}(t,s) - g_{\rho_1,\rho_2}(y_1,y_2) = \frac{\mid g(t,s) - g(y_1,y_2)\mid}{\mid t-y_1\mid^{\rho_1}\mid s-y_2\mid^{\rho_2}}; \quad (t,s),(y_1,y_2) \in \mathcal{I}^2. \tag{9}$$

Theorem 3.8. *Let $g \in \mathcal{L}_{\rho_1,\rho_2}(E\times E)$, then for any $\rho_1,\rho_2 \in [0,\infty)$, there exists $M > 0$ such that*

$$\begin{aligned} \mid H^*_{n_1,n_2}(g;y_1,y_2) - g(y_1,y_2)\mid &\leq M\Big\{\Big((d(y_1,E))^{\rho_1} + \left(\delta^2_{n_1,y_1}\right)^{\frac{\rho_1}{2}}\Big)\Big((d(y_2,E))^{\rho_2} + \left(\delta^2_{n_2,y_2}\right)^{\frac{\rho_2}{2}}\Big) \\ &+ (d(y_1,E))^{\rho_1}(d(y_2,E))^{\rho_2}\Big\}, \end{aligned}$$

where δ_{n_1,y_1} and δ_{n_2,y_2} defined by Theorem 3.5.

Proof. Take $\mid y_1 - x_0 \mid = d(y_1,E)$ and $\mid y_2 - y_0 \mid = d(y_2,E)$. For any $(y_1,y_2) \in \mathcal{I}^2$, and $(x_0,y_0) \in E\times E$ we let $d(y_1,E) = \inf\{\mid y_1 - y_2\mid : y_2 \in E\}$. Thus we can write here

$$\mid g(t,s) - g(y_1,y_2)\mid \leq M \mid g(t,s) - g(x_0,y_0)\mid + \mid g(x_0,y_0) - g(y_1,y_2)\mid . \tag{10}$$

Apply $H^*_{n_1,n_2}$, we obtain

$$\begin{aligned} \mid H^*_{n_1,n_2}(g;y_1,y_2) - g(y_1,y_2)\mid &\leq H^*_{n_1,n_2}(\mid g(y_1,y_2) - g(x_0,y_0)\mid + \mid g(x_0,y_0) - g(y_1,y_2)\mid) \\ &\leq MH^*_{n_1,n_2}(\mid t-x_0\mid^{\rho_1}\mid s-y_0\mid^{\rho_2};y_1,y_2) \\ &+ M\mid y_1-x_0\mid^{\rho_1}\mid y_2-y_0\mid^{\rho_2}. \end{aligned}$$

For all $A, B \geq 0$ and $\rho \in [0,\infty)$ we know inequality $(A+B)^\rho \leq A^\rho + B^\rho$, thus

$$\mid t-x_0\mid^{\rho_1} \leq \mid t-y_1\mid^{\rho_1} + \mid y_1-x_0\mid^{\rho_1},$$

$| s - y_0 |^{\rho_1} \leq | s - y_2 |^{\rho_2} + | y_2 - y_0 |^{\rho_2}$.

Therefore

$$\begin{aligned} | H^*_{n_1,n_2}(g; y_1, y_2) - g(y_1, y_2) | &\leq MH^*_{n_1,n_2}(| t - y_1 |^{\rho_1} | s - y_2 |^{\rho_2}; y_1, y_2) \\ &+ M | y_1 - x_0 |^{\rho_1} H^*_{n_1,n_2}(| s - y_2 |^{\rho_2}; y_1, y_2) \\ &+ M | y_2 - y_0 |^{\rho_2} H^*_{n_1,n_2}(| t - y_1 |^{\rho_1}; y_1, y_2) \\ &+ 2M | y_1 - x_0 |^{\rho_1} | y_2 - y_0 |^{\rho_2} H^*_{n_1,n_2}(\mu_{0,0}; y_1, y_2). \end{aligned}$$

On apply Hölder inequality on $H^*_{n_1,n_2}$, we get

$$\begin{aligned} H^*_{n_1,n_2}(| t - y_1 |^{\rho_1} | s - y_2 |^{\rho_2}; y_1, y_2) &= \mathcal{U}^{\alpha_1}_{n_1,k}(| t - y_1 |^{\rho_1}; y_1, y_2) \mathcal{V}^{\alpha_2}_{n_2,l}(| s - y_2 |^{\rho_2}; y_1, y_2) \\ &\leq \left(H^*_{n_1,n_2}(| t - y_1 |^2; y_1, y_2)\right)^{\frac{\rho_1}{2}} \left(H^*_{n_1,n_2}(\mu_{0,0}; y_1, y_2)\right)^{\frac{2-\rho_1}{2}} \\ &\times \left(H^*_{n_1,n_2}(| s - y_2 |^2; y_1, y_2)\right)^{\frac{\rho_2}{2}} \left(H^*_{n_1,n_2}(\mu_{0,0}; y_1, y_2)\right)^{\frac{2-\rho_2}{2}}. \end{aligned}$$

Thus we can obtain

$$\begin{aligned} | H^*_{n_1,n_2}(g; y_1, y_2) - g(y_1, y_2) | &\leq M\left(\delta^2_{n_1,y_1}\right)^{\frac{\rho_1}{2}} \left(\delta^2_{n_2,y_2}\right)^{\frac{\rho_2}{2}} + 2M (d(y_1, E))^{\rho_1} (d(y_2, E))^{\rho_2} \\ &+ M (d(y_1, E))^{\rho_1} \left(\delta^2_{n_2,y_2}\right)^{\frac{\rho_2}{2}} + L (d(y_2, E))^{\rho_2} \left(\delta^2_{n_1,y_1}\right)^{\frac{\rho_1}{2}}. \end{aligned}$$

We have complete the proof. □

Theorem 3.9. *If $g \in C'(\mathcal{I}^2)$, then for all $(y_1, y_2) \in \mathcal{I}^2$, operator $H^*_{n_1,n_2}$ satisfying*

$$| H^*_{n_1,n_2}(g; y_1, y_2) - g(y_1, y_2) | \leq \| g_{y_1} \|_{C(\mathcal{I}^2)} \left(\delta^2_{n_1,y_1}\right)^{\frac{1}{2}} + \| g_{y_2} \|_{C(\mathcal{I}^2)} \left(\delta^2_{n_2,y_2}\right)^{\frac{1}{2}},$$

where δ_{n_1,y_1} and δ_{n_2,y_2} are defined by Theorem 3.5.

Proof. Let $g \in C'(\mathcal{I}^2)$, and for any fixed $(y_1, y_2) \in \mathcal{I}^2$ we have

$$g(t, s) - g(y_1, y_2) = \int_{y_1}^{t} g_{\varrho}(\varrho, s) d\varrho + \int_{y_2}^{s} g_{\mu}(y_1, \mu) d\mu.$$

On apply $H^*_{n_1,n_2}$

$$H^*_{n_1,n_2}(g(t, s); y_1, y_2) - g(y_1, y_2) = H^*_{n_1,n_2}\left(\int_{y_1}^{t} g_{\varrho}(\varrho, s) d\varrho; y_1, y_2\right) + H^*_{n_1,n_2}\left(\int_{y_2}^{s} g_{\mu}(y_1, \mu) d\mu; y_1, y_2\right). \tag{11}$$

From the sup-norm on $\mathcal{I}^2$ we can see that

$$\left| \int_{y_1}^{t} g_{\varrho}(\varrho, s) d\varrho \right| \leq \int_{y_1}^{t} | g_{\varrho}(\varrho, s) d\varrho | \leq \| g_{y_1} \|_{C(\mathcal{I}^2)} | t - y_1 | \tag{12}$$

$$\left| \int_{y_2}^{s} g_{\mu}(y_1, \mu) d\mu \right| \leq \int_{y_2}^{s} | g_{\mu}(y_1, \mu) d\mu | \leq \| g_{y_2} \|_{C(\mathcal{I}^2)} | s - y_2 |. \tag{13}$$

In the view of (11), (12) and (13) we can obtain

$$\begin{aligned}
\mid H^*_{n_1,n_2}(g(y_1,y_2);y_1,y_2)-g(y_1,y_2)\mid &\leq H^*_{n_1,n_2}\left(\left|\int_{y_1}^{t} g_{\varrho}(\varrho,s)\mathrm{d}\varrho\right|;y_1,y_2\right)\\
&+ H^*_{n_1,n_2}\left(\left|\int_{y_2}^{s} g_{\mu}(y_1,\mu)\mathrm{d}\mu\right|;y_1,y_2\right)\\
&\leq \parallel g_{y_1}\parallel_{C(\mathcal{I}^2)} H^*_{n_1,n_2}(\mid t-y_1\mid;y_1,y_2)\\
&+ \parallel g_{y_2}\parallel_{C(\mathcal{I}^2)} H^*_{n_1,n_2}(\mid s-y_2\mid;y_1,y_2)\\
&\leq \parallel g_{y_1}\parallel_{C(\mathcal{I}^2)} \left(H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)H^*_{n_1,n_2}(1;y_1,y_2)\right)^{\frac{1}{2}}\\
&+ \parallel g_{y_2}\parallel_{C(\mathcal{I}^2)} \left(H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2)H^*_{n_1,n_2}(1;y_1,y_2)\right)^{\frac{1}{2}}\\
&= \parallel g_{y_1}\parallel_{C(\mathcal{I}^2)} \left(\delta^2_{n_1,y_1}\right)^{\frac{1}{2}} + \parallel g_{y_2}\parallel_{C(\mathcal{I}^2)} \left(\delta^2_{n_2,y_2}\right)^{\frac{1}{2}}.
\end{aligned}$$

□

Theorem 3.10. *For any $f \in C(\mathcal{I}^2)$, if we define an auxiliary operator such that*

$$R^{\alpha_1,\alpha_2}_{n_1,n_2}(f;y_1,y_2) = H^*_{n_1,n_2}(g;y_1,y_2)+f(y_1,y_2)-f\left(\mathcal{U}^{\alpha_1}_{n_1,k}(\mu_{1,0};y_1,y_2),\mathcal{V}^{\alpha_2}_{n_2,l}(\mu_{0,1};y_1,y_2)\right).$$

where, from Lemma 2.4, $\mathcal{U}^{\alpha_1}_{n_1,k}(\mu_{1,0};y_1,y_2)=\left(\frac{2}{n_1}(\alpha-1)y_1+\frac{\lambda+1}{n_1}\right)$ and $\mathcal{V}^{\alpha_2}_{n_2,l}(\mu_{0,1};y_1,y_2)=\frac{2}{n_2}(\alpha-1)y_2+\frac{\lambda+1}{n_2}$.

Then for all $g \in C'(\mathcal{I}^2)$, $R^{\alpha_1,\alpha_2}_{n_1,n_2}$ satisfying

$$\begin{aligned}
R^{\alpha_1,\alpha_2}_{n_1,n_2}(g;t,s)-g(y_1,y_2) &\leq \left\{\delta^2_{n_1,y_1}+\delta^2_{n_2,y_2}+\left(\frac{2}{n_1}(\alpha-1)y_1+\frac{\lambda+1}{n_1}-y_1\right)^2\right.\\
&+ \left.\left(\frac{2}{n_2}(\alpha-1)y_2+\frac{\lambda+1}{n_2}-y_2\right)^2\right\}\parallel g\parallel_{C^2(\mathcal{I}^2)}.
\end{aligned}$$

Proof. In the light of operator $R^{\alpha_1,\alpha_2}_{n_1,n_2}(f;y_1,y_2)$ and Lemma 2.4, we obtain $R^{\alpha_1,\alpha_2}_{n_1,n_2}(1;y_1,y_2)=1$, $R^{\alpha_1,\alpha_2}_{n_1,n_2}(t-y_1;y_1,y_2)=0$ and $R^{\alpha_1,\alpha_2}_{n_1,n_2}(s-y_2;y_1,y_2)=0$. For any $g \in C'(\mathcal{I}^2)$ the Taylor series give us:

$$\begin{aligned}
g(t,s)-g(y_1,y_2) &= \frac{\partial g(y_1,y_2)}{\partial y_1}(t-y_1)+\int_{y_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,y_2)}{\partial\lambda^2}\mathrm{d}\lambda\\
&+ \frac{\partial g(y_1,y_2)}{\partial y_2}(s-y_2)+\int_{y_2}^{s}(s-\psi)\frac{\partial^2 g(y_1,\psi)}{\partial\psi^2}\mathrm{d}\psi.
\end{aligned}$$

On apply $R^{\alpha_1,\alpha_2}_{n_1,n_2}$, we see that

$$R_{n_1,n_2}^{\alpha_1,\alpha_2}(g(t,s);y_1,y_2)-R_{n_1,n_2}^{\alpha_1,\alpha_2}(g(y_1,y_2)$$

$$\begin{aligned}
&= R_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\int_{y_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,y_2)}{\partial\lambda^2}\mathrm{d}\lambda;y_1,y_2\right)+R_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\int_{y_2}^{s}(s-\psi)\frac{\partial^2 g(y_1,\psi)}{\partial\psi^2}\mathrm{d}\psi;y_1,y_2\right)\\
&= H_{n_1,n_2}^{*}\left(\int_{y_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,y_2)}{\partial\lambda^2}\mathrm{d}\lambda;y_1,y_2\right)+H_{n_1,n_2}^{*}\left(\int_{y_2}^{s}(s-\psi)\frac{\partial^2 g(y_1,\psi)}{\partial\psi^2}\mathrm{d}\psi;y_1,y_2\right)\\
&- \int_{y_1}^{\frac{2}{n_1}(\alpha-1)y_1+\frac{\lambda+1}{n_1}}\left(\frac{2}{n_1}(\alpha-1)y_1+\frac{\lambda+1}{n_1}-\lambda\right)\frac{\partial^2 g(\lambda,y_2)}{\partial\lambda^2}\mathrm{d}\lambda\\
&- \int_{y_2}^{\frac{2}{n_2}(\alpha-1)y_2+\frac{\lambda+1}{n_2}}\left(\frac{2}{n_2}(\alpha-1)y_2+\frac{\lambda+1}{n_2}-\psi\right)\frac{\partial^2 g(y_1,\psi)}{\partial\psi^2}\mathrm{d}\psi.
\end{aligned}$$

From hypothesis we easily obtain

$$\left|\int_{y_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,y_2)}{\partial\lambda^2}\mathrm{d}\lambda\right|\le\int_{y_1}^{t}\left|(t-\lambda)\frac{\partial^2 g(\lambda,y_2)}{\partial\lambda^2}\mathrm{d}\lambda\right|\le\parallel g\parallel_{C^2(\mathcal{I}^2)}(t-y_1)^2,$$

$$\left|\int_{y_2}^{s}(s-\psi)\frac{\partial^2 g(y_1,\psi)}{\partial\psi^2}\mathrm{d}\psi\right|\le\int_{y_2}^{s}\left|(s-\psi)\frac{\partial^2 g(y_1,\psi)}{\partial\psi^2}\mathrm{d}\psi\right|\le\parallel g\parallel_{C^2(\mathcal{I}^2)}(s-y_2)^2,$$

$$\begin{aligned}
&\left|\int_{y_1}^{\frac{2}{n_1}(\alpha-1)y_1+\frac{\lambda+1}{n_1}}\left(\frac{2}{n_1}(\alpha-1)y_1+\frac{\lambda+1}{n_1}-\lambda\right)\frac{\partial^2 g(\lambda,y_2)}{\partial\lambda^2}\mathrm{d}\lambda\right|\\
\le\ &\parallel g\parallel_{C^2(\mathcal{I}^2)}\left(\frac{2}{n_1}(\alpha-1)y_1+\frac{\lambda+1}{n_1}-y_1\right)^2
\end{aligned}$$

$$\begin{aligned}
&\left|\int_{y_2}^{\frac{2}{n_2}(\alpha-1)y_2+\frac{\lambda+1}{n_2}}\left(\frac{2}{n_2}(\alpha-1)y_2+\frac{\lambda+1}{n_2}-\psi\right)\frac{\partial^2 g(y_1,\psi)}{\partial\psi^2}\mathrm{d}\psi\right|\\
\le\ &\parallel g\parallel_{C^2(\mathcal{I}^2)}\left(\frac{2}{n_2}(\alpha-1)y_2+\frac{\lambda+1}{n_2}-y_2\right)^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mid R_{n_1,n_2}^{\alpha_1,\alpha_2}(g;t,s)-g(y_1,y_2)\mid\ \le\ &\left\{H_{n_1,n_2}^{*}((t-y_1)^2;y_1,y_2)+H_{n_1,n_2}^{*}((s-y_2)^2;y_1,y_2)\right.\\
+\ &\left(\frac{2}{n_1}(\alpha-1)y_1+\frac{\lambda+1}{n_1}-y_1\right)^2\\
+\ &\left.\left(\frac{2}{n_2}(\alpha-1)y_2+\frac{\lambda+1}{n_2}-y_2\right)^2\right\}\parallel g\parallel_{C^2(\mathcal{I}^2)}.
\end{aligned}$$

We complete the proof of desired Theorem 3.10. □

4. Some approximation results in Bögel space

Take any function $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ for a real compact intervals of $\mathcal{I}_1 \times \mathcal{I}_2$. For all $(t,s),\ (y_1,y_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ suppose $\Delta^*_{(t,s)} g(y_1,y_2)$ denotes the bivariate mixed difference operators defined as follows:

$$\Delta^*_{(t,s)} g(y_1,y_2) = g(t,s) - g(t,y_2) - g(y_1,s) + g(y_1,y_2).$$

If at any point $(y_1,y_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ the function $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ defined on $\mathcal{I}_1 \times \mathcal{I}_2$, then $\lim_{(t,s)\to(y_1,y_2)} \Delta^*_{(t,s)} g(y_1,y_2) = 0$.

If set of all the space of all Bögel-continuous(B-continuous) denoted by $C_B(\mathcal{I}_1 \times \mathcal{I}_2)$ on $(y_1,y_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ and be defined such that $C_B(\mathcal{I}_1 \times \mathcal{I}_2) = \{g,$ such that $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ is $g,\ B-$bounded on $\mathcal{I}_1 \times \mathcal{I}_2\}$. Next, the Bögel-differentiable function on $(y_1,y_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ be $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ and limit exists finite defined by

$$\lim_{(t,s)\to(y_1,y_2),\ t\neq y_1,\ s\neq y_2} \frac{1}{(t-y_1)(s-y_2)} \left(\Delta^*_{(t,s)}\right) = D_B g(y_1,y_2) < \infty.$$

Let the classes of all Bögel-differentiable function denoted by $D_\varphi g(y_1,y_2)$ and be $D_\varphi(\mathcal{I}_1 \times \mathcal{I}_2) = \{g,$ such that $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ is g, B$-$differentiable on $\mathcal{I}_1 \times \mathcal{I}_2\}$. Suppose the function g is B-bounded on D and be $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$, then for all $(t,s),\ (y_1,y_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ there exists positive constant M such that $\mid \Delta^*_{(t,s)} g(y_1,y_2) \mid \leq M$. The classes of all B-continuous function is called a B-bounded on $\mathcal{I}_1 \times \mathcal{I}_2$, whene $\mathcal{I}_1 \times \mathcal{I}_2$ is compact subset. Let $B_\varphi(\mathcal{I}_1 \times \mathcal{I}_2)$ denote the classes of all B-bounded function defined on $\mathcal{I}_1 \times \mathcal{I}_2$ which equipped the norm on B as $\parallel g \parallel_B = \sup_{(t,s),(y_1,y_2)\in\mathcal{I}_1\times\mathcal{I}_2} \mid \Delta^*_{(t,s)} g(y_1,y_2) \mid$. As we know to approximate the degree for a set of all B-continuous function on positive linear operators, it is essential to use the properties of mixed-modulus of continuity. So we let for all $(t,s),\ (y_1,y_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ and $g \in B_\varphi(\mathcal{I}_{\alpha_n})$, the mixed-modulus of continuity of function g bt $\omega_B : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and be defined such as:

$$\omega_B(g;\delta_1,\delta_2) = \sup\{\Delta^*_{(t,s)} g(y_1,y_2) :\mid t-y_1 \mid\leq \delta_1,\ \mid s-y_2 \mid\leq \delta_2\}.$$

For any $\mathcal{I}^2 = [0,\infty) \times [0,\infty)$, we suppose the classes of all B-continuous function defined on $\mathcal{I}^2$ denoted by $C_\varphi(\mathcal{I}^2)$. Moreover, let set of all ordinary continuous function defined on $\mathcal{I}^2$ be $C(\mathcal{I}^2)$. For further details on space of Bögel functions related to this paper we propose the article [6, 7].

Let $(y_1,y_2) \in \mathcal{I}^2$ and $n_1,n_2 \in \mathbb{N}$ then for all $g \in C(\mathcal{I}^2)$ we define the GBS type operators for the positive linear operators $H^*_{n_1,n_2}$. Thus we suppose

$$K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);y_1,y_2) = H^*_{n_1,n_2}\Big(g(t,y_2) + g(y_1,s) - g(t,s);y_1,y_2\Big). \tag{14}$$

More precisely, the generalized GBS operator for bivariate function is defined as follows:

$$K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);y_1,y_2) = \sum_{k,l=0}^{\infty} \mathcal{S}^{\alpha_1,\alpha_2}_{n_1,n_2,k,l}(y_1,y_2) \int_0^\infty \int_0^\infty \mathcal{Q}_{n_1,n_2}(t,s) P_{y_1,y_2}(t,s) g(t,s)\,dtds, \tag{15}$$

where $P_{y_1,y_2}(t,s) = \Big(g(t,y_2) + g(y_1,s) - g(t,s)\Big)$.

Theorem 4.1. *For all $g \in C_\varphi(\mathcal{I}^2)$, it follows that*

$$\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);y_1,y_2) - g(y_1,y_2) \mid\leq 4\omega_B\left(g;\delta_{n_1,y_1},\delta_{n_2,y_2}\right),$$

where δ_{n_1,y_1} and δ_{n_2,y_2} are defined by Theorem 3.5.

Proof. Let $(t,s),(y_1,y_2)\in\mathcal{I}^2$. For all $n_1,n_2\in\mathbb{N}$ and $\delta_{n_1},\delta_{n_2}>0$, it follows that

$$\begin{aligned}|\Delta^*_{(y_1,y_2)}g(t,s)| &\le \omega_B(g;|t-y_1||s-y_2|)\\ &\le \left(1+\frac{t-y_1}{\delta_{n_1}}\right)\left(1+\frac{s-y_2}{\delta_{n_2}}\right)\omega_B(g;\delta_{n_1},\delta_{n_2}).\end{aligned}$$

From (14) and well-known Cauchy-Schwarz inequality, we easily conclude that

$$\begin{aligned}|K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);y_1,y_2)-g(y_1,y_2)| &\le H^*_{n_1,n_2}\left(|\Delta^*_{(y_1,y_2)}g(t,s)|;y_1,y_2\right)\\ &\le \left(H^*_{n_1,n_2}(\phi_{0,0};y_1,y_2)+\frac{1}{\delta_{n_1}}\left(H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)\right)^{\frac{1}{2}}\right.\\ &+ \frac{1}{\delta_{n_2}}\left(H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2)\right)^{\frac{1}{2}}\\ &+ \frac{1}{\delta_{n_1}}\left(H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)\right)^{\frac{1}{2}}\\ &\times \left.\frac{1}{\delta_{n_2}}\left(H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2)\right)^{\frac{1}{2}}\right)\omega_B(g;\delta_{n_1},\delta_{n_2}).\end{aligned}$$

In the view of Theorem 3.5 we easily get our results.
□

If we let $x=(t,s),\ y=(y_1,y_2)\in\mathcal{I}^2$, then the Lipschitz function in terms of B-continuous functions defined by

$$Lip^{\xi}_M=\left\{g\in C(\mathcal{I}^2):|\Delta^*_{(y_1,y_2)}g(x,y)|\le M\parallel x-y\parallel^{\xi},\right\}$$

where M is a positive constant, $0<\xi\le 1$, and Euclidean norm $\parallel x-y\parallel=\sqrt{(t-y_1)^2+(s-y_2)^2}$.

Theorem 4.2. *For all $g\in Lip^{\xi}_M$ operator $K^{\alpha_1,\alpha_2}_{n_1,n_2}$ satisfying*

$$|K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(x,y);y_1,y_2)-g(y_1,y_2)|\le M\{\delta^2_{n_1,y_1}+\delta^2_{n_2,y_2}\}^{\frac{\xi}{2}},$$

where δ_{n_1,y_1} and δ_{n_2,y_2} are defined by Theorem 3.5.

Proof. We easily see that

$$\begin{aligned}K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(x,y);y_1,y_2) &= H^*_{n_1,n_2}(g(y_1,y)+g(x,y_2)-g(x,s);y_1,y_2)\\ &= H^*_{n_1,n_2}\left(g(y_1,y_2)-\Delta^*_{(y_1,y_2)}g(x,s);y_1,y_2\right)\\ &= g(y_1,y_2)-H^*_{n_1,n_2}\left(\Delta^*_{(y_1,y_2)}g(x,s);y_1,y_2\right).\end{aligned}$$

Therefore,

$$\begin{aligned}|K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(x,y);y_1,y_2)-g(y_1,y_2)| &\le H^*_{n_1,n_2}\left(|\Delta^*_{(y_1,y_2)}g(x,y)|;y_1,y_2\right)\\ &\le MH^*_{n_1,n_2}\left(\parallel x-y\parallel^{\xi};y_1,y_2\right)\\ &\le M\{H^*_{n_1,n_2}\left(\parallel x-y\parallel^2;y_1,y_2\right)\}^{\frac{\xi}{2}}\\ &\le M\{H^*_{n_1,n_2}\left((t-y_1)^2;y_1,y_2\right)+H^*_{n_1,n_2}\left((s-y_2)^2;y_1,y_2\right)\}^{\frac{\xi}{2}}.\end{aligned}$$

□

Theorem 4.3. *If $g \in D_{\varphi}(\mathcal{I}^2)$ and $D_B g \in B(\mathcal{I}^2)$, then*

$$\begin{aligned} \mid K^{\alpha_1,\alpha_2}_{n_1,n_2}(g; y_1, y_2) - g(y_1, y_2) \mid &\leq C\Big\{3 \parallel D_B g \parallel_\infty + \varpi_{mixed}(D_B g; \delta_{n_1}, \delta_{n_2})\Big\}(y_1+1)(y_2+1) \\ &+ \Big\{1 + \sqrt{C_2}(y_1+1) + \sqrt{C_1}(y_2+1)\Big\} \\ &\times \varpi_{mixed}(D_B g; \delta_{n_1}, \delta_{n_2})(y_1+1)(y_2+1), \end{aligned}$$

where δ_{n_1}, δ_{n_2} defined by Theorem 3.5 and C is any positive constant.

Proof. Suppose $\rho \in (y_1, t)$, $\xi \in (y_2, s)$ and

$$\Delta^*_{(y_1,y_2)} g(t,s) = (t-y_1)(s-y_2) D_B g(\rho, \xi),$$

$$D_B g(\rho, \xi) = \Delta^*_{(y_1,y_2)} D_B g(\rho, \xi) + D_B g(\rho, y) + D_B g(x, \xi) - D_B g(y_1, y_2).$$

For all $D_B g \in B(\mathcal{I}^2)$, it follows that

$$\begin{aligned} \mid K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Delta^*_{(y_1,y_2)} g(t,s); y_1, y_2\right) \mid &= \mid K^{\alpha_1,\alpha_2}_{n_1,n_2}((t-y_1)(s-y_2) D_B g(\rho,\xi); y_1, y_2) \mid \\ &\leq K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\mid t-y_1 \mid\mid s-y_2 \mid\mid \Delta^*_{(y_1,y_2)} D_B g(\rho,\xi) \mid; y_1, y_2\right) \\ &+ K^{\alpha_1,\alpha_2}_{n_1,n_2}\Big(\mid t-y_1 \mid\mid s-y_2 \mid (\mid D_B g(\rho, y_2) \mid \\ &+ \mid D_B g(y_1, \xi) \mid + \mid D_B g(y_1, y_2) \mid); y_1, y_2\Big) \\ &\leq K^{\alpha_1,\alpha_2}_{n_1,n_2}\Big(\mid t-y_1 \mid\mid s-y_2 \mid \\ &\times \varpi_{mixed}(D_B g; \mid \rho - y_1 \mid, \mid \xi - y_2 \mid); y_1, y_2\Big) \\ &+ 3 \parallel D_B g \parallel_\infty K^{\alpha_1,\alpha_2}_{n_1,n_2}(\mid t-y_1 \mid\mid s-y_2 \mid; y_1, y_2). \end{aligned}$$

Here ϖ_{mixed}, is mixed-modulus of continuity and it follows that

$$\begin{aligned} &\varpi_{mixed}(D_B g; \mid \rho - y_1 \mid, \mid \xi - y_2 \mid) \\ &\quad\leq \varpi_{mixed}(D_B g; \mid t - y_1 \mid, \mid s - y_2 \mid) \\ &\quad\leq \left(1 + \delta^{-1}_{n_1} \mid t-y_1 \mid\right)\left(1 + \delta^{-1}_{n_2} \mid s-y_2 \mid\right) \varpi_{mixed}(D_B g; \mid \delta_{n_1}, \delta_{n_2}). \end{aligned}$$

Therefore, it is obvious that

$$\begin{aligned} \mid K^*_{n_1,n_2}(g; y_1, y_2) - g(y_1, y_2) \mid &= \mid \Delta^*_{(y_1,y_2)} g(t,s); y_1, y_2 \mid \\ &\leq 3 \parallel D_B g \parallel_\infty \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left((t-y_1)^2(s-y_2)^2; y_1, y_2\right)\right)^{\frac{1}{2}} \\ &+ \Big(K^{\alpha_1,\alpha_2}_{n_1,n_2}(\mid t-y_1 \mid\mid s-y_2 \mid; y_1, y_2) \\ &+ \delta^{-1}_{n_1} K^{\alpha_1,\alpha_2}_{n_1,n_2}\left((t-y_1)^2 \mid s-y_2 \mid; y_1, y_2\right)\Big) \\ &+ \delta^{-1}_{n_2} K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\mid t-y_1 \mid (s-y_2)^2; y_1, y_2\right) \\ &+ \delta^{-1}_{n_1}\delta^{-1}_{n_2} K^{\alpha_1,\alpha_2}_{n_1,n_2}\left((t-y_1)^2(s-y_2)^2; y_1, y_2\right) \varpi_{mixed}(D_B g; \delta_{n_1}, \delta_{n_2}); \end{aligned}$$

$$
\begin{aligned}
\mid K^{*}_{n_1,n_2}(g;y_1,y_2)-g(y_1,y_2)\mid &= \mid \Delta^{*}_{(y_1,y_2)}g(t,s);y_1,y_2 \mid \\
&\le 3\parallel D_B g\parallel_{\infty}\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,2}_{y_1,y_2};y_1,y_2\right)\right)^{\frac{1}{2}} \\
&+ \left\{\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,2}_{y_1,y_2};y_1,y_2\right)\right)^{\frac{1}{2}}\right. \\
&+ \delta^{-1}_{n_1}\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{4,2}_{y_1,y_2};y_1,y_2\right)\right)^{\frac{1}{2}}+\delta^{-1}_{n_2}\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,4}_{y_1,y_2};y_1,y_2\right)\right)^{\frac{1}{2}} \\
&+ \left.\delta^{-1}_{n_1}\delta^{-1}_{n_2}K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,2}_{y_1,y_2};y_1,y_2\right)\right\}\varpi_{mixed}\left(D_B g;\delta_{n_1},\delta_{n_2}\right).
\end{aligned}
$$

Which follows that

$$
\begin{aligned}
\mid K^{*}_{n_1,n_2}(g;y_1,y_2)-g(y_1,y_2)\mid &= 3\parallel D_B g\parallel_{\infty}\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{y_1,y_2};y_1,y_2\right)K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{y_1,y_2};y_1,y_2\right)\right)^{\frac{1}{2}} \\
&+ \left\{\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{y_1,y_2};y_1,y_2\right)K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{y_1,y_2};y_1,y_2\right)\right)^{\frac{1}{2}}\right. \\
&+ \delta^{-1}_{n_1}\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{4,0}_{y_1,y_2};y_1,y_2\right)K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{y_1,y_2};y_1,y_2\right)\right)^{\frac{1}{2}} \\
&+ \delta^{-1}_{n_2}\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{y_1,y_2};y_1,y_2\right)K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,4}_{y_1,y_2};y_1,y_2\right)\right)^{\frac{1}{2}} \\
&+ \left.\delta^{-1}_{n_1}\delta^{-1}_{n_2}K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{y_1,y_2};y_1,y_2\right)K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{y_1,y_2};y_1,y_2\right)\right\} \\
&\times \varpi_{mixed}\left(D_B g;\delta_{n_1},\delta_{n_2}\right).
\end{aligned}
$$

From Lemma 2.5, we demonstrate

$$
\begin{aligned}
\mid K^{*}_{n_1,n_2}(g;y_1,y_2)-g(y_1,y_2)\mid &\le 3\parallel D_B g\parallel_{\infty}\left(\sqrt{C_1C_2}(y_1+1)(y_2+1)\right) \\
&+ \left\{\left(\sqrt{C_1C_2}(y_1+1)(y_2+1)\right)\right. \\
&+ \delta^{-1}_{n_1}\left(\sqrt{C_2}\sqrt{o\left(\frac{1}{n_1}\right)}(y_1+1)^2(y_2+1)\right) \\
&+ \delta^{-1}_{n_2}\left(\sqrt{C_1}\sqrt{o\left(\frac{1}{n_2}\right)}(y_2+1)^2(y_1+1)\right) \\
&+ \left.\delta^{-1}_{n_1}\delta^{-1}_{n_2}\left(\sqrt{o\left(\frac{1}{n_1}\right)}\sqrt{o\left(\frac{1}{n_2}\right)}(y_1+1)(y_2+1)\right)\right\} \\
&\times \varpi_{mixed}\left(D_B g;\delta_{n_1},\delta_{n_2}\right).
\end{aligned}
$$

Which complete the proof of Theorem 4.3. □

5. Conclusion and Remarks

In this research article, we proposed a new bivariate sequence of Baskakov operators with aid of non negative parameter α. Further, We studied the bivariate properties of α-Baskakov operators with the help of modulus of continuity, mixed-modulus of continuity and then find the approximation results in Peetre's K-functional, Voronovskaja type theorem and Lipschitz maximal functions for these bivariate operators.

Next, we constructed the GBS type operator of these generalized operators and study approximation in Böogel continuous functions by use of mixed-modulus of continuity.

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