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Inequalities on the (p,q)-Mixed Volume Involving L_p Centroid Bodies and L_p Intersection Bodies

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Abstract. In this paper, applying for the Minkowski's and Hölder's integral inequalities, we obtain four theorems about the (p,q)-mixed volume involving the L_p centroid bodies and the L_p intersection bodies, respectively. The former two theorems reveal the convexity of the functionals related to the (p,q)-mixed volume, in terms of the dual Blaschke addition introduced in [Journal of Geometric Analysis, 30 (2020) 3026-3034], and the latter two theorems expose the monotonicity of the other functionals related to the (p,q)-mixed volume.

1. Introduction

The Brunn-Minkowski theory is a powerful apparatus for conquering problems involving metric quantities. As a cornerstone of such theory, the Brunn-Minkowski inequality has a closed relationship with other inequalities in geometry and analysis, and some applications (see e.g. [4]). In this paper, we intend to establish, in terms of the (p,q)-mixed volume, some related inequalities which characterize the convexity and the monotonicity of respectively functionals involving the L_p centroid bodies, the L_p intersection bodies and their polars.

Let $\mathcal{K}^n$ and $\mathcal{S}^n_o$ denote the set of all convex bodies (i.e., compact, convex subsets with nonempty interiors) and the set of all star bodies in the Euclidean n-space $\mathbb{R}^n$, respectively. Let $\mathcal{K}^n_o$ and $\mathcal{K}^n_s$ (resp. $\mathcal{S}^n_s$) denote the set of all convex bodies containing the origin in their interiors and the set of all convex bodies (resp. star bodies) that are origin symmetric, respectively. Denote by $V(K)$ the n-dimensional volume of a body K in $\mathbb{R}^n$, and ω_n the volume of the unit ball B^n. Let S^{n-1} be the unit sphere of $\mathbb{R}^n$.

Recently, a family of important L_p dual curvature measures (or the (p,q)-th dual curvature measures for $p,q\in\mathbb{R}$) was introduced by Lutwak, Yang and Zhang [17]. These measures are significant and they unify the previous three kinds of measures proposed in [10], [11] and [14]. Associated to such measures, the geometric quantity named as (p,q)-mixed volume, or L_p dual mixed volume, can be introduced as follows:

Definition 1.1 (cf. [17]). *Suppose $p,q\in\mathbb{R}$. If $K,L\in\mathcal{K}^n_o$ and $Q\in\mathcal{S}^n_o$, define the L_p dual mixed volume, or (p,q)-mixed volume, $\widetilde{V}_{p,q}(K,L,Q)$, by*

$$\widetilde{V}_{p,q}(K,L,Q)=\frac{1}{n}\int_{S^{n-1}}\left(\frac{h_L}{h_K}\right)^p(\alpha_K(u))\left(\frac{\rho_K}{\rho_Q}\right)^q(u)\rho_Q^n(u)du, \tag{1}$$

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where h_K, ρ_K and α_K denote the support function, radial function and radial Gauss map for K respectively, the integration is with respect to spherical Lebesgue measure.

The notion of the (p,q)-mixed volume unifies the L_p mixed volume and dual mixed volume ([17], Proposition 7.2). Moreover, Lutwak, Yang and Zhang proved a newly Minkowski inequality about this unified notion as well ([17], Theorem 7.4). Notice that, in terms of the classical mixed volume and the Minkowski addition, the Minkowski's first inequality and the Brunn-Minkowski inequality are equivalent. Due to their close relations, both inequalities are central topics in modern convex geometry, and that they have been studied extensively (see e.g. [1, 6, 20]).

The classical Brunn-Minkowski theory is mainly concerned with the analogues and generalizations of the Brunn-Minkowski inequality for geometric quantities. In 2018, Zou and Xiong [22] has formulated the L_p transference principle, and the L_p Brunn-Minkowski type inequalities they established, characterize the concavity of existing functionals, in terms of the L_p addition of convex bodies. This paper first focus on establishing Brunn-Minkowski type inequalities, in terms of the dual Blaschke addition (also called radial Blaschke sum) introduced by Guo-Jia [7].

Before presenting our results, we first fix the notations: Let $\star_m$ denote the m-radial Blaschke addition that will be given by Definition 2.1; Γ_p denote the L_p centroid operator given by Definition 2.2, and I_p denote the L_p intersection operator given by Definition 2.3; $\Gamma_p^* L$ denote the polar of the L_p centroid body $\Gamma_p L$, and $I_p^* L$ denote the polar of the L_p intersection body $I_p L$.

Then, our first result is the following Brunn-Minkowski type inequality about the L_p centroid bodies and their polars:

Theorem 1.1. *Let $K \in \mathcal{K}_o^n$, $L_1, L_2 \in \mathcal{S}_s^n$, $p \geq 1$ and $1 \leq m \leq n-1$.*
(i) *For $Q \in \mathcal{S}_o^n$ and $q \in \mathbb{R}$, it holds that*

$$\left[\widetilde{V}_{p,q}(K, \Gamma_p(L_1 \star_m L_2), Q) V(L_1 \star_m L_2)\right]^{\frac{m}{n+p}} \leq \left[\widetilde{V}_{p,q}(K, \Gamma_p L_1, Q) V(L_1)\right]^{\frac{m}{n+p}} + \left[\widetilde{V}_{p,q}(K, \Gamma_p L_2, Q) V(L_2)\right]^{\frac{m}{n+p}}; \qquad (2)$$

(ii) *For $Q \in \mathcal{K}_o^n$ and $q > n + \frac{mp}{n+p}$, it holds that*

$$\left[\widetilde{V}_{p,q}(K, Q, \Gamma_p^*(L_1 \star_m L_2))^{-\frac{p}{n-q}} V(L_1 \star_m L_2)\right]^{\frac{m}{n+p}} \leq \left[\widetilde{V}_{p,q}(K, Q, \Gamma_p^* L_1)^{-\frac{p}{n-q}} V(L_1)\right]^{\frac{m}{n+p}} + \left[\widetilde{V}_{p,q}(K, Q, \Gamma_p^* L_2)^{-\frac{p}{n-q}} V(L_2)\right]^{\frac{m}{n+p}}. \quad (3)$$

Moreover, the equality holds in each of the two inequalities (2) *and* (3) *if and only if L_1 and L_2 are dilations.*

The second result we obtained is the Brunn-Minkowski type inequality about the L_p intersection bodies and their polars:

Theorem 1.2. *Let $K \in \mathcal{K}_o^n$, $L_1, L_2 \in \mathcal{S}_s^n$, $0 < p < 1$ and $1 \leq m \leq n-1$.*
(i) *For $Q \in \mathcal{K}_o^n$ and $q < n - \frac{mp}{n-p}$, it holds that*

$$\widetilde{V}_{p,q}(K, Q, I_p(L_1 \star_m L_2))^{\frac{mp}{(n-p)(n-q)}} \leq \widetilde{V}_{p,q}(K, Q, I_p L_1)^{\frac{mp}{(n-p)(n-q)}} + \widetilde{V}_{p,q}(K, Q, I_p L_2)^{\frac{mp}{(n-p)(n-q)}}; \qquad (4)$$

(ii) *For $Q \in \mathcal{S}_o^n$ and $q \in \mathbb{R}$, it holds that*

$$\widetilde{V}_{-p,q}(K, I_p^*(L_1 \star_m L_2), Q)^{\frac{m}{n-p}} \leq \widetilde{V}_{-p,q}(K, I_p^* L_1, Q)^{\frac{m}{n-p}} + \widetilde{V}_{-p,q}(K, I_p^* L_2, Q)^{\frac{m}{n-p}}. \qquad (5)$$

Moreover, the equality holds in each of the two inequalities (4) *and* (5) *if and only if L_1 and L_2 are dilations.*

Applying for the above two theorems, we will characterize the convexity of four functionals in Theorem 3.1, in terms of the dual Blaschke addition.

Note that in each inequality of (2)-(5), the addition on the left hand side is for two star bodies. Next, about the (p,q)-mixed volume, we can prove, for two distinct real numbers $i, j \geq 1$, the monotonicity inequalities for the centroid bodies, and their polars respectively.

Theorem 1.3. *Let $K \in \mathcal{K}_o^n$, $L \in \mathcal{S}_o^n$ and $1 \le j < i$.*
(i) *For $Q \in \mathcal{S}_o^n$, $p > 0$ and $q \in \mathbb{R}$, it holds that*

$$\left[\frac{a_i \widetilde{V}_{p,q}(K, \Gamma_i L, Q)}{a_{i+j} \widetilde{V}_{p,q}(K, \Gamma_{i+j} L, Q)}\right]^{i^2} < \left[\frac{a_j \widetilde{V}_{p,q}(K, \Gamma_j L, Q)}{a_{i+j} \widetilde{V}_{p,q}(K, \Gamma_{i+j} L, Q)}\right]^{j^2}; \tag{6}$$

(ii) *For $Q \in \mathcal{K}_o^n$, $q > n$ and $p \in \mathbb{R}$, it holds that*

$$\left[\frac{b_i \widetilde{V}_{p,q}(K, Q, \Gamma_i^* L)}{b_{i+j} \widetilde{V}_{p,q}(K, Q, \Gamma_{i+j}^* L)}\right]^{i^2} < \left[\frac{b_j \widetilde{V}_{p,q}(K, Q, \Gamma_j^* L)}{b_{i+j} \widetilde{V}_{p,q}(K, Q, \Gamma_{i+j}^* L)}\right]^{j^2}. \tag{7}$$

Here, $a_i := (c_{n,i}(n+i))^{\frac{p}{i}}$, $b_i := (c_{n,i}(n+i))^{\frac{q-n}{i}}$, and $c_{n,i}$ is a positive constant given by Definition 2.2.

Analogously, for intersection bodies and their polars, we have

Theorem 1.4. *Let $K \in \mathcal{K}_o^n$, $L \in \mathcal{S}_o^n$ and $0 < j < i < i + j < 1$.*
(i) *For $Q \in \mathcal{K}_o^n$, $q < n$ and $p \in \mathbb{R}$, it holds that*

$$\left[\frac{c_i \widetilde{V}_{p,q}(K, Q, I_i L)}{c_{i+j} \widetilde{V}_{p,q}(K, Q, I_{i+j} L)}\right]^{i^2} < \left[\frac{c_j \widetilde{V}_{p,q}(K, Q, I_j L)}{c_{i+j} \widetilde{V}_{p,q}(K, Q, I_{i+j} L)}\right]^{j^2}; \tag{8}$$

(ii) *For $Q \in \mathcal{S}_o^n$, $p < 0$ and $q \in \mathbb{R}$, it holds that*

$$\left[\frac{d_i \widetilde{V}_{p,q}(K, I_i^* L, Q)}{d_{i+j} \widetilde{V}_{p,q}(K, I_{i+j}^* L, Q)}\right]^{i^2} < \left[\frac{d_j \widetilde{V}_{p,q}(K, I_j^* L, Q)}{d_{i+j} \widetilde{V}_{p,q}(K, I_{i+j}^* L, Q)}\right]^{j^2}. \tag{9}$$

Here, $c_i := (n-i)^{\frac{n-q}{i}}$ and $d_i := (n-i)^{-\frac{p}{i}}$.

Obviously, the case that $i + j$ is a constant in (6)-(9) yields, in terms of real $j < i$, the monotonicity of four functionals related to the (p, q)-mixed volume.

Remark 1.1. If taking $Q = K$ or $q = n$ in (2), (5)-(6) and (9), we can obtain inequalities for the L_p mixed volume; If taking $Q = K$ in (3)-(4), and $Q = K$ (or $p = 0$) in (7)-(8), we can get inequalities for the dual mixed volume.

2. Preliminaries

In this section, for our later purpose, we collect some basic facts from the Brunn-Minkowski theory. For more details we refer to Gardner [5] and Schneider [20].

2.1. *Support function, radial function and polar body*

Let $K \in \mathcal{K}^n$, its support function $h_K : \mathbb{R}^n \to \mathbb{R}$ is defined by $h_K(x) = \max\{x \cdot y : y \in K\}$ for $x \in \mathbb{R}^n$, where $x \cdot y$ denotes the standard inner product of x and y. Let $K \subset \mathbb{R}^n$ be a compact star-shaped set with respect to the origin, its radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is defined by $\rho_K(x) = \max\{\lambda \ge 0 : \lambda x \in K\}$ for $x \in \mathbb{R}^n \setminus \{0\}$.

A star body is a compact star-shaped set with respect to the origin whose radial function is positive and continuous. Two star bodies K and L are dilations (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of any $u \in S^{n-1}$.

It is easily seen that, on $\mathbb{R}^n \setminus \{0\}$, the support function of a convex body and the radial function of a star body are related by

$$\rho_K = 1/h_{K^*} \text{ and } h_K = 1/\rho_{K^*}, \tag{10}$$

where, $K^* := \{x \in \mathbb{R}^n : x \cdot y \le 1, y \in K\}$ is the polar body of K. It is easily seen that $(\lambda K)^* = 1/\lambda\, K^*$ for $\lambda > 0$, and $(K^*)^* = K$ for $K \in \mathcal{K}_o^n$.

2.2. Spherical image map, radial map and radial Gauss map

Let $K \in \mathcal{K}^n$ and $\sigma \subset \partial K$. Denote by $H_K(v)$ the support hyperplane to K with unit normal v. Then

$$\boldsymbol{\nu}_K(\sigma) = \{v \in S^{n-1} : x \in H_K(v) \text{ for some } x \in \sigma\} \subset S^{n-1}$$

is called the spherical image of σ. Let $\sigma_K \subset \partial K$ denote the set consisting of all $x \in \partial K$ for which the set $\boldsymbol{\nu}_K(\{x\})$ contains more than a single element. Then the spherical image map of K is given by $\nu_K : \partial K \setminus \sigma_K \to S^{n-1}$, which is defined such that, for each $x \in \partial K \setminus \sigma_K$, $\nu_K(x)$ is the unique element in $\boldsymbol{\nu}_K(\{x\})$.

For $K \in \mathcal{K}_o^n$, the radial map $r_K : S^{n-1} \to \partial K$ of K is defined by

$$r_K(u) = \rho_K(u)u \in \partial K \text{ for } u \in S^{n-1}.$$

Finally, we put $\omega_K = r_K^{-1}(\sigma_K)$. Then the radial Gauss map of the convex body K is given by $\alpha_K : S^{n-1} \setminus \omega_K \to S^{n-1}$ with $\alpha_K = \nu_K \circ r_K$. We refer to [10] for more details.

2.3. Radial Blaschke addition

In [2], Böröczky and Schneider showed that a star body is uniquely determined by the volumes and centroids of its hyperplane sections through the origin. Based on this unique result and Theorem 7.2.6 of [5], recently Guo-Jia [7] introduced the notions of radial Blaschke addition and the general m-radial Blaschke addition.

Definition 2.1 (cf. [7]). *Let $K, L \in \mathcal{S}_s^n$ and m be an integer with $1 \le m \le n-1$. The m-radial Blaschke sum of K and L, denoted by $K \star_m L$, is defined to be the unique star body symmetric about the origin such that*

$$V_m((K \star_m L) \cap E) = V_m(K \cap E) + V_m(L \cap E)$$

for all $E \in G(n, m)$. Here, $G(n, m)$ denotes the Grassmannian of m-dimensional linear subspaces of $\mathbb{R}^n$, and V_m denotes m-dimensional Hausdorff measure.

In particular, $K \star_{(n-1)} L$ is the dual Blaschke sum of K and L.

From the polar formula for the volume of sections and the uniqueness theorem for spherical Radon transform (cf. Lemma 3.1 of [7]), we immediately have

$$\rho^m_{K\star_m L}(u) = \rho^m_K(u) + \rho^m_L(u), \text{ for all } u \in S^{n-1}. \tag{11}$$

2.4. L_p centroid body and L_p intersection body

We first recall the notion of the L_p centroid body due to Lutwak and Zhang [18].

Definition 2.2 (cf. [18] and P.567 of [20]). *For $K \in \mathcal{S}_o^n$ and $p \ge 1$, the L_p centroid body $\Gamma_p K$ (which belongs to $\mathcal{K}_s^n$) is defined such that its support function $h_{\Gamma_p K}$ is given by*

$$h^p_{\Gamma_p K}(x) = \frac{1}{c_{n,p} V(K)} \int_K |x \cdot y|^p dy = \frac{1}{c_{n,p}(n+p)V(K)} \int_{S^{n-1}} |x \cdot v|^p \rho_K^{n+p}(v) dv, \text{ for } x \in \mathbb{R}^n, \tag{12}$$

where $c_{n,p} := \omega_{n+p}/\omega_2\omega_n\omega_{p-1}$.

Next, we recall the notion of L_p intersection body due to Haberl and Ludwig [9].

Definition 2.3 (cf. [8, 9] and P.581 of [20]). *For $K \in \mathcal{S}_o^n$ and $0 < p < 1$, the L_p intersection body $I_p K$ (which belongs to $\mathcal{S}_s^n$) is defined such that its radial function $\rho_{I_p K}$ is given by*

$$\rho^p_{I_p K}(u) = \int_K |u \cdot x|^{-p} dx = \frac{1}{n-p} \int_{S^{n-1}} |u \cdot v|^{-p} \rho_K^{n-p}(v) dv, \text{ for } u \in S^{n-1}. \tag{13}$$

It is worthy to point out that the notion of Lutwak's intersection body [13] is extremely useful, by which the famous Busemann-Petty problem was effectively solved (see, e.g. [3, 12, 21]); and that in the last several decades, the L_p centroid bodies and L_p intersection bodies have received great attention. See, e.g. [8, 16, 19].

3. Proofs of the Theorems

This section is devoted to the proofs of our main results. To achieve this goal, first of all we recall the following well-known Minkowski's and Hölder's integral inequalities. Let E be a measurable set and $L^p(E)$ denote the set of all functions defined on E which are in the L^p space. Then, we have:

Minkowski's integral inequality. Let $0 \neq p \in \mathbb{R}$ and $f, g \in L^p(E)$. If $p > 1$, it holds that

$$\Big(\int_E |f(x)|^p dx\Big)^{\frac{1}{p}} + \Big(\int_E |g(x)|^p dx\Big)^{\frac{1}{p}} \geq \Big(\int_E |f(x)+g(x)|^p dx\Big)^{\frac{1}{p}}; \tag{14}$$

If $p < 0$ or $0 < p < 1$, the inequality is reversed. Moreover, the equality holds if and only if there exist two constants c_1 and c_2 such that $c_1 f(x) = c_2 g(x)$.

Hölder's integral inequality. Let $0 \neq p, q \in \mathbb{R}$, $f \in L^p(E)$, $g \in L^q(E)$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $p < 0$, or $0 < p < 1$, then it holds that

$$\Big(\int_E |f(x)|^p dx\Big)^{\frac{1}{p}} \Big(\int_E |g(x)|^q dx\Big)^{\frac{1}{q}} \leq \int_E |f(x)g(x)| dx; \tag{15}$$

If $p > 1$, the inequality is reversed. Moreover, the equality holds if and only if there exist two constants c_1 and c_2 such that $c_1|f(x)|^p = c_2|g(x)|^q$.

Now, we are ready to prove each of our four theorems.

Proof of Theorem 1.1. Since $\frac{n+p}{m} > 1$, (14) implies that, for $u \in S^{n-1}$, it holds that

$$\Big[\int_{S^{n-1}} |u \cdot v|^p \Big(\rho_{L_1}^m(v) + \rho_{L_2}^m(v)\Big)^{\frac{n+p}{m}} dv\Big]^{\frac{m}{n+p}} \leq \Big[\int_{S^{n-1}} |u \cdot v|^p \rho_{L_1}^{n+p}(v) dv\Big]^{\frac{m}{n+p}} + \Big[\int_{S^{n-1}} |u \cdot v|^p \rho_{L_2}^{n+p}(v) dv\Big]^{\frac{m}{n+p}}. \tag{16}$$

Then, by using (11), (12) and (16), we obtain, for any $u \in S^{n-1}$,

$$\Big[h^p_{\Gamma_p(L_1 \star_m L_2)}(u) V(L_1 \star_m L_2)\Big]^{\frac{m}{n+p}} \leq \Big[h^p_{\Gamma_p L_1}(u) V(L_1)\Big]^{\frac{m}{n+p}} + \Big[h^p_{\Gamma_p L_2}(u) V(L_2)\Big]^{\frac{m}{n+p}}, \tag{17}$$

and the equality holds if and only if L_1 and L_2 are dilations.

In case (i), $Q \in \mathcal{S}^n_o$ and $q \in \mathbb{R}$, (1) and (17) with u replaced by $\alpha_K(u)$ yield

$$\begin{aligned} &n\widetilde{V}_{p,q}(K, \Gamma_p(L_1 \star_m L_2), Q) V(L_1 \star_m L_2) \\ &\leq \int_{S^{n-1}} \Big[\Big(\Big(\frac{h_{\Gamma_p L_1}}{h_K}\Big)^p (\alpha_K(u)) V(L_1)\Big)^{\frac{m}{n+p}} + \Big(\Big(\frac{h_{\Gamma_p L_2}}{h_K}\Big)^p (\alpha_K(u)) V(L_2)\Big)^{\frac{m}{n+p}}\Big]^{\frac{n+p}{m}} \rho_K^q(u) \rho_Q^{n-q}(u) du =: N_1, \end{aligned} \tag{18}$$

and, from the condition that the equality holds in (17), we see that the equality holds in (18) if and only if L_1 and L_2 are dilations.

On the other hand, according to (1) and (14), we get

$$\begin{aligned} N_1^{\frac{m}{n+p}} &\leq \Big[V(L_1) \int_{S^{n-1}} \Big(\frac{h_{\Gamma_p L_1}}{h_K}\Big)^p (\alpha_K(u)) \rho_K^q(u) \rho_Q^{n-q}(u) du\Big]^{\frac{m}{n+p}} + \Big[V(L_2) \int_{S^{n-1}} \Big(\frac{h_{\Gamma_p L_2}}{h_K}\Big)^p (\alpha_K(u)) \rho_K^q(u) \rho_Q^{n-q}(u) du\Big]^{\frac{m}{n+p}} \\ &= \Big[n\widetilde{V}_{p,q}(K, \Gamma_p L_1, Q) V(L_1)\Big]^{\frac{m}{n+p}} + \Big[n\widetilde{V}_{p,q}(K, \Gamma_p L_2, Q) V(L_2)\Big]^{\frac{m}{n+p}}. \end{aligned} \tag{19}$$

From (18) and (19), we get (2) as claimed.

Moreover, from the condition that the equality holds in (14), we see that the equality holds in (19) if and only if $h_{\Gamma_p L_1}$ is proportional to $h_{\Gamma_p L_2}$, or equivalently, $\Gamma_p L_1$ and $\Gamma_p L_2$ are dilations. From (12) and that L_1 and

L_2 are dilations, we get that $\Gamma_p L_1$ and $\Gamma_p L_2$ are also dilations. Since that the equality holds in (2) is equivalent to that the two equalities hold in (18) and (19), we finally obtain that the equality holds in (2) if and only if L_1 and L_2 are dilations.

To deal with case (ii), we use $q > n + \frac{mp}{n+p}$, which implies that $-\frac{(n+p)(n-q)}{mp} > 1$. It follows from (1), (10) and (17) that, for $Q \in \mathcal{K}_o^n$,

$$\begin{aligned}
& n\widetilde{V}_{p,q}(K, Q, \Gamma_p^*(L_1 \star_m L_2))V(L_1 \star_m L_2)^{-\frac{n-q}{p}} \\
&= \int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)^p (\alpha_K(u))\rho_K^q(u)\rho^{n-q}_{\Gamma_p^*(L_1\star_m L_2)}(u)V(L_1 \star_m L_2)^{-\frac{n-q}{p}}du \\
&\leq \int_{S^{n-1}} \left[\left(\rho^{-p}_{\Gamma_p^* L_1}(u)V(L_1)\right)^{\frac{m}{n+p}} + \left(\rho^{-p}_{\Gamma_p^* L_2}(u)V(L_2)\right)^{\frac{m}{n+p}}\right]^{-\frac{(n+p)(n-q)}{mp}} \left(\frac{h_Q}{h_K}\right)^p (\alpha_K(u))\rho_K^q(u)du =: N_2,
\end{aligned} \tag{20}$$

and the equality holds if and only if L_1 and L_2 are dilations, which follows again from the condition that the equality holds in (17).

Then, by using (14), we get

$$\begin{aligned}
N_2^{-\frac{mp}{(n+p)(n-q)}} \leq & \left[\int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)^p (\alpha_K(u))\rho_K^q(u)\rho^{n-q}_{\Gamma_p^* L_1}(u)du\right]^{-\frac{mp}{(n+p)(n-q)}} V(L_1)^{\frac{m}{n+p}} \\
& + \left[\int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)^p (\alpha_K(u))\rho_K^q(u)\rho^{n-q}_{\Gamma_p^* L_2}(u)du\right]^{-\frac{mp}{(n+p)(n-q)}} V(L_2)^{\frac{m}{n+p}} \\
= & \left[n\widetilde{V}_{p,q}(K, Q, \Gamma_p^* L_1)\right]^{-\frac{mp}{(n+p)(n-q)}} V(L_1)^{\frac{m}{n+p}} + \left[n\widetilde{V}_{p,q}(K, Q, \Gamma_p^* L_2)\right]^{-\frac{mp}{(n+p)(n-q)}} V(L_2)^{\frac{m}{n+p}}.
\end{aligned} \tag{21}$$

From (20) and (21), we get the desired inequality (3).

Similar to case (i), the equality holds in (21) if and only if $\Gamma_p^* L_1$ and $\Gamma_p^* L_2$ are dilations. From (12) and that L_1 and L_2 are dilations, we get that $\Gamma_p^* L_1$ and $\Gamma_p^* L_2$ are also dilations. Since that the equality holds in (3) is equivalent to that the two equalities hold in (20) and (21), we then come to the assertion that the equality holds in (3) if and only if L_1 and L_2 are dilations.

We have completed the proof of Theorem 1.1. □

Remark 3.1. We mention that for $p \geq 1$ not an even integer, the equality holds in (18) in case (19) becomes an equation. To see this, we first notice that the operator $\Gamma_p : \mathcal{S}_s^n \to \mathcal{K}_s^n$ is injective: For any $M, N \in \mathcal{S}_s^n$ satisfying $\Gamma_p M = \Gamma_p N$ and that $p \geq 1$ is not an even integer, by (12) and the fact that the p-cosine transformation is injective on even functions if and only if p is not an even integer (cf. P.435 of [5]), we know that, for any $v \in S^{n-1}$, the two even functions $\frac{1}{V(M)}\rho_M^{n+p}(v)$ and $\frac{1}{V(N)}\rho_N^{n+p}(v)$ are equal. Then the assertion $M = N$ follows from Proposition 1.11 of [15]. Now, (12) and the injectivity of $\Gamma_p : \mathcal{S}_s^n \to \mathcal{K}_s^n$ show that if the equality holds in (19) then L_1 and L_2 are dilations, so that the equality holds in (18).

Analogously, for $p \geq 1$ not an even integer and that if (21) is an equation, then the equality holds also in (20). This fact follows directly from the properties of the polar.

Proof of Theorem 1.2. The fact $\frac{n-p}{m} > 1$ and (14) imply that, for $u \in S^{n-1}$,

$$\left[\int_{S^{n-1}} |u \cdot v|^{-p}\left(\rho_{L_1}^m(v) + \rho_{L_2}^m(v)\right)^{\frac{n-p}{m}} dv\right]^{\frac{m}{n-p}} \leq \left[\int_{S^{n-1}} |u \cdot v|^{-p}\rho_{L_1}^{n-p}(v)dv\right]^{\frac{m}{n-p}} + \left[\int_{S^{n-1}} |u \cdot v|^{-p}\rho_{L_2}^{n-p}(v)dv\right]^{\frac{m}{n-p}}.$$

Then, according to (11) and (13), we have

$$\left[\rho_{I_p(L_1\star_m L_2)}(u)\right]^{\frac{mp}{n-p}} \leq \left[\rho_{I_p L_1}(u)\right]^{\frac{mp}{n-p}} + \left[\rho_{I_p L_2}(u)\right]^{\frac{mp}{n-p}}, \tag{22}$$

and, according to the condition that the equality holds in (14), we see that the equality holds in (22) if and only if ρ_{L_1} is proportional to ρ_{L_2}, or equivalently, L_1 and L_2 are dilations.

In case (i), since $Q \in \mathcal{K}_o^n$, and $q < n - \frac{mp}{n-p}$ implies that $\frac{(n-p)(n-q)}{mp} > 1$, from (1) and (22) we have

$$n\widetilde{V}_{p,q}(K, Q, I_p(L_1 \star_m L_2)) \leq \int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)^p (\alpha_K(u))\rho_K^q(u)\Big(\rho_{I_pL_1}^{\frac{mp}{n-p}}(u) + \rho_{I_pL_2}^{\frac{mp}{n-p}}(u)\Big)^{\frac{(n-p)(n-q)}{mp}} du =: N_3, \tag{23}$$

and the equality holds if and only if L_1 and L_2 are dilations, which follows from the condition that the equality holds in (22).

On the other hand, by using (14) and (1), we can show that

$$\begin{aligned} N_3^{\frac{mp}{(n-p)(n-q)}} &\leq \left[\int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)^p (\alpha_K(u))\rho_K^q(u)\rho_{I_pL_1}^{n-q}(u)du\right]^{\frac{mp}{(n-p)(n-q)}} + \left[\int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)^p (\alpha_K(u))\rho_K^q(u)\rho_{I_pL_2}^{n-q}(u)du\right]^{\frac{mp}{(n-p)(n-q)}} \\ &= \left[n\widetilde{V}_{p,q}(K, Q, I_pL_1)\right]^{\frac{mp}{(n-p)(n-q)}} + \left[n\widetilde{V}_{p,q}(K, Q, I_pL_2)\right]^{\frac{mp}{(n-p)(n-q)}}. \end{aligned} \tag{24}$$

From (23) and (24), we get the desired inequality (4).

From the condition that the equality holds in (14), we see that the equality holds in (24) if and only if $\rho_{I_pL_1}$ is proportional to $\rho_{I_pL_2}$, or equivalently, I_pL_1 and I_pL_2 are dilations.

As $L_i \in \mathcal{S}_s^n$ for $i = 1, 2$ imply that ρ_{L_i} is even, then Theorem 6 and equation (10) in [8] show that the operator $I_p : \mathcal{S}_s^n \to \mathcal{S}_s^n$ is injective. Thus, together with (13), the equality holds in (24) if and only if L_1 and L_2 are dilations.

This verifies the assertion that the equality holds in (4) if and only if L_1 and L_2 are dilations.

To deal with case (ii), we assume that $Q \in \mathcal{S}_o^n$ and $q \in \mathbb{R}$.

Then, by (1), (10) and (22), but with u replaced by $\alpha_K(u)$, we obtain

$$n\widetilde{V}_{-p,q}(K, I_p^*(L_1 \star_m L_2), Q) \leq \int_{S^{n-1}} \left[\left(\frac{h_{I_p^*L_1}}{h_K}\right)^{-\frac{mp}{n-p}}(\alpha_K(u)) + \left(\frac{h_{I_p^*L_2}}{h_K}\right)^{-\frac{mp}{n-p}}(\alpha_K(u))\right]^{\frac{n-p}{m}} \rho_K^q(u)\rho_Q^{n-q}(u)du =: N_4, \tag{25}$$

the equality holds if and only if L_1 and L_2 are dilations, which follows again from the condition that the equality holds in (22).

On the other hand, by using (14) and (1), we can show that

$$\begin{aligned} N_4^{\frac{m}{n-p}} &\leq \left[\int_{S^{n-1}} \left(\frac{h_{I_p^*L_1}}{h_K}\right)^{-p} (\alpha_K(u))\rho_K^q(u)\rho_Q^{n-q}(u)du\right]^{\frac{m}{n-p}} + \left[\int_{S^{n-1}} \left(\frac{h_{I_p^*L_2}}{h_K}\right)^{-p} (\alpha_K(u))\rho_K^q(u)\rho_Q^{n-q}(u)du\right]^{\frac{m}{n-p}} \\ &= \left[n\widetilde{V}_{-p,q}(K, I_p^*L_1, Q)\right]^{\frac{m}{n-p}} + \left[n\widetilde{V}_{-p,q}(K, I_p^*L_2, Q)\right]^{\frac{m}{n-p}}. \end{aligned} \tag{26}$$

Then the inequality (5) follows from (25) and (26).

Moreover, equality holds in (5) is equivalent to that both (25) and (26) become equality. Equality holds in (25) if and only if L_1 and L_2 are dilations. According to the condition such that the equality holds in (14), we see that the equality holds in (26) if and only if $h_{I_p^*L_1}$ is proportional to $h_{I_p^*L_2}$, or equivalently, $I_p^*L_1$ and $I_p^*L_2$ are dilations. By (12) and the property of the polar, if L_1 and L_2 are dilations, then $I_p^*L_1$ and $I_p^*L_2$ are dilations as well. This shows that equality holds in (5) if and only if L_1 and L_2 are dilations.

We have completed the proof of Theorem 1.2. □

Now, as applications of Theorems 1.1 and 1.2, we can show the convexity of the following four functionals $F_i : \mathcal{S}_s^n \to (0, \infty)$ for $i = 1, 2, 3, 4$.

(i) For $K \in \mathcal{K}_o^n$, $Q \in \mathcal{S}_o^n$, $1 \leq m \leq n-1$, $p \geq 1$ and $q \in \mathbb{R}$, define $F_1(L) := \left[\widetilde{V}_{p,q}(K, \Gamma_pL, Q)V(L)\right]^{\frac{m}{n+p}}$;

(ii) For $K, Q \in \mathcal{K}_o^n$, $1 \leq m \leq n-1$, $p \geq 1$ and $q > n + \frac{mp}{n+p}$, define $F_2(L) := \left[\widetilde{V}_{p,q}(K, Q, \Gamma_p^*L)^{-\frac{p}{n-q}} V(L)\right]^{\frac{m}{n+p}}$;

(iii) For $K, Q \in \mathcal{K}_o^n$, $1 \leq m \leq n-1$, $0 < p < 1$ and $q < n - \frac{mp}{n-p}$, define $F_3(L) := \widetilde{V}_{p,q}(K, Q, I_pL)^{\frac{mp}{(n-p)(n-q)}}$;

(iv) For $K \in \mathcal{K}_o^n$, $Q \in \mathcal{S}_o^n$, $1 \leq m \leq n-1$, $0 < p < 1$ and $q \in \mathbb{R}$, define $F_4(L) := \widetilde{V}_{-p,q}(K, I_p^*L, Q)^{\frac{m}{n-p}}$.

Theorem 3.1. *If $L_1, L_2 \in \mathcal{S}_s^n$, m is an integer with $1 \le m \le n-1$ and functions F_i for $i = 1, 2, 3, 4$ are defined as above, then each $F_i(\lambda L_1 \star_m (1-\lambda)L_2)$ is convex for $\lambda \in (0,1)$, in terms of the dual Blaschke addition.*

Proof. We only prove the case of F_1, the proof of other cases follows along the same line and hence is omitted.

From the definitions (1) and (12), it follows that $\Gamma_p(\lambda L) = \lambda \Gamma_p L$ and $\widetilde{V}_{p,q}(K, \lambda\Gamma_p L, Q) = \lambda^p \widetilde{V}_{p,q}(K, \Gamma_p L, Q)$, for $\lambda > 0$. These together with (2) give that for $\lambda \in (0,1)$,

$$\begin{aligned}
&F_1(\lambda L_1 \star_m (1-\lambda)L_2)\\
&= \Big[\widetilde{V}_{p,q}(K, \Gamma_p(\lambda L_1 \star_m (1-\lambda)L_2), Q)V(\lambda L_1 \star_m (1-\lambda)L_2)\Big]^{\frac{m}{n+p}}\\
&\le \Big[\widetilde{V}_{p,q}(K, \Gamma_p(\lambda L_1), Q)V(\lambda L_1)\Big]^{\frac{m}{n+p}} + \Big[\widetilde{V}_{p,q}(K, \Gamma_p((1-\lambda)L_2), Q)V((1-\lambda)L_2)\Big]^{\frac{m}{n+p}}\\
&= \lambda^m F_1(L_1) + (1-\lambda)^m F_1(L_2)\\
&\le \lambda F_1(L_1) + (1-\lambda)F_1(L_2),
\end{aligned}$$

which shows that $F_1(\lambda L_1 \star_m (1-\lambda)L_2)$ is convex for $\lambda \in (0,1)$, in terms of the dual Blaschke addition. □

In addition to the Theorems 1.1 and 1.2, we can also give the following Brunn-Minkowski type inequalities, motivated by the Theorem 3.3 of [7].

Proposition 3.1. *Let $K \in \mathcal{K}_o^n$, $L_1, L_2 \in \mathcal{S}_s^n$ and $1 \le m \le n-1$.*

(i) *For $Q \in \mathcal{K}_o^n$ and $p \in \mathbb{R}$, we have*

$$\Big[\widetilde{V}_{p,q}(K, Q, L_1 \star_m L_2)\Big]^{\frac{m}{n-q}} \le \Big[\widetilde{V}_{p,q}(K, Q, L_1)\Big]^{\frac{m}{n-q}} + \Big[\widetilde{V}_{p,q}(K, Q, L_2)\Big]^{\frac{m}{n-q}} \tag{27}$$

for $q < n-m$; and

$$\Big[\widetilde{V}_{p,q}(K, Q, L_1 \star_m L_2)\Big]^{\frac{m}{n-q}} \ge \Big[\widetilde{V}_{p,q}(K, Q, L_1)\Big]^{\frac{m}{n-q}} + \Big[\widetilde{V}_{p,q}(K, Q, L_2)\Big]^{\frac{m}{n-q}} \tag{28}$$

for $n-m < q \ne n$. Moreover, the equality holds in either (27) *or* (28) *if and only if L_1 and L_2 are dilations.*

(ii) *For $Q \in \mathcal{S}_o^n$ and $q \in \mathbb{R}$, we have*

$$\Big[\widetilde{V}_{p,q}(K, (L_1 \star_m L_2)^*, Q)\Big]^{-\frac{m}{p}} \le \Big[\widetilde{V}_{p,q}(K, L_1^*, Q)\Big]^{-\frac{m}{p}} + \Big[\widetilde{V}_{p,q}(K, L_2^*, Q)\Big]^{-\frac{m}{p}} \tag{29}$$

for $p < -m$; and

$$\Big[\widetilde{V}_{p,q}(K, (L_1 \star_m L_2)^*, Q)\Big]^{-\frac{m}{p}} \ge \Big[\widetilde{V}_{p,q}(K, L_1^*, Q)\Big]^{-\frac{m}{p}} + \Big[\widetilde{V}_{p,q}(K, L_2^*, Q)\Big]^{-\frac{m}{p}} \tag{30}$$

for $-m < p \ne 0$. Moreover, if $L_1, L_2 \in \mathcal{S}_s^n \setminus \mathcal{K}_s^n$ are dilations, then the equality holds in both (29) *and* (30)*; if, however, $L_1, L_2 \in \mathcal{K}_s^n$, then the equality holds in either* (29) *or* (30) *if and only if L_1 and L_2 are dilations.*

Proof. Since the proofs of the reverse inequalities (28) and (30) follow along the same lines, we shall prove only the inequalities (27) and (29).

In case (i), noting that $q < n-m$ implies $\frac{n-q}{m} > 1$, by (11) and (14), we have

$$\begin{aligned}
&\Big[\int_{S^{n-1}} \Big(\frac{h_Q}{h_K}\Big)^p (\alpha_K(u))\rho_K^q(u)\rho_{L_1\star_m L_2}^{n-q}(u)du\Big]^{\frac{m}{n-q}}\\
&\le \Big[\int_{S^{n-1}} \Big(\frac{h_Q}{h_K}\Big)^p (\alpha_K(u))\rho_K^q(u)\rho_{L_1}^{n-q}(u)du\Big]^{\frac{m}{n-q}} + \Big[\int_{S^{n-1}} \Big(\frac{h_Q}{h_K}\Big)^p (\alpha_K(u))\rho_K^q(u)\rho_{L_2}^{n-q}(u)du\Big]^{\frac{m}{n-q}},
\end{aligned}$$

which combining with (1) then yields (27). Moreover, from the condition that the equality holds in (14), we see that the equality holds in (27) if and only if ρ_{L_1} is proportional to ρ_{L_2}, or equivalently, L_1 and L_2 are dilations.

In case (ii), from (10), we can rewrite (11) as $h^{-m}_{(L_1\star_m L_2)^*}(u) = h^{-m}_{L_1^*}(u) + h^{-m}_{L_2^*}(u),$ for $u \in S^{n-1}$. Then, the fact $-\frac{p}{m} > 1$ and (14) imply that

$$\Big[\int_{S^{n-1}}\Big(\frac{h_{(L_1\star_m L_2)^*}}{h_K}\Big)^p(\alpha_K(u))\rho_K^q(u)\rho_Q^{n-q}(u)du\Big]^{-\frac{m}{p}}$$
$$\leq \Big[\int_{S^{n-1}}\Big(\frac{h_{L_1^*}}{h_K}\Big)^p(\alpha_K(u))\rho_K^q(u)\rho_Q^{n-q}(u)du\Big]^{-\frac{m}{p}} + \Big[\int_{S^{n-1}}\Big(\frac{h_{L_2^*}}{h_K}\Big)^p(\alpha_K(u))\rho_K^q(u)\rho_Q^{n-q}(u)du\Big]^{-\frac{m}{p}},$$

which combining with (1) yields (29).

Moreover, from the condition that the equality holds in (14), we see that the equality holds in (29) if and only if $h_{L_1^*}$ is proportional to $h_{L_2^*}$, or equivalently, L_1^* and L_2^* are dilations. Now, for $L_1, L_2 \in \mathcal{S}_s^n \setminus \mathcal{K}_s^n$, by the properties of the polar, we know that if L_1 and L_2 are dilations, then L_1^* and L_2^* are dilations; while for $L_1, L_2 \in \mathcal{K}_s^n$, L_1^* and L_2^* are dilations if and only if L_1 and L_2 are dilations. Then the remaining assertion of Proposition 3.1 immediately follows. □

Remark 3.2. If $p = 0$ (resp. $Q = K$) in (27)-(28) and if $q = n$ (resp. $Q = K$) in (29)-(30), then Proposition 7.2 of [17] implies that our results yield the Brunn-Minkowski type inequalities for the L_p mixed volume and the dual mixed volume.

Remark 3.3. On the condition that (27) and (29) hold respectively, the following two functionals:

$$F_5(\lambda L_1 \star_m (1-\lambda)L_2) := \Big[\widetilde{V}_{p,q}(K, Q, \lambda L_1 \star_m (1-\lambda)L_2)\Big]^{\frac{m}{n-q}},$$
$$F_6(\lambda L_1 \star_m (1-\lambda)L_2) := \Big[\widetilde{V}_{p,q}(K, (\lambda L_1 \star_m (1-\lambda)L_2)^*, Q)\Big]^{-\frac{m}{p}}$$

are all convex for $\lambda \in (0,1)$, in terms of the dual Blaschke addition.

Remark 3.4. Let $\lambda \in (0,1)$, the geometric-arithmetic mean inequality and (28) yield that, if $n - m < q < n$, then

$$\widetilde{V}_{p,q}(K, Q, \lambda^{\frac{1}{m}} L_1 \star_m (1-\lambda)^{\frac{1}{m}} L_2) \geq \Big[\widetilde{V}_{p,q}(K, Q, L_1)\Big]^{\lambda}\Big[\widetilde{V}_{p,q}(K, Q, L_2)\Big]^{1-\lambda};$$

if $q > n$, then this inequality is reversed; the equality holds in each case if and only if $L_1 = L_2$. Similar argument can be used in inequality (30).

Next, in terms of subscripts p of the centroid body $\Gamma_p L$ and the intersection body $I_p L$, we will prove four monotonicity inequalities introduced in Theorems 1.3 and 1.4. In the proof, the Hölder's integral inequality will be used many times.

Proof of Theorem 1.3. The Hölder's integral inequality (15) shows that for $i, j \geq 1$, $i \neq j$, and $u \in S^{n-1}$,

$$\int_{S^{n-1}} |u\cdot v|^{i+j}\rho_L^{n+i+j}(v)dv = \int_{S^{n-1}}\Big(|u\cdot v|^i\rho_L^{n+i}(v)\Big)^{\frac{i}{i-j}}\Big(|u\cdot v|^j\rho_L^{n+j}(v)\Big)^{-\frac{j}{i-j}}dv$$
$$\geq \Big[\int_{S^{n-1}}|u\cdot v|^i\rho_L^{n+i}(v)dv\Big]^{\frac{i}{i-j}}\Big[\int_{S^{n-1}}|u\cdot v|^j\rho_L^{n+j}(v)dv\Big]^{-\frac{j}{i-j}}.$$

From this and (12), it follows that

$$c_{n,i+j}(n+i+j)h^{i+j}_{\Gamma_{i+j}L}(u) \geq \Big[c_{n,i}(n+i)h^i_{\Gamma_i L}(u)\Big]^{\frac{i}{i-j}}\Big[c_{n,j}(n+j)h^j_{\Gamma_j L}(u)\Big]^{-\frac{j}{i-j}}, \tag{31}$$

for $u \in S^{n-1}$, and from the condition that the equality holds in (15), we see the equality holds in (31) if and only if $|u \cdot v|^{i-j}\rho_L^{i-j}(v) = \lambda$, $\lambda > 0$. By (10), this is equivalent to $h_{L^*}(v) = \lambda^{\frac{1}{j-i}}|u \cdot v|$, or, L^* is an origin-symmetric line segment in the direction u.

In case (i), $Q \in \mathcal{S}_o^n$, $p > 0$ and $q \in \mathbb{R}$, from (1) and (31), but with u replaced by $\alpha_K(u)$, we see that

$$\begin{aligned} &n(c_{n,i+j}(n+i+j))^{\frac{p}{i+j}}\widetilde{V}_{p,q}(K,\Gamma_{i+j}L,Q) \\ &\geq \int_{S^{n-1}}\left[\left(c_{n,i}(n+i)\right)^{\frac{p}{i}}\left(\frac{h_{\Gamma_i L}}{h_K}\right)^p(\alpha_K(u))\right]^{\frac{i^2}{i^2-j^2}}\left[\left(c_{n,j}(n+j)\right)^{\frac{p}{j}}\left(\frac{h_{\Gamma_j L}}{h_K}\right)^p(\alpha_K(u))\right]^{-\frac{j^2}{i^2-j^2}}\rho_K^q(u)\rho_Q^{n-q}(u)du =: N_5, \end{aligned} \tag{32}$$

if equality holds in (32), then L^* depends on the variables $\alpha_K(u)$. Hence the inequality (32) is strict. Then, as $i > j$, from (1) and (15) it follows immediately that

$$\begin{aligned} N_5^{i^2-j^2} \geq &\left[\int_{S^{n-1}}\left(c_{n,i}(n+i)\right)^{\frac{p}{i}}\left(\frac{h_{\Gamma_i L}}{h_K}\right)^p(\alpha_K(u))\rho_K^q(u)\rho_Q^{n-q}(u)du\right]^{i^2} \\ &\cdot\left[\int_{S^{n-1}}\left(c_{n,j}(n+j)\right)^{\frac{p}{j}}\left(\frac{h_{\Gamma_j L}}{h_K}\right)^p(\alpha_K(u))\rho_K^q(u)\rho_Q^{n-q}(u)du\right]^{-j^2} \\ = &\left[n\left(c_{n,i}(n+i)\right)^{\frac{p}{i}}\widetilde{V}_{p,q}(K,\Gamma_i L,Q)\right]^{i^2}\left[n\left(c_{n,j}(n+j)\right)^{\frac{p}{j}}\widetilde{V}_{p,q}(K,\Gamma_j L,Q)\right]^{-j^2}. \end{aligned}$$

This shows that (6) holds.

In case (ii), noticing that $Q \in \mathcal{K}_o^n$, $q > n$ and $p \in \mathbb{R}$, by (10), (1) and (31) we obtain

$$\begin{aligned} &n\widetilde{V}_{p,q}(K,Q,\Gamma_{i+j}^*L)(c_{n,i+j}(n+i+j))^{\frac{q-n}{i+j}} \\ &= \int_{S^{n-1}}\left[c_{n,i+j}(n+i+j)\rho_{\Gamma_{i+j}^*L}^{-(i+j)}(u)\right]^{\frac{(i-j)(q-n)}{i^2-j^2}}\left(\frac{h_Q}{h_K}\right)^p(\alpha_K(u))\rho_K^q(u)du \\ &\geq \int_{S^{n-1}}\left[\left(c_{n,i}(n+i)\rho_{\Gamma_i^*L}^{-i}(u)\right)^i\left(c_{n,j}(n+j)\rho_{\Gamma_j^*L}^{-j}(u)\right)^{-j}\right]^{\frac{q-n}{i^2-j^2}}\left(\frac{h_Q}{h_K}\right)^p(\alpha_K(u))\rho_K^q(u)du =: N_6. \end{aligned} \tag{33}$$

If the equality holds in (33), then L^* depends on the variables u. Hence the inequality (33) is strict.

As $i > j$, by using (1), (15) and (33), we have

$$\begin{aligned} N_6^{i^2-j^2} \geq &\left[\int_{S^{n-1}}\left(c_{n,i}(n+i)\right)^{\frac{q-n}{i}}\left(\frac{h_Q}{h_K}\right)^p(\alpha_K(u))\rho_K^q(u)\rho_{\Gamma_i^*L}^{n-q}(u)du\right]^{i^2} \\ &\cdot\left[\int_{S^{n-1}}\left(c_{n,j}(n+j)\right)^{\frac{q-n}{j}}\left(\frac{h_Q}{h_K}\right)^p(\alpha_K(u))\rho_K^q(u)\rho_{\Gamma_j^*L}^{n-q}(u)du\right]^{-j^2} \\ = &n^{i^2-j^2}\left(c_{n,i}(n+i)\right)^{i(q-n)}(c_{n,j}(n+j))^{-j(q-n)}\widetilde{V}_{p,q}(K,Q,\Gamma_i^*L)^{i^2}\widetilde{V}_{p,q}(K,Q,\Gamma_j^*L)^{-j^2}. \end{aligned} \tag{34}$$

Combined with the above (33) and (34) directly, we can conclude the desired result (7). □

Remark 3.5. For $x, y \in \mathbb{R}^n$, we denote by $[x, y]$ the closed segment with two endpoints x and y. From the fact $h_{[-u,u]}(v) = |u \cdot v|$ for $u, v \in S^{n-1}$, we see that equality holds in (31) if and only if $L^* = \lambda^{\frac{1}{j-i}}[-u, u]$.

Proof of Theorem 1.4. Assume that $0 < i, j < i + j < 1$ and $i \neq j$. By (13) and (15), we can easily show that, for $u \in S^{n-1}$,

$$(n-i-j)\rho_{I_{i+j}L}^{i+j}(u) \geq ((n-i)\rho_{I_i L}^i(u))^{\frac{i}{i-j}}((n-j)\rho_{I_j L}^j(u))^{-\frac{j}{i-j}}, \tag{35}$$

and the condition that the equality holds in (15) implies that the equality holds in (35) if and only if $|u \cdot v|^{i-j}\rho_L^{i-j}(v) = \lambda$, $\lambda > 0$. By (10), this is equivalent to $h_{L^*}(v) = \lambda^{\frac{1}{j-i}}|u \cdot v|$, or, L^* is an origin-symmetric line segment in the direction u.

In case (i), $Q \in \mathcal{K}_o^n$, $q < n$ and $p \in \mathbb{R}$. By using (1) and (35), we have

$$n\widetilde{V}_{p,q}(K, Q, I_{i+j}L)(n-i-j)^{\frac{n-q}{i+j}} \geq \int_{S^{n-1}} \left[\left((n-i)\rho_{I_iL}^i(u)\right)^{\frac{i}{i-j}}\left((n-j)\rho_{I_jL}^j(u)\right)^{-\frac{j}{i-j}}\right]^{\frac{n-q}{i+j}} \left(\frac{h_Q}{h_K}\right)^p (\alpha_K(u))\rho_K^q(u)du =: N_7. \quad (36)$$

Note that if equality holds in (36), then L^* depends on the variables u. Hence the inequality (36) is strict. As $i > j$, combining (1), (36) and (15), we obtain

$$\begin{aligned} N_7^{i^2-j^2} \geq &(n-i)^{i(n-q)}\Big[\int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)^p (\alpha_K(u))\rho_K^q(u)\rho_{I_iL}^{n-q}(u)du\Big]^{i^2} \\ &\cdot (n-j)^{-j(n-q)}\Big[\int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)^p (\alpha_K(u))\rho_K^q(u)\rho_{I_jL}^{n-q}(u)du\Big]^{-j^2} \\ =&n^{i^2-j^2}(n-i)^{i(n-q)}(n-j)^{-j(n-q)}\widetilde{V}_{p,q}(K, Q, I_iL)^{i^2}\widetilde{V}_{p,q}(K, Q, I_jL)^{-j^2}. \end{aligned} \quad (37)$$

Then, (36) and (37) yield the desired conclusion (8).

In case (ii), $Q \in \mathcal{S}_o^n$, $p < 0$ and $q \in \mathbb{R}$. By (1), (10) and (35), but with u replaced by $\alpha_K(u)$, we have that

$$\begin{aligned} &n\widetilde{V}_{p,q}(K, I_{i+j}^*L, Q)(n-i-j)^{-\frac{p}{i+j}} \\ \geq &\int_{S^{n-1}} \left[\left((n-i)\left(\frac{h_{I_i^*L}}{h_K}\right)^{-i}(\alpha_K(u))\right)^{\frac{i}{i-j}}\left((n-j)\left(\frac{h_{I_j^*L}}{h_K}\right)^{-j}(\alpha_K(u))\right)^{-\frac{j}{i-j}}\right]^{-\frac{p}{i+j}} \rho_K^q(u)\rho_Q^{n-q}(u)du =: N_8. \end{aligned} \quad (38)$$

Similar to (36), the inequality (38) is also strict.

As $i > j$, by (1), (38) and (15) we can verify that

$$N_8^{i^2-j^2} \geq n^{i^2-j^2}(n-i)^{-pi}(n-j)^{pj}\widetilde{V}_{p,q}(K, I_i^*L, Q)^{i^2}\widetilde{V}_{p,q}(K, I_j^*L, Q)^{-j^2}. \quad (39)$$

Then, the desired inequality (9) follows from (38) and (39). □

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