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The wf-Rank of Topological Spaces

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Abstract. The class of well-filtered spaces is one of the important class of topological spaces in domain theory. Recently, the existence of well-filterification of T_0-spaces has been established. In this paper, we introduce the definition of wf-rank and prove that for any ordinal α, there exists a T_0-space whose wf-rank equals α.

1. Introduction

In the theory of lattices and partially ordered sets, completions of various types play a fundamental role, such as the Dedekind-MacNeille completion ([11]), the Jónsson-Tarski's two-sided completion for Boolean algebras ([8, 9]), and Δ_1-completions ([3]). For domain theory, the salient type of completeness is directed completeness, i.e., all directed subsets of a poset have a supremum. Directed complete posets are called dcpo's. A topological analogue of a dcpo, called a *d*-space, lives in the world of T_0-spaces, where one requires that all directed sets in the order of specialization have a supremum, and that a directed set always converges topologically to its supremum ([14]). Such spaces are also called monotone convergence spaces in [4], p. 183. Wyler was the first to invent *d*-spaces, seeing it as a virtue out of necessity[1)] to use *d*-spaces as a natural generalization of sober spaces ([14]). Here, sober spaces refer to those topological spaces in which every non-empty closed irreducible set is the closure of a unique singleton. Since every directed set of a T_0 space with respect to the specialization order is irreducible, it holds at once that every sober space is a *d*-space. Additionally, *d*-completion is to *d*-spaces as sobrification is to sober spaces, which is exactly what Wyler established in [14].

Elsewhere and independently, Ershov proved that for any subspace X_0 of a *d*-spaces X, there is a *d*-completion of X_0 ([1]). Later in a recent work ([2]), Ershov gave another construction for *d*-completion of a space through the use of what he called the *d-rank*, which is an ordinal that measures how far a T_0 space is from being a *d*-space. We now make this precise. For a given T_0 space (X, τ), let $\overline{D}(X)$ be the family of all

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1) The original phrase in German "Wir machen aus der Not eine Tugend" appears in the abstract of [14].

directed subspaces with respect to the specialization order in X. Endow an equivalence relation on $\overline{D}(X)$ as follows:

$$S_1 \sim S_2 \overset{\text{def}}{\iff} \forall U \in \tau.\ (S_1 \cap U \neq \emptyset \iff S_2 \cap U \neq \emptyset),$$

and denote the equivalence class induced by $\sim$ that contains S by $[S]$. Furthermore, define

$$\begin{aligned} D(X) &:= \{[S] : S \in \overline{D}(X)\} \\ U^* &:= \{[S] : S \cap U \neq \emptyset\}, \ \text{ where } U \in \tau; \\ \tau^* &:= \{U^* : U \in \tau\}. \end{aligned}$$

It can be easily shown that $(D(X), \tau^*)$ is a T_0 space into which one can embed X as a subspace. Define a sequence of T_0 spaces by transfinite induction as follows:

- $D_0(X) := X$.
- $D_{\alpha+1}(X) := D(D_\alpha(X))$.
- If α is a limit ordinal, then $D_\alpha(X) := \bigcup_{\beta<\alpha} D_\beta(X)$, where we identify $D_\beta(X)$ with the corresponding subspace of $D_\alpha(X)$ for all ordinals $\beta \leq \alpha$.

The *d-rank* of a T_0 space X is the least ordinal α such that $D_\alpha(X) = D_{\alpha+1}(X)$. In [1], Ershov proved that (i) for every T_0 space X there exists a least ordinal α such that $D_\alpha(X)$ $(= D_{\alpha+1}(X))$ gives the d-completion of X, and in [2] he further showed that (ii) for any ordinal α, there exists a T_0-space whose d-rank equals α.

The main purpose of this paper is to make similar considerations, along the aforementioned development by Ershov in [2], in the context of another important class of topological spaces – the *well-filtered* spaces – which is attracting an increasing amount of attention in recent years ([12, 13, 15–19]). So, what are well-filtered spaces? A T_0-space X is said to be well-filtered if for any open set U of X and any filtered family $\mathcal{F}$ of compact saturated sets, whenever $\bigcap \mathcal{F} \subseteq U$ then already $F \subseteq U$ for some $F \in \mathcal{F}$. Because every directed set D of a T_0 space (with respect to the specialization) induces a filtered family of compact saturated sets, namely, $\{\uparrow d \mid d \in D\}$, it is immediately clear that every well-filtered space is a d-space. Also, by the Hofmann-Mislove Theorem ([6]), one can prove that every sober space is well-filtered. Consequently, well-filtered spaces are situated between the sober spaces and the d-spaces in the sense that

the class of sober spaces $\subset$ the class of well-filtered spaces $\subset$ the class of d-spaces.

Sobrification (respectively, d-completion) exists in that the category of sober spaces (respectively, d-spaces) is reflective in the category of all T_0 spaces. It is therefore natural to ask if the category of well-filtered spaces, which is situated between these, is also reflective in the category of all T_0 spaces. Recently, Wu et al. settled this question in the positive by demonstrating the existence of well-filterification ([13]). Guided by Ershov's method ([1]) of constructing the d-completion of a T_0-space, Shen et al. ([12]) introduced the notion of KF-sets so as to give another more direct (and explicit) construction of the well-filterification of any given T_0-space. For us, we continue along this line of research by taking up the task of defining the wf-rank of a T_0-space – à la Ershov – which is an ordinal that indicates how far a given T_0 space is from being well-filtered. In particular, we prove in Section 3 that for any ordinal α there exists a T_0-space whose wf-rank equals α.

2. Preliminaries

Let us gather at one place some basic definitions and useful results concerning well-filtered spaces.

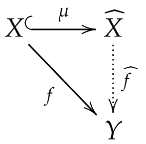
Throughout the paper, we use the symbols $(X, \tau(X))$ to denote a topological space with underlying set X and topology $\tau(X)$ endowed on it. When there is no confusion, we just write X or (X, τ) instead of $(X, \tau(X))$. The default underlying partial order of a T_0 space X is its specialization order, $\leqslant_X$, unless otherwise stated.

For instance, $\downarrow_X x$ refers to the lower part of an element $x \in X$ with respect to the specialization order, and $x <_X y$ means that x is strictly below y in the specialization order of X.

Given a T_0-space X, denote by $K(X)$ the set of all nonempty compact saturated subsets of X. We write $\mathcal{K} \subseteq_{\text{filt}} K(X)$ to denote that $\mathcal{K}$ is a filtered subfamily of $K(X)$, i.e., for any K_1 and $K_2 \in \mathcal{K}$ there exists $K_3 \in \mathcal{K}$ such that $K_3 \subseteq K_1 \cap K_2$.

Definition 2.1 (Well-filtered space, [5]). A T_0-space X is called *well-filtered* if for any open set U of X and any filtered family $\mathcal{F} \subseteq_{\text{filt}} K(X)$, $\bigcap \mathcal{F} \subseteq U$ implies $F \subseteq U$ for some $F \in \mathcal{F}$.

Definition 2.2 (Well-filterification of a space, [13]). Let X be a T_0-space. A *well-filterication* of X is a pair $(\widehat{X}, \mu)$ consisting of a well-filtered space $\widehat{X}$ and a continuous map $\mu : X \to \widehat{X}$ satisfying the condition that for any continuous map $f : X \longrightarrow Y$ to a well-filtered space Y, there exists a unique continuous map $\widehat{f} : \widehat{X} \longrightarrow Y$ such that the following diagram commutes:



i.e., $\widehat{f} \circ \mu = f$. Whenever a well-filterification of a space X exists, we denote it by $H_{\text{wf}}(X)$.

Definition 2.3 (KF property and KF-set, [12]). Let X be a T_0-space. A nonempty subset A of X is said to have a *compactly filtered property* (*KF property*) if there exists $\mathcal{K} \subseteq_{\text{filt}} K(X)$ such that $\text{cl}_X(A)$ is a minimal closed set that intersects all members of $\mathcal{K}$. Such an A is called a KF-*set*. Denote by KF(X) the set of all closed KF-subsets of X.

Remark 2.4. ([12]) Let X be a T_0-space. Then the following statements hold:

(1) Every singleton subset is a KF-subset.

(2) Every directed subset is a KF-subset.

(3) Every KF-subset is an irreducible subset of X.

The role played by KF-subsets with respect to a well-filtered space is the same as that played by irreducible subsets with respect to a sober space in the sense that:

Lemma 2.5. ([12]) *The following statements are equivalent for a T_0-space X:*

1. *X is well-filtered.*
2. *For all $A \in \text{KF}(X)$, there exists $x \in X$ such that $A = \downarrow_X x$.*

Without further ado, let us perform the well-filterication of a T_0-space X using a bottom-up approach. The idea here is to modify a given space X in stages to reach its well-filterification by adding points at each stage.

For any T_0-space X, denote by $\overline{W}(X)$ the family of all KF-sets in X. Consider the equivalence relation $\sim$ on $\overline{W}(X)$ defined as follows:

$$A_1 \sim A_2 \stackrel{\text{def}}{\iff} \forall U \in \tau.\ (A_1 \cap U \neq \emptyset \iff A_2 \cap U \neq \emptyset).$$

The following notations are in place:

$$\begin{aligned}
[A] &:= \{A' \in \overline{W}(X) : A \sim A'\},\ A \in \overline{W}(X), \\
W(X) &:= \{[A] : A \in \overline{W}(X)\}, \\
U^* &:= \{[A] : A \cap U \neq \emptyset\},\ U \in \tau, \\
\tau^* &:= \{U^* : U \in \tau\}.
\end{aligned}$$

Expectedly, a one-step modification via W (W-*modification*, for short) of a T_0-space yields a T_0-space.

Theorem 2.6. ([10]) *For any given T_0-space X, $(W(X), \tau^*)$ is a T_0 space. Moreover, we may regard X as a subspace of $W(X)$ in the sense that the map*

$$\lambda : X \to W(X),\ \lambda(x) = [\{x\}].$$

is a homeomorphic embedding.

Lemma 2.7. ([10]) *Let X be a T_0 space. Then the following are equivalent:*

1. *X is a well-filtered space.*
2. *X and $W(X)$ are homeomorphic, i.e., $X \cong W(X)$.*

From the above lemma, one can see that a single W-modification of a well-filtered space X leaves it untouched.

Let X be a T_0-space and X' a sober space that has X as a subspace (for instance, when X' is the sobrification of X). Since X' is a sober space, it is well-filtered. According to Lemma 2.7, we have $W(X') \cong X'$.

For each ordinal β, we may identify $W_\delta(X)$ with the corresponding subspace of $W_\beta(X)$ for all ordinals $\delta < \beta$, and identify $W_\beta(X)$ with the corresponding subspace of X' for all ordinal β. The transfinite sequence of extensions $(W_\alpha(X))_\alpha$ is constructed as follows:

1. $W_0(X) = X$.
2. $W_{\beta+1}(X) = W(W_\beta(X))$.
3. $W_\beta(X) = \bigcup_{\delta<\beta} W_\delta(X)$ if β is a limit ordinal.

Experts of transfinite induction will not be surprised that iterated modifications stabilize in the long run, that is:

Theorem 2.8. ([10]) *For every T_0-space X, there exists a well-filterification of X. More precisely, there exists an ordinal α such that $H_{\mathrm{wf}}(X) = W_\alpha(X) = W_{\alpha+1}(X)$.*

A topological space X is said to as *irreducible* if the set X is itself an irreducible subset of X. For any irreducible topological space X, let

$$X^\top := \begin{cases} X & \text{if } X \text{ has a greatest element;} \\ (X \cup \{\top\}, \tau^\top) & \text{otherwise,} \end{cases}$$

where $\tau^\top := \{\emptyset\} \cup \{U \cup \{\top\} : \emptyset \neq U \in \tau\}$. Sometimes, we refer to the creation of the new space $X^\top$ out of an irreducible space X as *topping-up* the space X.

Topping up the (irreducible) cofinite topological space of natural numbers yields its well-filterification.

Lemma 2.9. ([10]) *Let (N, τ_{cof}) be the cofinite topology defined on the set N of natural numbers. Then $H_{\mathrm{wf}}(N) = N^\top \cong W(N)$.*

For irreducible spaces, the topping-up construction $(-)^\top$ and the W-modification commute up to homeomorphism.

Lemma 2.10. ([10]) *Let X be an irreducible T_0-space. Then*

$$W(X^\top) \cong W(X)^\top.$$

Apart from the topping-up construction, another topological construction we frequently perform in this paper is the so-called fibred sum which we describe below.

Let $(X, \tau(X))$ be a T_0-space, and $(Y_x)_{x\in X}$ an X-indexed family of T_0-spaces, where $Y_x := (Y_x, \tau(Y_x))$. Define

$$Z := \bigcup_{x\in X} (Y_x \times \{x\}),$$

and

$$\tau := \{U \subseteq Z : (U)_x \in \tau(Y_x) \text{ for any } x \in X \text{ and } (U)_X \in \tau(X)\},$$

where $(U)_x := \{y \in Y_x : (y, x) \in U\}$ for each $x \in X$, and $(U)_X := \{x \in X : U_x \neq \emptyset\}$.

Lemma 2.11. ([2]) *Let X be a T_0-space and an X-indexed family $(Y_x)_{x\in X}$ of irreducible T_0 spaces. The following statements hold:*

(1) (Z, τ) is a T_0-topological space.

(2) The map $(y \mapsto (y, x))$ is a homeomorphic embedding of Y_x into Z for each $x \in X$.

(3) If the space X is irreducible, so is the space Z.

Hereafter, the T_0-space (Z, τ) in Lemma 2.11 will be denoted by $\sum_{x\in X} Y_x$, and we call this space the *fibred sum* of $(Y_x)_{x\in X}$. An illustration of the relevant sets mentioned in the above definition of a fibred sum is given in Figure 1. At this point, we advise the reader to conscientiously suppress the urge to misread the fibred sum, Z, as the set of integers.

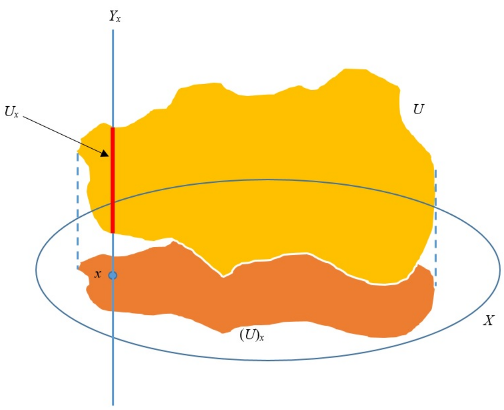


Figure 1: Fibred sum of $(Y_x)_{x\in X}$: $\sum_{x\in X} Y_x$

For any subset $A \subseteq Z$, we adopt the following notations:

$$(A)_x := \{y \in Y_x : (y, x) \in A\}, \text{ and } (A)_X := \{x \in X : (A)_x \neq \emptyset\}.$$

Lemma 2.12. ([2]) *Let X be a T_0-space and $(Y_x)_{x\in X}$ an X-indexed family of irreducible T_0-spaces. Then the specialization order $\leqslant$ on $\sum_{x\in X} Y_x$ is characterized as follows: for any (y_0, x_0), $(y_1, x_1) \in Z$, $(y_0, x_0) \leqslant_Z (y_1, x_1)$ if and only if one of the following two alternatives holds:*

(1) $x_0 = x_1$ and $y_0 \leqslant_{Y_{x_0}} y_1$;

(2) $x_0 <_X x_1$ and $y_1 = \top_{x_1}$ is the greatest element in Y_{x_1}.

Example 2.13. Let $X = (N, \tau_{\text{cof}})$, $(Y_x)_{x \in X}$ an X-indexed family of irreducible T_0-spaces, and $Z = \sum_{x \in X} Y_x$. Then $(y_0, x_0) \leqslant_Z (y_1, x_1)$ if and only if $x_0 = x_1$ and $y_0 \leqslant_{Y_{x_0}} y_1$.

Lemma 2.14. ([10]) *Let (N, τ_{cof}) be the cofinite topology defined on the set N of natural numbers, $(X_n)_{n \in N}$ a countable family of irreducible T_0-spaces such that at most finitely many X_n's have a greatest element, and $Z = \sum_{n \in N} X_n$. Then the following statements hold:*

(1) For each compact set A of Z, there exists a finite subset $F \subseteq N$ such that $A \subseteq \bigcup_{n \in F} (X_n \times \{n\})$.

(2) For every subset $A \subseteq Z$, A is compact if and only if $(A)_N$ is finite and $(A)_k$ is a compact set of X_k for each $k \in (A)_N$.

(3) For any KF*-set A of Z, there exists a unique $n \in N$ such that A is contained in $X_n \times \{n\}$.*

(4) A is a KF*-set of Z if and only if there exists a unique $n \in N$ such that $A \subseteq X_n \times \{n\}$ and $\{y \in X_n : (y, x) \in A\}$ is a* KF*-set of X_n.*

Lemma 2.15. ([10]) *Let (N, τ_{cof}) be the cofinite topology defined on the set N of natural numbers, $(X_n)_{n \in N}$ a countable family of irreducible T_0-spaces such that at most finitely many X_n's have a greatest element, and $Z = \sum_{n \in N} X_n$. Then the spaces $W(Z)$ and $Z' = \sum_{n \in N} W(X_n)$ are homeomorphic.*

Lemma 2.16. ([10]) *Let (N, τ_{cof}) be the cofinite topology defined on the set N of natural numbers, $(X_n)_{n \in N}$ a countable family of well-filtered spaces such that X_n has a greatest element for all $n \in N$, and $Z = \sum_{n \in N} X_n$. Then every compact set K of Z takes one of the following forms:*

(i) $K = \{(\top_n, n) : n \in M\}$, where M is some subset of N, and $\top_n$ is the greatest element of X_n for each $n \in N$.

(ii) $\{n \in (K)_N : (K)_n \cap (X_n - \{\top_n\}) \neq \emptyset\}$ is a finite subset of N.

Lemma 2.17. ([10]) *Let (N, τ_{cof}) be the cofinite topology defined on the set N of natural numbers, $(X_n)_{n \in N}$ a countable family of well-filtered T_0-spaces such that X_n has a greatest element for all $n \in N$, and $Z = \sum_{n \in N} X_n$. Then every* KF*-set A of Z takes one of the following forms:*

(i) There exists $n \in N$ such that $A \subseteq X_n \times \{n\}$.

(ii) $A \in [Z]$.

Lemma 2.18. ([10]) *Let (N, τ_{cof}) be the cofinite topology defined on the set N of natural numbers, $(X_n)_{n \in N}$ a countable family of well-filtered T_0-spaces such that X_n has a greatest element for all $n \in N$, and $Z = \sum_{n \in N} X_n$. Then $Z^\top$ is a well-filtered space and $W(Z) \cong Z^\top$.*

Lemma 2.19. *Let $X = (N, \tau_{\text{cof}})$, $(Y_x)_{x \in X}$ be an X-indexed family of irreducible T_0-spaces. Further, let $Z = \sum_{x \in X} Y_x$, and $Z' = \sum_{x \in X^\top} Y_x$, where $Y_\top = (\{\top\}, \{\emptyset, \{\top\}\})$. Then*

$$Z^\top \cong Z'.$$

Proof. Define a map $f : Z^\top \longrightarrow Z'$ by setting

$$f(z) = \begin{cases} z & \text{if } z \in Z; \\ (\top, \top) & \text{if } z = \top. \end{cases}$$

Claim 1. The map f is a bijection.

It is obvious that f is injective. For any $z \in Z$, we have $f^{-1}(z) = z$. And for $(\top, \top) \in Z'$, we have $f^{-1}(\top, \top) = \top$. Thus, the map f is a bijection.

Claim 2. The map f is continuous.

Let V be a nonempty open set of Z'. We aim to prove that $f^{-1}(V)$ is open in $Z^{\top}$. By the topology on Z', there exists $U \in \tau_{\text{cof}}$ such that $V_{N^{\top}} = U \cup \{\top\}$. For any $x \in V_{N^{\top}}$, we have $V_x \in \tau(Y_x)$ for $x \neq \top$ and $V_{\top} = \{\top\}$. Let $W = (\bigcup_{x \in U} V_x \times \{x\}) \cup \{\top\}$. Then W is an open set of $Z^{\top}$ by the topology on $Z^{\top}$. Thus, it remains for us to prove that $W = f^{-1}(V)$. For any $z \in W$, if $z \in \bigcup_{x \in U} V_x \times \{x\}$, we have $f(z) = z \in \bigcup_{x \in U} V_x \times \{x\} \in V$. If $z = \top$, then $f(z) = (\top, \top) \in V$. In either case, we have $f(z) \in V$, i.e., $z \in f^{-1}(V)$. Now for the reverse inclusion, let $z \in f^{-1}(V)$ be given, we need to show $z \in W$. Clearly, $f(z) \in V$. So we have two cases to consider:

Case 1. If $z \in Z$, since $z = f(z)$ and $f(z) \in V$, we have $z \in \bigcup_{x \in U} V_x \times \{x\} \subseteq W$.

Case 2. If $z = \top$, then of course $z \in W$.

In both cases, we see that $z \in W$, and so $f^{-1}(V) \subseteq W$.

Claim 3. The map f is open.

Let U be a nonempty open set of Z. Then $U \cup \{\top\}$ is an open set of $Z^{\top}$. In the proof of Claim 2, we have seen that $U \cup \{\top\} = f^{-1}(V)$, where $(V)_x = (U)_x$ for each $x \in (U)_N$ and $(V)_{\top} = \{\top\}$. Since f is one-to-one, it follows that $f(U \cup \{\top\}) = V$, which we know is open. Thus, f is open. □

3. α-special spaces

In this section, we begin by defining an appropriate index, called the wf-rank of a T_0-space, that indicates how far a given T_0-space is from being a well-filtered space. Let X be a topological space, if there is a topological space Y such that for all $x \in X$, $Y_x = Y$, then $\sum_{x \in X} Y_x$ is denoted by $\sum_{x \in X} Y$.

Definition 3.1 (wf-rank). The wf-*rank* of a topological T_0-space X is the least ordinal α such that $W_\alpha(X) = W_{\alpha+1}(X)$. We denote the wf-rank of a space X by $\text{rank}_{\text{wf}}(X)$.

Remark 3.2. By Theorem 2.8, the wf-rank of a T_0-space X is well-defined.

A natural question to ask in the reverse is for an arbitrarily given ordinal α whether there exists a T_0-space whose wf-rank is exactly α. The remaining of this section is to develop certain topological constructions that will answer this question in the positive. In our ensuing development, we make heavy use of transfinite induction. Though the arguments seem repetitive, it is important to remember that all lemmas in Section 2 are indispensable pieces that fit together in the intricate assembly of the transfinite jigsaw, culminating in the main result, i.e., Theorem 3.13.

An indispensable concept in our development is that of α-special spaces.

Definition 3.3 (α-special space). Let α be an ordinal. A T_0-space X is said to be *α-special* if the following two conditions are satisfied:

(1) $\text{rank}_{\text{wf}}(X) = \alpha$, and

(2) α is the least ordinal for which $W_\alpha(X)$ has a greatest element.

It is always reassuring to know that 0-special T_0-spaces exist.

Lemma 3.4. *The singleton space $T = (\{\top\}, \{\emptyset, \{\top\}\})$ is 0-special.*

Proof. Obvious. □

We now demonstrate that $(m+1)$-special T_0 spaces exist for any finite ordinal $m \in N$.

For any $m \in N$, let $Y^0 = N$, and we define

$$Y^m := \sum_{n \in N} X_n \text{ (in the sense defined in Lemma 2.11),}$$

where $X_n = Y^{m-1}$ for all $n \in N$. Simply, Y^m is denoted by $\sum_{n\in N} Y^{m-1}$, i.e., $Y^m := \sum_{n\in N} Y^{m-1}$.

Lemma 3.5. *The space Y^m is $(m+1)$-special for each $m \in N$.*

Proof. We proceed by induction on m.

Base case. For $m = 0$, the statement follows from Lemma 3.4.

Inductive step. Assume for all $n < m$ the space Y^n is $(n+1)$-special. We must prove that the statement holds for m.

First we show that the following homeomorphism of spaces hold

$$W_k(Y^m) \cong \sum_{n \in N} X_n$$

for all $k \leqslant m$, where $X_n = W_k(Y^{m-1})$ for all $n \in N$. This is achieved by induction on k as follows:

1. Base case. For $k = 0$, the statement follows from the definition of the space Y^m.
2. Inductive step. Let k be such that $k + 1 \leqslant m$, and assume that $W_k(Y^m) \cong \sum_{n\in N} X_n$, where $X_n = W_k(Y^{m-1})$ for all $n \in N$. We proceed to prove that the statement holds for $k + 1$.

$$\begin{aligned} W_{k+1}(Y^m) &\cong W(W_k(Y^m)) && \text{(by definition of } W_{k+1}(Y^m))\\ &\cong W(\sum_{n\in N} W_k(Y^{m-1})) && \text{(by induction hypothesis)}\\ &\cong \sum_{n\in N} W_{k+1}(Y^{m-1}) && \text{(by Lemma 2.15).} \end{aligned}$$

Particularly, $W_m(Y^m) \cong \sum_{n\in N} W_m(Y^{m-1})$. Since Y^{m-1} is m-special, then $W_m(Y^{m-1}) = H_{\mathrm{wf}}(Y^{m-1})$. Therefore,

$$\begin{aligned} W_{m+1}(Y^m) &\cong W(W_m(Y^m)) && \text{(by definition of } W_{m+1}(Y^m))\\ &\cong W(\sum_{n\in N} H_{\mathrm{wf}}(Y^{m-1}))\\ &\cong (\sum_{n\in N} H_{\mathrm{wf}}(Y^{m-1}))^\top && \text{(by Lemma 2.18).} \end{aligned}$$

According to Lemma 2.18, $(\sum_{n\in N} H_{\mathrm{wf}}(Y^{m-1}))^\top$ is a well-filtered space. Thus,

$$\begin{aligned} W_{m+2}(Y^m) &\cong W(W_{m+1}(Y^m)) && \text{(by definition of } W_{m+2}(Y^m))\\ &\cong W((\sum_{n\in N} H_{\mathrm{wf}}(Y^{m-1}))^\top)\\ &\cong (\sum_{n\in N} H_{\mathrm{wf}}(Y^{m-1}))^\top && \text{(by Lemma 2.7)}\\ &\cong W_{m+1}(Y^m). \end{aligned}$$

For any $k < m+1$, we have $W_k(Y^m) \cong \sum_{n\in N} W_k(Y^{m-1})$. Crucially, Example 2.13 informs us that $\sum_{n\in N} W_k(Y^{m-1})$ does not have a greatest element, and so doesn't $W_k(Y^m)$ for each $k < m+1$. Let $\top$ be the greatest element of $H_{\mathrm{wf}}(Y^{m-1})$. By Lemma 2.16, $\{(\top, n) : n \in N\}$ is a KF-set of $\sum_{n\in N} H_{\mathrm{wf}}(Y^{m-1})$. But $\mathrm{cl}(\{(\top, n) : n \in N\}) \neq \downarrow x$ for each $x \in \sum_{n\in N} H_{\mathrm{wf}}(Y^{m-1})$. Thus, $\sum_{n\in N} H_{\mathrm{wf}}(Y^{m-1})$ is not a well-filtered space by Lemma 2.5. Since $W_m(Y^m) \cong \sum_{n\in N} H_{\mathrm{wf}}(Y^{m-1})$, it follows that $W_m(Y^m)$ is not well-filtered. Therefore, Y^m is $(m+1)$-special. □

At the juncture, let us pause and reflect on the role of the wf-rank. While we have advertised earlier that the wf-rank is an index that informs us how far a T_0 space is from being well-filtered, it is not immediately clear what good, if at all, such an index serve. So let us, for goodness sake, put to rest this small pesky question before we move on to business proper by looking at the example below.

Example 3.6. On one hand, consider the set $J = N \times (N \cup \{\infty\})$ with the partial order $\leqslant$ defined by

$$(j,k) \leqslant (m,n) \iff j = m \text{ and } k \leqslant n, \text{ or } n = \infty \text{ and } k \leqslant m.$$

$(J, \leqslant)$ is a famous counterexample first constructed by P. T. Johnstone to demonstrate that there is a dcpo with non-sober Scott topology ([7]). We denote by J the Scott topological space on J. In fact, J is not even well-filtered.

On the other hand, consider the space Y^1. Clearly, Y^1 is not well-filtered too. The question here is to determine whether J and Y^1 are homeomorphic (or not).

Now, it is easy to verify that $\mathrm{rank_{wf}}(J) = 1$, while Lemma 3.5 asserts that $\mathrm{rank_{wf}}(Y^1) = 2$. It thus follows that J and Y^1 are not homeomorphism because homeomorphic spaces must have the same wf-rank.

Moving up the ladder of ordinals, we now show that there are $(\omega + 1)$-special T_0-spaces.

Theorem 3.7. *Let $Z = \sum_{n\in N} X_n$, where $X_n = Y^n$ for each $n \in N$. Then Z is $(\omega + 1)$-special.*

Proof. We shall proceed by stages as one would expect. First we prove that for any $\delta < \omega$, the following homeomorphism of spaces

$$W_\delta(Z) \cong \sum_{n\in N} W_n^\delta$$

holds by the induction on δ, where the collection, $\{W_n^k\}_{k\in N}$, of T_0-spaces is given by:

- $W_n^0 = Y^n, \ 0 \leqslant n < \omega,$
- $W_n^1 = \begin{cases} W_1(Y^n) & \text{if } n \geqslant 1; \\ H_{\mathrm{wf}}(Y^n) & \text{if } n < 1, \end{cases}$
- $W_n^2 = \begin{cases} W_2(Y^n) & \text{if } n \geqslant 2, \\ H_{\mathrm{wf}}(Y^n) & \text{if } n < 2, \end{cases}$ and $\cdots$
- $W_n^\delta = \begin{cases} W_\delta(Y^n) & \text{if } n \geqslant \delta, \\ H_{\mathrm{wf}}(Y^n) & \text{if } n < \delta. \end{cases}$

Basic case. For $\delta = 0$, the statement follows immediately from the definition of Z.

Inductive step. Let δ be such that $\delta + 1 < \omega$. Suppose that $W_\delta(Z) \cong \sum_{n\in N} W_n^\delta$. We must prove that the statement holds for $\delta + 1$.

$$\begin{aligned} W_{\delta+1}(Z) &\cong W(W_\delta(Z)) && \text{(by definition of } W_{\delta+1}(Z)) \\ &\cong W(\textstyle\sum_{n\in N} W_n^\delta) && \text{(by induction hypothesis)} \\ &\cong \textstyle\sum_{n\in N} W(W_n^\delta) && \text{(by Lemma 2.15).} \end{aligned}$$

It is obvious that

$$W(W_n^\delta) = \begin{cases} W(W_\delta(Y^n)) & \text{if } \delta \leqslant n, \\ W(H_{\mathrm{wf}}(Y^n)) & \text{if } n < \delta \end{cases} = \begin{cases} W_{\delta+1}(Y^n) & \text{if } \delta \leqslant n, \\ W(H_{\mathrm{wf}}(Y^n)) & \text{if } n < \delta. \end{cases}$$

According to Lemma 2.7, we have $W(H_{\rm wf}(Y^n)) \cong H_{\rm wf}(Y^n)$, and according to Lemma 3.5, we have $W_{\delta+1}(Y^n) = H_{\rm wf}(Y^n)$ when $\delta = n$. These facts then imply that

$$W(W_n^\delta) \cong \begin{cases} W_{\delta+1}(Y^n) & \text{if } \delta < n, \\ H_{\rm wf}(Y^n) & \text{if } n \leqslant \delta \end{cases} = \begin{cases} W_{\delta+1}(Y^n) & \text{if } \delta+1 \leqslant n, \\ H_{\rm wf}(Y^n) & \text{if } n < \delta+1 \end{cases} = W_n^{\delta+1}.$$

Thus $W_{\delta+1}(Z) \cong \sum_{n\in N} W(W_n^\delta) \cong \sum_{n\in N} W_n^{\delta+1}$. Then,

$$\begin{aligned} W_\omega(Z) &= \bigcup_{\delta<\omega} W_\delta(Z) && \text{(by definition of } W_\omega(Z)) \\ &\cong \bigcup_{\delta<\omega} \sum_{n\in N} W_n^\delta && \text{(by induction hypothesis)} \\ &\cong \sum_{n\in N} \bigcup_{\delta<\omega} W_n^\delta && \text{(by the property of } \textstyle\bigcup) \\ &\cong \sum_{n\in N} H_{\rm wf}(Y^n) && \text{(by definition of } W_n^\delta) \end{aligned}$$

Furthermore,

$$\begin{aligned} W_{\omega+1}(Z) &= W(W_\omega(Z)) && \text{(by definition of } W_{\omega+1}(Z)) \\ &\cong W(\sum_{n\in N} H_{\rm wf}(Y^n)) && \text{(by the above homeomorphism } W_\omega(Z) \cong \textstyle\sum_{n\in N} H_{\rm wf}(Y^n)) \\ &\cong (\sum_{n\in N} H_{\rm wf}(Y^n))^\top && \text{(by Lemma 2.18)} \\ &\cong (W_\omega(Z))^\top && \text{(by the above homeomorphism } W_\omega(Z) \cong \textstyle\sum_{n\in N} H_{\rm wf}(Y^n)). \end{aligned}$$

Thus,

$$\begin{aligned} W_{\omega+2}(Z) &= W(W_{\omega+1}(Z)) && \text{(by definition of } W_{\omega+2}(Z)) \\ &\cong W((W_\omega(Z))^\top) && \\ &\cong (W_\omega(Z))^\top && \text{(by Lemma 2.7)} \\ &\cong W_{\omega+1}(Z). && \end{aligned}$$

For any $\delta < \omega$, we have proven that $W_\delta(Z) \cong \sum_{n\in N} W_n^\delta$ and $W_\omega(Z) \cong \sum_{n\in N} H_{\rm wf}(Y^n)$. According to Example 2.13, it is obvious that $\sum_{n\in N} H_{\rm wf}(Y^n)$ and $\sum_{n\in N} W_n^\delta$ do not have a greatest element for each $\delta < \omega$. Then $W_\delta(Z)$ does not have a greatest element for each $\delta \leqslant \omega$. Let $\top_n$ be the greatest element of $H_{\rm wf}(Y^n)$ for any $n \in N$. According to Lemma 2.16, $\{(\top_n, n) : n \in N\}$ is a KF-set of $\sum_{n\in N} H_{\rm wf}(Y^n)$. But ${\rm cl}(\{(\top_n, n) : n \in N\}) \neq \downarrow x$ for each $x \in \sum_{n\in N} H_{\rm wf}(Y^n)$. Therefore, $\sum_{n\in N} H_{\rm wf}(Y^n)$ is not a well-filtered space by Lemma 2.5. Since $\sum_{n\in N} H_{\rm wf}(Y^n) \cong W_\omega(Z)$, we conclude that $W_\omega(Z)$ is not well-filtered. Therefore, Z is $(\omega+1)$-special. □

Theorem 3.8. *Let $Z = \sum_{n\in N} Y^n$ as in Theorem 3.7. Then* ${\rm rank}_{\rm wf}(Z^\top) = \omega$.

Proof. We shall prove by induction on δ that for any $\delta \leqslant \omega$,

$$W_\delta(Z^\top) \cong \sum_{n\in N^\top} W_n^\delta,$$

where for any $n \in N^\top$,

- $W_n^0 = \begin{cases} Y^n & \text{if } 0 \leqslant n < \omega; \\ T & \text{if } n = \top, \end{cases}$

- $W_n^1 = \begin{cases} W_1(Y^n) & \text{if } 1 \leqslant n < \omega; \\ H_{\rm wf}(Y^n) & \text{if } n < 1 \leqslant \omega;, \cdots, \text{and} \\ T & \text{if } n = \top, \end{cases}$

- $W_n^\delta = \begin{cases} W_\delta(Y^n) & \text{if } \delta \leqslant n < \omega, \\ H_{\mathrm{wf}}(Y^n) & \text{if } n < \delta \leqslant \omega, \\ T & \text{if } n = \top, \end{cases}$

where $T = (\{\top\}, \{\emptyset, \{\top\}\})$.

Basic step. For $\delta = 0$, $Z^\top \cong \sum_{n\in N^\top} Y^n$ (by Lemma 2.19).

Inductive step. Suppose now that $W_\delta(Z^\top) \cong \sum_{n\in N^\top} W_n^\delta$ holds for $\delta < \omega$. We must prove that the statement holds for $\delta + 1$.

$$\begin{aligned} W_{\delta+1}(Z^\top) &= W(W_\delta(Z^\top)) && \text{(by definition of } W_{\delta+1}(Z^\top)) \\ &\cong W(\sum_{n\in N^\top} W_n^\delta) && \text{(by induction hypothesis)} \\ &\cong W((\sum_{n\in N} (W_n^\delta))^\top) && \text{(by Lemma 2.19)} \\ &\cong (W(\sum_{n\in N} (W_n^\delta)))^\top && \text{(by Lemma 2.10)} \\ &\cong (\sum_{n\in N} W(W_n^\delta))^\top && \text{(by Lemma 2.15)} \\ &\cong \sum_{n\in N^\top} W(W_n^\delta) && \text{(by Lemma 2.10)} \\ &\cong \sum_{n\in N^\top} W_n^{\delta+1}. \end{aligned}$$

Thus,

$$W_\omega(Z^\top) = \bigcup\nolimits_{\delta<\omega} W_\delta(Z^\top) \cong \bigcup\nolimits_{\delta<\omega} \sum_{n\in N^\top} W_n^\delta \cong \sum_{n\in N^\top} \bigcup\nolimits_{\delta<\omega} W_n^\delta \cong \sum_{n\in N^\top} W_n^\omega. \tag{1}$$

The definition of W_n^δ bears upon us that $W_n^\omega = H_{\mathrm{wf}}(Y^n)$ for $n < \omega$, and $W_\top^\omega = T$. Finally,

$$\begin{aligned} W_{\omega+1}(Z^\top) &\cong W(W_\omega(Z^\top)) && \text{(by definition)} \\ &\cong W(\sum_{n\in N^\top} W_n^\omega) && \text{(by (1))} \\ &\cong W((\sum_{n\in N} W_n^\omega)^\top) && \text{(by Lemma 2.19)} \\ &\cong (W(\sum_{n\in N} W_n^\omega))^\top && \text{(by Lemma 2.10)} \\ &\cong (\sum_{n\in N} W(W_n^\omega))^\top && \text{(by Lemma 2.15)} \\ &\cong (\sum_{n\in N} W_n^\omega)^\top && \text{(by Lemma 2.7)} \\ &\cong \sum_{n\in N^\top} W_n^\omega && \text{(by Lemma 2.19)} \\ &\cong W_\omega(Z^\top). \end{aligned}$$

From the preceding discussion, we gather that for any $\delta < \omega$ it holds that $W_\delta(Z^\top) \cong \sum_{n\in N^\top} W_n^\delta \cong (\sum_{n\in N} W_n^\delta)^\top$. By the definition of W_n^δ, there are finitely many W_n^δ's which are well-filtered spaces. More precisely, these spaces are W_0^δ, $W_1^\delta, \ldots, W_{\delta-1}^\delta$ and T.

Since W_n^δ is not well-filtered for any $\delta \leqslant n$, by Lemma 2.5 there exists some KF-set A such that for any $x \in W_n^\delta$, $\mathrm{cl}_{W_n^\delta} A \neq \downarrow_{W_n^\delta} x$. Invoking Lemma 2.14(4), $A \times \{n\}$ is a KF-set of $\sum_{n\in N} W_n^\delta$, and so $A \times \{n\}$ is a KF-set of $(\sum_{n\in N} W_n^\delta)^\top$. Since $\mathrm{cl}(A \times \{n\}) \subseteq W_n^\delta \times \{n\}$, it follows that $\mathrm{cl}(A) \neq \downarrow(x, n)$ for any $(x, n) \in W_n^\delta \times \{n\}$. According to Example 2.13, we have $\mathrm{cl}(A) \neq \downarrow(y, m)$ for any $(y, m) \in \sum_{n\in N} W_n^\delta$. Thus, we have $\mathrm{cl}(A) \neq \downarrow(y, m)$ for any $(y, m) \in (\sum_{n\in N} W_n^\delta)^\top$. Consequently, $(\sum_{n\in N} W_n^\delta)^\top$ is not a well-filtered space by Lemma 2.5. Hence $W_\delta(Z^\top)$ is not a well-filtered space for each $\delta < \omega$. Therefore, $\mathrm{rank}_{\mathrm{wf}}(Z^\top) = \omega$. □

We now climb up to the next rung of ordinals $\alpha = \omega + n$, where $n = 1, 2, 3, \cdots$.

Lemma 3.9. *For each $n \in N$, there exists an $(\omega + n)$-special T_0-space.*

Proof. Again we proceed by induction on n.

Basic step. For $n = 0$, the statement follows from Theorem 3.8.

Inductive step. Let m be such that $m < n$, and suppose that the space X is $(\omega + m)$-special. We shall prove that the statement holds for n.

To this end, let $Y = (Y, \tau)$ be $(\omega + n - 1)$-special and Z denotes the fibred sum $\sum_{n\in N} Y$. We now prove that $\mathrm{rank}_{\mathrm{wf}}(Z) = \omega + n$.

First, we show that the space $W_k(Z) \cong \sum_{n\in N} W_k(Y)$ holds for any $k \leqslant \omega + n - 1$. Here, we prove by induction on k.

(1) Basic step. For $k = 0$, the statement follows from the definition of the space Z.

(2) Inductive step. Let k be such that $k + 1 \leqslant \omega + n - 1$, and suppose that $W_k(Z) \cong \sum_{n\in N} W_k(Y)$. We shall prove that the statement holds for $k + 1$.

$$\begin{array}{rcll} W_{k+1}(Z) & \cong & W(W_k(Z)) & \text{(by definition of } W_{k+1}(Y^m)) \\ & \cong & W(\sum_{n\in N} W_k(Y)) & \text{(by induction hypothesis)} \\ & \cong & \sum_{n\in N} W_{k+1}(Y) & \text{(by Lemma 2.15)} \end{array}$$

In particular, $W_{\omega+n-1}(Z) \cong \sum_{n\in N} W_{\omega+n-1}(Y)$. Since Y is $(\omega + n - 1)$-special, it follows that $W_{\omega+n-1}(Y) = H_{\mathrm{wf}}(Y)$. Therefore,

$$\begin{array}{rcll} W_{\omega+n}(Z) & = & W(W_{\omega+n-1}(Z)) & \text{(by definition of } W_{\omega+n-1}(Z)) \\ & \cong & W(\sum_{n\in N} H_{\mathrm{wf}}(Y)) & \\ & \cong & (\sum_{n\in N} H_{\mathrm{wf}}(Y))^{\top} & \text{(by Lemma 2.18)} \end{array}$$

According Lemma 2.18, $(\sum_{n\in N} H_{\mathrm{wf}}(Y))^{\top}$ is a well-filtered space. Thus, $W((\sum_{n\in N} H_{\mathrm{wf}}(Y))^{\top}) \cong (\sum_{n\in N} H_{\mathrm{wf}}(Y))^{\top}$ by Lemma 2.7. Hence

$$W_{\omega+n+1}(Z) \cong W(W_{\omega+n}(Z)) \cong W((\sum_{n\in N} H_{\mathrm{wf}}(Y))^{\top}) \cong (\sum_{n\in N} H_{\mathrm{wf}}(Y))^{\top} \cong W_{\omega+n}(Z).$$

For any $\delta < \omega + n - 1$, we have $W_\delta(Z) \cong \sum_{n\in N} W_\delta(Y)$ and $W_{\omega+n-1}(Z) \cong \sum_{n\in N} H_{\mathrm{wf}}(Y)$. From Example 2.13, we know that $W_\delta(Z)$ does not have a greatest element for any $\delta < \omega + n$. Let $\top$ be the greatest element of $H_{\mathrm{wf}}(Y)$. According to Lemma 2.16, $\{(\top, n) : n \in N\}$ is a KF-set of $\sum_{n\in N} H_{\mathrm{wf}}(Y)$. But $\mathrm{cl}(\{(\top, n) : n \in N\}) \neq \downarrow x$ for any $x \in \sum_{n\in N} H_{\mathrm{wf}}(Y)$. Moreover, $\sum_{n\in N} H_{\mathrm{wf}}(Y)$ is not a well-filtered space by Lemma 2.5. Since $W_{\omega+n-1}(Z) \cong \sum_{n\in N} H_{\mathrm{wf}}(Y)$, we have that $W_{\omega+n-1}(Z)$ is not well-filtered. Therefore, the space Z is $(\omega + n)$-special as desired. □

The ascension along the hierarchy of ordinals can, of course, be continued, and is in fact somewhat repetitive in the manner of argument used in the proof of the following result:

Theorem 3.10. *If for each $n \in N$ the T_0-space X_n is $(\omega+n)$-special, then the fibred sum $Z = \sum_{n\in N} X_n$ is $(2\omega+1)$-special and* $\mathrm{rank}_{\mathrm{wf}}(Z^{\top}) = 2\omega$.

Proof. By induction on δ, we prove that for any $\delta < 2\omega$, the homeomorphism of spaces hold:

$$W_\delta(Z) \cong \sum_{n\in N} W_n^\delta,$$

where

- $W_0^\delta = \begin{cases} W_\delta(X_0) & \text{if } \delta < \omega < 2\omega; \\ H_{\mathrm{wf}}(X_0) & \text{if } \omega \leqslant \delta < 2\omega, \end{cases}$

- $W_1^\delta = \begin{cases} W_\delta(X_1) & \text{if } \delta < \omega + 1 < 2\omega; \\ H_{\mathrm{wf}}(X_1) & \text{if } \omega + 1 \leqslant \delta < 2\omega, \end{cases}, \cdots$, and

- for any $n \in N$, $W_n^\delta = \begin{cases} W_\delta(X_n) & \text{if } \delta < \omega + n < 2\omega; \\ H_{\mathrm{wf}}(X_n) & \text{if } \omega + n \leqslant \delta < 2\omega. \end{cases}$

Basic step. For $\delta = 0$, the statement follows from the definition of Z.
Inductive step: There are two case to consider:

Case 1. Let δ be a successor ordinal such that $\delta + 1 < 2\omega$ and suppose that $W_\delta(Z) \cong \sum_{n\in N} W_n^\delta$, then according to Lemma 2.15,

$$W_{\delta+1}(Z) \cong W(W_\delta(Z)) \cong W(\sum_{n\in N} W_n^\delta) \cong \sum_{n\in N} W(W_n^\delta) \cong \sum_{n\in N} W_n^{\delta+1}.$$

Case 2. Let $\delta < 2\omega$ be a limit ordinal and and suppose that $W_\delta(Z) \cong \sum_{n\in N} W_n^\delta$ holds for any ordinal $\delta' < \delta$, then we have

$$W_\delta(Z) = \bigcup_{\delta'<\delta} W_{\delta'}(Z) \cong \bigcup_{\delta'<\delta} \sum_{n\in N} W_n^{\delta'} \cong \sum_{n\in N} \bigcup_{\delta'<\delta} W_n^{\delta'} \cong \sum_{n\in N} W_n^\delta.$$

Then $W_\delta(Z) \cong \sum_{n\in N} W_n^\delta$ holds for any $\delta < 2\omega$. Thus,

$$W_{2\omega}(Z) = \bigcup_{\delta<2\omega} W_\delta(Z) \cong \bigcup_{\delta<2\omega} \sum_{n\in N} W_n^\delta \cong \sum_{n\in N} \bigcup_{\delta<2\omega} W_n^\delta \cong \sum_{n\in N} H_{\mathrm{wf}}(X_n).$$

According to Lemma 2.18,

$$W_{2\omega+1}(Z) \cong W(W_{2\omega}(Z)) \cong W(\sum_{n\in N} H_{\mathrm{wf}}(X_n)) \cong (\sum_{n\in N} H_{\mathrm{wf}}(X_n))^\top \cong (W_{2\omega}(Z))^\top,$$

and

$$W_{2\omega+2}(Z) \cong W(W_{2\omega+1}(Z)) \cong W((W_{2\omega}(Z))^\top) \cong (W_{2\omega}(Z))^\top \cong W_{2\omega+1}(Z).$$

For any $\delta < 2\omega$, we have $W_\delta(Z) \cong \sum_{n\in N} W_n^\delta$ and $W_{2\omega}(Z) \cong \sum_{n\in N} H_{\mathrm{wf}}(X_n)$. In Example 2.13, it is clear that $W_\delta(Z)$ does not have a greatest element for each $\delta < 2\omega + 1$. Let $\top_n$ be the greatest element of $H_{\mathrm{wf}}(X_n)$ for any $n \in N$. According to Lemma 2.16, $\{(\top_n, n) : n \in N\}$ is a KF-set of $\sum_{n\in N} H_{\mathrm{wf}}(X_n)$. But $\mathrm{cl}(\{(\top_n, n) : n \in N\}) \neq \downarrow x$ for every $x \in \sum_{n\in N} H_{\mathrm{wf}}(X_n)$. Therefore, $\sum_{n\in N} H_{\mathrm{wf}}(X_n)$ is not a well-filtered space by Lemma 2.5. Since $\sum_{n\in N} H_{\mathrm{wf}}(X_n) \cong W_{2\omega}(Z)$, we have $W_{2\omega}(Z)$ is not well-filtered. Therefore, Z is $(2\omega+1)$-special.

Next, we prove that for any $\delta \leqslant 2\omega$, $W_\delta(Z^\top)$ is homeomorphic to $\sum_{n\in N^\top} W_n^\delta$, where

- $W_0^\delta = \begin{cases} W_\delta(X_0) & \text{if } \delta < \omega < 2\omega, \\ H_{\mathrm{wf}}(X_0) & \text{if } \omega \leqslant \delta \leqslant 2\omega; \\ T & \text{if } n = \top, \end{cases}$

- $W_1^\delta = \begin{cases} W_\delta(X_1) & \text{if } \delta < \omega + 1 < 2\omega; \\ H_{\rm wf}(X_1) & \text{if } \omega + 1 \leqslant \delta \leqslant 2\omega; \\ T & \text{if } n = \top, \end{cases}$, $\cdots$, and

- $W_n^\delta = \begin{cases} W_\delta(X_n) & \text{if } \delta < \omega + n < 2\omega; \\ H_{\rm wf}(X_n) & \text{if } \omega + n \leqslant \delta \leqslant 2\omega; \\ T & \text{if } n = \top. \end{cases}$

The proof is by induction on δ.

Basic step. If $\delta = 0$, then $Z^\top \cong \sum_{n\in N^\top} X_n$.

Inductive step. Suppose now that $W_\delta(Z^\top) \cong \sum_{n\in N^\top} W_n^\delta$ holds for $\delta < 2\omega$.

There are two cases to consider:

Case 1. If δ is a successor ordinal such that $\delta + 1 < 2\omega$, then

$$\begin{aligned}
W_{\delta+1}(Z^\top) &\cong W(W_\delta(Z^\top)) && \text{(by definition of } W_{\delta+1}(Z^\top)) \\
&\cong W(\sum_{n\in N^\top} W_n^\delta) && \text{(by induction hypothesis)} \\
&\cong W((\sum_{n\in N} (W_n^\delta))^\top) && \text{(by Lemma 2.19)} \\
&\cong W(\sum_{n\in N} (W_n^\delta))^\top && \text{(by Lemma 2.10)} \\
&\cong (\sum_{n\in N} W(W_n^\delta))^\top && \text{(by Lemma 2.15)} \\
&\cong \sum_{n\in N^\top} W(W_n^\delta) && \text{(by Lemma 2.10)}
\end{aligned}$$

It is obvious that $W(W_n^\delta) = \begin{cases} W(W_\delta(Y^n)) & \text{if } \delta \leqslant n; \\ W(H_{\rm wf}(Y^n)) & \text{if } n < \delta \end{cases} = \begin{cases} W_{\delta+1}(Y^n) & \text{if } \delta \leqslant n; \\ W(H_{\rm wf}(Y^n)) & \text{if } n < \delta. \end{cases}$

since $W(W_n^\delta) = \begin{cases} W(W_\delta(X_n)) & \text{if } \delta < \omega + n < 2\omega; \\ W(H_{\rm wf}(X_n)) & \text{if } \omega + n \leqslant \delta \leqslant 2\omega; \\ W(T) & \text{if } n = \top. \end{cases} = \begin{cases} W_{\delta+1}(X_n) & \text{if } \delta < \omega + n < 2\omega; \\ W(H_{\rm wf}(X_n)) & \text{if } \omega + n \leqslant \delta \leqslant 2\omega; \\ W(T) & \text{if } n = \top. \end{cases}$

According to Lemma 2.7, we have $W(H_{\rm wf}(X_n)) \cong H_{\rm wf}(X_n)$ and $W(T) = T$. Then

$$W(W_n^\delta) = \begin{cases} W_{\delta+1}(X_n) & \text{if } \delta < \omega + n < 2\omega; \\ H_{\rm wf}(X_n) & \text{if } \omega + n \leqslant \delta \leqslant 2\omega; \\ T & \text{if } n = \top. \end{cases} = W_n^{\delta+1}.$$

Therefore, $W_{\delta+1}(Z^\top) \cong \sum_{n\in N^\top} W_n^{\delta+1}$.

Case 2. If $\delta < 2\omega$ is a limit ordinal and that the statement holds for any ordinal $\delta' < \delta$, we have

$$W_\delta(Z^\top) = \bigcup_{\delta'<\delta} W_{\delta'}(Z^\top) \cong \bigcup_{\delta'<\delta} \sum_{n\in N^\top} W_n^{\delta'} \cong \sum_{n\in N^\top} \bigcup_{\delta'<\delta} W_n^{\delta'} \cong \sum_{n\in N^\top} W_n^\delta.$$

Thus,

$$W_{2\omega}(Z^\top) = \bigcup_{\delta<2\omega} W_\delta(Z^\top) \cong \bigcup_{\delta<2\omega} \sum_{n\in N^\top} W_n^\delta \cong \sum_{n\in N^\top} \bigcup\nolimits_{\delta<2\omega} W_n^\delta \cong \sum_{n\in N^\top} W_n^{2\omega}. \tag{2}$$

By the definition of W_n^δ, we know that $W_n^{2\omega} = H_{\mathrm{wf}}(X^n)$ for $n < \omega$, and $W_\top^{2\omega} = T$. Finally,

$$\begin{array}{rcll} W_{2\omega+1}(Z^\top) & \cong & W(W_{2\omega}(Z^\top)) & \text{(by definition of } W_{2\omega+1}(Z^\top)) \\ & \cong & W(\sum\limits_{n\in N^\top} W_n^{2\omega}) & \text{(by (2))} \\ & \cong & W((\sum\limits_{n\in N} W_n^{2\omega})^\top) & \text{(by Lemma 2.19)} \\ & \cong & (W(\sum\limits_{n\in N} W_n^{2\omega}))^\top & \text{(by Lemma 2.10)} \\ & \cong & (\sum\limits_{n\in N} W(W_n^{2\omega}))^\top & \text{(by Lemma 2.15)} \\ & \cong & (\sum\limits_{n\in N} W_n^{2\omega})^\top & \text{(by Lemma 2.7)} \\ & \cong & \sum\limits_{n\in N^\top} W_n^{2\omega} & \text{(by Lemma 2.19)} \\ & \cong & W_{2\omega}(Z^\top). & \end{array}$$

According to the above discussion, for any $\delta < 2\omega$, we have $W_\delta(Z^\top) \cong \sum_{n\in N^\top} W_n^\delta \cong (\sum_{n\in N} W_n^\delta)^\top$. By the definition of W_n^δ, we conclude that there are at most finitely many W_n^δ's which are well-filtered spaces. Since W_n^δ is not well-filtered for any $\delta < \omega + n$, by Lemma 2.5 there exists some KF-set A such that for any $x \in W_n^\delta$, $\mathrm{cl}_{W_n^\delta} A \neq {\downarrow}_{W_n^\delta} x$. According to Lemma 2.14 (4), we have that $A\times\{n\}$ is a KF-set of $\sum_{n\in N} W_n^\delta$, then $A\times\{n\}$ is a KF-set of $(\sum_{n\in N} W_n^\delta)^\top$. Since $\mathrm{cl}(A\times\{n\}) \subseteq W_n^\delta \times \{n\}$, then $\mathrm{cl}(A) \neq {\downarrow}(x, n)$ for any $(x, n) \in W_n^\delta \times \{n\}$. Example 2.13 informs us that $\mathrm{cl}(A) \neq {\downarrow}(y, m)$ for any $(y, m) \in \sum_{n\in N} W_n^\delta$. Then we have $\mathrm{cl}(A) \neq {\downarrow}(y, m)$ for all $(y, m) \in (\sum_{n\in N} W_n^\delta)^\top$. Thus, $(\sum_{n\in N} W_n^\delta)^\top$ is not a well-filtered space by Lemma 2.5. Then $W_\delta(Z^\top)$ is not a well-filtered space for each $\delta < 2\omega$. Therefore, the $\mathrm{rank}_{\mathrm{wf}}(Z^\top) = 2\omega$. □

Lemma 3.11. *Let $\alpha > 0$ be an ordinal.*

(i) If γ is not a limit ordinal, and a T_0-space X_n is γ-special for all $n \in N$, then the fibred sum $Z = \sum_{n\in N} X_n$ is $(\gamma + 1)$-special.

(ii) If α is limit ordinal, and a T_0-space X_n is $(\bar{\alpha} + n)$-special for all $n \in N$, then the fibred sum $Z = \sum_{n\in N} X_n$ is $(\alpha + 1)$-special and $\mathrm{rank}_{\mathrm{wf}}(Z^\top) = \alpha$, where $\bar{\alpha} = 0$ if $\alpha = \omega$, otherwise, $\bar{\alpha}$ denotes the largest limit ordinal less than α.

Proof. (i) First we show that $W_\delta(Z) \cong \sum_{n\in N} W_\delta(X_n)$ for any ordinal $\delta \leqslant \gamma$. We use induction on δ.

(1) Basic step. For $\delta = 0$, the statement follows from the definition of space Z.

(2) Hypothesis. Let δ be such that $\delta + 1 \leqslant \gamma$, and suppose that $W_\delta(Z) \cong \sum_{n\in N} W_\delta(X_n)$. There are two cases to consider:

Case1. If δ is a successor ordinal, then,

$$\begin{array}{rcll} W_{\delta+1}(Z) & \cong & W(W_\delta(Z)) & \text{(by definition of } W_{\delta+1}(Z)) \\ & \cong & W(\sum\limits_{n\in N} W_\delta(X_n)) & \text{(by induction hypothesis)} \\ & \cong & \sum\limits_{n\in N} W(W_\delta(X_n)) & \text{(by Lemma 2.15)} \\ & \cong & \sum\limits_{n\in N} W_{\delta+1}(X_n) & \end{array}$$

Case 2. If $\delta \leqslant \gamma$ is a limit ordinal and suppose that $W_\delta(Z) \cong \sum_{n\in N} W_\delta(X_n)$ holds for any ordinal $\delta' < \delta$. Then,

$$\begin{array}{rcll} W_\delta(Z) & = & \bigcup_{\delta'<\delta} W_{\delta'}(Z) & \text{(by definition of } W_\delta(Z)) \\ & \cong & \bigcup_{\delta'<\delta} \sum_{n\in N} W_{\delta'}(X_n) & \text{(by induction hypothesis)} \\ & \cong & \sum_{n\in N} \bigcup_{\delta'<\delta} W_{\delta'}(X_n) & \text{(by the property of } \bigcup) \\ & \cong & \sum_{n\in N} W_\delta(X_n) & \text{(by definition of } W_\delta(X_n)) \end{array}$$

Thus we have proved that $W_\gamma(Z) \cong \sum_{n\in N} W_\gamma(X_n) \cong \sum_{n\in N} H_{\mathrm{wf}}(X_n)$. Finally,

$$\begin{array}{rcll} W_{\gamma+1}(Z) & \cong & W(W_\gamma(Z)) & \text{(by definition of } W_{\gamma+1}(Z)) \\ & \cong & W(\sum_{n\in N} H_{\mathrm{wf}}(X_n)) & \text{(by above formular)} \\ & \cong & (\sum_{n\in N} H_{\mathrm{wf}}(X_n))^\top & \text{(by Lemma 2.18)} \end{array}$$

Thus,

$$\begin{array}{rcll} W_{\gamma+2}(Z) & \cong & W(W_{\gamma+1}(Z)) & \text{(by definition of } W_{\gamma+2}(Z)) \\ & \cong & W((\sum_{n\in N} H_{\mathrm{wf}}(X_n))^\top) & \\ & \cong & (\sum_{n\in N} H_{\mathrm{wf}}(X_n))^\top & \text{(by Lemma 2.7)} \\ & \cong & W_{\gamma+1}(Z). & \end{array}$$

For any $\delta < \gamma$, we have $W_\delta(Z) \cong \sum_{n\in N} W_\delta(X_n)$ and $W_\gamma(Z) \cong \sum_{n\in N} H_{\mathrm{wf}}(X_n)$. According to Example 2.13, it is obvious that $W_\delta(Z)$ does not have a greatest element for any $\delta < \gamma + 1$. Let $\top_n$ be the greatest element of $H_{\mathrm{wf}}(X_n)$ for any $n \in N$. According to Lemma 2.16, $\{(\top_n, n) : n \in N\}$ is a KF-set of $\sum_{n\in N} H_{\mathrm{wf}}(X_n)$. But $\mathrm{cl}(\{(\top_n, n) : n \in N\}) \neq \downarrow x$ for any $x \in \sum_{n\in N} H_{\mathrm{wf}}(X_n)$. Therefore, $\sum_{n\in N} H_{\mathrm{wf}}(X_n)$ is not a well-filtered space by Lemma 2.5. Since $\sum_{n\in N} H_{\mathrm{wf}}(X_n) \cong W_\gamma(Z)$, we have $W_\gamma(Z)$ is not well-filtered. Therefore, the space Z is $(\gamma + 1)$-special.

(ii) Using the induction on δ, we prove that for any $\delta < \alpha$, the spaces $W_\delta(Z) \cong \sum_{n\in N} W_n^\delta$, where

$$W_n^\delta = \begin{cases} W_\delta(X_n), & \text{if } \delta < \bar{\alpha} + n < \alpha, \\ H_{\mathrm{wf}}(X_n), & \text{if } \bar{\alpha} + n \leqslant \delta < \alpha. \end{cases}$$

For $\delta = 0$, the statement follows from the definition of Z.

Let δ be an ordinal such that $\delta + 1 < \alpha$, and suppose that $W_\delta(Z) \cong \sum_{n\in N} W_n^\delta$ holds for any $\delta + 1 < \alpha$. There are two cases to consider:

Case 1. If δ is an successor ordinal such that $W_\delta(Z) \cong \sum_{n\in N} W_n^\delta$ holds for any $\delta + 1 < \alpha$, then

$$\begin{aligned} W_{\delta+1}(Z) &= W(W_\delta(Z)) && \text{(by definition of } W_{\delta+1}(Z)) \\ &\cong W(\sum_{n\in N} W_n^\delta) && \text{(by induction hypothesis)} \\ &\cong \sum_{n\in N} W(W_n^\delta) && \text{(by Lemma 2.15)} \\ &\cong \sum_{n\in N} W_n^{\delta+1}. \end{aligned}$$

Case 2. If $\delta < \alpha$ is a limit ordinal such that $W_\delta(Z) \cong \sum_{n\in N} W_n^\delta$ holds for any ordinal $\delta' < \delta$, then we have

$$\begin{aligned} W_\delta(Z) &= \bigcup_{\delta'<\delta} W_{\delta'}(Z) && \text{(by definition of } W_\delta(Z)) \\ &\cong \bigcup_{\delta'<\delta} \sum_{n\in N} W_n^{\delta'} && \text{(by induction hypothesis)} \\ &\cong \sum_{n\in N} \bigcup_{\delta'<\delta} W_n^{\delta'} && \text{(by the property of } \textstyle\bigcup) \\ &\cong \sum_{n\in N} W_n^\delta && \text{(by definition of } W_n^\delta) \end{aligned}$$

Thus $W_\alpha(Z) = \bigcup_{\delta<\alpha} W_\delta(Z) \cong \bigcup_{\delta<\alpha} \sum_{n\in N} W_n^\delta \cong \sum_{n\in N} \bigcup_{\delta<\alpha} W_n^\delta \cong \sum_{n\in N} H_{\mathrm{wf}}(X_n)$. Furthermore,

$$\begin{aligned} W_{\alpha+1}(Z) &= W(W_\alpha(Z)) && \text{(by definition of } W_{\alpha+1}(Z)) \\ &\cong W(\sum_{n\in N} H_{\mathrm{wf}}(Y^n)) \\ &\cong (\sum_{n\in N} H_{\mathrm{wf}}(Y^n))^\top && \text{(by Lemma 2.18)} \\ &\cong (W_\alpha(Z))^\top. \end{aligned}$$

Thus,

$$\begin{aligned} W_{\alpha+2}(Z) &= W(W_{\alpha+1}(Z)) && \text{(by definition of } W_{\alpha+2}(Z)) \\ &\cong W((W_\alpha(Z))^\top) \\ &\cong (W_\alpha(Z))^\top && \text{(by Lemma 2.7)} \\ &\cong W_{\alpha+1}(Z). \end{aligned}$$

For any $\delta < \alpha$, we have $W_\delta(Z) \cong \sum_{n\in N} W_n^\delta$ and $W_\alpha(Z) \cong \sum_{n\in N} H_{\mathrm{wf}}(X_n)$. According to Example 2.13, it is obvious that $W_\delta(Z)$ does not have a greatest element for any $\delta < \alpha + 1$. Let $\top_n$ be the greatest element of $H_{\mathrm{wf}}(X_n)$ for any $n \in N$. According to Lemma 2.16, $\{(\top_n, n) : n \in N\}$ is a KF-set of $\sum_{n\in N} H_{\mathrm{wf}}(X_n)$. But $\mathrm{cl}(\{(\top_n, n) : n \in N\}) \neq \downarrow x$ for any $x \in \sum_{n\in N} H_{\mathrm{wf}}(X_n)$. Therefore, $\sum_{n\in N} H_{\mathrm{wf}}(X_n)$ is not a well-filtered space by Lemma 2.5. Since $\sum_{n\in N} H_{\mathrm{wf}}(X_n) \cong W_\alpha(Z)$, we have $W_\alpha(Z)$ is not well-filtered. Therefore, Z is $(\alpha + 1)$-special.

Next, we prove that for any $\delta \leqslant \alpha$, the space $W_\delta(Z^\top)$ is homeomorphic to a space $\sum_{n\in N^\top} W_n^\delta$, where

$$W_n^\delta = \begin{cases} W_\delta(X_n) & \text{if } \delta < \bar{\alpha} + n < \alpha; \\ H_{\mathrm{wf}}(X_n), & \text{if } \bar{\alpha} + n \leqslant \delta \leqslant \alpha; . \\ T & \text{if } n = \top. \end{cases}$$

As usual, the proof is given by induction on δ. Base case. If $\delta = 0$, then $Z^\top \cong \sum_{n\in N^\top} X_n$.

Inductive step. Suppose that $W_\delta(Z^\top) \cong \sum_{n\in N^\top} W_n^\delta$ holds for any $\delta + 1 < \alpha$. There are two cases to consider:

Case 1. If δ is a successor ordinal such that $W_\delta(Z^\top) \cong \sum_{n\in N^\top} W_n^\delta$ holds for any $\delta + 1 < \alpha$, then

$$\begin{aligned}
W_{\delta+1}(Z^\top) &\cong W(W_\delta(Z^\top)) && \text{(by definition of } W_{\delta+1}(Z^\top))\\
&\cong W(\sum_{n\in N^\top} W_n^\delta) && \text{(by induction hypothesis)}\\
&\cong W((\sum_{n\in N} (W_n^\delta))^\top) && \text{(by Lemma 2.19)}\\
&\cong W(\sum_{n\in N} (W_n^\delta))^\top && \text{(by Lemma 2.10)}\\
&\cong (\sum_{n\in N} W(W_n^\delta))^\top && \text{(by Lemma 2.15)}\\
&\cong \sum_{n\in N^\top} W(W_n^\delta) && \text{(by Lemma 2.10)}\\
&\cong \sum_{n\in N^\top} W_n^{\delta+1}.
\end{aligned}$$

Case 2. If δ is a limit ordinal such that $W_\delta(Z^\top) \cong \sum_{n\in N^\top} W_n^\delta$ holds for any ordinal $\delta' < \delta$, we have

$$W_\delta(Z^\top) = \bigcup_{\delta'<\delta} W_{\delta'}(Z^\top) \cong \bigcup_{\delta'<\delta} \sum_{n\in N^\top} W_n^{\delta'} \cong \sum_{n\in N^\top} \bigcup_{\delta'<\delta} W_n^{\delta'} \cong \sum_{n\in N^\top} W_n^\delta.$$

Thus, $W_\alpha(Z^\top) = \bigcup_{\delta<\alpha} W_\delta(Z^\top) \cong \bigcup_{\delta<\alpha} \sum_{n\in N^\top} W_n^\delta \cong \sum_{n\in N^\top} \bigcup_{\delta<\alpha} W_n^\delta \cong \sum_{n\in N^\top} W_n^\alpha$. From the definition of W_n^δ, we know that $W_n^\alpha = H_{\mathrm{wf}}(X^n)$ for $n < \omega$, and $W_\top^\alpha = T$. Finally,

$$\begin{aligned}
W_{\alpha+1}(Z^\top) &\cong W(W_\alpha(Z^\top)) && \text{(by definition of } W_{\alpha+1}(Z^\top))\\
&\cong W(\sum_{n\in N^\top} W_n^\alpha) && \text{(by induction hypothesis)}\\
&\cong W((\sum_{n\in N} W_n^\alpha)^\top) && \text{(by Lemma 2.19)}\\
&\cong (W(\sum_{n\in N} W_n^\alpha))^\top && \text{(by Lemma 2.10)}\\
&\cong (\sum_{n\in N} W(W_n^\alpha))^\top && \text{(by Lemma 2.15)}\\
&\cong (\sum_{n\in N} W_n^\alpha)^\top && \text{(by Lemma 2.7)}\\
&\cong \sum_{n\in N^\top} W_n^\alpha && \text{(by Lemma 2.19)}\\
&\cong W_\alpha(Z^\top),
\end{aligned}$$

From the preceding discussion, for any $\delta < \alpha$, we have $W_\delta(Z^\top) \cong \sum_{n\in N^\top} W_n^\delta \cong (\sum_{n\in N} W_n^\delta)^\top$. The definition of W_n^δ bears upon us that there are at most finitely many of the W_n^δ's are well-filtered spaces.

Since W_n^δ is not well-filtered for any $\delta < \bar{\alpha} + n$, by Lemma 2.5, there exists some KF-set A such that for any $x \in W_n^\delta$, $\mathrm{cl}_{W_n^\delta} A \neq \downarrow_{W_n^\delta} x$. By Lemma 2.14(4), we have that $A \times \{n\}$ is a KF-set of $\sum_{n\in N} W_n^\delta$. Then it follows that $A \times \{n\}$ is a KF-set of $(\sum_{n\in N} W_n^\delta)^\top$. Since $\mathrm{cl}(A \times \{n\}) \subseteq W_n^\delta \times \{n\}$, we deduce that $\mathrm{cl}(A) \neq \downarrow(x, n)$ for all $(x, n) \in W_n^\delta \times \{n\}$. According to Example 2.13, we have that $\mathrm{cl}(A) \neq \downarrow (y, m)$ for all $(y, m) \in \sum_{n\in N} W_n^\delta$. Thus, $\mathrm{cl}(A) \neq \downarrow (y, m)$ for any $(y, m) \in (\sum_{n\in N} W_n^\delta)^\top$. Hence $(\sum_{n\in N} W_n^\delta)^\top$ is not a well-filtered space by Lemma 2.5. Therefore, $W_\delta(Z^\top)$ is not well-filtered for any $\delta < \alpha$. Hence $\mathrm{rank}_{\mathrm{wf}}(Z^\top) = \alpha$, as desired. □

With all the preparatory work completed, we finally reap the fruits of our labour.

Theorem 3.12. *For any non-limit ordinal α, there exists an α-special T_0-space.*

Proof. The proof is by induction on α. For $\alpha = 0$, the required statement follows from Lemma 3.4. Suppose that $\alpha = \gamma + 1$ and that the statement of the theorem is valid for any non-limit ordinal $\beta \leqslant \gamma$. There are two cases to consider:

Case 1. Let γ be a limit ordinal. In view of the induction hypothesis, there exists a $(\bar{\gamma} + n)$-special space X_n for any $n \in N$. By Lemma 3.11(ii), the fibred sum $\sum_{n\in N} X_n$ is $(\gamma + 1)$-special.

Case 2. Let γ be a non-limit ordinal. In view of the induction hypothesis, there exists a γ-special space X. Let $X_n = X$ for each $n \in N$. By Lemma 3.11(i), the fibred sum $\sum_{n\in N} X_n$ is $(\gamma + 1)$-special.

□

Theorem 3.13. *For any ordinal α, there exists an irreducible T_0-space X whose* wf*-rank is equal to α.*

Proof. If α is a nonlimit ordinal, then the result follows directly from Theorem 3.12. If α is a limit ordinal, then the statement follows from Theorem 3.12 and Lemma 3.11(ii). □

4. Conclusion

In this paper, a parallel development of Ershov's results pertaining to d-spaces in [1, 2] has been worked out pertaining to well-filtered spaces. The key result is that for each ordinal α there exists a T_0-space whose wf-rank is exactly α – a result that is parallel to the existence of a T_0-space whose d-rank is exactly α.

D-spaces and well-filtered spaces are two generalizations of sober spaces. Apart from these, bounded sobriety also generalizes sobriety. Recall that a T_0-space is *bounded sober* if every upper-bounded closed irreducible set is the closure of a unique singleton ([20]). Along the same vein, one may also define bounded well-filtered space, i.e., a T_0-space is *bounded well-filtered* if every upper-bounded closed KF-set is the closure of a unique singleton. Here we may ask whether our current methods can be extended to cope with all these generalizations of sobriety in a *uniform* manner? In particular, can one define suitable notions of *bounded sobriety rank* and *bounded* wf *rank*? If so, how do these definitions relate to bounded sobrification and bounded well-filterification? In fact, it is not even clear whether bounded well-filterification exists, though bounded sobrification does (see [20]).

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