



A Study on q -Analogues of Catalan-Daehee Numbers and Polynomials

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Abstract. Catalan-Daehee numbers and polynomials, generating functions of which can be expressed as p -adic Volkenborn integrals on \mathbb{Z}_p , were studied previously. The aim of this paper is to introduce q -analogues of the Catalan-Daehee numbers and polynomials with the help of p -adic q -integrals on \mathbb{Z}_p . We derive, among other things, some explicit expressions for the q -analogues of the Catalan-Daehee numbers and polynomials.

1. Introduction and preliminaries

In recent years, many special numbers and polynomials have been studied by using several different tools such as combinatorial methods, generating functions, p -adic analysis, umbral calculus, differential equations, probability theory, special functions and analytic number theory. Catalan-Daehee numbers and polynomials were studied in [10] and several properties and identities associated with those numbers and polynomials were derived by utilizing umbral calculus techniques. The family of linear differential equations arising from the generating function of Catalan-Daehee numbers were considered in [11] in order to derive some explicit identities involving Catalan-Daehee numbers and Catalan numbers. In [6], w -Catalan polynomials were introduced as a generalization of Catalan polynomials and many symmetric identities in three variables related to the w -Catalan polynomials and analogues of alternating power sums were obtained by means of p -adic fermionic integrals. In [18], the authors constructed a family of modified p -adic twisted functions, which interpolate the modified twisted q -Bernoulli polynomials and the generalized twisted q -Bernoulli numbers at negative integers. The main purpose of [21] is to present a systematic study of some families of multiple q -zeta functions and basic (or q -) L -series. Indeed, the authors constructed some p -adic q -zeta functions, which interpolate at negative integers the q -Bernoulli numbers and polynomials. In addition, they constructed some basic (or q -) L -series, which interpolate at negative integers the Barnes' type Changhee q -Bernoulli numbers with attached Dirichlet character.

The aim of this paper is to introduce q -analogues of the Catalan-Daehee numbers and polynomials with the help of p -adic q -integrals on \mathbb{Z}_p , and derive some explicit expressions and identities related to

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those numbers and polynomials. For the rest of this section, we recall the necessary facts that are needed throughout this paper.

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p$ and \mathbb{C}_p denote respectively the ring of p -adic integers, the field of p -adic rational numbers and the completion of the algebraic closure of \mathbb{Q}_p . The p -adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let q be an indeterminate in \mathbb{C}_p with $|1 - q|_p < p^{-\frac{1}{p-1}}$. The q -analogue of x is defined by $[x]_q = \frac{1 - q^x}{1 - q}$. Note that $\lim_{q \rightarrow 1} [x]_q = x$.

Let f be a uniformly differentiable function on \mathbb{Z}_p . Then the p -adic q -integral on \mathbb{Z}_p is defined by Kim as

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x) d\mu_q(x) &= \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) \mu_q(x + p^N \mathbb{Z}_p) \\ &= \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (\text{see [7, 8]}). \end{aligned} \tag{1}$$

From (1), we have

$$q \int_{\mathbb{Z}_p} f(x + 1) d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) + (q - 1)f(0) + \frac{q - 1}{\log q} f'(0), \tag{2}$$

where

$$f'(0) = \left. \frac{df}{dx} \right|_{x=0}, \quad (\text{see [1, 2, 7, 8]}).$$

Let us take $f(x) = e^{xt}$. Then, by (2), we get

$$\frac{(q - 1) + \frac{q-1}{\log q} t}{qe^t - 1} = \int_{\mathbb{Z}_p} e^{xt} d\mu_q(x). \tag{3}$$

The q -Bernoulli numbers are defined, in light of (3), by

$$\frac{(q - 1) + \frac{q-1}{\log q} t}{qe^t - 1} = \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{n!}. \tag{4}$$

From (4), we note that

$$q(B_q + 1)^n - B_{n,q} = \begin{cases} q - 1, & \text{if } n = 0, \\ \frac{q-1}{\log q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{5}$$

with the usual convention about replacing B_q^n by $B_{n,q}$.

For $|t|_p < p^{-\frac{1}{p-1}}$, the (q, λ) -Daehee polynomials are defined by

$$\sum_{n=0}^{\infty} D_{n,q}(x|\lambda) \frac{t^n}{n!} = \frac{2(q - 1) + \lambda \frac{q-1}{\log q} \log(1 + t)}{q^2(1 + t)^\lambda - 1} (1 + t)^{\lambda x}, \quad (\text{see [3, 12 - 17]}). \tag{6}$$

When $x = 0$, $D_{n,q}(\lambda) = D_{n,q}(0|\lambda)$ are called (q, λ) -Daehee numbers.

In particular, $D_{0,q}(0|1) = \frac{2}{[2]_q}$.

The Catalan-Daehee numbers are defined by

$$\frac{\frac{1}{2} \log(1 - 4t)}{\sqrt{1 - 4t} - 1} = \sum_{n=0}^{\infty} d_n t^n, \quad (\text{see [5, 10]}). \tag{7}$$

We note that

$$\sqrt{1 + t} = \sum_{m=0}^{\infty} (-1)^{m-1} \binom{2m}{m} \left(\frac{1}{4}\right)^m \left(\frac{1}{2m - 1}\right) t^m. \tag{8}$$

By replacing t by $-4t$ in (8), we get

$$\sqrt{1 - 4t} = 1 - 2 \sum_{m=0}^{\infty} \binom{2m}{m} \frac{1}{m + 1} t^{m+1} = 1 - 2 \sum_{m=0}^{\infty} C_m t^{m+1}, \tag{9}$$

where C_m is the Catalan number.

From (7) and (9), we have

$$d_n = \begin{cases} 1, & \text{if } n = 0, \\ \frac{4^n}{n + 1} - \sum_{m=0}^{n-1} \frac{4^{n-m-1}}{n - m} C_m, & \text{if } n \geq 1. \end{cases} \tag{10}$$

When $q = 1$, by (2), we get

$$\int_{\mathbb{Z}_p} (1 - 4t)^{\frac{x}{2}} d\mu_1(x) = \frac{\frac{1}{2} \log(1 - 4t)}{\sqrt{1 - 4t} - 1} = \sum_{n=0}^{\infty} d_n t^n. \tag{11}$$

2. q -analogues of Catalan-Daehee numbers and polynomials

For $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, we have

$$\int_{\mathbb{Z}_p} (1 - 4t)^{\frac{x}{2}} d\mu_q(x) = \frac{q - 1 + \frac{q-1}{\log q} \frac{1}{2} \log(1 - 4t)}{q \sqrt{1 - 4t} - 1}. \tag{12}$$

In view of (11) and (12), we define the q -analogue of Catalan-Daehee numbers by

$$\frac{q - 1 + \frac{q-1}{\log q} \frac{1}{2} \log(1 - 4t)}{q \sqrt{1 - 4t} - 1} = \sum_{n=0}^{\infty} d_{n,q} t^n. \tag{13}$$

Note that $\lim_{q \rightarrow 1} d_{n,q} = d_n, (n \geq 0)$.

From (6) and (13), we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} d_{n,q} t^n &= \frac{1}{2} \left(\frac{2(q-1) + \frac{q-1}{\log q} \log(1-4t)}{q^2(1-4t) - 1} \right) (q \sqrt{1-4t} + 1) \\
 &= \frac{1}{2} \left(\sum_{l=0}^{\infty} (-4)^l D_{l,q}(0|1) \frac{t^l}{l!} \right) \left(1 + q - 2q \sum_{m=0}^{\infty} C_m t^{m+1} \right) \\
 &= \frac{[2]_q}{2} \sum_{n=0}^{\infty} (-4)^n \frac{D_{n,q}(0|1)}{n!} t^n - q \sum_{n=1}^{\infty} \left(\sum_{m=0}^{n-1} \frac{(-4)^{n-m-1}}{(n-m-1)!} D_{n-m-1,q}(0|1) C_m \right) t^n \\
 &= 1 + \sum_{n=1}^{\infty} \frac{[2]_q}{2} (-4)^n \frac{D_{n,q}(0|1)}{n!} t^n - q \sum_{n=1}^{\infty} \left(\sum_{m=0}^{n-1} \frac{(-4)^{n-m-1}}{(n-m-1)!} D_{n-m-1,q}(0|1) C_m \right) t^n \\
 &= 1 + \sum_{n=1}^{\infty} \left(\frac{[2]_q}{2} \frac{(-4)^n}{n!} D_{n,q}(0|1) - q \sum_{m=0}^{n-1} \frac{(-4)^{n-m-1}}{(n-m-1)!} D_{n-m-1,q}(0|1) C_m \right) t^n.
 \end{aligned}
 \tag{14}$$

Therefore, by comparing the coefficients on both sides of (14), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$, we have

$$d_{n,q} = \begin{cases} 1, & \text{if } n = 0, \\ \frac{[2]_q}{2} \frac{(-4)^n}{n!} D_{n,q}(0|1) - q \sum_{m=0}^{n-1} \frac{(-4)^{n-m-1}}{(n-m-1)!} D_{n-m-1,q}(0|1) C_m, & \text{if } n \geq 1. \end{cases}$$

From (13) and (14), we have

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} x^n d\mu_q(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{xt} d\mu_q(x) = \frac{(q-1) + \frac{q-1}{\log q} t}{q e^t - 1} = \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{n!}.
 \tag{15}$$

Thus, by (15), we get

$$\int_{\mathbb{Z}_p} x^n d\mu_q(x) = B_{n,q}, \quad (n \geq 0).
 \tag{16}$$

Now, we observe that

$$\begin{aligned}
 \sum_{n=0}^{\infty} d_{n,q} t^n &= \frac{q-1 + \frac{q-1}{\log q} \frac{1}{2} \log(1-4t)}{q \sqrt{1-4t} - 1} = \int_{\mathbb{Z}_p} (1-4t)^{\frac{x}{2}} d\mu_q(x) \\
 &= \sum_{m=0}^{\infty} \left(\frac{1}{2} \right)^m \frac{1}{m!} (\log(1-4t))^m \int_{\mathbb{Z}_p} x^m d\mu_q(x) \\
 &= \sum_{m=0}^{\infty} \left(\frac{1}{2} \right)^m B_{m,q} \sum_{n=m}^{\infty} S_1(n, m) \frac{1}{n!} (-4t)^n \\
 &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n 2^{2n-m} (-1)^n B_{m,q} S_1(n, m) \right) \frac{t^n}{n!},
 \end{aligned}
 \tag{17}$$

where $S_1(n, m)$, $(n, m \geq 0)$ are the Stirling numbers of the first kind defined by

$$(x)_n = \sum_{l=0}^n S_1(n, l) x^l, \quad (n \geq 0), \quad (\text{see [1 - 17]}).$$

Here $(x)_0 = 1, (x)_n = x(x - 1) \cdots (x - n + 1), (n \geq 1)$.

Therefore, by (17), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$, we have

$$(-1)^n d_{n,q} = \frac{1}{n!} \sum_{m=0}^n 2^{2^{n-m}} B_{m,q} S_1(n, m).$$

By binomial expansion, we get

$$\int_{\mathbb{Z}_p} (1 - 4t)^{\frac{x}{2}} d\mu_q(x) = \sum_{n=0}^{\infty} (-4)^n \int_{\mathbb{Z}_p} \binom{\frac{x}{2}}{n} d\mu_q(x) t^n. \tag{18}$$

From (12), (17) and (18), we obtain the following corollary.

Corollary 2.3. For $n \geq 0$, we have

$$\int_{\mathbb{Z}_p} \binom{\frac{x}{2}}{n} d\mu_q(x) = (-1)^n 2^{-2^n} d_{n,q} = \frac{1}{n!} \sum_{m=0}^n \left(\frac{1}{2}\right)^m B_{m,q} S_1(n, m).$$

The q -analogue of λ -Daehee polynomials are given by the following p -adic q -integral on \mathbb{Z}_p :

$$\begin{aligned} \int_{\mathbb{Z}_p} (1 + t)^{\lambda y + x} d\mu_q(y) &= \frac{(q - 1) + \lambda \frac{q-1}{\log q} \log(1 + t)}{q(1 + t)^\lambda - 1} (1 + t)^x \\ &= \sum_{n=0}^{\infty} D_{n,q,\lambda}(x) \frac{t^n}{n!}. \end{aligned} \tag{19}$$

When $x = 0, D_{n,q,\lambda} = D_{n,q,\lambda}(0), (n \geq 0)$, are called the q -analogue of λ -Daehee numbers.

Here, we note that

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n 4^n D_{n,q,\frac{1}{2}} \frac{t^n}{n!} &= \frac{q - 1 + \frac{1}{2} \frac{q-1}{\log q} \log(1 - 4t)}{q(1 - 4t)^{\frac{1}{2}} - 1} \\ &= \sum_{n=0}^{\infty} d_{n,q} t^n. \end{aligned} \tag{20}$$

Thus, by (20), we get

$$d_{n,q} = (-1)^n \frac{4^n}{n!} D_{n,q,\frac{1}{2}}, \quad (n \geq 0).$$

Replacing t by $\frac{1}{4}(1 - e^{2t})$ in (13), we have

$$\begin{aligned} \sum_{k=0}^{\infty} d_{k,q} \left(\frac{1}{4}\right)^k (1 - e^{2t})^k &= \frac{q - 1 + \frac{q-1}{\log q} t}{qe^t - 1} = \int_{\mathbb{Z}_p} e^{xt} d\mu_q(x) \\ &= \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{n!}. \end{aligned} \tag{21}$$

On the other hand,

$$\begin{aligned} \sum_{k=0}^{\infty} d_{k,q} \left(\frac{1}{4}\right)^k (1 - e^{2t})^k &= \sum_{k=0}^{\infty} k! d_{k,q} \left(-\frac{1}{4}\right)^k \frac{1}{k!} (e^{2t} - 1)^k \\ &= \sum_{k=0}^{\infty} k! d_{k,q} \left(-\frac{1}{4}\right)^k \sum_{n=k}^{\infty} S_2(n, k) 2^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^k k! d_{k,q} 2^{n-2k} S_2(n, k) \right) \frac{t^n}{n!}, \end{aligned} \tag{22}$$

where $S_2(n, k)$, $(n, k \geq 0)$, are the Stirling numbers of the second kind defined by

$$x^n = \sum_{l=0}^n S_2(n, l) (x)_l, \quad (n \geq 0).$$

Therefore, by (21) and (22), we obtain the following theorem.

Theorem 2.4. For $n \geq 0$, we have

$$B_{n,q} = \sum_{k=0}^n (-1)^k 2^{n-2k} k! S_2(n, k) d_{k,q}.$$

Now, we observe that

$$\int_{\mathbb{Z}_p} (1 - 4t)^{\frac{x+y}{2}} d\mu_q(y) = \frac{(q-1) + \frac{q-1}{\log q} \frac{1}{2} \log(1-4t)}{q \sqrt{1-4t} - 1} (1 - 4t)^{\frac{x}{2}}.$$

We define the Catalan-Daehee polynomials by

$$\frac{q-1 + \frac{q-1}{\log q} \frac{1}{2} \log(1-4t)}{q \sqrt{1-4t} - 1} (1 - 4t)^{\frac{x}{2}} = \sum_{n=0}^{\infty} d_{n,q}(x) t^n. \tag{23}$$

Note that

$$\begin{aligned} (1 - 4t)^{\frac{x}{2}} &= \sum_{l=0}^{\infty} \binom{x}{2} \frac{1}{l!} (\log(1-4t))^l = \sum_{l=0}^{\infty} \binom{x}{2} \sum_{m=l}^{\infty} S_1(m, l) \frac{(-4)^m t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{l=0}^m S_1(m, l) \frac{(-4)^m}{m!} \binom{x}{2} \right) t^m. \end{aligned} \tag{24}$$

Thus, by (13), (23) and (24), we get

$$\begin{aligned} \sum_{n=0}^{\infty} d_{n,q}(x) t^n &= \frac{q-1 + \frac{q-1}{\log q} \frac{1}{2} \log(1-4t)}{q \sqrt{1-4t} - 1} (1 - 4t)^{\frac{x}{2}} \\ &= \sum_{k=0}^{\infty} d_{k,q} t^k \sum_{m=0}^{\infty} \left(\sum_{l=0}^m S_1(m, l) \frac{(-4)^m}{m!} \binom{x}{2} \right) t^m \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \sum_{l=0}^m S_1(m, l) \frac{(-4)^m}{m!} d_{n-m,q} \binom{x}{2} \right) t^n. \end{aligned} \tag{25}$$

By comparing the coefficients on both sides (25), we obtain the following theorem.

Theorem 2.5. For $n \geq 0$, we have

$$d_{n,q}(x) = \sum_{l=0}^n \left(\sum_{m=l}^n (-1)^m \frac{2^{2m-l}}{m!} S_1(m, l) d_{n-m,q} \right) x^l.$$

3. Conclusion

Quite a few special numbers and polynomials have been studied by employing various different tools. Previously, the Catalan-Daehee numbers and polynomials were introduced by means of p -adic Volkenborn integrals and some interesting results for them were obtained by using generating functions, differential equations, umbral calculus and p -adic Volkenborn integrals. In this paper, we introduced q -analogues of the Catalan-Daehee numbers and polynomials and obtained several explicit expressions and identities related to them. In more detail, we expressed the Catalan-Daehee numbers in terms of the (q, λ) -Daehee numbers, and of the q -Bernoulli polynomials and Stirling numbers of the first kind. We obtained an identity involving q -Bernoulli number, q -analogues of Catalan-Daehee numbers and Stirling numbers of the second kind. In addition, we got an explicit expression for the q -analogues of Catalan-Daehee polynomials which involve the q -analogues of Catalan-Daehee numbers and Stirling numbers of the first kind.

Here we would like to point out the triviality and inconsequential nature of the current trend of translating known q -calculus by using the so-called “ (p, q) -calculus” in which the forced-in parameter p is obviously redundant. In this connection, the interested reader may want to refer, for details, to page 340 of [19] and pages 1511-1512 of [20].

It has been our constant interest to find q -analogues of some interesting special numbers and polynomials and to study their arithmetic and combinatorial properties and their applications. We would like to continue to study this line of research in the future.

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