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Solution of Equations with q-Derivatives and Ward's Derivatives Using an Operational Method

Gabriel Bengochea[a], Luis Verde-Star[b], Manuel Ortigueira[c]

[a]*Colegio de Ciencia y Tecnología, Universidad Autónoma de la Ciudad de México, Ciudad de México, México.*
[b]*Departamento de Matemáticas, Universidad Autónoma Metropolitana, Iztapalapa, Apartado 55–534, Ciudad de México, México.*
[c]*CTS–UNINOVA/Departament of Electrical Enginering, Faculdade de Ciéncias e Tecnologia da Universidade Nova de Lisboa, Campus da FCT da UNL, Quinta da Torre 2825–149 Monte da Caparica, Portugal.*

Abstract. We show that several types of differential equations that involve q-derivatives, Fibonacci derivatives, and other Ward's derivatives, can be solved by an algebraic operational method that does not use integrals nor integral transforms. We deal with extensions of the Ward's derivatives that can be applied to formal Laurent series. Several examples of linear and nonlinear equations are presented.

1. Introduction

It is well-known that formal power series play a central role in the resolution of ordinary differential equations. This is mainly due to the fact that the usual derivative D acts on the monomials x^n in a simple way, that is, $Dx^n = nx^{n-1}$. In order to extend this property of D, Appell [1] defined polynomial sequences, called Appell sequences, $\{p_n(x)\}_{n\in\mathbb{N}_0}$, that satisfy $Dp_n(x) = np_{n-1}(x)$ and such that the degree of $p_n(x)$ is equal to n, for all $n \geq 0$. The Appell sequences constitute a useful tool for solving several kinds of differential and difference equations. Many properties of Appell sequences have been studied using the theory of umbral calculus. See the books [7] and [8].

In an attempt to generalize previous work of Jackson [3], Ward [11] introduced several "derivatives", that is, generalizations of the usual derivative, by replacing the ordinary binomial coefficients by the generalized binomial coefficients

$$[n,r] = \frac{h_n h_{n-1}\cdots h_{n-r+1}}{h_1 h_2 \cdots h_r},$$

where $\{h_k\}_{k=0}^{\infty}$ is a fixed sequence of complex numbers such that $h_0 = 0$, $h_1 = 1$, and $h_k \neq 0$, for $k > 1$. In others words, for the usual derivative we have $Dx^n = \binom{n}{1}x^{n-1}$, and for the new derivative, denoted by $\mathscr{D}_h$, we have $\mathscr{D}_h x^n = [n,1]x^{n-1}$. Each sequence $\{h_k\}_{k=0}^{\infty}$ yields a new derivative. Among the most important are

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Corresponding author: Gabriel Bengochea
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Email addresses: gabriel.bengochea@uacm.edu.mx (Gabriel Bengochea), verde@xanum.uam.mx (Luis Verde-Star), mdo@fct.unl.pt (Manuel Ortigueira)

the Fibonacci derivative, the zero Jackson derivative, and the Jackson derivative, or q-derivative, which we will consider in this paper. It is important to mention that the Ward's calculus and the Appell sequences are closely related. A sequence of the form $\left\{\frac{x^n}{h_1 h_2 \cdots h_n}\right\}_{n\geq 0}$ is an Appell sequence with respect to Ward's derivative $\mathscr{D}_h$, that is,

$$\mathscr{D}_h \frac{x^n}{h_1 h_2 \cdots h_n} = \frac{x^{n-1}}{h_1 h_2 \cdots h_{n-1}}, \quad n \geq 0,$$

and $\mathscr{D}_h 1 = 0$.

Recently, Luzón, Morón, and Ramírez [6] extended the theory of Ward's calculus and studied its connections with Sheffer polynomial sequences and Riordan matrices.

In this paper, we study the extension of generalized difference operators from formal series to formal Laurent series. For this, we will define the h-factorial function for any integer. An important case of such generalized operators is the Ward's derivative. We obtain properties of generalized difference operators showing that the Ward's derivative satisfies $\mathscr{D}_h p_n = p_{n-1}$, for every nonzero integer n, and $\mathscr{D}_h p_0 = 0$, when $p_n = \frac{x^n}{h_1 h_2 \cdots h_n}$. Consequently, the derivative $\mathscr{D}_h$ can be applied to any formal Laurent series of the form

$$a = \sum_{k\geq v(a)}^{\infty} a_k p_k, \quad v(a) \in \mathbb{Z}, \quad a_{v(a)} \neq 0.$$

This is the key property that allows us to apply the general algebraic operational calculus, that we introduced in [2], to solve equations with Ward's derivatives, without using neither integrals nor integral transforms.

The paper is organized as follows. Section 2 presents the basic theory of the operational method introduced in [2]. In Section 3 we study the generalized difference operator and extend its domain to the complex vector space of formal Laurent series. We consider some particular cases that include the Ward's derivative, the Fibonacci derivative, the zero Jackson derivative, and the q-derivative. The application of the operational method to Ward's calculus is presented in Section 4. Finally, Section 5 contains the main conclusions.

2. The operational method

In this section, we summarize the main results obtained in [2]. Let $\{p_k : k \in \mathbb{Z}\}$ be a group with the multiplication defined by $p_k p_n = p_{k+n}$, for $k,\ n \in \mathbb{Z}$. Let $\mathscr{F}$ be the set of all the formal series of the form

$$a = \sum_{k\in\mathbb{Z}} a_k p_k,$$

where the coefficients a_k are complex numbers and, either, all the a_k are equal to zero, or there exists an integer $v(a)$ such that $a_k = 0$ whenever $k < v(a)$, and $a_{v(a)} \neq 0$. In the first case we write $a = 0$ and define $v(0) = \infty$. Addition in $\mathscr{F}$ and multiplication by complex numbers are defined in the usual way.

The usual Cauchy product series is used to extend the multiplication of the group $\{p_k : k \in \mathbb{Z}\}$ to a multiplication on $\mathscr{F}$ as follows. If $a = \sum a_k p_k$ and $b = \sum b_k p_k$ are elements of $\mathscr{F}$ then $ab = c = \sum c_n p_n$, where the coefficients c_n are defined by

$$c_n = \sum_{v(a)\leq k\leq n-v(b)} a_k b_{n-k}.$$

Note that $v(ab) = v(a) + v(b)$. This multiplication in $\mathscr{F}$ is associative and commutative and p_0 is its unit element. Define $\mathscr{F}_n = \{a \in \mathscr{F} : v(a) \geq n\}$, for $n \in \mathbb{Z}$. It was proved in [2] that $\mathscr{F}$ is a field and that $\mathscr{F}_0$ is a subring of $\mathscr{F}$.

Some important properties of the series in $\mathscr{F}$ are the following.

- For $x \in \mathbb{C}$

$$(p_0 - xp_1)\sum_{n\geq 0} x^n p_n = p_0.$$

The series $\sum_{k\geq 0} x^k p_k$ is denoted by $e_{x,0}$ and called the *geometric series* associated with x.

- For $x \in \mathbb{C}$ and $k > 0$ we define

$$e_{x,k} = \frac{p_k}{(p_0 - xp_1)^{k+1}}.$$

Then we have

$$e_{x,k} = p_k(e_{x,0})^{k+1} = \frac{D_x^k}{k!}e_{x,0} = \sum_{n\geq k}\binom{n}{k}x^{n-k}p_n, \qquad k \geq 0, \tag{1}$$

where D_x denotes the usual differentiation operator with respect to x.

- For $x, y \in \mathbb{C}$ such that $x \neq y$

$$p_1 e_{x,m} e_{y,n} = \sum_{k=0}^{m} \frac{\binom{n+k}{k}(-1)^k e_{x,m-k}}{(x-y)^{1+n+k}} + \sum_{j=0}^{n} \frac{\binom{m+j}{j}(-1)^j e_{y,n-j}}{(y-x)^{1+m+j}}, \quad n, m \in \mathbb{N}.$$

Note that the right-hand side is a linear combination of the elements of the sets $\{e_{x,k} : 0 \leq k \leq m\}$ and $\{e_{y,j} : 0 \leq j \leq n\}$. A simple particular case is

$$p_1 e_{x,0} e_{y,0} = \frac{e_{x,0} - e_{y,0}}{x - y}. \tag{2}$$

Denote by P_n the projection on $\langle p_n \rangle$, the subspace generated by p_n, that is, if a is in $\mathscr{F}$ then $P_n a = a_n p_n$. We define a linear operator L on $\mathscr{F}$ as follows. $Lp_k = p_{-1}p_k = p_{k-1}$ for $k \neq 0$, and $Lp_0 = 0$. Then, for a in $\mathscr{F}$ we have

$$La = L\sum_{k\geq v(a)} a_k p_k = p_{-1}(a - a_0 p_0) = p_{-1}(I - P_0)a,$$

and

$$L^k a = p_{-k}\left(I - \sum_{j=0}^{k-1} P_j\right)a, \qquad k \geq 1, \tag{3}$$

where I is the identity operator. We call L the *modified left shift*. Note that L is not invertible, since its kernel is the subspace $\langle p_0 \rangle$.

For $k \geq 0$ let $\mathscr{F}_{[0,k]} = \mathrm{Ker}(P_0 + P_1 + \cdots + P_k)$. If $k = 0$, we write $\mathscr{F}_{[0]}$ instead of $\mathscr{F}_{[0,0]}$.

Let

$$w(t) = \prod_{j=0}^{r}(t - x_j)^{m_j+1},$$

where $x_0, x_1, \ldots, x_r$ are distinct complex numbers, $m_0, m_1, \ldots, m_r$ are nonnegative integers, and we set $n+1 = \sum_j (m_j + 1)$. We define the operator

$$w(L) = (L - x_0 I)^{m_0+1}(L - x_1 I)^{m_1+1} \cdots (L - x_r I)^{m_r+1}. \tag{4}$$

Theorem 2.1 ([2], p. 339.). *, Let $w(L)$ be as defined in* (4)*. Define*

$$d_w = p_{r+1} e_{x_0,m_0} e_{x_1,m_1} \cdots e_{x_r,m_r},$$

and

$$K_w = \langle e_{x_j,i} : 0 \le j \le r,\ 0 \le i \le m_j \rangle.$$

Then g is in the image of $w(L)$ if and only if $d_w g \in \mathscr{F}_{[0,n]}$, $K_w = \mathrm{Ker}(w(L))$, and for every $g \in \mathrm{Im}(w(L))$ we have $w(L)(d_w g) = g$ and thus

$$\{f \in \mathscr{F} : w(L)f = g\} = \{d_w g + h : h \in K_w\}.$$

3. Generalized difference operators and Ward's derivatives

Let $\{u_k(t)\}_{k\ge0}$ be a sequence of monic polynomials in $\mathbb{C}[t]$ such that u_k has degree k for $k \ge 0$. We say that $\{u_k(t)\}_{k\ge0}$ is a *polynomial sequence.* It is clear that every polynomial sequence is a basis for the vector space $\mathbb{C}[t]$.

Let $(h_k)_{k=0}^{\infty} = (h_0, h_1, h_2, \ldots)$ be a sequence of complex numbers such that $h_0 = 0$ and $h_k \ne 0$ for $k \ge 1$, and let $\{u_k(t)\}_{k\ge0}$ be a polynomial sequence. We define the *generalized difference operator* $\mathscr{D}_h$ on the vector space of polynomials through $\mathscr{D}_h u_k = h_k u_{k-1}$ for $k \ge 0$. If $u_k(t) = t^k$ for $k \ge 0$ then $\mathscr{D}_h$ is called the *Ward's derivative* associated with the sequence $(h_k)_{k\ge0}$. The usual derivative is obtained when $h_k = k$ for $k \ge 0$.

The usual difference operator with increment $d \ne 0$ is obtained when the polynomial sequence $u_k(t)$ is defined by $u_0(t) = 1$ and

$$u_k(t) = t(t-d)(t-2d)\cdots(t-(k-1)d), \qquad k \ge 1.$$

In this case we have

$$\frac{u_k(t+d) - u_k(t)}{d} = kd\, u_{k-1}(t), \qquad k \ge 0,$$

and therefore, if we define $h_k = kd$ for $k \ge 0$, the generalized difference associated with $\{u_k\}$ and (h_k) is the usual difference operator

$$\mathscr{D}_h p(t) = \frac{p(t+d) - p(t)}{d}, \qquad p \in \mathbb{C}[t].$$

Note that this difference operator is not a Ward's derivative.

If $\mathscr{D}_h$ is the generalized difference operator associated with a polynomial sequence $\{u_k\}_{k\ge0}$ and sequence $(h_k)_{k\ge0}$ then $\mathscr{D}_h$ can be extended to a linear operator on the vector space of formal series of the form $\sum_{k=0}^{\infty} a_k\, u_k(t)$, where the coefficients a_k are complex numbers, by defining

$$\mathscr{D}_h \sum_{k=0}^{\infty} a_k\, u_k(t) = \sum_{k=1}^{\infty} a_k\, h_k\, u_{k-1}(t).$$

Now let $\{w_k(t) : k \ge 0\}$ be another polynomial sequence and extend the sequence $(h_k)_{k\ge0}$ to a bilateral sequence $(h_k)_{k\in\mathbb{Z}}$, with $h_k \ne 0$ if $k \ne 0$. Define the rational functions

$$u_{-k}(t) = \frac{1}{w_k(t)}, \qquad k \ge 1.$$

Let $\mathscr{L}_u$ be the complex vector space of the formal Laurent series of the form

$$a = \sum_{k\in\mathbb{Z}} a_k u_k(t),$$

where the coefficients a_k are complex numbers, and if not all the a_k are zero then there exists an integer $v(a)$ such that $a_k = 0$ whenever $k < v(a)$ and $a_{v(a)} \neq 0$. We extend the generalized difference $\mathscr{D}_h$ to an operator on $\mathscr{L}_u$, which we also denote by $\mathscr{D}_h$, as follows

$$\mathscr{D}_h a = \mathscr{D}_h \sum_{k \geq v(a)} a_k u_k(t) = \sum_{k \in \mathbb{Z}} a_k h_k u_{k-1}(t), \qquad a \in \mathscr{L}_u.$$

If $\mathscr{D}_h$ is a Ward's derivative, then $u_k(t) = t^k$ for $k \geq 0$ and we take $w_k(t) = t^k$ for $k \geq 0$. Therefore in that case, $\mathscr{L}_u$ is the usual vector space of formal Laurent power series, that we denote by $\mathscr{L}$, and the extension of $\mathscr{D}_h$ to $\mathscr{L}$ is given by

$$\mathscr{D}_h a = \sum_{k \geq v(a)} a_k h_k\, t^{k-1}, \qquad a \in \mathscr{L}.$$

If $(h_k)_{k\in\mathbb{Z}}$ is a bilateral sequence such that $h_0 = 0$ and $h_k \neq 0$ for $k \neq 0$ then we define the *h-factorial function* c_k as follows, $c_0 = 1$, $c_{-1} = 1$,

$$c_k = h_1 h_2 \cdots h_k, \qquad k > 0, \tag{5}$$

and

$$c_k = \frac{1}{h_{-1} h_{-2} \cdots h_{k+1}}, \qquad k < -1. \tag{6}$$

If $\mathscr{D}_h$ is the generalized difference operator associated with the sequence $\{u_k\}_{k\in\mathbb{Z}}$ and the bilateral sequence $(h_k)_{k\in\mathbb{Z}}$ then we have

$$\mathscr{D}_h \frac{u_n(t)}{c_n} = \frac{u_{n-1}(t)}{c_{n-1}}, \qquad n \neq 0, \tag{7}$$

and if $\mathscr{D}_h$ is the Ward's derivative associated with $(h_k)_{k\in\mathbb{Z}}$ then we have

$$\mathscr{D}_h \frac{t^n}{c_n} = \frac{t^{n-1}}{c_{n-1}}, \qquad n \neq 0. \tag{8}$$

From (7), we see that if we take $p_n = \frac{u_n(t)}{c_n}$ then the generalized difference operator $\mathscr{D}_h$ plays the role of the modified left shift L and therefore we can use the operational method presented in Section 2 to solve equations of the form $w(\mathscr{D}_h)f = g$, where w is a polynomial. The multiplication $p_k p_n = p_{k+n}$ becomes a convolution product on the space $\mathscr{L}$ defined by

$$\frac{u_k(t)}{c_k} * \frac{u_n(t)}{c_n} = \frac{u_{k+n}(t)}{c_{k+n}}, \qquad k, n \in \mathbb{Z}.$$

The generalized difference operators include a large class of difference and differential operators with variable coefficients and different orders.

From this point on, we restrict our attention to the case when $u_n(t) = t^n$, for $n \in \mathbb{Z}$, that is, when $\mathscr{D}_h$ is a Ward's derivative on the space $\mathscr{L}$ of formal Laurent power series. We present next some particular examples of such Ward's derivatives on the space $\mathscr{L}$.

Example 3.1. *The finite Fibonacci derivative $\mathscr{D}_F$ [5] is obtained when $(h_k)_{k\geq 0}$ is the sequence of Fibonacci numbers*

$$(F_k)_{k=0}^{\infty} = (0, 1, 1, 2, 3, 5, 8, \dots),$$

that satisfy the difference equation $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$, with initial conditions $F_0 = 1$, $F_1 = 1$. Using the same recurrence relation, written as $F_{n-2} = F_n - F_{n-1}$, we extend the sequence (F_n) to the bilateral sequence

$$(F_k)_{k=-\infty}^{\infty} = (\dots, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, \dots),$$

and then the Fibonacci derivative acts on formal Laurent power series as follows

$$\mathscr{D}_F a = \mathscr{D}_F \sum_{n=v(a)}^{\infty} a_n t^n = \sum_{n=v(a)}^{\infty} F_n a_n t^{n-1}.$$

Example 3.2. *The usual derivative D is obtained when $h_k = k$ for $k \in \mathbb{Z}$, and it acts on formal Laurent power series as*

$$D \sum_{n=v(a)}^{\infty} a_n t^n = \sum_{n=v(a)}^{\infty} n a_n t^{n-1}.$$

In the paper [6] the authors prove that the only derivative $\mathscr{D}_h$ on the space of formal power series that satisfies both the Leibniz rule and the chain rule is the usual derivative. Their proof can be adapted to show that the same result holds for derivative operators $\mathscr{D}_h$ on the space of formal Laurent power series.

Example 3.3. *The zero Jackson derivative $\mathscr{D}_0$ is the Ward's derivative associated with the sequence*

$$(z_k)_{k=1}^{\infty} = (1, 1, 1, \dots).$$

Extending this sequence to the bilateral sequence

$$(z_k)_{k=-\infty}^{\infty} = (\dots, -1, -1, -1, 0, 1, 1, 1, \dots),$$

we obtain the extension of $\mathscr{D}_0$ to the space of formal Laurent power series. For $a = \sum_{n=v(a)}^{\infty} a_n t^n$ we have

$$\mathscr{D}_0 a = \sum_{n=1}^{\infty} a_n t^{n-1} - \sum_{n=v(a)}^{-1} a_n t^{n-1}.$$

Example 3.4. *The q-derivative $\mathscr{D}_q$, also called Jackson derivative, is the Ward's derivative associated with the sequence*

$$(J_k)_{k=1}^{\infty} = ([1]_q, [2]_q, [3]_q, \dots), \qquad [n]_q = \frac{q^n - 1}{q - 1}.$$

When $q = 1$, $\mathscr{D}_q$ becomes the usual derivative, and when $q = 0$ we obtain the zero Jackson derivative. Since the definition of q-integer $[n]_q$ makes sense for $n \in \mathbb{Z}$, the bilateral sequence

$$(J_k)_{k=-\infty}^{\infty} = (\dots, [-3]_q, [-2]_q, [-1]_q, [0]_q, [1]_q, [2]_q, [3]_q, \dots),$$

gives us the extension of $\mathscr{D}_q$ to the space of formal Laurent power series. Observe that $[-n]_q = -[n]_q q^{-n}$. For $a = \sum_{n=v(a)}^{\infty} a_n t^n$ we have $\mathscr{D}_q a = \sum_{n=v(a)}^{\infty} [n]_q a_n t^{n-1}$.

Some authors use the (p, q)-integers, defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \qquad n \in \mathbb{Z},$$

to replace the q-integers and they define an operator $\mathscr{D}_{p,q}$, called the (p, q)-derivative, or difference. The theory of such operators is not a generalization of the theory of the Jackson derivative, and the additional parameter p turns out to be redundant. The results obtained in the (p, q)-calculus can be easily produced, by straightforward parametric and argument variations, from the corresponding results in the q-calculus. For example,

$$[n]_{p,q} = p^{n-1} [n]_{\frac{q}{p}}, \qquad [n]_q = p^{1-n} [n]_{p,pq},$$

and

$$(\mathscr{D}_{p,q} f)(z) = \left(\mathscr{D}_{\frac{q}{p}} f\right)(pz), \quad (\mathscr{D}_q f)(z) = (\mathscr{D}_{p,pq} f)\left(\frac{z}{p}\right), \quad 0 < q < p \leq 1.$$

See [9] and [10, Sect. 5].

Remark 3.5. *It is important to note that for every derivative operator $\mathscr{D}_h$ on the space $\mathscr{L}$ of formal Laurent power series we have $\mathscr{D}_h \frac{t^n}{c_n} = \frac{t^{n-1}}{c_{n-1}}$, $n \neq 0$, and $\mathscr{D}_h \frac{t^0}{c_0} = 0$, where c_n is the corresponding h-factorial function.*

4. Application of the operational method to Ward's derivatives

In this section we apply the operational method presented in Section 2 to solve equations that involve Ward's derivatives.

Let $\mathscr{D}_h$ be the derivative operator on the space $\mathscr{L}$ of formal Laurent power series associated with the bilateral sequence $(h_k)_{k\in\mathbb{Z}}$. Define $L = \mathscr{D}_h$ and $p_n = \frac{t^n}{c_n}$ with $n \in \mathbb{Z}$, where c_n is the h-factorial function defined in (5) and (6). From (8) we see that $Lp_n = p_{n-1}$ for $n \neq 0$, and $Lp_0 = 0$, and therefore we can use the results in Section 2 to solve h-differential equations, that is, equations constructed with the derivative operator $\mathscr{D}_h$.

The geometric series is in this case

$$e_{x,0} = \sum_{k=0}^{\infty} x^k \frac{t^k}{c_k}, \qquad x \in \mathbb{C},$$

and

$$e_{x,k} = \sum_{n=k}^{\infty} \binom{n}{k} x^{n-k} \frac{t^n}{c_n}, \qquad x \in \mathbb{C}.$$

The multiplication of the p_k gives us the convolution on $\mathscr{L}$ defined by

$$\frac{t^k}{c_k} * \frac{t^n}{c_n} = \frac{t^{k+n}}{c_{k+n}}, \qquad k, n \in \mathbb{Z}.$$

Let us note that our geometric series $e_{x,0}$ coincides with the one introduced in [6].

Remark 4.1. *The set $\{e_{x,k} : x \in \mathbb{C},\ k \in \mathbb{N}\}$ is a basis for the vector space $\mathscr{E}_h$ of the h-exponential polynomials. By Theorem 2.1 the space of solutions of every homogeneous equation of the form $w(\mathscr{D}_h)f = 0$, where w is a polynomial, is a finite dimensional subspace of $\mathscr{E}_h$ whose dimension is equal to the degree of w. When $h_n = n$, for $n \in \mathbb{Z}$, the derivative $\mathscr{D}_h$ is the usual operator of differentiation with respect to t, and the h-exponential polynomials are the usual exponential polynomials, also called pseudo-polynomials.*

We present next some examples that illustrate how the operational method is used to solve equations involving particular derivative operators $\mathscr{D}_h$.

4.1. Fibonacci derivative

For the Fibonacci derivative $\mathscr{D}_F$, the F-factorial function is given by $c_0 = c_{-1} = 1$,

$$c_k = F_1 F_2 \cdots F_k, \qquad k \geq 1,$$

and

$$c_k = F_{-1} F_{-2} \cdots F_{k+1}, \qquad k < -1,$$

where the F_k are the Fibonacci numbers.

The geometric series are

$$e_{x,0} = \sum_{n=0}^{\infty} x^n \frac{t^n}{c_n} \tag{9}$$

and

$$e_{x,k} = \frac{D_x^k}{k!} e_{x,0} = \sum_{n \geq k} \binom{n}{k} x^{n-k} \frac{t^n}{c_n}$$

Remark 4.2. *The series* (9) *coincides with the Fibonacci exponential function* e_F^{xt} *presented in [5, 6].*

Example 4.3. *Consider the equation*

$$\mathscr{D}_F^2 y(t) + 2\mathscr{D}_F y(t) + y(t) = e_F^t.$$

This equation can be rewritten as

$$(\mathscr{D}_F + I)^2 y(t) = e_F^t.$$

Observe that $e_F^t = e_{1,0}$. *Suppose that* $y = \sum y_n \frac{t^n}{c_n} = \sum y_n p_n$. *By Theorem 2.1, a particular solution is given by*

$$d_w e_{1,0} = p_1 e_{-1,1} e_{1,0} = -\frac{1}{2} e_{-1,1} - \frac{1}{4} e_{-1,0} + \frac{1}{4} e_{1,0}.$$

and the space of solutions of the homogeneous equation is

$$K_w = \langle e_{-1,0}, e_{-1,1} \rangle.$$

Finally, we obtain the general solution

$$y = \left(d_1 - \frac{1}{2}\right) e_{-1,1} + \left(d_2 - \frac{1}{4}\right) e_{-1,0} + \frac{1}{4} e_{1,0},$$

where d_1 *and* d_2 *are arbitrary constants. In terms of the concrete realization we have*

$$\begin{aligned} y(t) &= C_1 \sum_{n=1}^{\infty} n(-1)^{n-1} \frac{t^n}{c_n} + C_2 \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{c_n} + \sum_{n=0}^{\infty} \frac{t^n}{c_n} \\ &= \sum_{n=0}^{\infty} \left(n(-1)^{n-1} C_1 + (-1)^n C_2 + 1\right) \frac{t^n}{c_n}. \end{aligned}$$

The constants C_1 *and* C_2 *allow us to work with initial conditions.*

4.2. *Zero Jackson derivative*

Recall that the bilateral sequence associated with the zero Jackson derivative is given by

$$(\ldots, -1, -1, -1, 0, 1, 1, 1, \ldots).$$

In this case the h-factorial function is given by

$$c_k = 1, \qquad k \geq 0,$$

and

$$c_k = (-1)^{-k+1}, \qquad k < 0.$$

The geometric series $e_{x,0}$ becomes

$$e_{x,0} = \sum_{n=0}^{\infty} x^n t^n = \frac{1}{1 - xt}, \tag{10}$$

and from (1)

$$e_{x,k} = \frac{D_x^k}{k!} e_{x,0} = \sum_{n \geq k} \binom{n}{k} x^{n-k} t^n = \frac{t^k}{(1 - xt)^{k+1}}$$

where D_x is the usual derivation with respect to x.

Remark 4.4. *Our geometric series* (10) *coincides with the zero exponential function e_0^{xt} presented in [6].*

Example 4.5. *Consider the equation*

$$\mathscr{D}_0 y(t) - y(t) = t^3 + e_0^{-3t}.$$

This equation can be transformed into

$$(L - I)y(t) = p_3 + e_{-3,0}.$$

Suppose that $y = \sum y_n t^n = \sum y_n p_n$. Applying Theorem 2.1, we obtain the particular solution

$$\begin{aligned} d_w(p_3 + e_{-3,0}) &= p_1 e_{1,0}(p_3 + e_{-3,0}) \\ &= p_4 e_{1,0} + p_1 e_{1,0} e_{-3,0} \\ &= p_4 e_{1,0} + \frac{e_{1,0} - e_{-3,0}}{4}, \end{aligned}$$

and the space of solutions of the homogeneous equation is $K_w = \langle e_{1,0} \rangle$. Finally, we obtain the general solution

$$y = \left(d_1 + \frac{1}{4}\right) e_{1,0} + p_4 e_{1,0} - \frac{1}{4} e_{-3,0},$$

where d_1 is an arbitrary constant.

In terms of the concrete realization, we have

$$y(t) = \sum_{n=0}^{\infty} \left(d_1 + \frac{1}{4} - \frac{(-3)^n}{4}\right) t^n + \sum_{n=4}^{\infty} t^n.$$

4.3. Jackson derivative (q-derivative)

We recall that the bilateral sequence associated with the Jackson derivative $\mathscr{D}_q$ is the sequence of q-integers $[k]_q$ and therefore the h-factorial function is given by $c_0 = c_{-1} = 1$,

$$c_k = [1]_q [2]_q \cdots [k]_q, \qquad k \geq 1,$$

and

$$c_k = \frac{1}{[-1]_q [-2]_q \cdots [k+1]_q}, \qquad k < -1.$$

The q-analogue of the binomial coefficients is given by

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{c_n}{c_j c_{n-j}}. \tag{11}$$

The geometric series $e_{x,0}$ is in this case

$$e_{x,0} = \sum_{n=0}^{\infty} x^n \frac{t^n}{c_n}, \tag{12}$$

and from (1) we obtain

$$e_{x,k} = \frac{D_x^k}{k!} e_{x,0} = \sum_{n \geq k}^{\infty} \binom{n}{k} x^{n-k} \frac{t^n}{c_n},$$

where D_x is the usual derivative with respect to x.

Remark 4.6. *The series* (12) *coincides with the q-analogue of the exponential function e_q^{xt} presented in [4].*

In this concrete realization, the q-trigonometric functions can be expressed as linear combinations of geometric series, this is

$$\sin_q(\omega t) = \frac{e_{i\omega,0} - e_{-i\omega,0}}{2i}$$

and

$$\cos_q(\omega t) = \frac{e_{i\omega,0} + e_{-i\omega,0}}{2}.$$

The next examples can be easily solved by the direct application of Theorem 2.1, but we prefer to show another procedure based in the same theory, which helps us to understand better some aspects of the theory.

Example 4.7. *Several problems appearing in physics can be described by means of the classical q-oscillator equation which is*

$$\mathscr{D}_q^2 y(t) + \omega^2 y(t) = 0. \tag{13}$$

Suppose that the solution y can be represented as a sum of the form $\sum y_n p_n$. Equation (13) *can be written as*

$$(L^2 + \omega^2 I)y = 0.$$

From (3)*, $L^2 = p_{-2}(p_0 - P_0 - P_1)$, and then*

$$(p_{-2}(p_0 - P_0 - P_1) + \omega^2 p_0)) \sum y_n p_n = 0.$$

After some algebraic manipulations (remember that $P_0 y = y_0 p_0$ and $P_1 y = y_1 p_1$) and simplifications we get

$$(p_0 + i\omega p_1)(p_0 - i\omega p_1)y = y_0 p_0 + y_1 p_1.$$

The multiplicative inverses of $(p_0 + i\omega p_1)$ and $(p_0 - i\omega p_1)$ are $e_{-i\omega,0}$ and $e_{i\omega,0}$, respectively. Then

$$y = (y_0 p_0 + y_1 p_1) e_{-i\omega,0} e_{i\omega,0}.$$

From (2) *we obtain*

$$e_{i\omega,0} e_{-i\omega,0} = p_{-1} \frac{e_{i\omega,0} - e_{-i\omega,0}}{2i\omega},$$

and thus

$$\begin{aligned} y &= y_0 p_{-1} \frac{e_{i\omega,0} - e_{-i\omega,0}}{2i\omega} + y_1 \frac{e_{i\omega,0} - e_{-i\omega,0}}{2i\omega} \\ &= y_0 \frac{e_{i\omega,0} + e_{-i\omega,0}}{2} + \frac{y_1}{\omega} \frac{e_{i\omega,0} - e_{-i\omega,0}}{2i}, \end{aligned}$$

where y_0 and y_1 are arbitrary values that are determined by the initial conditions. In terms of the concrete realization we have

$$y(t) = C_1 \cos_q(\omega t) + C_2 \sin_q(\omega t).$$

We can also solve non-linear equations using a slightly different procedure. In the next example, we consider a q-analogue of the logistic equation.

Example 4.8. *Consider the q-analogue of the logistic equation*

$$\mathscr{D}_q y(t) - \beta y(t) = -\beta y^2(t). \tag{14}$$

Before we solve the equation, we need to deal with the non-linear term. If we suppose that the solution $y(t)$ can be written as the q-Taylor series $\sum_{n\geq 0} y_n \frac{t^n}{c_n}$, then

$$y^2(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{c_n},$$

where

$$b_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q y_j y_{n-j}. \tag{15}$$

In terms of the p_k's, we have $y = \sum_{n\geq 0} y_n p_n$ and $y^2 = \sum_{n=0}^{\infty} b_n p_n$.

On the other hand, it is clear that (14) *can be written as*

$$(L - \beta I)y = -\beta y^2,$$

and in terms of the p_k's this is

$$(p_{-1}(p_0 - P_0) - \beta p_0)y = -\beta y^2,$$

which can be rewritten as

$$(p_0 - \beta p_1)y = y_0 p_0 - \beta p_1 y^2.$$

Since $e_{\beta,0}$ is the multiplicative inverse of $(p_0 - \beta p_1)$, we obtain

$$\begin{aligned} y &= y_0 e_{\beta,0} - \beta p_1 e_{\beta,0} y^2 \\ &= y_0 \left(\sum_{n=0}^{\infty} \beta^n p_n \right) - \beta \left(\sum_{n=0}^{\infty} \beta^n p_n \right) \left(\sum_{n=0}^{\infty} b_n p_{n+1} \right) \\ &= \sum_{n=0}^{\infty} \left(y_0 \beta^n - \sum_{j=0}^{n-1} \beta^{j+1} b_{n-j-1} \right) p_n, \end{aligned}$$

where $\sum_{j=0}^{n-1} \beta^{j+1} b_{n-j} = 0$, for $n = 0$. This result leads us to the recursive formula

$$y_n = y_0 \beta^n - \sum_{j=0}^{n-1} \beta^{j+1} b_{n-j-1},$$

where b_{n-j-1} is computed by means of (15)*, and y_0 is the initial condition. A few of the initial coefficients are the following*

$$\begin{aligned} y_0 &= y_0 \\ y_1 &= \beta y_0 - \beta b_0 = \beta y_0 - \beta y_0^2 \\ y_2 &= \beta y_1 - \beta(b_1 + \beta b_0) = \beta y_1 - \beta(2 y_0 y_1 + y_0^2) \\ y_3 &= \beta y_2 - \beta(b_2 + \beta b_1 + \beta^2 b_0) = \beta y_2 - \beta \left(2 y_0 y_2 + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q y_1^2 + 2\beta y_0 y_1 + \beta^2 y_0^2 \right) \\ &\vdots \qquad \vdots \end{aligned}$$

Finally, the solution in terms of the concrete realization is given by

$$y(t) = \sum_{n=0}^{\infty} \left(y_0 \beta^n - \sum_{j=0}^{n-1} \beta^{j+1} b_{n-j-1} \right) \frac{t^n}{c_n}. \tag{16}$$

For $q = 1$, $\beta = 1$ *and* $y_0 = \frac{1}{2}$, (16) *can be written in closed form as* $y(t) = \frac{1}{1+e^{-t}}$. *It would be interesting to investigate if* (16) *is related with* $\frac{1}{1+e_q^{-t}}$.

5. Conclusions

Based on the theory presented in [2], we developed an algebraic method to solve several kinds of h-differential equations in Ward's calculus. We extended the domain of the Ward's derivatives from the space of formal power series to the space of formal Laurent series. We showed through several examples that the theory can be applied in two ways. The first one uses Theorem 2.1, which gives us quickly the solution, but dealing with initial conditions is not simple. The second one is more laborious, but the incorporation of initial conditions is more natural. A main feature of this theory is that the concept of integral in Ward's calculus is not required. Finally, we leave for future works the application of the operational method in the more general case of equations with generalized difference operators and *delta operators*.

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