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On Quantum Hermite-Hadamard Inequalities for Differentiable Convex Functions

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Abstract. In this paper, we establish some new quantum Hermite-Hadamard type inequalities for differentiable convex functions by using the q^{κ_2}-quantum integral. The results presented in this paper extend the results of Bermudo et al. (On q-Hermite-Hadamard inequalities for general convex functions, Acta Mathematica Hungarica, 2020, 162, 363-374). Finally, we give some examples to show validation of new results of this paper.

1. Introduction

The Hermite-Hadamard inequality discovered by C. Hermite and J. Hadamard (see, e.g., [17], [29, p.137]) is one of the most well-established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that if $\mathcal{F} : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$, then

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)d\varkappa \leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if $\mathcal{F}$ is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied. In recent years, inequalities have been studied for different types of fractional integrals, different types of convexity, (see, [23, 30–33]).

On the other hand, several works in the field of q-analysis are being carried out, beginning with Euler, in order to achieve mastery in the mathematics that drives quantum computing. The link between physics and mathematics is referred to as q-calculus. It has a wide range of applications in mathematics, including number theory, combinatorics, orthogonal polynomials, basic hyper geometric functions, and other disciplines, as well as mechanics, relativity theory, and quantum theory [19, 22]. Quantum calculus also has many applications in quantum information theory which is an interdisciplinary area that encompasses computer

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science, information theory, philosophy, and cryptography, among other areas [11, 12]. Euler is thought to be the inventor of this significant branch of mathematics. In Newton's work on infinite series, he used the q-parameter. Jackson [18, 20] was the first to present the q-calculus that knew without limits calculus in a methodical manner. In 1966, Al-Salam [9] introduced a q-analogue of the q-fractional integral and q-Riemann-Liouville fractional. Since then, the related research has gradually increased. In particular, in 2013, Tariboon and Ntouyas introduced ${}_{\kappa_1}D_q$-difference operator and q_{κ_1}-integral in [36]. In 2020, Bermudo et al. introduced the notion of ${}^{\kappa_2}D_q$ derivative and q^{κ_2}-integral in [10].

Many integral inequalities have been studied using quantum and post-quantum integrals for various types of functions. For example, in [1, 5, 7, 8, 10, 13, 14, 21, 25, 26], the authors used ${}_{\kappa_1}D_q$,${}^{\kappa_2}D_q$-derivatives and $q_{\kappa_1}, q^{\kappa_2}$-integrals to prove Hermite-Hadamard integral inequalities and their left-right estimates for convex and coordinated convex functions. In [27], Noor et al. presented a generalized version of quantum integral inequalities. For generalized quasi-convex functions, Nwaeze et al. proved certain parameterized quantum integral inequalities in [28]. Khan et al. proved quantum Hermite-Hadamard inequality using the green function in [24]. Budak et al. [15], Ali et al. [2, 3] and Vivas-Cortez et al. [37] developed new quantum Simpson's and quantum Newton's type inequalities for convex and coordinated convex functions. For quantum Ostrowski's inequalities for convex and co-ordinated convex functions on can consult [4, 6, 16].

Inspired by the ongoing studies, we use the q-integrals to develop some new Hermite-Hadamard type inequalities for differentiable functions. We also discuss some special case of newly established inequalities and obtain new inequalities.

The following is the structure of this paper: Section 2 provides a brief overview of the fundamentals of q-calculus as well as other related studies in this field. In Section 3, we prove different refinements of Hermite-Hadamard type inequalities for differentiable convex functions. In Section 4, we give some examples to show the validation of newly established inequalities. Section 5 concludes with some recommendations for future research.

2. Preliminaries of q-Calculus and Some Inequalities

In this section, we recall some basic notions, notations, and results about the quantum calculus. Additionally, here and further we use the following notation (see, [22]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1}, \;\; q \in (0,1).$$

In [20], Jackson gave the q-Jackson integral from 0 to κ_2 for $0 < q < 1$ as follows:

$$\int_0^{\kappa_2} \mathcal{F}(\varkappa) \; d_q\varkappa \; = (1-q)\,\kappa_2 \sum_{n=0}^{\infty} q^n \mathcal{F}(\kappa_2 q^n) \tag{2}$$

provided the sum converge absolutely. Moreover, he gave the q-Jackson integral in a generic interval $[\kappa_1, \kappa_2]$ as:

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \; d_q\varkappa \; = \int_0^{\kappa_2} \mathcal{F}(\varkappa) \; d_q\varkappa \; - \int_0^{\kappa_1} \mathcal{F}(\varkappa) \; d_q\varkappa \; .$$

Definition 2.1. *[36] For a continuous function $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$, the q_{κ_1}-derivative of $\mathcal{F}$ at $\varkappa \in [\kappa_1, \kappa_2]$ is characterized by the expression*

$${}_{\kappa_1}D_q\mathcal{F}(\varkappa) \;\; = \frac{\mathcal{F}(\varkappa) - \mathcal{F}(q\varkappa + (1-q)\,\kappa_1)}{(1-q)(\varkappa - \kappa_1)}, \; \varkappa \neq \kappa_1. \tag{3}$$

For $\varkappa = \kappa_1$, we state ${}_{\kappa_1}D_q\mathcal{F}(\kappa_1) = \lim_{\varkappa \to \kappa_1} {}_{\kappa_1}D_q\mathcal{F}(\varkappa)$ if it exists and it is finite.

Definition 2.2. *[10] For a continuous function $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$, the q^{κ_2}-derivative of $\mathcal{F}$ at $\varkappa \in [\kappa_1, \kappa_2]$ is characterized by the expression*

$$^{\kappa_2}D_q\mathcal{F}(\varkappa) = \frac{\mathcal{F}(q\varkappa + (1-q)\kappa_2) - \mathcal{F}(\varkappa)}{(1-q)(\kappa_2 - \varkappa)}, \ \varkappa \neq \kappa_2.$$

For $\varkappa = \kappa_2$, we state $^{\kappa_2}D_q\mathcal{F}(\kappa_1) = \lim_{\varkappa\to\kappa_2} {}^{\kappa_2}D_q\mathcal{F}(\varkappa)$ if it exists and it is finite.

Definition 2.3. *[36] Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a continuous function. Then, the q_{κ_1}-definite integral on $[\kappa_1, \kappa_2]$ is defined as:*

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}_{\kappa_1}d_q\varkappa = (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\kappa_2 + (1-q^n)\kappa_1) = (\kappa_2-\kappa_1)\int_0^1 \mathcal{F}((1-t)\kappa_1 + t\kappa_2)\ d_qt\ .$$

In [7], Alp et al. proved the following q_{κ_1}-Hermite-Hadamard inequalities for convex functions in the setting of quantum calculus:

Theorem 2.4. *Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex differentiable function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then q-Hermite-Hadamard inequalities are as follows:*

$$\mathcal{F}\left(\frac{q\kappa_1 + \kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}_{\kappa_1}d_q\varkappa \leq \frac{q\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{[2]_q}. \tag{4}$$

In [7] and [26], the authors established some bounds for the left and right hand sides of the inequality (4).
On the other hand, Bermudo et al. gave the following new definition and related Hermite-Hadamard type inequalities:

Definition 2.5. *[10] Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a continuous function. Then, the q^{κ_2}-definite integral on $[\kappa_1, \kappa_2]$ is defined as:*

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa = (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\kappa_1 + (1-q^n)\kappa_2) = (\kappa_2-\kappa_1)\int_0^1 \mathcal{F}(t\kappa_1 + (1-t)\kappa_2)\ d_qt\ .$$

Theorem 2.6. *[10] Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then, q-Hermite-Hadamard inequalities are as follows:*

$$\mathcal{F}\left(\frac{\kappa_1 + q\kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa \leq \frac{\mathcal{F}(\kappa_1) + q\mathcal{F}(\kappa_2)}{[2]_q}. \tag{5}$$

From Theorem 2.4 and Theorem 2.6, one can the following inequalities:

Corollary 2.7. *[10] For any convex function $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ and $0 < q < 1$, we have*

$$\mathcal{F}\left(\frac{q\kappa_1 + \kappa_2}{[2]_q}\right) + \mathcal{F}\left(\frac{\kappa_1 + q\kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2 - \kappa_1}\left\{\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}_{\kappa_1}d_q\varkappa + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa\right\} \leq \mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2) \tag{6}$$

and

$$\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{2(\kappa_2 - \kappa_1)}\left\{\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}_{\kappa_1}d_q\varkappa + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa\right\} \leq \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}. \tag{7}$$

3. Main Results

In this section, we present some new Hermite-Hadamard type inequalities for differentiable convex functions by using the q^{κ_2}-quantum integral.

Theorem 3.1. *Suppose that $\mathcal{F} : [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}$ is a differentiable convex function on (κ_1, κ_2) for $\kappa_3 \in (\kappa_1, \kappa_2)$, and let q be a constant with $0 < q < 1$. Then, we have the following inequalities for the q^{κ_2}-quantum integral*

$$\begin{aligned} & \mathcal{F}\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right) \\ & +\mathcal{F}'\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right)\left(\frac{q(2\kappa_2-\kappa_1-\kappa_3)+\kappa_1-\kappa_2}{[2]_q}\right) \\ \leq\ & \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa \\ \leq\ & \frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}. \end{aligned} \tag{8}$$

Proof. Since the function $\mathcal{F}$ is differentiable on (κ_1, κ_2), there exists a tangent line at the point $\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q} \in (\kappa_1, \kappa_2)$, we have

$$\begin{aligned} h(\varkappa) = & \mathcal{F}\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right) \\ & +\mathcal{F}'\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right)\left(\varkappa-\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right) \end{aligned} \tag{9}$$

Since $\mathcal{F}$ is a convex function on $[\kappa_1, \kappa_2]$, it follows that $h(\varkappa) \leq \mathcal{F}(\varkappa)$ for all $\varkappa \in [\kappa_1, \kappa_2]$. After q^{κ_2}-integrating to (9) on $[\kappa_1, \kappa_2]$, we have

$$\begin{aligned} & \int_{\kappa_1}^{\kappa_2} h(\varkappa)\ {}^{\kappa_2}d_q\varkappa \\ =\ & \int_{\kappa_1}^{\kappa_2}\left[\mathcal{F}\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right)+\mathcal{F}'\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right)\right. \\ & \left.\times\left(\varkappa-\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right)\right]\ {}^{\kappa_2}d_q\varkappa \\ =\ & (\kappa_2-\kappa_1)\mathcal{F}\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right) \\ & +\mathcal{F}'\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right)\left(\int_{\kappa_1}^{\kappa_2}\varkappa\ {}^{\kappa_2}d_q\varkappa-(\kappa_2-\kappa_1)\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right) \\ =\ & (\kappa_2-\kappa_1)\mathcal{F}\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right) \\ & +\mathcal{F}'\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right)\left((\kappa_2-\kappa_1)\frac{(\kappa_1+q\kappa_2)}{[2]_q}-(\kappa_2-\kappa_1)\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right) \end{aligned}$$

$$= (\kappa_2-\kappa_1)\left[\mathcal{F}\left(\frac{(1-q)(\kappa_1+\kappa_3)+q\kappa_2}{[2]_q}\right)+\mathcal{F}'\left(\frac{(1-q)(\kappa_1+\kappa_3)+q\kappa_2}{[2]_q}\right)\left(\frac{q(2\kappa_2-\kappa_1-\kappa_3)+\kappa_1-\kappa_2}{[2]_q}\right)\right]$$

$$\leq \int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,{}^{\kappa_2}d_q\varkappa.$$

On the other hand, since $\mathcal{F}$ is a convex function, we obtain

$$\begin{aligned}
\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,{}^{\kappa_2}d_q\varkappa &= \frac{1}{\kappa_2-\kappa_1}\left[(1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\kappa_1+(1-q^n)\kappa_2)\right] \\
&= (1-q)\sum_{n=0}^{\infty}q^n\mathcal{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right) \\
&\leq (1-q)\sum_{n=0}^{\infty}q^n\left[q^n\mathcal{F}(\kappa_1)+(1-q^n)\mathcal{F}(\kappa_2)\right] \\
&= (1-q)\left[\frac{\mathcal{F}(\kappa_1)}{1-q^2}+\frac{\mathcal{F}(\kappa_2)}{1-q}+\frac{\mathcal{F}(\kappa_2)}{1-q^2}\right] \\
&= \frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}.
\end{aligned}$$

The proof is complete. □

Corollary 3.2. *In Theorem 8, if* $q\in\left(0,\frac{\kappa_2-\kappa_1}{2\kappa_2-\kappa_1-\kappa_3}\right]$ *and* $\mathcal{F}'(\kappa_3)=0$, *we can reduce the left-hand side of Theorem 8 as:*

$$\mathcal{F}\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right)\leq\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,{}^{\kappa_2}d_q\varkappa\leq\frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}.$$

Proof. Since $\mathcal{F}'(\kappa_3)=0$ and $q\in\left(0,\frac{\kappa_2-\kappa_1}{2\kappa_2-\kappa_1-\kappa_3}\right]$ then $\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\in(\kappa_1,\kappa_3)$. Thus we have

$$\mathcal{F}'\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right)\left(\frac{q(2\kappa_2-\kappa_1-\kappa_3)+\kappa_1-\kappa_2}{[2]_q}\right)\geq 0.$$

This completes the proof. □

Theorem 3.3. *Suppose that* $\mathcal{F}:[\kappa_1,\kappa_2]\longrightarrow\mathbb{R}$ *is a differentiable convex function on* (κ_1,κ_2) *such that* $\mathcal{F}'(\kappa_3)=0$ *for* $\kappa_3\in(\kappa_1,\kappa_2)$, *and let q be a constant with* $0<q<1$. *Then, we have the following inequalities for the* q^{κ_2}*-quantum integral*

$$\mathcal{F}\left(\frac{(1-q)\kappa_1+q(\kappa_3+\kappa_2)}{[2]_q}\right)+\mathcal{F}'\left(\frac{(1-q)\kappa_1+q(\kappa_3+\kappa_2)}{[2]_q}\right)\left(\frac{q(\kappa_1-\kappa_3)}{[2]_q}\right) \tag{10}$$

$$\leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,{}^{\kappa_2}d_q\varkappa$$

$$\leq \frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}.$$

Proof. Since the function $\mathcal{F}$ is differentiable on (κ_1, κ_2), there exists a tangent line at the point $\frac{(1-q)\kappa_1+q(\kappa_3+\kappa_2)}{[2]_q} \in (\kappa_1, \kappa_2)$, we get

$$k(\varkappa) = \mathcal{F}\left(\frac{(1-q)\,\kappa_1 + q(\kappa_3+\kappa_2)}{[2]_q}\right) + \mathcal{F}'\left(\frac{(1-q)\,\kappa_1 + q(\kappa_3+\kappa_2)}{[2]_q}\right)\left(\varkappa - \frac{(1-q)\,\kappa_1 + q(\kappa_3+\kappa_2)}{[2]_q}\right). \tag{11}$$

Since $\mathcal{F}$ is a convex function on $[\kappa_1, \kappa_2]$, it follows that $k(\varkappa) \leq \mathcal{F}(\varkappa)$ for all $\varkappa \in [\kappa_1, \kappa_2]$. After q^{κ_2}-integrating to (11) on $[\kappa_1, \kappa_2]$, we have

$$\begin{aligned}
&\int_{\kappa_1}^{\kappa_2} k(\varkappa)\, {}^{\kappa_2}d_q\varkappa \\
=\ & \int_{\kappa_1}^{\kappa_2} \left[\mathcal{F}\left(\frac{(1-q)\,\kappa_1 + q\,(\kappa_3+\kappa_2)}{[2]_q}\right) + \mathcal{F}'\left(\frac{(1-q)\,\kappa_1 + q\,(\kappa_3+\kappa_2)}{[2]_q}\right)\right. \\
& \left.\times\left(\varkappa - \frac{(1-q)\,\kappa_1 + q\,(\kappa_3+\kappa_2)}{[2]_q}\right)\right] {}^{\kappa_2}d_q\varkappa \\
=\ & (\kappa_2-\kappa_1)\mathcal{F}\left(\frac{(1-q)\,\kappa_1 + q\,(\kappa_3+\kappa_2)}{[2]_q}\right) \\
& + \mathcal{F}'\left(\frac{(1-q)\,\kappa_1 + q\,(\kappa_3+\kappa_2)}{[2]_q}\right)\int_{\kappa_1}^{\kappa_2}\left(\varkappa\, {}^{\kappa_2}d_q\varkappa - (\kappa_2-\kappa_1)\frac{(1-q)\,\kappa_1 + q\,(\kappa_3+\kappa_2)}{[2]_q}\right) \\
=\ & (\kappa_2-\kappa_1)\mathcal{F}\left(\frac{(1-q)\,\kappa_1 + q\,(\kappa_3+\kappa_2)}{[2]_q}\right) \\
& + \mathcal{F}'\left(\frac{(1-q)\,\kappa_1 + q\,(\kappa_3+\kappa_2)}{[2]_q}\right)\left((\kappa_2-\kappa_1)\left(\frac{(\kappa_1+q\kappa_2)}{[2]_q} - \frac{(1-q)\,\kappa_1 + q\,(\kappa_3+\kappa_2)}{[2]_q}\right)\right) \\
=\ & (\kappa_2-\kappa_1)\left[\mathcal{F}\left(\frac{(1-q)\,\kappa_1 + q\,(\kappa_3+\kappa_2)}{[2]_q}\right) + \mathcal{F}'\left(\frac{(1-q)\,\kappa_1 + q\,(\kappa_3+\kappa_2)}{[2]_q}\right)\right. \\
& \left.\times\left(\frac{q(\kappa_1-\kappa_3)}{[2]_q}\right)\right] \\
\leq\ & \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\, {}^{\kappa_2}d_q\varkappa.
\end{aligned}$$

The proof is completed. □

Corollary 3.4. *In Theorem 10, if* $q \in \left(0, \frac{\kappa_3-\kappa_1}{\kappa_2-\kappa_1}\right]$ *and* $\mathcal{F}'(\kappa_3) = 0$*, We can reduce the left-hand side of Theorem 10 as:*

$$\mathcal{F}\left(\frac{(1-q)\,\kappa_1 + q\,(\kappa_3+\kappa_2)}{[2]_q}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\, {}^{\kappa_2}d_q\varkappa \leq \frac{\mathcal{F}(\kappa_1) + q\mathcal{F}(\kappa_2)}{[2]_q}.$$

Proof. Since $\mathcal{F}'(\kappa_3) = 0$ and $q \in \left(0, \frac{\kappa_3-\kappa_1}{\kappa_2-\kappa_1}\right]$ then $\frac{(1-q)\kappa_1+q(\kappa_3+\kappa_2)}{[2]_q} \in (\kappa_1, \kappa_3)$. Thus, we have $\mathcal{F}'\left(\frac{(1-q)\kappa_1+q(\kappa_3+\kappa_2)}{[2]_q}\right)\left(\frac{q(\kappa_1-\kappa_3)}{[2]_q}\right) \geq 0$, which completes the proof. □

Theorem 3.5. *[Generalized q^{κ_2}-Hermite-Hadamard inequality for convex differentiable functions].Let $\mathcal{F} : [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}$ be a differentiable convex function on (κ_1, κ_2) such that $\mathcal{F}'(\kappa_3) = 0$ for $\kappa_3 \in (\kappa_1, \kappa_2)$, and $0 < q < 1$. Then, we have the following inequalities for the q^{κ_2}-quantum integral*

$$\max\{I_1, I_2\} \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa \leq \frac{\mathcal{F}(\kappa_1) + q\mathcal{F}(\kappa_2)}{[2]_q} \tag{12}$$

where

$$\begin{aligned} I_1 &= \mathcal{F}\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right) + \mathcal{F}'\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right)\left(\frac{q(2\kappa_2-\kappa_1-\kappa_3)+\kappa_1-\kappa_2}{[2]_q}\right) \\ I_2 &= \mathcal{F}\left(\frac{(1-q)\kappa_1+q(\kappa_3+\kappa_2)}{[2]_q}\right) + \mathcal{F}'\left(\frac{(1-q)\kappa_1+q(\kappa_3+\kappa_2)}{[2]_q}\right)\left(\frac{q(\kappa_1-\kappa_3)}{[2]_q}\right) \end{aligned}$$

Proof. A combination (8) and(10) yields (12). Thus, the proof is complete. □

4. Examples

In this section, we give some examples to show the validation of newly established inequalities in the previous section.

Example 4.1. *Define the function $f(x) = x^2$ on $[-1, 3]$, and let $q \in (0, 1)$. Applying Theorem 3.1 with $\kappa_1 = -1$, $\kappa_2 = 3$, and $\kappa_3 = 0$, the left-hand side becomes:*

$$\begin{aligned} & \mathcal{F}\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right) \\ & +\mathcal{F}'\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right)\left(\frac{q(2\kappa_2-\kappa_1-\kappa_3)+\kappa_1-\kappa_2}{[2]_q}\right) \\ & -\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa \\ = {} & \mathcal{F}\left(\frac{3-4q}{1+q}\right) + f'\left(\frac{3-4q}{1+q}\right)\left(\frac{7q-4}{1+q}\right) - \frac{1}{4}\int_{-1}^{3} x^2\ {}^3d_q\varkappa \\ = {} & -\frac{1}{(q+1)^2(q^2+q+1)}\left(49q^4 - 7q^3 + 9q^2 - 24q + 16\right) \leq 0. \end{aligned}$$

For the right hand side, we have

$$\begin{aligned} & \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa - \frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q} \\ = {} & -16\frac{q^2}{q^3+2q^2+2q+1} \leq 0. \end{aligned}$$

Example 4.2. *Define function $f(x) = x^2$ on $[-1, 1]$, and let $q \in (0, 1)$. Applying Corollary 3.2 with $\kappa_1 = -1$, $\kappa_2 = 1$ and $\kappa_3 = 0$, the left hand-side becomes:*

$$\mathcal{F}\left(\frac{q(\kappa_1+\kappa_3)+(1-q)\kappa_2}{[2]_q}\right) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa$$

$$= \quad -\frac{q}{(q+1)^2(q^2+q+1)}\left(6-3q^3-q^2-q\right) \leq 0.$$

The right hand side is:

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa - \frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}$$

$$= \quad -4\frac{q^2}{q^3+2q^2+2q+1} \leq 0.$$

Example 4.3. *Define functions $f(x) = x^2$ on $[-3, 1]$, and let $q \in (0, 1)$. Applying Theorem 3.3 with $\kappa_1 = -3$, $\kappa_2 = 1$, and $\kappa_3 = 0$, the left-hand side becomes:*

$$\mathcal{F}\left(\frac{(1-q)\kappa_1+q(\kappa_3+\kappa_2)}{[2]_q}\right)+\mathcal{F}'\left(\frac{(1-q)\kappa_1+q(\kappa_3+\kappa_2)}{[2]_q}\right)\left(\frac{q(\kappa_1-\kappa_3)}{[2]_q}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa$$

$$= \quad -\frac{q}{(q+1)^2(q^2+q+1)}\left(9q^3+9q^2+9q+16\right) \leq 0.$$

The right hand side is:

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa - \frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}$$

$$= \quad -16\frac{q^2}{q^3+2q^2+2q+1} \leq 0.$$

Example 4.4. *Define function $f(x) = x^2$ on $[-1, 1]$, and let $q \in (0, 1)$. Applying Corollary 3.4 with $\kappa_1 = -1$, $\kappa_2 = 1$ and $\kappa_3 = 0$, the left hand-side becomes:*

$$\mathcal{F}\left(\frac{(1-q)\kappa_1+q(\kappa_3+\kappa_2)}{[2]_q}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa$$

$$= \quad -\frac{q}{(q+1)^2(q^2+q+1)}\left(6-3q^3-q^2-q\right) \leq 0.$$

The right hand side is:

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa - \frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}$$

$$= \quad -4\frac{q^2}{q^3+2q^2+2q+1} \leq 0.$$

5. Conclusions

In this paper, using the q^{κ_2}-integrals, we developed certain Hermite-Hadamard type inequalities for differentiable convex functions. The findings of this study expand previously established results in the area of quantum Hermite-Hadamard inequalities. In Section 4, we also demonstrated the validation of newly proven results using some examples. It is an intriguing and novel problem, and future scholars will be able to demonstrate analogous inequalities for many types of convexities in their future study.

We would like to point out that in [34, p. 340] and [35, pp. 1511-1512], the authors show that since the simple and forced-in parameter p is unnecessary, converting known q-calculus results to (p, q)-calculus is unnecessary.

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