Filomat 36:5 (2022), 1433–1450
https://doi.org/10.2298/FIL2205433W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# $L$-topological Derived Neighborhood Relations and $L$-topological Derived Remotehood Relations

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**Abstract.** Fuzzy relations and fuzzy derived operators are useful tools to characterize fuzzy mathematical structures such as fuzzy topology, fuzzy convexity, fuzzy matroid and fuzzy convergence structure. In this paper, notions of $L$-topological neighborhood relation space, $L$-topological derived neighborhood relation space and $L$-topological derived neighborhood space are introduced. It is proved that all of these spaces are categorically isomorphic to $L$-topological internal relation space and $L$-topological neighborhood space. Also, notions of $L$-topological remotehood relation space, $L$-topological derived remotehood relation space and $L$-topological derived remotehood space are introduced. It is proved that all of these spaces are categorically isomorphic to $L$-topological enclosed relation space and $L$-topological remotehood space.

## 1. Introduction

Since the concept of fuzzy set was introduced in 1965 [37], many classic mathematical structures such as topology, matroid, convergence structure and convex structure have been extended into fuzzy setting [1, 10, 18, 19, 21, 22, 25]. In order to describe these structures, a great many papers have being devoting on characterizations of these structures such as fuzzy topology [3, 9, 35, 36, 39], fuzzy convergence structure [5–7, 11, 13, 14, 34], fuzzy matroid [4, 19, 31, 40] and fuzzy convex structure [12, 13, 15–17, 21, 22, 27–29, 32, 33, 38].

Fuzzy relations and fuzzy derived operators are useful tools to characterize fuzzy mathematical structures. Shi et al introduced $L$-topological internal relation and $L$-topological enclosed relation by which they characterized $L$-topology [23]. Later, they further introduced $(L, M)$-fuzzy topological internal relation and $(L, M)$-fuzzy topological enclosed relation by which they characterized $(L, M)$-fuzzy topology [24]. Liao et al introduced $L$-convex enclosed relation and characterized $L$-convex structure. Meanwhile, they further introduced $L$-topological-convex enclosed relation by which they characterized $L$-topological-convex structure [8]. Wu et al introduced $(L, M)$-fuzzy convex enclosed relation and characterized $(L, M)$-fuzzy

2020 *Mathematics Subject Classification*. Primary 54A40; Secondary 26A51
*Keywords*. $L$-topological derived internal space, $L$-topological derived enclosed space, $L$-topological derived interior space, $L$-topological derived closure space
Received: 23 March 2021; Revised: 12 July 2021; Accepted: 15 July 2021
Communicated by Ljubiša D.R. Kočinac
Research supported by University Science Research Project of Anhui Province (KJ2020A0056), Doctoral Scientific Research Foundation of Anhui Normal University (751966), Hunan Educational Committee Project (18A474,19C0822) and Science Foundation of Hunan Province (2018JJ3192;2019JJ40089).
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convex structure. In addition, they further introduced $(L,M)$-fuzzy topological-convex enclosed relation and characterized $(L,M)$-fuzzy topological-convex structure [29]. Chen and Shen introduced $M$-fuzzifying derived operator by which they characterize $M$-fuzzifying convex structure [2, 17]. Xin and Zhong introduced $M$-fuzzifying derived operator by which they characterize $M$-fuzzifying matroid [31, 40]. Recently, Wu et al introduced $L$-topological derived internal relation and $L$-topological derived enclosed relation by which they characterized $L$-topology [30].

As mentioned above, $L$-topology can be characterized by both $L$-topological internal relation and $L$-topological enclosed relation. Then, a natural question arises: is it possible to define $L$-topological neighborhood relation or $L$-topological derived neighborhood relation which can be used to characterize $L$-topological internal relation or $L$-topological neighborhood system? Similarly, is it possible to define $L$-topological remotehood relation or $L$-topological derived remotehood relation which can be used to characterize $L$-topological enclosed relation or $L$-topological remotehood system?

The aim of this paper is to solve the above problems. The arrangement of this paper is as follows. In Section 2, we recall some basic notions related to $L$-topological spaces. In Section 3, we introduce $L$-topological neighborhood relation space by which we characterize $L$-topological internal relation space and $L$-topological neighborhood space. In Section 4, we introduce $L$-topological derived neighborhood relation space and $L$-topological derived neighborhood space by which we characterize $L$-topological neighborhood relation space and $L$-topological derived internal relation space. In Section 5, we introduce $L$-topological remotehood relation space by which we characterize $L$-topological enclosed relation space and $L$-topological remotehood space. In Section 6, we introduce $L$-topological derived remotehood relation space and $L$-topological derived remotehood space by which we characterize $L$-topological derived remotehood relation space and $L$-topological derived enclosed relation space. In the conclusion section, we present a simple example to show different relations mentioned.

## 2. Preliminaries

In this paper, $X$ and $Y$ are nonempty sets. The power set of $X$ is denoted by $2^X$. $L$ is a completely distributive lattice with an inverse involution $'$. The smallest (resp. largest) element in $L$ is denoted by $\bot$ (resp. $\top$). An element $a \in L$ is called a co-prime, if for all $b, c \in L$, $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. The set of all co-primes in $L\backslash\{\bot\}$ is denoted by $J(L)$. For any $a \in L$, there is an $L_1 \subseteq J(L)$ such that $a = \bigvee_{b \in L_1} b$. A binary relation $\prec$ on $L$ is defined by $a \prec b$ if and only if for each $L_1 \subseteq L$, $b \leq \bigvee L_1$ implies some $d \in L_1$ with $a \leq d$. The mapping $\beta : L \to 2^L$, defined by $\beta(a) = \{b : b \prec a\}$, satisfies $\beta(\bigvee_{i\in I} a_i) = \bigcup_{i\in I} \beta(a_i)$ for any $\{a_i\}_{i\in I} \subseteq L$. For any $a \in L$, we denote $\beta^*(a) = \beta(a) \cap J(L)$. We have $a = \bigvee \beta(a) = \bigvee \beta^*(a)$, $\beta(a) = \bigcup_{b\in\beta^*(a)} \beta(b)$ and $\beta^*(a) = \bigcup_{b\in\beta^*(a)} \beta^*(b)$ [20, 26].

An $L$-fuzzy set on $X$ is a mapping $A : X \to L$. The set of all $L$-fuzzy sets on $X$ is denoted by $L^X$. The smallest (resp, largest) element in $L^X$ is denoted by $\underline{\bot}$ (resp. $\underline{\top}$). An $L$-fuzzy point $x_\lambda$ ($\lambda \in L\backslash\{\bot\}$) is an $L$-fuzzy set defined by $x_\lambda(x) = \lambda$ and $x_\lambda(y) = \bot$ for any $y \in X\backslash\{x\}$. The set of all $L$-fuzzy points on $L^X$ is denoted by $Pt(L^X)$. In addition, we denote $J(L^X) = \{x_\lambda \in Pt(L^X) : \lambda \in J(L)\}$. For a mapping $f : X \to Y$, the $L$-fuzzy mapping $f_L^{\to} : L^X \to L^Y$ is defined by $f_L^{\to}(A)(y) = \bigvee\{A(x) : f(x) = y\}$ for $A \in L^X$ and $y \in Y$, and the mapping $f_L^{\leftarrow} : L^Y \to L^X$ is defined by $f_L^{\leftarrow}(B)(x) = B(f(x))$ for $B \in L^Y$ and $x \in X$ [20, 25].

Next, we recall some basic notions and results related to $L$-topological spaces.

**Definition 2.1.** ([25]) A subset $\mathcal{T} \subseteq L^X$ is called an $L$-topology on $L^X$ and $(X, \mathcal{T})$ is called an $L$-topological space if

(LT1) $\underline{\top}, \underline{\bot} \in \mathcal{T}$;

(LT2) $\forall \{A_i\}_{i\in I} \subseteq \mathcal{T}$, $\bigvee_{i\in I} A_i \in \mathcal{T}$;

(LT3) $\forall A, B \in \mathcal{T}$, $A \wedge B \in \mathcal{T}$.

**Theorem 2.2.** ([25]) *Let $(X, \mathcal{T})$ be an $L$-topological space.*

*(1) The $L$-topological closure operator $Cl_{\mathcal{T}} : L^X \to L^X$ of $\mathcal{T}$ is defined by $Cl_{\mathcal{T}}(A) = \bigwedge\{B \in L^X : A \leq B, B' \in \mathcal{T}\}$ for any $A \in L^X$. It satisfies*

*(LTCl1) $Cl_{\mathcal{T}}(\underline{\bot}) = \underline{\bot}$;*

*(LTCl2)* $A \leq Cl_{\mathcal{T}}(A)$;

*(LTCl3)* $Cl_{\mathcal{T}}(Cl_{\mathcal{T}}(A)) = Cl_{\mathcal{T}}(A)$;

*(LTCl4)* $Cl_{\mathcal{T}}(A \vee B) = Cl_{\mathcal{T}}(A) \vee Cl_{\mathcal{T}}(B)$.

*Conversely, if an operator* $Cl : L^X \to L^X$ *satisfies (LTCl1)–(LTCl4), then the set* $\mathcal{T}_{Cl} = \{A \in L^X : Cl(A') = A'\}$ *is an L-topology satisfying* $Cl_{\mathcal{T}_{Cl}} = Cl$.

*(2) The L-topological interior operator* $Int_{\mathcal{T}} : L^X \to L^X$ *of* $\mathcal{T}$ *is defined by* $Int_{\mathcal{T}}(A) = \bigvee\{B \in \mathcal{T} : B \leq A\}$ *for any* $A \in L^X$. *It satisfies*

*(LTInt1)* $Int_{\mathcal{T}}(\underline{\top}) = \underline{\top}$;

*(LTInt2)* $Int_{\mathcal{T}}(A) \leq A$;

*(LTInt3)* $Int_{\mathcal{T}}(Int_{\mathcal{T}}(A)) = Int_{\mathcal{T}}(A)$;

*(LTInt4)* $Int_{\mathcal{T}}(A \wedge B) = Int_{\mathcal{T}}(A) \wedge Int_{\mathcal{T}}(B)$.

*Conversely, if an operator* $Int : L^X \to L^X$ *satisfies (LTInt1)–(LTInt4), then the set* $\mathcal{T}_{Int} = \{A \in L^X : Int(A) = A\}$ *is an L-topology satisfying* $Int_{\mathcal{T}_{Int}} = Int$.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be $L$-topological spaces. A mapping $f : X \to Y$ is called an $L$-continuous mapping, if $f_L^{\leftarrow}(A) \in \mathcal{T}_X$ for any $A \in \mathcal{T}_Y$. The category of $L$-topological spaces and $L$-continuous mappings is denoted by $L$-**TOP** [20].

**Definition 2.3.** ([20]) A family $\mathcal{N} = \{\mathcal{N}_{x_\lambda} \subseteq L^X : x_\lambda \in J(L^X)\}$ is called an $L$-topological neighborhood system on $L^X$ and the pair $(X, \mathcal{N})$ is called an $L$-topological neighborhood space, if for any $x_\lambda \in J(L^X)$,

(LTN1) $\underline{\top} \in \mathcal{N}_{x_\lambda}$ and $\underline{\bot} \notin \mathcal{N}_{x_\lambda}$;

(LTN2) $A \in \mathcal{N}_{x_\lambda}$ implies $x_\lambda \leq A$;

(LTN3) $A \in \mathcal{N}_{x_\lambda}$ implies some $B \in \mathcal{N}_{x_\lambda}$ such that $B \in \mathcal{N}_{y_\mu}$ for any $y_\mu \in \beta^*(B)$;

(LTN4) $A \wedge B \in \mathcal{N}_{x_\lambda}$ if and only if $A, B \in \mathcal{N}_{x_\lambda}$.

Let $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$ be $L$-topological neighborhood spaces. A mapping $f : X \to Y$ is called an $L$-topological neighborhood preserving mapping if $B \in \mathcal{N}_{f_L^{\rightarrow}(x_\lambda)}$ implies $f_L^{\leftarrow}(B) \in \mathcal{N}_{x_\lambda}$ for all $x_\lambda \in J(L^X)$ and $B \in L^Y$. The category of $L$-topological neighborhood spaces and $L$-topological neighborhood preserving mapping is denoted by $L$-**TNS** [20].

**Definition 2.4.** ([20]) A family $\mathcal{R} = \{\mathcal{R}_{x_\lambda} \subseteq L^X : x_\lambda \in J(L^X)\}$ is called an $L$-topological remotehood system on $L^X$ and the pair $(X, \mathcal{R})$ is called an $L$-topological remotehood space, if for any $x_\lambda \in J(L^X)$,

(LTRN1) $\underline{\bot} \in \mathcal{R}_{x_\lambda}$ and $\underline{\top} \notin \mathcal{R}_{x_\lambda}$;

(LTRN2) $A \in \mathcal{R}_{x_\lambda}$ implies $x_\lambda \nleq A$;

(LTRN3) $A \in \mathcal{R}_{x_\lambda}$ implies some $B \in \mathcal{R}_{x_\lambda}$ such that $A \leq B \in \mathcal{R}_{y_\mu}$ for any $y_\mu \nleq B$;

(LTRN4) $A \vee B \in \mathcal{R}_{x_\lambda}$ if and only if $A, B \in \mathcal{R}_{x_\lambda}$.

Let $(X, \mathcal{R}_X)$ and $(Y, \mathcal{R}_Y)$ be $L$-topological remotehood spaces. A mapping $f : X \to Y$ is called an $L$-topological remotehood preserving mapping if $f_L^{\leftarrow}(B) \in \mathcal{R}_{x_\lambda}$ for all $x_\lambda \in J(L^X)$ and $B \in \mathcal{R}_{f_L^{\rightarrow}(x_\lambda)}$. The category of $L$-topological remotehood spaces and $L$-topological remotehood preserving mapping is denoted by $L$-**TRNS** [20].

**Definition 2.5.** ([23]) A binary relation $\preccurlyeq$ on $L^X$ is called an $L$-topological enclosed relation and the pair $(X, \preccurlyeq)$ is called an $L$-topological enclosed relation space, if $\preccurlyeq$ satisfies

(LTER1) $\underline{\bot} \preccurlyeq \underline{\bot}$;

(LTER2) $A \preccurlyeq B$ implies $A \leq B$;

(LTER3) $A \preccurlyeq \bigwedge_{i \in I} B_i$ if and only if $A \preccurlyeq B_i$ for all $i \in I$;

(LTER4) $A \preccurlyeq B$ implies some $C \in L^X$ such that $A \preccurlyeq C \preccurlyeq B$;

(LTER5) $A \vee B \preccurlyeq C$ if and only if $A \preccurlyeq C$ and $B \preccurlyeq C$.

Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be $L$-topological enclosed relation spaces. A mapping $f : X \to Y$ is called an $L$-topological enclosed relation preserving mapping, if $f_L^{\leftarrow}(A) \preccurlyeq_X f_L^{\leftarrow}(B)$ for all $A, B \in L^Y$ with $A \preccurlyeq_Y B$. The category of $L$-topological enclosed relation spaces and $L$-topological enclosed relation preserving mappings is denoted by $L$-**TERS** [23].

**Theorem 2.6.** ([23]) *(1) For an L-topological enclosed relation space* $(X, \preccurlyeq)$*, the operator* $Cl_{\preccurlyeq} : L^X \to L^X$*, defined by* $Cl_{\preccurlyeq}(A) = \bigwedge\{B \in L^X : A \preccurlyeq B\}$ *for any* $A \in L^X$*, is the L-topological closure operator of some L-topology* $\mathcal{T}_{\preccurlyeq}$*.*

*(2) For an L-topological space* $(X, \mathcal{T})$*, the binary operator* $\preccurlyeq_{\mathcal{T}}$*, defined by* $A \preccurlyeq_{\mathcal{T}} B$ *if and only if* $Cl_{\mathcal{T}}(A) \le B$ *for all* $A, B \in L^X$*, is an L-topological enclosed relation.*

*(3) L-***TOP** *is isomorphic to L-***TERS***.*

**Definition 2.7.** ([23]) A binary relation $\preccurlyeq$ on $L^X$ is called an $L$-topological internal relation and the pair $(X, \preccurlyeq)$ is called an $L$-topological internal relation space, if $\preccurlyeq$ satisfies

(LTIR1) $\underline{\top} \preccurlyeq \underline{\top}$;

(LTIR2) $A \preccurlyeq B$ implies $A \le B$;

(LTIR3) $\bigvee_{i \in I} A_i \preccurlyeq B$ if and only if $A_i \preccurlyeq B$ for all $i \in I$;

(LTIR4) $A \preccurlyeq B$ implies some $C \in L^X$ such that $A \preccurlyeq C \preccurlyeq B$;

(LTIR5) $A \preccurlyeq B \wedge C$ if and only if $A \preccurlyeq B$ and $A \preccurlyeq C$.

Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be $L$-topological internal relation spaces. A mapping $f : X \to Y$ is called an $L$-topological internal relation preserving mapping, if $f_L^{\leftarrow}(A) \preccurlyeq_X f_L^{\leftarrow}(B)$ for all $A, B \in L^Y$ with $A \preccurlyeq_Y B$. The category of $L$-topological internal relation spaces and $L$-topological internal relation preserving mappings is denoted by $L$-**TIRS** [23].

**Theorem 2.8.** ([23]) *(1) For an L-topological internal relation space* $(X, \preccurlyeq)$*, the operator* $Int_{\preccurlyeq} : L^X \to L^X$*, defined by* $Int_{\preccurlyeq}(A) = \bigvee\{B \in L^X : B \preccurlyeq A\}$ *for any* $A \in L^X$*, is the L-topological interior operator of some L-topology* $\mathcal{T}_{\preccurlyeq}$*.*

*(2) For an L-topological space* $(X, \mathcal{T})$*, the binary operator* $\preccurlyeq_{\mathcal{T}}$*, defined by* $A \preccurlyeq_{\mathcal{T}} B$ *if and only if* $A \le Int_{\mathcal{T}}(B)$ *for all* $A, B \in L^X$*, is an L-topological internal relation.*

*(3) L-***TOP** *is isomorphic to L-***TIRS***.*

## 3. *L*-topological neighborhood relation spaces

In this section, we introduce $L$-topological neighborhood relation by which we characterize $L$-topological internal relation space and $L$-topological neighborhood space.

**Definition 3.1.** A binary relation $\sqsubseteq$ on $J(L^X) \times L^X$ is called an $L$-topological neighborhood relation on $L^X$ and the pair $(X, \sqsubseteq)$ is called an $L$-topological neighborhood relation space if for all $x_\lambda \in J(L^X)$ and $A, B \in L^X$,

(LTNR1) $x_\lambda \sqsubseteq \underline{\top}$;

(LTNR2) $x_\lambda \sqsubseteq A$ if and only if $x_\lambda \le C \le A$ for some $C \in \psi_{\sqsubseteq}(L^X)$, where $\psi_{\sqsubseteq}(L^X) = \{C \in L^X : \forall y_\mu \in \beta^*(C),\ y_\mu \sqsubseteq C\}$;

(LTNR3) $x_\lambda \sqsubseteq A \wedge B$ if and only if $x_\lambda \sqsubseteq A$ and $x_\lambda \sqsubseteq B$.

Let $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$ be $L$-topological neighborhood relation spaces. A mapping $f : X \to Y$ is called an $L$-topological neighborhood relation preserving mapping, if $f_L^{\rightarrow}(x_\lambda) \sqsubseteq_Y B$ implies $x_\lambda \sqsubseteq_X f_L^{\leftarrow}(B)$ for all $x_\lambda \in J(L^X)$ and $B \in L^Y$.

The category of $L$-topological neighborhood relation spaces and $L$-topological neighborhood relation preserving mappings is denoted by $L$-**TNRS**. Next, we discuss relations between $L$-**TNRS** and $L$-**TIRS**.

**Lemma 3.2.** *Let* $(X, \sqsubseteq)$ *be an L-topological neighborhood relation space. Let* $x_\lambda, x_\eta \in J(L^X)$ *and* $A, B \in L^X$*. We have*

*(1)* $x_\eta \le x_\lambda \sqsubseteq A \le B$ *implies* $x_\eta \sqsubseteq B$*;*

*(2)* $x_\lambda \sqsubseteq A$ *if and only if* $x_\mu \sqsubseteq A$ *for any* $\mu \in \beta^*(\lambda)$*.*

*Proof.* (1) Notice that $x_\lambda \sqsubseteq A$. By (LTNR2), there is a set $D \in \psi_{\sqsubseteq}(L^X)$ such that $x_\lambda \le D \le A$. Thus $x_\eta \le D \le B$. Hence $x_\eta \sqsubseteq B$ by (LTNR2).

(2) Let $x_\lambda \sqsubseteq A$. By (1), we have $x_\mu \sqsubseteq A$ for any $\mu \in \beta^*(\lambda)$. Conversely, assume that $x_\mu \sqsubseteq A$ for any $\mu \in \beta^*(\lambda)$. By (LTNR2), for any $\mu \in \beta^*(\lambda)$ there is a set $D_\mu \in \psi_{\sqsubseteq}(L^X)$ such that $x_\mu \le D_\mu \le A$. Let $D = \bigvee_{\mu \in \beta^*(\lambda)} D_\mu$. To prove that $D \in \psi_{\sqsubseteq}(L^X)$, let $z_\eta \in \beta^*(D)$. Then there is a $\mu \in \beta^*(\lambda)$ such that $z_\eta \in \beta^*(D_\mu)$. By $D_\mu \in \psi_{\sqsubseteq}(L^X)$, we have $z_\eta \sqsubseteq D_\mu \le D$. Hence $z_\eta \sqsubseteq D$ by (1). Therefore $D \in \psi_{\sqsubseteq}(L^X)$. Further, since $x_\lambda \le D \le A$ and $D \in \psi_{\sqsubseteq}(L^X)$, we have $x_\lambda \sqsubseteq A$ by (LTNR2). □

**Theorem 3.3.** *Let $(X, \sqsubseteq)$ be an L-topological neighborhood relation space. Define a binary relation $\preccurlyeq_{\sqsubseteq}$ on $L^X$ by*

$$\forall A, B \in L^X, \quad A \preccurlyeq_{\sqsubseteq} B \quad \Leftrightarrow \quad \forall x_\lambda \in \beta^*(A),\ x_\lambda \sqsubseteq B.$$

*Then $\preccurlyeq_{\sqsubseteq}$ is an L-topological internal relation.*

*Proof.* We check that $\preccurlyeq_{\sqsubseteq}$ satisfies (LTIR1)–(LTIR5).

(LTIR1). For any $x_\lambda \in \beta^*(\underline{\top})$, we have $x_\lambda \sqsubseteq \underline{\top}$ by (LTNR1). Thus $\underline{\top} \preccurlyeq_{\sqsubseteq} \underline{\top}$.

(LTIR2). If $A \preccurlyeq_{\sqsubseteq} B$, then $x_\lambda \sqsubseteq B$ for any $x_\lambda \in \beta^*(A)$. Thus $x_\lambda \leq B$ by (LTNR2). Hence $A = \bigvee_{x_\lambda \in \beta^*(A)} x_\lambda \leq B$.

(LTIR3). For all $\{A_i\}_{i\in I} \subseteq L^X$ and $B \in L^X$, we have

$$\begin{aligned}\bigvee_{i\in I} A_i \preccurlyeq_{\sqsubseteq} B \quad &\Leftrightarrow \quad \forall x_\lambda \in \beta^*(\bigvee_{i\in I} A_i) = \bigcup_{i\in I} \beta^*(A_i),\ x_\lambda \sqsubseteq B\\ &\Leftrightarrow \quad \forall i \in I, \forall x_\lambda \in \beta^*(A_i),\ x_\lambda \sqsubseteq B\\ &\Leftrightarrow \quad \forall i \in I,\ A_i \preccurlyeq_{\sqsubseteq} B.\end{aligned}$$

(LTIR4). Let $A \preccurlyeq_{\sqsubseteq} B$. We need to find some $D \in L^X$ such that $A \preccurlyeq_{\sqsubseteq} D \preccurlyeq_{\sqsubseteq} B$.

For any $x_\lambda \in \beta^*(A)$, there is a point $x_\mu \in \beta^*(A)$ such that $x_\lambda \in \beta^*(x_\mu)$. By $x_\mu \in \beta^*(A)$ and $A \preccurlyeq_{\sqsubseteq} B$, we have $x_\mu \sqsubseteq B$. Further, by (LTNR2), there is a set $D_{x_\lambda} \in \psi_{\sqsubseteq}(L^X)$ such that $x_\mu \leq D_{x_\lambda} \leq B$. Thus $x_\lambda \in \beta^*(D_{x_\lambda})$ which implies $x_\lambda \sqsubseteq D_{x_\lambda}$. Let $D = \bigvee_{x_\lambda \in \beta^*(A)} D_{x_\lambda}$. We next prove that $A \preccurlyeq_{\sqsubseteq} D \preccurlyeq_{\sqsubseteq} B$.

For any $x_\lambda \in \beta^*(A)$, we have $x_\lambda \sqsubseteq D_{x_\lambda} \leq D$. Thus $x_\lambda \sqsubseteq D$ by (1) of Lemma 3.2. Hence $A \preccurlyeq_{\sqsubseteq} D$. Also, for any $y_\mu \in \beta^*(D)$, we have $y_\mu \prec D_{x_\lambda}$ for some $x_\lambda \in \beta^*(A)$. Since $D_{x_\lambda} \in \psi_{\sqsubseteq}(L^X)$, we have $y_\mu \sqsubseteq D_{x_\lambda} \leq D$ followed by $y_\mu \sqsubseteq D$. Thus $D \preccurlyeq_{\sqsubseteq} D \leq B$ which implies $D \preccurlyeq_{\sqsubseteq} B$. Therefore $A \preccurlyeq_{\sqsubseteq} D \preccurlyeq_{\sqsubseteq} B$ as desired.

(LTIR5). For all $A, B, C \in L^X$, we obtain from (LTNR3) that

$$\begin{aligned}A \preccurlyeq_{\sqsubseteq} B \wedge C \quad &\Leftrightarrow \quad \forall x_\lambda \in \beta^*(A),\ x_\lambda \sqsubseteq B \wedge C\\ &\Leftrightarrow \quad \forall x_\lambda \in \beta^*(A),\ x_\lambda \sqsubseteq B \text{ and } x_\lambda \sqsubseteq C\\ &\Leftrightarrow \quad A \preccurlyeq_{\sqsubseteq} B \text{ and } A \preccurlyeq_{\sqsubseteq} C.\end{aligned}$$

Therefore $\preccurlyeq_{\sqsubseteq}$ is an $L$-topological internal relation. □

**Theorem 3.4.** *Let $(X, \sqsubseteq_X)$ and $(X, \sqsubseteq_Y)$ be L-topological neighborhood relation spaces. If $f : X \to Y$ is an L-topological neighborhood relation preserving mapping, then $f : (X, \preccurlyeq_{\sqsubseteq_X}) \to (Y, \preccurlyeq_{\sqsubseteq_Y})$ is an L-topological internal relation preserving mapping.*

*Proof.* Let $A, B \in L^Y$ with $A \preccurlyeq_{\sqsubseteq_Y} B$. To prove that $f_L^{\leftarrow}(A) \preccurlyeq_{\sqsubseteq_X} f_L^{\leftarrow}(B)$, let $x_\lambda \in \beta^*(f_L^{\leftarrow}(A))$. Then $f_L^{\rightarrow}(x_\lambda) \in \beta^*(A)$ and $f_L^{\rightarrow}(x_\lambda) \sqsubseteq_Y B$. Hence $x_\lambda \sqsubseteq_X f_L^{\leftarrow}(B)$. By the arbitrariness of $x_\lambda \in \beta^*(A)$, we have $f_L^{\leftarrow}(A) \preccurlyeq_{\sqsubseteq_X} f_L^{\leftarrow}(B)$. Therefore $f$ is an $L$-topological internal relation preserving mapping. □

**Theorem 3.5.** *Let $(X, \preccurlyeq)$ be an L-topological internal relation space. Define a binary relation $\sqsubseteq_{\preccurlyeq}$ on $J(L^X) \times L^X$ by*

$$\forall x_\lambda \in J(L^X), \forall A \in L^X, \quad x_\lambda \sqsubseteq_{\preccurlyeq} A \quad \Leftrightarrow \quad x_\lambda \preccurlyeq A.$$

*Then $\sqsubseteq_{\preccurlyeq}$ is an L-topological neighborhood relation on $L^X$.*

*Proof.* We check that $\sqsubseteq_{\preccurlyeq}$ satisfies (LTNR1)–(LTNR3).

(LTNR1). Since $x_\lambda \leq \underline{\top} \preccurlyeq \underline{\top}$ by (LTIR1), we have $x_\lambda \preccurlyeq \underline{\top}$. Thus $x_\lambda \sqsubseteq_{\preccurlyeq} \underline{\top}$.

(LTNR2). Let $x_\lambda \sqsubseteq_{\preccurlyeq} A$. We need to find some $D \in \psi_{\sqsubseteq_{\preccurlyeq}}(L^X)$ such that $x_\lambda \leq D \leq A$.

By $x_\lambda \sqsubseteq_{\preccurlyeq} A$, we have $x_\lambda \preccurlyeq A$. Further, by (LTIR4), there is a set $C \in L^X$ such that $x_\lambda \preccurlyeq C \preccurlyeq A$. Let $D = \bigvee\{E \in L^X : x_\lambda \preccurlyeq E \preccurlyeq A\}$. We have $x_\lambda \leq D \leq A$. Next, we prove that $D \in \psi_{\sqsubseteq_{\preccurlyeq}}(L^X)$.

Let $y_\mu \in \beta^*(D)$. Then there is a set $E \in L^X$ such that $x_\lambda \preccurlyeq E \preccurlyeq A$ and $y_\mu \leq E$. Notice that $E \preccurlyeq A$. By (LTIR4), there is a set $F \in L^X$ such that $E \preccurlyeq F \preccurlyeq A$. Thus $y_\mu \leq E \preccurlyeq F \preccurlyeq A$ which implies that $y_\mu \preccurlyeq F$. Further, we have $y_\mu \preccurlyeq F \leq D$ by $x_\lambda \preccurlyeq E \preccurlyeq F \preccurlyeq A$. Hence $y_\mu \preccurlyeq D$ which implies $y_\mu \sqsubseteq_{\preccurlyeq} D$. Therefore $D \in \psi_{\sqsubseteq_{\preccurlyeq}}(L^X)$.

Conversely, assume that there is a set $D \in \psi_{\sqsubseteq_{\preccurlyeq}}(L^X)$ such that $x_\lambda \leq D \leq A$. We need to prove that $x_\lambda \sqsubseteq_{\preccurlyeq} A$.

For any $\mu \in \beta^*(\lambda)$, we have $x_\mu \in \beta^*(D)$. Thus $x_\mu \sqsubseteq_{\preccurlyeq} D$. Hence $x_\mu \preccurlyeq D \leq A$ followed by $x_\mu \preccurlyeq A$. Therefore $x_\lambda = \bigvee_{\mu \in \beta^*(\lambda)} x_\mu \preccurlyeq A$ by (LTIR3). This implies that $x_\lambda \sqsubseteq_{\preccurlyeq} A$.

(LTNR3). Let $A, B \in L^X$. By (LTIR5), we have

$$x_\lambda \sqsubseteq_{\preccurlyeq} A \wedge B \iff x_\lambda \preccurlyeq A \wedge B \iff x_\lambda \preccurlyeq A \text{ and } x_\lambda \preccurlyeq B \iff x_\lambda \sqsubseteq_{\preccurlyeq} A \text{ and } x_\lambda \sqsubseteq_{\preccurlyeq} B.$$

Therefore $\sqsubseteq_{\preccurlyeq}$ is an $L$-topological neighborhood relation. □

**Theorem 3.6.** *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be $L$-topological internal relation spaces. If $f : X \to Y$ is an $L$-topological internal relation preserving mapping, then $f : (X, \sqsubseteq_{\preccurlyeq_X}) \to (Y, \sqsubseteq_{\preccurlyeq_Y})$ is an $L$-topological neighborhood relation preserving mapping.*

*Proof.* Let $x_\lambda \in J(L^X)$ and $B \in L^Y$. If $f_L^{\rightarrow}(x_\lambda) \sqsubseteq_{\preccurlyeq_Y} B$, then $x_\lambda \leq f_L^{\leftarrow}(f_L^{\rightarrow}(x_\lambda)) \preccurlyeq_X f_L^{\leftarrow}(B)$ which implies that $x_\lambda \preccurlyeq_X f_L^{\leftarrow}(B)$. Thus $x_\lambda \sqsubseteq_{\preccurlyeq_X} f_L^{\leftarrow}(B)$. So $f$ is an $L$-topological neighborhood relation preserving mapping. □

**Theorem 3.7.** *We have $\sqsubseteq_{\preccurlyeq_\sqsubseteq} = \sqsubseteq$ for any $L$-topological neighborhood relation space $(X, \preccurlyeq)$ and $\preccurlyeq_{\sqsubseteq_\preccurlyeq} = \preccurlyeq$ for any $L$-topological internal relation space $(X, \preccurlyeq)$.*

*Proof.* Let $(X, \sqsubseteq)$ be an $L$-topological neighborhood relation space. By (2) of Lemma 3.2, we have

$$x_\lambda \sqsubseteq A \iff \forall \mu \in \beta^*(\lambda),\ x_\mu \sqsubseteq A \iff x_\lambda \preccurlyeq_\sqsubseteq A \iff x_\lambda \sqsubseteq_{\preccurlyeq_\sqsubseteq} A.$$

In conclusion, for any $x_\lambda \in J(L^X)$ and any $A \in L^X$, we have $x_\lambda \sqsubseteq_{\preccurlyeq_\sqsubseteq} A$ if and only if $x_\lambda \sqsubseteq A$. That is, $\sqsubseteq_{\preccurlyeq_\sqsubseteq} = \sqsubseteq$.

Let $(X, \preccurlyeq)$ be an $L$-topological internal relation space. By (LTIR3) of $\preccurlyeq$, we have

$$A \preccurlyeq B \iff \forall x_\lambda \in \beta^*(A),\ x_\lambda \preccurlyeq B \iff \forall x_\lambda \in \beta^*(A),\ x_\lambda \sqsubseteq_\preccurlyeq B \iff A \preccurlyeq_{\sqsubseteq_\preccurlyeq} B.$$

In conclusion, for all $A, B \in L^X$, we have $A \preccurlyeq_{\sqsubseteq_\preccurlyeq} B$ if and only if $A \preccurlyeq B$. That is, $\preccurlyeq_{\sqsubseteq_\preccurlyeq} = \preccurlyeq$. □

Based on Theorems 3.3 and 3.4, we obtain a functor $\mathbb{F} : L$-**TNRS** $\to L$-**TIRS** by

$$\mathbb{F}((X, \sqsubseteq)) = (X, \preccurlyeq_\sqsubseteq), \quad \mathbb{F}(f) = f.$$

Based on Theorems 3.3–3.7, we find that $\mathbb{F}$ is an isomorphic functor. Thus we have the following conclusion.

**Theorem 3.8.** *The category $L$-**TNRS** is isomorphic to the category $L$-**TIRS**.*

**Remark 3.9.** Relations between $L$-topological neighborhood relations and $L$-topological neighborhood systems can be checked directly as follows.

(1) Let $(X, \sqsubseteq)$ be an $L$-topological neighborhood relation space. For any $x_\lambda \in J(L^X)$, we define

$$(\mathcal{N}_\sqsubseteq)_{x_\lambda} = \{A \in L^X : x_\lambda \sqsubseteq A\}.$$

Then $\mathcal{N}_\sqsubseteq = \{(\mathcal{N}_\sqsubseteq)_{x_\lambda} : x_\lambda \in J(L^X)\}$ is an $L$-topological neighborhood systems on $L^X$.

(2) Let $(X, \mathcal{N})$ be an $L$-topological neighborhood space. Define a binary mapping $\sqsubseteq_\mathcal{N}$ by

$$\forall x_\lambda \in J(L^X), \forall A \in L^X,\ \ x_\lambda \sqsubseteq_\mathcal{N} A \iff A \in \mathcal{N}_{x_\lambda}.$$

Then $\sqsubseteq_\mathcal{N}$ is an $L$-topological neighborhood system.

(3) The category $L$-**TNRS** is isomorphic to the category $L$-**TNS**.

## 4. *L*-topological derived neighborhood relation spaces

In this section, we introduce $L$-topological derived neighborhood relation space and $L$-topological derived neighborhood space by which we characterize $L$-topological neighborhood relation space.

**Definition 4.1.** A binary relation $\sqsubseteq^d$ on $J(L^X) \times L^X$ is called an $L$-topological derived neighborhood relation on $L^X$ and the pair $(X, \sqsubseteq^d)$ is called an $L$-topological derived neighborhood relation space, if for all $x_\lambda \in J(L^X)$ and $A, B \in L^X$,

(LTDNR1) $x_\lambda \sqsubseteq^d \underline{\top}$;

(LTDNR2) $x_\lambda \sqsubseteq^d A$ if and only if any $\mu \in \beta^*(\lambda)$ implies some $D \in \psi_{\sqsubseteq^d}(L^X)$ such that $x_\mu \leq D \leq A \vee x_\mu$, where $\psi_{\sqsubseteq^d}(L^X) = \{B \in L^X : \forall y_\eta \in \beta^*(B),\ y_\eta \sqsubseteq^d B\}$.

(LTDNR3) $x_\lambda \sqsubseteq^d A \wedge B$ if and only if $x_\lambda \sqsubseteq^d A$ and $x_\lambda \sqsubseteq^d B$.

Let $(X, \sqsubseteq^d_X)$ and $(Y, \sqsubseteq^d_Y)$ be $L$-topological derived neighborhood relation spaces. A mapping $f : X \to Y$ is called an $L$-topological derived neighborhood relation preserving mapping, if $f_L^{\rightarrow}(x_\lambda) \sqsubseteq^d_Y B$ implies $x_\lambda \sqsubseteq^d_X f_L^{\leftarrow}(B \vee f_L^{\rightarrow}(x_\lambda))$ for all $x_\lambda \in J(L^X)$ and $B \in L^Y$.

The category of $L$-topological derived neighborhood relation spaces and $L$-topological derived neighborhood relation preserving mappings is denoted by $L$-**TDNRS**. Next, we discuss relations between $L$-**TNRS** and $L$-**TDNRS**.

**Lemma 4.2.** *Let $(X, \sqsubseteq^d)$ be an $L$-topological derived neighborhood relation space. For all $x_\lambda, x_\eta \in J(L^X)$ and $A, B \in L^X$, we have*

*(1) $x_\lambda \leq x_\eta \sqsubseteq^d B \leq A$ implies $x_\lambda \sqsubseteq^d A$;*

*(2) $x_\lambda \sqsubseteq^d A$ if and only if $x_\mu \sqsubseteq^d A$ for any $\mu \in \beta^*(\lambda)$;*

*(3) $A, B \in \psi_{\sqsubseteq^d}(L^X)$ implies $A \wedge B \in \psi_{\sqsubseteq^d}(L^X)$;*

*(4) $A \in \psi_{\sqsubseteq^d}(L^X)$ and $x_\lambda \sqsubseteq^d A$ implies $x_\lambda \vee A \in \psi_{\sqsubseteq^d}(L^X)$.*

*Proof.* (1) Let $x_\lambda \leq x_\eta \sqsubseteq^d B \leq A$. If $\mu \in \beta^*(\lambda)$, then $\mu \in \beta^*(\eta)$. By $x_\eta \sqsubseteq^d B$, we obtain from (LTDNR2) that there is a set $D \in \psi_{\sqsubseteq^d}(L^X)$ such that $x_\mu \leq D \leq B \vee x_\mu \leq A \vee x_\mu$. Hence $x_\lambda \sqsubseteq^d A$ by (LTDNR2).

(2) If $x_\lambda \sqsubseteq^d A$, then $x_\mu \sqsubseteq^d A$ for any $\mu \in \beta^*(\lambda)$ by (1). Conversely, assume that $x_\mu \sqsubseteq^d A$ for any $\mu \in \beta^*(\lambda)$. For any $\theta \in \beta^*(\lambda)$, there is a $\mu \in \beta^*(\lambda)$ such that $\theta \in \beta^*(\mu)$. By the assumption, we have $x_\mu \sqsubseteq^d A$. By (LTDNR2), there is a set $B \in \psi_{\sqsubseteq^d}(L^X)$ such that $x_\theta \leq B \leq A \vee x_\theta$. Thus $x_\lambda \sqsubseteq^d A$ by (LTDNR2).

(3) Let $A, B \in \psi_{\sqsubseteq^d}(L^X)$. For any $x_\lambda \in \beta^*(A \wedge B)$, we have $x_\lambda \in \beta^*(A)$ and $x_\lambda \in \beta^*(B)$. Thus $x_\lambda \sqsubseteq^d A$ and $x_\lambda \sqsubseteq^d B$. By (LTDNR3), we have $x_\lambda \sqsubseteq^d A \wedge B$. Therefore $A \wedge B \in \psi_{\sqsubseteq^d}(L^X)$.

(4) Let $y_\mu \in \beta^*(A \vee x_\lambda) = \beta^*(A) \cup \beta^*(x_\lambda)$. If $y_\mu \in \beta^*(A)$, then $y_\mu \sqsubseteq^d A$ by $A \in \psi_{\sqsubseteq^d}(L^X)$. If $y_\mu \in \beta^*(x_\lambda)$, then $y_\mu = x_\mu$. Since $A \in \psi_{\sqsubseteq^d}(L^X)$ and $x_\lambda \sqsubseteq^d A$, we have $x_\mu \sqsubseteq^d A$ by (2). Thus we have $y_\mu \sqsubseteq^d A \leq A \vee x_\lambda$ in either case. Hence $y_\mu \sqsubseteq^d A \vee x_\lambda$ by (1). Therefore $A \vee x_\lambda \in \psi_{\sqsubseteq^d}(L^X)$. □

**Theorem 4.3.** *Let $(X, \sqsubseteq)$ be an $L$-topological neighborhood relation space. Define a binary relation $\sqsubseteq^d_{\sqsubseteq}$ on $J(L^X) \times L^X$ by*

$$x_\lambda \sqsubseteq^d_{\sqsubseteq} A \quad \Leftrightarrow \quad \forall \mu \in \beta^*(\lambda),\ x_\mu \sqsubseteq A \vee x_\mu.$$

*Then $\sqsubseteq^d_{\preccurlyeq}$ is an $L$-topological derived neighborhood relation on $L^X$.*

*Proof.* We check that $\sqsubseteq^d_{\sqsubseteq}$ satisfies (LTDNR1)–(LTDNR3).

(LTDNR1). By (LTNR1), we have $x_\mu \sqsubseteq \underline{\top} = \underline{\top} \vee x_\mu$ for any $\mu \in \beta^*(\lambda)$. Thus $x_\lambda \sqsubseteq^d_{\sqsubseteq} \underline{\top}$.

(LTDNR2). Let $x_\lambda \sqsubseteq^d_{\sqsubseteq} A$ and let $\mu \in \beta^*(\lambda)$. We need to find some $D \in \psi_{\sqsubseteq^d_{\sqsubseteq}}(L^X)$ such that $x_\mu \leq D \leq A \vee x_\mu$.

Since $\mu \in \beta^*(\lambda)$, we have $x_\mu \sqsubseteq A \vee x_\mu$. By (LTNR2), there is a set $B \in \psi_{\sqsubseteq}(L^X)$ such that $x_\mu \leq B \leq A \vee x_\mu$. To prove that $B \in \psi_{\sqsubseteq^d_{\sqsubseteq}}(L^X)$, we prove that $y_\eta \sqsubseteq^d_{\sqsubseteq} B$ for any $y_\eta \in \beta^*(B)$. Indeed, for any $\theta \in \beta^*(\eta)$, we have $y_\theta \in \beta^*(B)$ followed by $y_\theta \sqsubseteq B = B \vee y_\theta$. Thus $y_\eta \sqsubseteq^d_{\sqsubseteq} B$. Hence $B \in \psi_{\sqsubseteq^d_{\sqsubseteq}}(L^X)$. Therefore $B$ is what as desired.

Conversely, assume that any $\mu \in \beta^*(\lambda)$ implies some $B \in \psi_{\sqsubseteq^d_{\sqsubseteq}}(L^X)$ such that $x_\mu \leq B \leq A \vee x_\mu$. To prove that $x_\lambda \sqsubseteq^d_{\sqsubseteq} A$, we need to prove that $x_\mu \sqsubseteq A \vee x_\mu$ for any $\mu \in \beta^*(\lambda)$.

Let $\mu \in \beta^*(\lambda)$. By the assumption, there is $B \in \psi_{\sqsubseteq^d_{\sqsubseteq}}(L^X)$ such that $x_\mu \leq B \leq A \vee x_\mu$. For any $\eta \in \beta^*(\mu)$, we have $x_\eta \sqsubseteq^d_{\sqsubseteq} B$. Thus $x_\theta \sqsubseteq B \vee x_\theta = B$ for any $\theta \in \beta^*(\eta)$. Hence $x_\eta \sqsubseteq B$ by (2) of Lemma 3.2. Again, by arbitrariness of $\eta \in \beta^*(\mu)$ and (2) of Lemma 3.2, we have $x_\mu \sqsubseteq B$. Since $B \leq A \vee x_\mu$, we have $x_\mu \sqsubseteq A \vee x_\mu$ by (1) of Lemma 3.2. Therefore $x_\lambda \sqsubseteq^d_{\sqsubseteq} A$.

(LTDNR3). Let $A, B \in L^X$. By (LTNR3), we have

$$\begin{aligned} x_\lambda \sqsubseteq^d_{\sqsubseteq} A \wedge B \quad &\Leftrightarrow \quad \forall \mu \in \beta^*(\lambda),\ x_\mu \sqsubseteq (A \wedge B) \vee x_\mu = (A \vee x_\mu) \wedge (B \vee x_\mu) \\ &\Leftrightarrow \quad \forall \mu \in \beta^*(\lambda),\ x_\mu \sqsubseteq A \vee x_\mu \text{ and } x_\mu \sqsubseteq B \vee x_\mu \\ &\Leftrightarrow \quad x_\lambda \sqsubseteq^d_{\sqsubseteq} A \text{ and } x_\mu \sqsubseteq^d_{\sqsubseteq} B. \end{aligned}$$

Therefore $\sqsubseteq^d_{\sqsubseteq}$ is an $L$-topological derived neighborhood relation. □

**Theorem 4.4.** *Let $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$ be an L-topological neighborhood relation spaces. If $f : X \to Y$ is an L-topological neighborhood relation preserving mapping, then $f : (X, \sqsubseteq^d_{\sqsubseteq_X}) \to (Y, \sqsubseteq^d_{\sqsubseteq_Y})$ is an L-topological derived neighborhood relation preserving mapping.*

*Proof.* Let $x_\lambda \in J(L^X)$ and $B \in L^Y$ with $f^\rightarrow_L(x_\lambda) \sqsubseteq^d_{\sqsubseteq_Y} B$. For any $\mu \in \beta^*(\lambda)$, we have $f^\rightarrow_L(x_\mu) \in \beta^*(f^\rightarrow_L(x_\lambda))$ Thus $f^\rightarrow_L(x_\mu) \sqsubseteq^d_{\sqsubseteq_X} B$. So $f^\rightarrow_L(x_\eta) \sqsubseteq_Y B \vee f^\rightarrow_L(x_\eta) \leq B \vee f^\rightarrow_L(x_\mu)$ for any $\eta \in \beta^*(\mu)$. By (2) of Lemma 3.2, we have

$$f^\rightarrow_L(x_\mu) \sqsubseteq_Y B \vee f^\leftarrow_L(x_\mu) \leq B \vee f^\leftarrow_L(x_\lambda).$$

Hence $f^\rightarrow_L(x_\mu) \sqsubseteq_Y B \vee f^\leftarrow_L(x_\lambda)$ followed by $x_\mu \sqsubseteq_X f^\leftarrow_L(B \vee f^\rightarrow_L(x_\lambda))$. By the arbitrariness if $\mu \in \beta^*(\lambda)$, we have $x_\lambda \sqsubseteq^d_{\sqsubseteq_X} f^\rightarrow_L(B \vee f^\rightarrow_L(x_\lambda))$. So $f$ is an $L$-topological derived neighborhood relation preserving mapping. □

**Theorem 4.5.** *Let $(X, \sqsubseteq^d)$ be an L-topological derived neighborhood relation space. Define a binary relation $\sqsubseteq_{\sqsubseteq^d}$ on $L^X$ by*

$$\forall x_\lambda \in J(L^X), \forall A \in L^X,\ x_\lambda \sqsubseteq_{\sqsubseteq^d} A \quad \Leftrightarrow \quad \exists B \in \psi_{\sqsubseteq^d}(L^X),\ x_\lambda \sqsubseteq^d B,\ x_\lambda \vee B \leq A.$$

*Then $\sqsubseteq_{\sqsubseteq^d}$ is an L-topological neighborhood relation.*

*Proof.* We check that $\sqsubseteq_{\sqsubseteq^d}$ satisfies (LTNR1)–(LTNR3).

(LTNR1). We have $\underline{\top} \in \psi_{\sqsubseteq^d}(L^X)$ with $x_\lambda \sqsubseteq^d \underline{\top}$ and $x_\lambda \vee \underline{\top} = \underline{\top}$. Thus $x_\lambda \sqsubseteq_{\sqsubseteq^d} \underline{\top}$.

(LTNR2). If $x_\lambda \sqsubseteq_{\sqsubseteq^d} A$, then there is a set $B \in \psi_{\sqsubseteq^d}(L^X)$ such that $x_\lambda \sqsubseteq^d B$ and $x_\lambda \vee B \leq A$. Let $D = B \vee x_\lambda$. We have $x_\lambda \leq D \leq A$. Next, we prove that $D \in \psi_{\sqsubseteq_{\sqsubseteq^d}}(L^X)$.

Let $y_\mu \in \beta^*(D) = \beta^*(B) \cup \beta^*(x_\lambda)$. If $y_\mu \in \beta^*(B)$, then $y_\mu \sqsubseteq^d B$ by $B \in \psi_{\sqsubseteq^d}(L^X)$. In addition, $y_\mu \vee B = B \leq D$. Thus $y_\mu \sqsubseteq_{\sqsubseteq^d} D$. If $y_\mu \in \beta^*(x_\lambda)$, then $y_\mu = x_\mu \in \beta^*(x_\lambda)$. Since $x_\lambda \sqsubseteq^d B$, we have $x_\mu \sqsubseteq^d B$ by (2) of Lemma 4.2. By this result, $B \in \psi_{\sqsubseteq^d}(L^X)$ and $B \vee x_\mu \leq D$, we have $x_\mu \sqsubseteq_{\sqsubseteq^d} D$. Therefore $D \in \psi_{\sqsubseteq_{\sqsubseteq^d}}(L^X)$.

Conversely, assume that there is a $D \in \psi_{\sqsubseteq_{\sqsubseteq^d}}(L^X)$ such that $x_\lambda \leq D \leq A$. Next, we prove that $D \in \psi_{\sqsubseteq^d}(L^X)$.

Let $y_\mu \in \beta^*(D)$. Since $D \in \psi_{\sqsubseteq_{\sqsubseteq^d}}(L^X)$, we have $y_\mu \sqsubseteq_{\sqsubseteq^d} D$. Thus there is a set $B \in \psi_{\sqsubseteq^d}(L^X)$ such that $y_\mu \sqsubseteq^d B$ and $y_\mu \vee B \leq D$. Since $y_\mu \sqsubseteq^d B \leq D$, we have $y_\mu \sqsubseteq^d D$ by (1) of Lemma 4.2. Therefore $D \in \psi_{\sqsubseteq^d}(L^X)$.

Notice that $x_\lambda \leq D$ and $D \in \psi_{\sqsubseteq^d}(L^X)$. For any $\mu \in \beta^*(\lambda)$, we have $x_\mu \sqsubseteq^d D$. Thus $x_\lambda \sqsubseteq^d D$ by (2) of Lemma 4.2. Further, since $D \vee x_\lambda = D \leq A$, we have $x_\lambda \sqsubseteq_{\sqsubseteq^d} A$.

(LTNR3). For all $A, B \in L^X$, we have

$$\begin{aligned} x_\lambda \sqsubseteq_{\sqsubseteq^d} A \wedge B \quad &\Leftrightarrow \quad \exists D \in \psi_{\sqsubseteq^d}(L^X),\ x_\lambda \sqsubseteq^d D,\ x_\lambda \vee D \leq A \wedge B \\ &\Leftrightarrow \quad \exists D \in \psi_{\sqsubseteq^d}(L^X),\ x_\lambda \sqsubseteq^d D,\ x_\lambda \vee D \leq A \text{ and } x_\lambda \vee D \leq B \\ &\Rightarrow \quad x_\lambda \sqsubseteq_{\sqsubseteq^d} A \text{ and } x_\lambda \sqsubseteq_{\sqsubseteq^d} B. \end{aligned}$$

Conversely, from (LTDNR3) and (3) of Lemma 4.2, we have

$$
\begin{aligned}
x_\lambda \sqsubseteq_{\sqsubseteq^d} A \text{ and } x_\lambda \sqsubseteq_{\sqsubseteq^d} B \quad &\Leftrightarrow \quad \exists U, V \in \psi_{\sqsubseteq^d}(L^X),\ x_\lambda \sqsubseteq^d U,\ x_\lambda \sqsubseteq^d V,\ x_\lambda \vee U \leq A \text{ and } x_\lambda \vee V \leq B \\
&\Leftrightarrow \quad \exists U, V \in \psi_{\sqsubseteq^d}(L^X),\ x_\lambda \sqsubseteq^d U \wedge V,\ x_\lambda \vee (U \wedge V) \leq A \wedge B \\
&\Rightarrow \quad \exists D \in \psi_{\sqsubseteq^d}(L^X),\ x_\lambda \sqsubseteq^d D,\ x_\lambda \vee D \leq A \wedge B \\
&\Leftrightarrow \quad x_\lambda \sqsubseteq_{\sqsubseteq^d} A \wedge B.
\end{aligned}
$$

Therefore $\sqsubseteq_{\sqsubseteq^d}$ is an $L$-topological neighborhood relation. □

**Theorem 4.6.** *Let $(X, \sqsubseteq^d_X)$ and $(Y, \sqsubseteq^d_Y)$ be L-topological derived neighborhood relation spaces. If $f : X \to Y$ is an L-topological derived neighborhood relation preserving mapping, then $f : (X, \sqsubseteq_{\sqsubseteq^d_X}) \to (Y, \sqsubseteq_{\sqsubseteq^d_Y})$ is an L-topological neighborhood relation preserving mapping.*

*Proof.* Let $x_\lambda \in J(L^X)$ and $B \in L^Y$. If $f_L^{\rightarrow}(x_\lambda) \sqsubseteq_{\sqsubseteq^d_Y} B$, then there is a set $D \in \psi_{\sqsubseteq^d_Y}(L^Y)$ such that $f_L^{\rightarrow}(x_\lambda) \sqsubseteq^d_Y D$ and $D \vee f_L^{\rightarrow}(x_\lambda) \leq B$. Thus $x_\lambda \sqsubseteq^d_X f_L^{\leftarrow}(D \vee f_L^{\rightarrow}(x_\lambda)) \leq f_L^{\leftarrow}(B)$. Next, we prove that $f_L^{\leftarrow}(D \vee f_L^{\rightarrow}(x_\lambda)) \in \psi_{\sqsubseteq^d_X}(L^X)$.

If $y_\eta \in \beta^*(f_L^{\leftarrow}(D \vee f_L^{\rightarrow}(x_\lambda)))$, then $y_\eta \in \beta^*(f_L^{\leftarrow}(D))$ or $y_\eta \in \beta^*(f_L^{\leftarrow}(f_L^{\rightarrow}(x_\lambda)))$. Since $D \in \psi_{\sqsubseteq^d_Y}(L^Y)$ and $f_L^{\rightarrow}(x_\lambda) \sqsubseteq^d_Y D$, we have $f_L^{\rightarrow}(y_\eta) \sqsubseteq^d_Y D$ in either case. Thus

$$y_\eta \sqsubseteq^d_X f_L^{\leftarrow}(D \vee f_L^{\rightarrow}(y_\eta)) \leq f_L^{\leftarrow}(D \vee f_L^{\rightarrow}(x_\lambda)).$$

Hence $y_\eta \sqsubseteq^d_X f_L^{\leftarrow}(D \vee f_L^{\rightarrow}(x_\lambda))$. Therefore $f_L^{\leftarrow}(D \vee f_L^{\rightarrow}(x_\lambda)) \in \psi_{\sqsubseteq^d_X}(L^X)$.

By the above result, $x_\lambda \sqsubseteq^d_X f_L^{\leftarrow}(D \vee f_L^{\rightarrow}(x_\lambda))$ and $f_L^{\leftarrow}(D \vee f_L^{\rightarrow}(x_\lambda)) \vee x_\lambda \leq f_L^{\leftarrow}(B)$. We have $x_\lambda \sqsubseteq_{\sqsubseteq^d_X} f_L^{\leftarrow}(B)$. Therefore $f$ is an $L$-topological neighborhood relation preserving mapping. □

**Lemma 4.7.** *Let $(X, \sqsubseteq)$ be an L-topological neighborhood relation space. For all $x_\lambda \in J(L^X)$ and $A \in L^X$, we have $A \in \psi_{\sqsubseteq^d_{\sqsubseteq}}(L^X)$ if and only if $A \in \psi_{\sqsubseteq}(L^X)$.*

*Proof.* Let $A \in \psi_{\sqsubseteq^d_{\sqsubseteq}}(L^X)$. For any $y_\mu \in \beta^*(A)$, we have $y_\mu \sqsubseteq^d_{\sqsubseteq} A$. Thus $y_\eta \sqsubseteq A \vee y_\eta = A$ for any $y_\eta \in \beta^*(\mu)$. Hence $y_\mu \sqsubseteq A$ by (2) of Lemma 3.2. Therefore $A \in \psi_{\sqsubseteq}(L^X)$.

Conversely, let $A \in \psi_{\sqsubseteq}(L^X)$. For any $y_\mu \in \beta^*(A)$, we have $y_\mu \sqsubseteq A$. Thus $y_\eta \sqsubseteq A = A \vee y_\eta$ for any $\eta \in \beta^*(\mu)$ by (2) of Lemma 3.2. Hence $y_\mu \sqsubseteq^d_{\sqsubseteq} A$. Therefore $A \in \psi_{\sqsubseteq^d_{\sqsubseteq}}(L^X)$. □

**Theorem 4.8.** *We have $\sqsubseteq_{\sqsubseteq^d_{\sqsubseteq}} = \sqsubseteq$ for any L-topological neighborhood relation space $(X, \sqsubseteq)$ and $\sqsubseteq^d_{\sqsubseteq_{\sqsubseteq^d}} = \sqsubseteq^d$ for any L-topological derived neighborhood relation space $(X, \sqsubseteq^d)$.*

*Proof.* Let $(X, \sqsubseteq)$ be an $L$-topological neighborhood relation space. Let $x_\lambda \in J(L^X)$ and $A \in L^X$.

If $x_\lambda \sqsubseteq_{\sqsubseteq^d_{\sqsubseteq}} A$, then there is a set $B \in \psi_{\sqsubseteq^d_{\sqsubseteq}}(L^X)$ such that $x_\lambda \sqsubseteq^d_{\sqsubseteq} B$ and $B \vee x_\lambda \leq A$. By (4) of Lemma 4.2, we have $B \vee x_\lambda \in \psi_{\sqsubseteq^d_{\sqsubseteq}}(L^X)$. Thus $B \vee x_\lambda \in \psi_{\sqsubseteq}(L^X)$ by Lemma 4.7. Hence $x_\mu \sqsubseteq B \vee x_\lambda \leq A$ for any $\mu \in \beta^*(\lambda)$. So $x_\mu \sqsubseteq A$ by (1) of Lemma 3.2. Therefore $x_\lambda \sqsubseteq A$ by (2) of Lemma 3.2.

Conversely, assume that $x_\lambda \sqsubseteq A$. By (LTNR2), there is a set $B \in \psi_{\sqsubseteq}(L^X)$ such that $x_\lambda \leq B \leq A$. Thus $B \in \psi_{\sqsubseteq^d_{\sqsubseteq}}(L^X)$ by Lemma 4.7. For any $\mu \in \beta^*(\lambda)$, we have $x_\mu \sqsubseteq B$ and so $x_\mu \sqsubseteq B \leq A$. Hence $x_\mu \sqsubseteq A = A \vee x_\mu$. By arbitrariness of $\mu \in \beta^*(\lambda)$, we have $x_\lambda \sqsubseteq^d_{\sqsubseteq} A$. For any $\mu \in \beta^*(\lambda)$, we have $x_\mu \sqsubseteq^d_{\sqsubseteq} A$ by (2) of Lemma 4.2. That is, $x_\mu \sqsubseteq^d_{\sqsubseteq} A = A \vee x_\mu$ for any $\mu \in \beta^*(\lambda)$. Therefore $x_\lambda \sqsubseteq_{\sqsubseteq^d_{\sqsubseteq}} A$.

In conclusion, we have $x_\lambda \sqsubseteq_{\sqsubseteq^d_{\sqsubseteq}} A$ if and only if $x_\lambda \sqsubseteq A$ for any $x_\lambda \in J(L^X)$ and any $A \in L^X$. That is, $\sqsubseteq_{\sqsubseteq^d_{\sqsubseteq}} = \sqsubseteq$.

Let $(X, \sqsubseteq^d)$ be an $L$-topological derived neighborhood relation space. Let $x_\lambda \in J(L^X)$ and $A \in L^X$.

If $x_\lambda \sqsubseteq^d_{\sqsubseteq_{\sqsubseteq^d}} A$, then $x_\mu \sqsubseteq_{\sqsubseteq^d} A \vee x_\mu$ for any $\mu \in \beta^*(\lambda)$. Thus there is a set $B \in \psi_{\sqsubseteq^d}(L^X)$ such that $x_\mu \sqsubseteq^d B$ and $B \vee x_\mu \leq A \vee x_\mu$. Further, by (4) of Lemma 4.2, we have $B \vee x_\mu \in \psi_{\sqsubseteq^d}(L^X)$. Hence $x_\mu \sqsubseteq^d B \vee x_\mu$ by (2) of Lemma 4.2. So $x_\mu \sqsubseteq^d A \vee x_\mu$ by (1) of Lemma 4.2. Therefore $x_\lambda \sqsubseteq^d A$ by (LTDNR2).

Conversely, let $x_\lambda \sqsubseteq^d A$. By (2) of Lemma 4.2, we have $x_\mu \sqsubseteq^d A$ for any $\mu \in \beta^*(\lambda)$. By (LTDNR2), there is a set $B \in \psi_{\sqsubseteq^d}(L^X)$ such that $x_\mu \leq B \leq A \vee x_\mu$. For any $\eta \in \beta^*(\mu)$, we have $x_\eta \in \beta^*(B)$. Thus $x_\eta \sqsubseteq^d B$. Hence

$x_\mu \sqsubseteq^d B$ by (2) of Lemma 4.2. By this result and $B \vee x_\mu \leq A \vee x_\mu$, we have $x_\mu \preccurlyeq_{\sqsubseteq^d} A \vee x_\mu$. Further, by the arbitrariness of $\mu \in \beta^*(\lambda)$, we have $x_\lambda \sqsubseteq^d_{\sqsubseteq_{\sqsubseteq^d}} A$.

In conclusion, for any $A \in L^X$ and any $x_\lambda \in J(L^X)$, we have $x_\lambda \sqsubseteq^d_{\sqsubseteq_{\sqsubseteq^d}} A$ if and only if $x_\lambda \sqsubseteq^d A$. That is, $\sqsubseteq^d_{\sqsubseteq_{\sqsubseteq^d}} = \sqsubseteq^d$. $\square$

Based on Theorems 4.5 and 4.6, we obtain a functor $\mathbb{G}$ :$L$-**TDNRS** $\to$ $L$-**TNRS** by

$$\mathbb{G}((X, \sqsubseteq^d)) = (X, \sqsubseteq_{\sqsubseteq^d}), \quad \mathbb{G}(f) = f.$$

Based on Theorems 4.3–4.8, we find that $\mathbb{G}$ is an isomorphic functor. Thus we have the following conclusion.

**Theorem 4.9.** *The category $L$-**TDNRS** is isomorphic to the category $L$-**TNRS**.*

In Section 3, we find that there is a one-to-one correspondence between $L$-topological neighborhood spaces and $L$-topological neighborhood relation spaces. Actually, we have a similar result with $L$-topological derived neighborhood relations spaces. To show this, we present the following notion.

**Definition 4.10.** A set $\mathcal{N}^d = \{\mathcal{N}^d_{x_\lambda} \subseteq L^X : x_\lambda \in J(L^X)\}$ is called an $L$-topological derived neighborhood system on $L^X$ and the pair $(X, \mathcal{N}^d)$ is called an $L$-topological derived neighborhood space, if for all $A, B \in L^X$ and $x_\lambda \in J(L^X)$,

(LTDN1) $\underline{\top} \in \mathcal{N}^d_{x_\lambda}$;

(LTDN2) $A \in \mathcal{N}^d_{x_\lambda}$ if and only if any $\mu \in \beta^*(\lambda)$ implies some $D \in \mathcal{N}^d_{x_\lambda}$ such that $x_\mu \leq D \leq A \vee x_\mu$ and $D \in \mathcal{N}^d_{y_\eta}$ for any $y_\eta \in \beta^*(D)$;

(LTDN3) $A \wedge B \in \mathcal{N}^d_{x_\lambda}$ if and only if $A, B \in \mathcal{N}^d_{x_\lambda}$.

Let $(X, \mathcal{N}^d_X)$ and $(Y, \mathcal{N}^d_Y)$ be $L$-topological derived neighborhood spaces. A mapping $f : X \to Y$ is called an $L$-topological derived neighborhood preserving mapping, if $B \in \mathcal{N}^d_{f^{\to}_L(x_\lambda)}$ implies $f^{\leftarrow}_L(B \vee f^{\to}_L(x_\lambda)) \in \mathcal{N}^d_{x_\lambda}$ for any $x_\lambda \in J(L^X)$ and $B \in L^Y$.

The category of $L$-topological derived neighborhood spaces and $L$-topological derived neighborhood preserving mappings is denoted by $L$-**TDNS**. Similar to Remark 3.9, we have the following result.

**Remark 4.11.** (1) Let $(X, \sqsubseteq^d)$ be an $L$-topological derived neighborhood relation space. For any $x_\lambda \in J(L^X)$, we define

$$(\mathcal{N}^d_{\sqsubseteq^d})_{x_\lambda} = \{A \in L^X : x_\lambda \sqsubseteq^d A\}.$$

Then $\mathcal{N}^d_{\sqsubseteq^d} = \{(\mathcal{N}^d_{\sqsubseteq^d})_{x_\lambda} : x_\lambda \in J(L^X)\}$ is an $L$-topological derived neighborhood system on $L^X$.

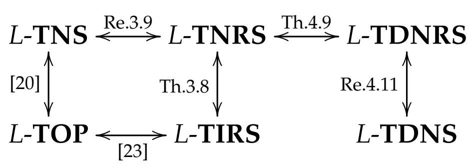
(2) Let $(X, \mathcal{N}^d)$ be an $L$-topological derived neighborhood space. Define a binary relation $\sqsubseteq^d_{\mathcal{N}^d}$ on $J(L^X) \times L^X$ by

$$\forall x_\lambda \in J(L^X), \forall A \in L^X, \ \ x_\lambda \sqsubseteq^d_{\mathcal{N}^d} A \quad \Leftrightarrow \quad A \in \mathcal{N}^d_{x_\lambda}.$$

Then $\sqsubseteq^d_{\mathcal{N}^d}$ is an $L$-topological derived neighborhood relation on $L^X$.

(3) The category $L$-**TDNRS** is isomorphic to the category $L$-**TDNS**.

Isomorphisms among the categories mentioned in Sections 3 and 4 are presented by as follows.



## 5. *L*-topological remotehood relation spaces

In this section, we introduce *L*-topological remotehood relation space by which we characterize *L*-topological enclosed relation space and *L*-topological remotehood space.

**Definition 5.1.** A binary relation $\overline{\sqsubset}$ on $J(L^X) \times L^X$ is called an *L*-topological remotehood relation on $L^X$ and the pair $(X, \overline{\sqsubset})$ is called an *L*-topological remotehood relation space, if for all $x_\lambda \in J(L^X)$ and $A, B \in L^X$,

(LTRNR1) $x_\lambda \overline{\sqsubset} \perp$;

(LTRNR2) $x_\lambda \overline{\sqsubset} A$ if and only if $x_\lambda \not\leq B \geq A$ for some $B \in \psi_{\overline{\sqsubset}}(L^X)$, where $\psi_{\overline{\sqsubset}}(L^X) = \{D \in L^X : \forall y_\mu \not\leq D,\ y_\mu \overline{\sqsubset} D\}$;

(LTRNR3) $x_\lambda \overline{\sqsubset} A \vee B$ if and only if $x_\lambda \overline{\sqsubset} A$ and $x_\lambda \overline{\sqsubset} B$.

Let $(X, \overline{\sqsubset}_X)$ and $(Y, \overline{\sqsubset}_Y)$ be *L*-topological remotehood relation spaces. A mapping $f : X \to Y$ is called an *L*-topological remotehood relation preserving mapping, if $f_L^{\to}(x_\lambda) \overline{\sqsubset}_Y A$ implies $x_\lambda \overline{\sqsubset}_X f_L^{\leftarrow}(A)$ for all $x_\lambda \in J(L^X)$ and $A \in L^Y$.

The category of *L*-topological remotehood relation spaces and *L*-topological remotehood relation preserving mappings is denoted by *L*-**TRNRS**. Next, we discuss relations between *L*-**TRNRS** and *L*-**TERS**.

**Theorem 5.2.** *Let $(X, \overline{\sqsubset})$ be an L-topological remotehood relation space. Define a binary relation $\lessdot_{\overline{\sqsubset}}$ on $L^X$ by*

$$\forall A, B \in L^X,\ \ A \lessdot_{\overline{\sqsubset}} B \quad \Leftrightarrow \quad \forall x_\lambda \not\leq B,\ x_\lambda \overline{\sqsubset} A.$$

*Then $\lessdot_{\overline{\sqsubset}}$ is an L-topological enclosed relation.*

*Proof.* It is sufficient to check that $\lessdot_{\overline{\sqsubset}}$ satisfies (LTER1)–(LTER5).

(LTER1) For any $x_\lambda \in J(L^X)$, we have $x_\lambda \overline{\sqsubset} \perp$ by (LTRNR1). Thus $\perp \lessdot_{\overline{\sqsubset}} \perp$.

(LTER2) Let $A \lessdot_{\overline{\sqsubset}} B$. For any $x_\lambda \not\leq B$, we have $x_\lambda \overline{\sqsubset} A$. Thus $x_\lambda \not\leq A$ by (LTRNR2). Hence $A \leq B$.

(LTER3) Let $\{B_i\}_{i\in I} \subseteq L^X$. Assume that $A \lessdot_{\overline{\sqsubset}} \bigwedge_{i\in I} B_i$. For any $i \in I$ and any $x_\lambda \not\leq B_i$, we have $x_\lambda \not\leq \bigwedge_{i\in I} B_i$. Thus $x_\lambda \overline{\sqsubset} A$. Hence $A \lessdot_{\overline{\sqsubset}} B_i$ for any $i \in I$.

Conversely, assume that $A \lessdot_{\overline{\sqsubset}} B_i$ for any $i \in I$. For any $x_\lambda \not\leq \bigwedge_{i\in I} B_i$, there is an index $i \in I$ such that $x_\lambda \not\leq B_i$. Thus $x_\lambda \overline{\sqsubset} A$ by $A \lessdot_{\overline{\sqsubset}} B_i$. By the arbitrariness of $x_\lambda \not\leq \bigwedge_{i\in I} B_i$, we have $A \lessdot_{\overline{\sqsubset}} \bigwedge_{i\in I} B_i$.

(LTER4) Let $A \lessdot_{\overline{\sqsubset}} B$. We need to find some $C \in L^X$ such that $A \lessdot_{\overline{\sqsubset}} C \lessdot_{\overline{\sqsubset}} B$.

For any $x_\lambda \not\leq B$, we have $x_\lambda \overline{\sqsubset} A$ by $A \lessdot_{\overline{\sqsubset}} B$. Further, by (LTRNR2), there is a set $C_{x_\lambda} \in \psi_{\overline{\sqsubset}}(L^X)$ such that $x_\lambda \not\leq C_{x_\lambda} \geq A$. Let $C = \bigwedge_{y_\mu \not\leq B} C_{y_\mu}$. We have $x_\lambda \not\leq C \geq A$ for any $x_\lambda \not\leq B$. Next, we prove that $A \lessdot_{\overline{\sqsubset}} C \lessdot_{\overline{\sqsubset}} B$.

For any $z_\eta \not\leq C$, there is a point $y_\mu \not\leq B$ such that $z_\eta \not\leq C_{y_\mu}$. Thus $z_\eta \not\leq C_{y_\mu} \geq A$. Hence $z_\eta \overline{\sqsubset} A$ by (LTRNR2). Therefore $A \lessdot_{\overline{\sqsubset}} C$. Also, for any $u_\theta \not\leq B$, we have $u_\theta \not\leq C$. Thus there is a point $v_\sigma \not\leq B$ such that $u_\theta \not\leq C_{v_\sigma}$. Since $C_{v_\sigma} \in \psi_{\overline{\sqsubset}}(L^X)$, we have $u_\theta \overline{\sqsubset} C_{v_\sigma}$. Hence $u_\theta \not\leq C_{v_\sigma} \geq C$. By (LTRNR2), we have $u_\theta \overline{\sqsubset} C$. Therefore $C \lessdot_{\overline{\sqsubset}} B$. In conclusion, we have $A \lessdot_{\overline{\sqsubset}} C \lessdot_{\overline{\sqsubset}} B$ as desired.

(LTER5) Let $A, B, C \in L^X$. By (LTRNR3), we have

$$\begin{aligned} A \vee B \lessdot_{\overline{\sqsubset}} C \quad &\Leftrightarrow \quad \forall x_\lambda \not\leq C,\ x_\lambda \overline{\sqsubset} A \vee B \\ &\Leftrightarrow \quad \forall x_\lambda \not\leq C,\ x_\lambda \overline{\sqsubset} A \text{ and } x_\lambda \overline{\sqsubset} B \\ &\Leftrightarrow \quad A \lessdot_{\overline{\sqsubset}} C \text{ and } B \lessdot_{\overline{\sqsubset}} C. \end{aligned}$$

Therefore $\lessdot_{\overline{\sqsubset}}$ is an *L*-topological enclosed relation. □

**Theorem 5.3.** *Let $(X, \overline{\sqsubset}_X)$ and $(Y, \overline{\sqsubset}_X)$ be an L-topological remotehood relation spaces. If $f : X \to Y$ is an L-topological remotehood relation preserving mapping, then $f : (X, \lessdot_{\overline{\sqsubset}_X}) \to (Y, \lessdot_{\overline{\sqsubset}_Y})$ is an L-topological enclosed relation preserving mapping.*

*Proof.* Let $A, B \in L^Y$ with $A \lessdot_{\overline{\sqsubset}_Y} B$. To prove $f_L^{\leftarrow}(A) \lessdot_{\overline{\sqsubset}_X} f_L^{\leftarrow}(B)$, let $x_\lambda \not\leq f_L^{\leftarrow}(B)$. We prove that $x_\lambda \overline{\sqsubset}_X f_L^{\leftarrow}(A)$.

By $x_\lambda \not\leq f_L^{\leftarrow}(B)$, we have $f_L^{\to}(x_\lambda) \not\leq B$. By $A \lessdot_{\overline{\sqsubset}_Y} B$, we have $f_L^{\to}(x_\lambda) \overline{\sqsubset}_Y A$. Thus $x_\lambda \overline{\sqsubset}_X f_L^{\leftarrow}(A)$. Hence $f_L^{\leftarrow}(A) \lessdot_{\overline{\sqsubset}_X} f_L^{\leftarrow}(B)$. Therefore $f$ is an *L*-topological enclosed relation preserving mapping. □

**Theorem 5.4.** *Let $(X, \preccurlyeq)$ be an L-topological enclosed relation space. Define a binary relation $\overline{\sqsubset}_{\preccurlyeq}$ on $J(L^X) \times L^X$ by*

$$\forall x_\lambda \in J(L^X), \forall A \in L^X, \quad x_\lambda \overline{\sqsubset}_{\preccurlyeq} A \quad \Leftrightarrow \quad \exists B \in L^X,\ A \preccurlyeq B \not\geq x_\lambda.$$

*Then $\overline{\sqsubset}_{\preccurlyeq}$ is an L-topological remotehood relation.*

*Proof.* It is sufficient to check that (LTRNR1)–(LTRNR3) holds for $\overline{\sqsubset}_{\preccurlyeq}$.

(LTRNR1) For any $x_\lambda \in J(L^X)$, we obtain from (LTER1) that $\perp \preccurlyeq \perp \not\geq x_\lambda$. Thus $x_\lambda \overline{\sqsubset}_{\preccurlyeq} \perp$.

(LTRNR2). Assume that $x_\lambda \overline{\sqsubset}_{\preccurlyeq} A$. We need to find some $E \in \psi_{\overline{\sqsubset}_{\preccurlyeq}}(L^X)$ such that $x_\lambda \not\leq E \geq A$.

By $x_\lambda \overline{\sqsubset}_{\preccurlyeq} A$, there is a set $B \in L^X$ such that $A \preccurlyeq B \not\geq x_\lambda$. By $A \preccurlyeq B$ and (LTER4), there is a set $D \in L^X$ such that $A \preccurlyeq D \preccurlyeq B$. Let $E = \bigwedge\{D \in L^X : A \preccurlyeq D \preccurlyeq B\}$. We have $A \preccurlyeq E \leq B$ by (LTER3) and (LTER2). Thus $x_\lambda \not\leq E \geq A$.

To prove that $E \in \psi_{\overline{\sqsubset}_{\preccurlyeq}}(L^X)$, let $y_\mu \not\leq E$. We need to prove that $y_\mu \overline{\sqsubset}_{\preccurlyeq} E$. By $y_\mu \not\leq E$, there is a set $D \in L^X$ such that $A \preccurlyeq D \preccurlyeq B$ and $y_\mu \not\leq D$. Further, by $A \preccurlyeq E$ and (LTER4), there is a set $G \in L^X$ such that $A \preccurlyeq G \preccurlyeq E$. Thus $G \leq E$ by (LTER2). Further, since $A \preccurlyeq G \preccurlyeq E \leq B$, we have $A \preccurlyeq G \preccurlyeq B$. Thus $E \leq G$. So $G = E$ followed by $E \preccurlyeq E \not\geq y_\mu$. Hence $y_\mu \overline{\sqsubset}_{\preccurlyeq} E$. Therefore $E \in \psi_{\overline{\sqsubset}_{\preccurlyeq}}(L^X)$ as desired.

Conversely, assume that there is a set $D \in \psi_{\overline{\sqsubset}_{\preccurlyeq}}(L^X)$ such that $x_\lambda \not\leq D \geq A$. We aim to prove that $x_\lambda \overline{\sqsubset}_{\preccurlyeq} A$.

Since $D \in \psi_{\overline{\sqsubset}_{\preccurlyeq}}(L^X)$, we have $y_\mu \overline{\sqsubset}_{\preccurlyeq} D$ for any $y_\mu \not\leq D$. Thus there is a set $B_{y_\mu} \in L^X$ such that $D \preccurlyeq B_{y_\mu} \not\geq y_\mu$. Let $H = \bigwedge_{y_\mu \not\leq D} B_{y_\mu}$. By (LTER3), we have $D \preccurlyeq H$. Hence $A \leq D \preccurlyeq H \not\geq x_\lambda$ which implies that $A \preccurlyeq H \not\geq x_\lambda$. Therefore $x_\lambda \overline{\sqsubset}_{\preccurlyeq} A$.

(LTRNR3) Let $A, B \in L^X$. On one have, by (LTER5), it is clear that $x_\lambda \overline{\sqsubset}_{\preccurlyeq} A \vee B$ implies $x_\lambda \overline{\sqsubset}_{\preccurlyeq} A$ and $x_\lambda \overline{\sqsubset}_{\preccurlyeq} B$. On the other hand, we have

$$\begin{aligned}
x_\lambda \overline{\sqsubset}_{\preccurlyeq} A \text{ and } x_\lambda \overline{\sqsubset}_{\preccurlyeq} B \quad &\Leftrightarrow \quad \exists C, D \in L^X,\ A \preccurlyeq C \not\geq x_\lambda \text{ and } B \preccurlyeq D \not\geq x_\lambda \\
&\Rightarrow \quad \exists C, D \in L^X,\ A \preccurlyeq C \vee D \not\geq x_\lambda \text{ and } B \preccurlyeq C \vee D \not\geq x_\lambda \\
&\Rightarrow \quad \exists C, D \in L^X,\ A \vee B \preccurlyeq C \vee D \not\geq x_\lambda \\
&\Rightarrow \quad \exists H \in L^X,\ A \vee B \preccurlyeq H \not\geq x_\lambda \\
&\Leftrightarrow \quad x_\lambda \overline{\sqsubset}_{\preccurlyeq} A \vee B.
\end{aligned}$$

Therefore $\overline{\sqsubset}_{\preccurlyeq}$ is an $L$-topological remotehood relation. □

**Theorem 5.5.** *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-topological enclosed relation spaces. If $f : X \to Y$ is an L-topological enclosed relation preserving mapping, then $f : (X, \overline{\sqsubset}_{\preccurlyeq_X}) \to (Y, \overline{\sqsubset}_{\preccurlyeq_Y})$ is an L-topological remotehood relation preserving mapping.*

*Proof.* Let $x_\lambda \in J(L^X)$ and $A \in L^Y$ with $f_L^{\rightarrow}(x_\lambda) \overline{\sqsubset}_{\preccurlyeq_Y} A$. Then there is a set $B \in L^Y$ such that $A \preccurlyeq_Y B \not\geq f_L^{\rightarrow}(x_\lambda)$. Thus $f_L^{\leftarrow}(A) \preccurlyeq_X f_L^{\leftarrow}(B) \not\geq x_\lambda$. Hence $x_\lambda \overline{\sqsubset}_{\preccurlyeq_X} f_L^{\leftarrow}(A)$. Therefore $f$ is an $L$-topological remotehood relation preserving mapping. □

**Theorem 5.6.** *We have $\overline{\sqsubset}_{\preccurlyeq_{\overline{\sqsubset}}} = \overline{\sqsubset}$ for any L-topological remotehood relation space $(X, \overline{\sqsubset})$ and $\preccurlyeq_{\overline{\sqsubset}_{\preccurlyeq}} = \preccurlyeq$ for any L-topological enclosed relation space $(X, \preccurlyeq)$.*

*Proof.* Let $(X, \overline{\sqsubset})$ be an $L$-topological remotehood relation space. Let $x_\lambda \in J(L^X)$ and $A \in L^X$.

If $x_\lambda \overline{\sqsubset}_{\preccurlyeq_{\overline{\sqsubset}}} A$, then there is a set $B \in L^X$ such that $A \preccurlyeq_{\overline{\sqsubset}} B \not\geq x_\lambda$. This implies that $x_\lambda \overline{\sqsubset} A$. Conversely, assume that $x_\lambda \overline{\sqsubset} A$. By (LTRNR2), there is a set $D \in \psi_{\overline{\sqsubset}}(L^X)$ such that $x_\lambda \not\leq D \geq A$. To prove that $A \preccurlyeq_{\overline{\sqsubset}} D$, let $y_\mu \not\leq D$. Since $D \in \psi_{\overline{\sqsubset}}(L^X)$, we have $y_\mu \overline{\sqsubset} D$. Further, by $D \in \psi_{\overline{\sqsubset}}(L^X)$ and $y_\mu \not\leq D \geq A$, we obtain from (LTRNR2) that $y_\mu \overline{\sqsubset} A$. Thus $A \preccurlyeq_{\overline{\sqsubset}} D \not\geq x_\lambda$ which implies that $x_\lambda \overline{\sqsubset}_{\preccurlyeq_{\overline{\sqsubset}}} A$.

In conclusion, for any $x_\lambda \in J(L^X)$ and any $A \in L^X$, we have $x_\lambda \overline{\sqsubset} A$ if and only if $x_\lambda \overline{\sqsubset}_{\preccurlyeq_{\overline{\sqsubset}}} A$. That is, $\overline{\sqsubset}_{\preccurlyeq_{\overline{\sqsubset}}} = \overline{\sqsubset}$.

Let $(X, \preccurlyeq)$ be an $L$-topological enclosed relation space. Let $A, B \in L^X$.

If $A \preccurlyeq_{\overline{\sqsubset}_{\preccurlyeq}} B$, then $x_\lambda \overline{\sqsubset}_{\preccurlyeq} A$ for any $x_\lambda \not\leq B$. By $x_\lambda \overline{\sqsubset}_{\preccurlyeq} A$, there is a set $D_{x_\lambda} \in L^X$ such that $A \preccurlyeq D_{x_\lambda} \not\geq x_\lambda$. Let $D = \bigwedge_{x_\lambda \not\leq B} D_{x_\lambda}$. Then $D \leq B$. In addition, we have $A \preccurlyeq D$ by (LTER3). Thus $A \preccurlyeq B$.

Conversely, assume that $A \preccurlyeq B$. By (LTER4), there is a set $D \in L^X$ such that $A \preccurlyeq D \preccurlyeq B$. Let $E = \bigwedge\{C \in L^X : A \preccurlyeq C \preccurlyeq B\}$. We have $A \preccurlyeq E \leq B$ by (LTER3) and (LTER2). This implies that $A \preccurlyeq E \not\geq x_\lambda$ for any $x_\lambda \not\leq B$. Thus $x_\lambda \overline{\sqsubset}_{\preccurlyeq} A$ for any $x_\lambda \not\leq B$. Therefore $A \preccurlyeq_{\overline{\sqsubset}_{\preccurlyeq}} B$.

In conclusion, for all $A, B \in L^X$, we have $A \preccurlyeq_{\overline{\sqsubset}_{\preccurlyeq}} B$ if and only if $A \preccurlyeq B$. That is, $\preccurlyeq_{\overline{\sqsubset}_{\preccurlyeq}} = \preccurlyeq$. □

Based on Theorems 5.2 and 5.3, we obtain a functor $\mathbb{H}$ :$L$-**TRNRS** $\to$ $L$-**TERS** by

$$\mathbb{H}((X,\overline{\sqsubset})) = (X, \preccurlyeq_{\overline{\sqsubset}}), \quad \mathbb{H}(f) = f.$$

Based on Theorems 5.2–5.6, we find that $\mathbb{H}$ is an isomorphic functor. Thus we have the following conclusion.

**Theorem 5.7.** *The category* $L$-**TRNRS** *is isomorphic to the category* $L$-**TERS**.

**Remark 5.8.** Relations between $L$-topological remotehood relation spaces and $L$-topological remotehood spaces can be checked directly as follows.

(1) Let $(X,\overline{\sqsubset})$ be an $L$-topological remotehood relation space. For any $x_\lambda \in J(L^X)$, we define

$$(\mathcal{R}_{\overline{\sqsubset}})_{x_\lambda} = \{A \in L^X : x_\lambda \overline{\sqsubset} A\}.$$

Then $\mathcal{R}_{\overline{\sqsubset}} = \{(\mathcal{R}_{\overline{\sqsubset}})_{x_\lambda} : x_\lambda \in J(L^X)\}$ is an $L$-topological remotehood system on $X$.

(2) Let $(X,\mathcal{R})$ be an $L$-topological remotehood space. Define a binary relation $\overline{\sqsubset}_{\mathcal{R}}$ by

$$\forall x_\lambda \in J(L^X), \forall A \in L^X, \ x_\lambda \overline{\sqsubset}_{\mathcal{R}} A \iff A \in \mathcal{R}_{x_\lambda}.$$

Then $\overline{\sqsubset}_{\mathcal{R}}$ is an $L$-topological remotehood relation.

(3) The category $L$-**TRNRS** is isomorphic to the category $L$-**TRNS**.

## 6. *L*-topological derived remotehood relation spaces

In this section, we introduce $L$-topological derived remotehood relation space by which we characterize $L$-topological remote neighborhood relation space. For this, we recall the following denotations.

For $A \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$, we denote $A_{x_\lambda} = \bigvee\{y_\mu \in \beta^*(A) : x_\lambda \not\leq y_\mu\}$ and $\beta^*_\lambda(L) = \{\mu \in \beta^*(\top) : \lambda \in \beta^*(\mu)\}$ [30]. We have the following results.

**Proposition 6.1.** ([30]) *For all* $x_\lambda, y_\eta \in \beta^*(\underline{\top})$, $A \in L^X$ *and* $\{A_i\}_{i\in I} \subseteq L^X$, *we have*

*(1)* $x_\lambda \not\leq A$ *implies* $A_{x_\lambda} = A$;

*(2)* $A \leq B$ *implies* $A_{x_\lambda} \leq B_{x_\lambda}$;

*(3)* $(A_{x_\lambda})_{x_\lambda} = A_{x_\lambda}$;

*(4)* $\mu \in \beta^*_\lambda(L)$ *implies* $A_{x_\lambda} \leq A_{x_\mu}$ *and* $(A_{x_\mu})_{x_\lambda} = (A_{x_\lambda})_{x_\mu} = A_{x_\lambda}$;

*(5)* $(\bigvee_{i\in I} A_i)_{x_\lambda} = \bigvee_{i\in I}(A_i)_{x_\lambda}$.

**Definition 6.2.** A binary relation $\overline{\sqsubset}^d$ on $J(L^X) \times L^X$ is called an $L$-topological derived remotehood relation on $L^X$ and the pair $(X,\overline{\sqsubset}^d)$ is called an $L$-topological derived remotehood relation space, if for all $x_\lambda \in J(L^X)$ and $A, B \in L^X$,

(LTDRNR1) $x_\lambda \overline{\sqsubset}^d \underline{\perp}$;

(LTDRNR2) $x_\lambda \overline{\sqsubset}^d A$ if and only if any $\mu \in \beta^*(\lambda)$ implies some $B \in \psi_{\overline{\sqsubset}^d}(L^X)$ such that $x_\lambda \not\leq B \geq A_{x_\mu}$, where $\psi_{\overline{\sqsubset}^d}(L^X) = \{D \in L^X : \forall y_\mu \not\leq D, \ y_\mu \overline{\sqsubset}^d D\}$;

(LTDRNR3) $x_\lambda \overline{\sqsubset}^d A \vee B$ if and only if $x_\lambda \overline{\sqsubset}^d A$ and $x_\lambda \overline{\sqsubset}^d B$.

Let $(X,\overline{\sqsubset}^d_X)$ and $(Y,\overline{\sqsubset}^d_Y)$ be $L$-topological derived remotehood relation spaces. A mapping $f : X \to Y$ is called an $L$-topological derived remotehood relation preserving mapping if $f^{\to}_L(x_\lambda)\overline{\sqsubset}^d_Y B$ and $f^{\to}_L(x_\lambda) \not\leq B$ imply $x_\lambda \overline{\sqsubset}^d_X f^{\leftarrow}_L(B)$ for all $x_\lambda \in J(L^X)$ and $B \in L^Y$.

The category of $L$-topological derived remotehood relation spaces and $L$-topological derived remote neighborhood relation preserving mappings is denoted by $L$-**TDRNRS**.

**Lemma 6.3.** *Let* $(X,\overline{\sqsubset}^d)$ *be an L-topological derived remotehood relation space. For* $x_\lambda \in J(L^X)$ *and* $A, B \in L^X$,

*(1)* $x_\lambda \overline{\sqsubset}^d B \geq A$ *implies* $x_\lambda \overline{\sqsubset}^d A$;

*(2)* $A, B \in \psi_{\overline{\sqsubset}^d}(L^X)$ *implies* $A \vee B \in \psi_{\overline{\sqsubset}^d}(L^X)$;

*(3)* $x_\lambda \overline{\sqsubset}^d A$ *if and only if* $x_\lambda \overline{\sqsubset}^d A_{x_\mu}$ *for any* $\mu \in \beta^*(\lambda)$.

*Proof.* From (LTDRNR2), (LTDRNR3) and (2) of Proposition 6.1, (1) and (2) are clear.

(3) Let $x_\lambda \overline{\sqsubset}^d A$. For any $\eta \in \beta^*(\lambda)$, we have $A_{x_\eta} \leq A$. Thus $x_\lambda \overline{\sqsubset}^d A_{x_\eta}$ by (1). Conversely, assume that $x_\lambda \overline{\sqsubset}^d A_{x_\eta}$ for any $\eta \in \beta^*(\lambda)$. For any $\mu \in \beta^*(\lambda)$, there is an element $\eta \in \beta^*(\lambda)$ such that $\mu \in \beta^*(\eta)$. By $x_\lambda \overline{\sqsubset}^d A_{x_\eta}$ and (LTDRNR2), for any $\theta \in \beta^*(\lambda)$ there is a set $B \in \psi_{\overline{\sqsubset}^d}(L^X)$ such that $x_\lambda \not\leq B \geq (A_{x_\eta})_{x_\theta}$. In particular, we have $x_\lambda \not\leq B \geq (A_{x_\eta})_{x_\mu} = A_{x_\mu}$. By the arbitrariness of $\mu \in \beta^*(\lambda)$, we have $x_\lambda \overline{\sqsubset}^d A$. □

**Theorem 6.4.** *Let $(X, \overline{\sqsubset}^d)$ be an L-topological derived remotehood relation space. Define a binary relation $\overline{\sqsubset}_{\overline{\sqsubset}^d}$ by*

$$\forall x_\lambda \in J(L^X), \forall B \in L^X,\ x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d} B \quad \Leftrightarrow \quad \exists D \in \psi_{\overline{\sqsubset}^d}(L^X),\ x_\lambda \overline{\sqsubset}^d D \text{ and } x_\lambda \not\leq D \geq B,$$

*where $\psi_{\overline{\sqsubset}^d}(L^X) = \{D \in L^X : \forall y_\mu \not\leq D, y_\mu \overline{\sqsubset}^d D\}$. Then $\overline{\sqsubset}_{\overline{\sqsubset}^d}$ is an L-topological remotehood relation on $L^X$.*

*Proof.* (LTRNR1) We have $\underline{\perp} \in \psi_{\overline{\sqsubset}^d}(L^X)$, $x_\lambda \not\leq \underline{\perp}$ and $x_\lambda \overline{\sqsubset}^d \underline{\perp}$ by (LTDRNR1). Thus $x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d} \underline{\perp}$.

(LTRNR2) If $x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d} A$, then there is a set $D \in \psi_{\overline{\sqsubset}^d}(L^X)$ such that $x_\lambda \overline{\sqsubset}^d D$ and $x_\lambda \not\leq D \geq A$. Further, for any $y_\mu \not\leq D$, we have $y_\mu \overline{\sqsubset}^d D$ by $D \in \psi_{\overline{\sqsubset}^d}(L^X)$. Hence $D \in \psi_{\overline{\sqsubset}_{\overline{\sqsubset}^d}}(L^X)$. So the necessity of (LTRNR2) holds.

Conversely, assume that $x_\lambda \not\leq D \geq A$ for some $D \in \psi_{\overline{\sqsubset}_{\overline{\sqsubset}^d}}(L^X)$. We need to prove that $x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d} A$.

For any $y_\mu \not\leq D$, we have $y_\mu \overline{\sqsubset}_{\overline{\sqsubset}^d} D$ by $D \in \psi_{\overline{\sqsubset}_{\overline{\sqsubset}^d}}(L^X)$. Then there is a set $E \in \psi_{\overline{\sqsubset}^d}(L^X)$ such that $y_\mu \overline{\sqsubset}^d E$ and $y_\mu \not\leq E \geq D$. Thus $y_\mu \overline{\sqsubset}^d D$ by (1) of Lemma 6.3. Hence $D \in \psi_{\overline{\sqsubset}^d}(L^X)$. From this result and $x_\lambda \not\leq D \geq A$, we have $x_\lambda \overline{\sqsubset}^d D \geq A$. Therefore $x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d} A$ as desired.

(LTRNR3) We have

$$\begin{aligned} x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d} A \vee B \quad &\Leftrightarrow \quad \exists D \in \psi_{\overline{\sqsubset}^d}(L^X),\ x_\lambda \overline{\sqsubset}^d D \text{ and } x_\lambda \not\leq D \geq A \vee B \\ &\Rightarrow \quad \exists D \in \psi_{\overline{\sqsubset}^d}(L^X),\ x_\lambda \overline{\sqsubset}^d D,\ x_\lambda \not\leq D \geq A \text{ and } x_\lambda \not\leq D \geq B \\ &\Rightarrow \quad x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d} A \text{ and } x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d} B. \end{aligned}$$

Conversely, by (LTDRNR3) and (2) of Lemma 6.3, we have

$$\begin{aligned} x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d} A \text{ and } x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d} B \quad &\Leftrightarrow \quad \exists D, E \in \psi_{\overline{\sqsubset}^d}(L^X),\ x_\lambda \overline{\sqsubset}^d D, x_\lambda \overline{\sqsubset}^d E \text{ and } x_\lambda \not\leq D \vee E \geq A \vee B \\ &\Rightarrow \quad \exists D \vee E \in \psi_{\overline{\sqsubset}^d}(L^X),\ x_\lambda \overline{\sqsubset}^d D \vee E \text{ and } x_\lambda \not\leq D \vee E \geq A \vee B \\ &\Rightarrow \quad x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d} A \vee B. \end{aligned}$$

Therefore $\overline{\sqsubset}_{\overline{\sqsubset}^d}$ is an $L$-topological remotehood relation. □

**Theorem 6.5.** *Let $(X, \overline{\sqsubset}^d_X)$ and $(Y, \overline{\sqsubset}^d_Y)$ be L-topological derived remotehood relation spaces. If $f : X \to Y$ is an L-topological derived remotehood relation preserving mapping, then $f : (X, \overline{\sqsubset}_{\overline{\sqsubset}^d_X}) \to (Y, \overline{\sqsubset}_{\overline{\sqsubset}^d_Y})$ is an L-topological remotehood relation preserving mapping.*

*Proof.* Let $x_\lambda \in J(L^X)$ and $B \in L^Y$. Let $f_L^{\rightarrow}(x_\lambda) \overline{\sqsubset}_{\overline{\sqsubset}^d_Y} B$. Then there is a set $D \in \psi_{\overline{\sqsubset}^d_Y}(L^X)$ such that $f_L^{\rightarrow}(x_\lambda) \overline{\sqsubset}^d_Y D$ and $f_L^{\rightarrow}(x_\lambda) \not\leq D \geq B$. Thus $x_\lambda \overline{\sqsubset}^d_X f_L^{\leftarrow}(D) \geq f_L^{\leftarrow}(B)$. Further, for any $y_\eta \not\leq f_L^{\leftarrow}(D)$, we have $f_L^{\rightarrow}(y_\eta) \not\leq D$ and $f_L^{\rightarrow}(y_\eta) \overline{\sqsubset}^d_Y D$ by $D \in \psi_{\overline{\sqsubset}^d_Y}(L^X)$. Thus $y_\eta \overline{\sqsubset}^d_X f_L^{\leftarrow}(D)$ which implies that $f_L^{\leftarrow}(D) \in \psi_{\overline{\sqsubset}^d_X}(L^X)$. Hence $x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d_X} f_L^{\leftarrow}(B)$. Therefore $f$ is an $L$-topological remotehood relation preserving mapping. □

**Theorem 6.6.** *Let $(X, \overline{\sqsubset})$ be an L-topological derived remotehood relation space. Define a binary relation $\overline{\sqsubset}^d_{\overline{\sqsubset}}$ on $J(L^X) \times L^X$ by*

$$\forall x_\lambda \in J(L^X), \forall A \in L^X,\ x_\lambda \overline{\sqsubset}^d_{\overline{\sqsubset}} A \quad \Leftrightarrow \quad \forall \mu \in \beta^*(\lambda),\ x_\lambda \overline{\sqsubset} A_{x_\mu}.$$

*Then $\overline{\sqsubset}^d_{\overline{\sqsubset}}$ is an L-topological derived remotehood relation on $L^X$.*

*Proof.* (LTDRNR1) For any $\mu \in \beta^*(\lambda)$, we have $x_\lambda \overline{\sqsubset} \underline{\perp} = \underline{\perp}_{x_\mu}$ by (LTRNR1). Thus $x_\lambda \overline{\sqsubset}^d_{\overline{\sqsubset}} \underline{\perp}$.

(LTDRNR2) For any $B \in L^X$, we check that $B \in \psi_{\overline{\sqsubset}^d_{\overline{\sqsubset}}}(L^X)$ if and only if $B \in \psi_{\overline{\sqsubset}}(L^X)$.

Let $B \in \psi_{\overline{\sqsubset}^d_{\overline{\sqsubset}}}(L^X)$. For any $z_\eta \not\leq B$, we have $z_\eta \overline{\sqsubset}^d_{\overline{\sqsubset}} B$ by $B \in \psi_{\overline{\sqsubset}^d_{\overline{\sqsubset}}}(L^X)$. By $z_\eta \not\leq B$, there is an element $\theta \in \beta^*(\eta)$ such that $z_\theta \not\leq B$. Further, by $z_\eta \overline{\sqsubset}^d_{\overline{\sqsubset}} B$, we have $z_\eta \overline{\sqsubset} B_{z_\theta} = B$. Hence $B \in \psi_{\overline{\sqsubset}}(L^X)$. Conversely, let $B \in \psi_{\overline{\sqsubset}}(L^X)$. For any $y_\eta \not\leq B$, we have $y_\theta \not\leq B$ for some $\theta \in \beta^*(\eta)$. Thus $y_\theta \overline{\sqsubset} B$ by $B \in \psi_{\overline{\sqsubset}}(L^X)$. Further, since $y_\theta \leq y_\eta$ and $B \geq B_{y_\delta}$ for any $\delta \in \beta^*(\eta)$, we obtain from (LTRNR2) that $y_\eta \overline{\sqsubset} B_{y_\delta}$. Thus $y_\eta \overline{\sqsubset}^d_{\overline{\sqsubset}} B$. Therefore $B \in \psi_{\overline{\sqsubset}^d_{\overline{\sqsubset}}}(L^X)$.

Now, by the above fact and (LTRNR2), we have

$$\begin{aligned} x_\lambda \overline{\sqsubset}^d_{\overline{\sqsubset}} A \quad &\Leftrightarrow \quad \forall \mu \in \beta^*(\lambda),\ x_\lambda \overline{\sqsubset} A_{x_\mu} \\ &\Leftrightarrow \quad \forall \mu \in \beta^*(\lambda),\ \exists B \in \psi_{\overline{\sqsubset}}(L^X),\ x_\lambda \not\leq B \geq A_{x_\mu} \\ &\Leftrightarrow \quad \forall \mu \in \beta^*(\lambda),\ \exists B \in \psi_{\overline{\sqsubset}^d_{\overline{\sqsubset}}}(L^X),\ x_\lambda \not\leq B \geq A_{x_\mu}. \end{aligned}$$

So (LTDRNR2) holds for $\overline{\sqsubset}^d_{\overline{\sqsubset}}$.

(LTDRNR3) By (LTRNR3), we have

$$\begin{aligned} x_\lambda \overline{\sqsubset}^d_{\overline{\sqsubset}} A \vee B \quad &\Leftrightarrow \quad \forall \mu \in \beta^*(\lambda),\ x_\lambda \overline{\sqsubset} (A \vee B)_{x_\mu} \\ &\Leftrightarrow \quad \forall \mu \in \beta^*(\lambda),\ x_\lambda \overline{\sqsubset} A_{x_\mu} \vee B_{x_\mu} \\ &\Leftrightarrow \quad \forall \mu \in \beta^*(\lambda),\ x_\lambda \overline{\sqsubset} A_{x_\mu} \text{ and } x_\lambda \overline{\sqsubset} B_{x_\mu} \\ &\Leftrightarrow \quad x_\lambda \overline{\sqsubset}^d_{\overline{\sqsubset}} A \text{ and } x_\lambda \overline{\sqsubset}^d_{\overline{\sqsubset}} B. \end{aligned}$$

Therefore $\overline{\sqsubset}^d_{\overline{\sqsubset}}$ is an $L$-topological derived remotehood relation. □

**Theorem 6.7.** *Let $(X, \overline{\sqsubset}_X)$ and $(Y, \overline{\sqsubset}_Y)$ be L-topological remotehood relation spaces. If $f : X \to Y$ is an L-topological remotehood relation preserving mapping, then $f : (X, \overline{\sqsubset}^d_{\overline{\sqsubset}_X}) \to (Y, \overline{\sqsubset}^d_{\overline{\sqsubset}_Y})$ is an L-topological derived remotehood relation preserving mapping.*

*Proof.* Let $x_\lambda \in J(L^X)$ and $B \in L^Y$. Let $f_L^{\rightarrow}(x_\lambda) \overline{\sqsubset}^d_{\overline{\sqsubset}_Y} B$ and $x_\lambda \not\leq f_L^{\leftarrow}(B)$. Then there is a $\mu \in \beta^*(\lambda)$ such that $x_\mu \not\leq f_L^{\leftarrow}(B)$. Thus $f_L^{\rightarrow}(x_\mu) \not\leq B$. By $f_L^{\rightarrow}(x_\lambda) \overline{\sqsubset}^d_{\overline{\sqsubset}_Y} B$, we have $f_L^{\rightarrow}(x_\lambda) \overline{\sqsubset}_Y B_{f_L^{\rightarrow}(x_\mu)} = B$. Thus $x_\lambda \overline{\sqsubset}_X f_L^{\leftarrow}(B)$. Hence $x_\lambda \overline{\sqsubset}^d_{\overline{\sqsubset}_X} f_L^{\leftarrow}(B) \geq f_L^{\leftarrow}(B)_{x_\eta}$ for any $\eta \in \beta^*(\lambda)$. This implies that $x_\lambda \overline{\sqsubset}^d_{\overline{\sqsubset}_X} f_L^{\leftarrow}(B)_{x_\eta}$ for any $\eta \in \beta^*(\lambda)$. Therefore $f$ is an $L$-topological derived remotehood relation preserving mapping. □

**Theorem 6.8.** *We have $\overline{\sqsubset}^d_{\overline{\sqsubset}_{\overline{\sqsubset}^d}} = \overline{\sqsubset}^d$ for any L-topological derived remotehood relation space $(X, \overline{\sqsubset}^d)$ and $\overline{\sqsubset}_{\overline{\sqsubset}^d_{\overline{\sqsubset}}} = \overline{\sqsubset}$ for any L-topological remotehood relation space $(X, \overline{\sqsubset})$.*

*Proof.* Let $(X, \overline{\sqsubset})$ be an $L$-topological remotehood relation space.

Let $x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d_{\overline{\sqsubset}}} A$. Then $x_\lambda \not\leq A$ by (LTRNR2). Thus there is a $\mu \in \beta^*(\lambda)$ such that $x_\mu \not\leq A$. So $A_{x_\mu} = A$. Since $x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d_{\overline{\sqsubset}}} A$, there is a set $B \in \psi_{\overline{\sqsubset}^d_{\overline{\sqsubset}}}(L^X)$ such that $x_\lambda \not\leq B \geq A$. This implies that $x_\lambda \overline{\sqsubset}^d_{\overline{\sqsubset}} B \geq A$. Hence $x_\lambda \overline{\sqsubset}^d_{\overline{\sqsubset}} A$ by (1) of Lemma 6.3. Therefore $x_\lambda \overline{\sqsubset} A_{x_\mu} = A$.

Conversely, assume that $x_\lambda \overline{\sqsubset} A$. By (LTRNR2), there is a set $D \in \psi_{\overline{\sqsubset}}(L^X)$ such that $x_\lambda \not\leq D \geq A$. Since $D \in \psi_{\overline{\sqsubset}}(L^X)$, we have $x_\lambda \overline{\sqsubset} D$. In addition, for any $\mu \in \beta^*(\lambda)$, we have $x_\lambda \overline{\sqsubset} D \geq D_{x_\mu} \geq A_{x_\mu}$. Thus $x_\lambda \overline{\sqsubset} A_{x_\mu}$ by (LTRNR2). Hence $x_\lambda \overline{\sqsubset}^d_{\overline{\sqsubset}} A$. Further, by the proof of Theorem 6.6, we have $D \in \psi_{\overline{\sqsubset}^d_{\overline{\sqsubset}}}(L^X)$. Therefore $x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d_{\overline{\sqsubset}}} A$.

In conclusion, for any $x_\lambda \in J(L^X)$ any $A \in L^X$, we have $x_\lambda \overline{\sqsubset} A$ if and only if $x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d_{\overline{\sqsubset}}} A$. That is, $\overline{\sqsubset} = \overline{\sqsubset}_{\overline{\sqsubset}^d_{\overline{\sqsubset}}}$.

Let $(X, \overline{\sqsubset}^d)$ be an $L$-topological derived remotehood relation space.

Let $x_\lambda \overline{\sqsubset}^d_{\overline{\sqsubset}_{\overline{\sqsubset}^d}} A$. For any $\mu \in \beta^*(\lambda)$, we have $x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d} A_{x_\mu}$. Then there is a set $D \in \psi_{\overline{\sqsubset}^d}(L^X)$ such that $x_\lambda \not\leq D \geq A_{x_\mu}$. Since $D \in \psi_{\overline{\sqsubset}^d}(L^X)$, we have $x_\lambda \overline{\sqsubset}^d D \geq A_{x_\mu}$. Thus $x_\lambda \overline{\sqsubset}^d A_{x_\mu}$. Hence $x_\lambda \overline{\sqsubset}^d A$ by (3) of Lemma 6.3.

Conversely, let $x_\lambda \overline{\sqsubset}^d A$. By (LTDRNR2), for any $\mu \in \beta^*(\lambda)$ there is a set $D \in \psi_{\overline{\sqsubset}^d}(L^X)$ such that $x_\lambda \overline{\sqsubset}^d D \geq A_{x_\mu}$. That is, $x_\lambda \overline{\sqsubset}_{\overline{\sqsubset}^d} A_{x_\mu}$ for any $\mu \in \beta^*(\lambda)$. Hence $x_\lambda \overline{\sqsubset}^d_{\overline{\sqsubset}_{\overline{\sqsubset}^d}} A$.

In conclusion, for any $x_\lambda \in J(L^X)$ any $A \in L^X$, we have $x_\lambda \overline{\sqsubset}^d A$ if and only if $x_\lambda \overline{\sqsubset}^d_{\overline{\sqsubset}_{\overline{\sqsubset}^d}} A$. That is, $\overline{\sqsubset}^d = \overline{\sqsubset}^d_{\overline{\sqsubset}_{\overline{\sqsubset}^d}}$. □

Based on Theorems 6.4 and 6.5, we obtain a functor $\mathbb{U}$ :$L$-**CRNRS** → $L$-**CERS** by

$$\mathbb{U}((X, \overline{\sqsubset}^d)) = (X, \overline{\sqsubset}_{\overline{\sqsubset}^d}), \quad \mathbb{U}(f) = f.$$

Based on Theorems 6.4–6.8, we find that $\mathbb{U}$ is an isomorphic functor. Thus we have the following conclusion.

**Theorem 6.9.** *The category $L$-**TDRNRS** is isomorphic to the category $L$-**TRNRS**.*

In Remark 4.11, we established connections between $L$-topological derived neighborhood relation space and $L$-topological derived neighborhood space. Actually, we can introduce $L$-topological derived remotehood space and discuss its connections with $L$-topological derived remotehood relation space.

**Definition 6.10.** A set $\mathcal{R}^d = \{\mathcal{R}^d_{x_\lambda} \subseteq L^X : x_\lambda \in J(L^X)\}$ is called an $L$-topological derived remotehood system on $L^X$ and the pair $(X, \mathcal{R}^d)$ is called an $L$-topological derived remotehood space, if for all $A, B \in L^X$ and $x_\lambda \in J(L^X)$,

(LTDRN1) $\perp \in \mathcal{R}^d_{x_\lambda}$;

(LTDRN2) $A \in \mathcal{R}^d_{x_\lambda}$ if and only if any $\mu \in \beta^*(\lambda)$ implies some $D \in \mathcal{R}^d_{x_\lambda}$ such that $x_\lambda \not\leq D \geq A_{x_\mu}$ and $D \in \mathcal{R}^d_{y_\eta}$ for any $y_\eta \not\leq D$;

(LTDRN3) $A \vee B \in \mathcal{R}^d_{x_\lambda}$ if and only if $A, B \in \mathcal{R}^d_{x_\lambda}$.

Let $(X, \mathcal{R}^d_X)$ and $(Y, \mathcal{R}^d_Y)$ be $L$-topological derived remotehood spaces. A mapping $f : X \to Y$ is called an $L$-topological derived remotehood preserving mapping, if $B \in \mathcal{R}^d_{f_L^{\to}(x_\lambda)}$ and $f_L^{\to}(x_\lambda) \not\leq B$ imply $f_L^{\leftarrow}(B) \in \mathcal{R}^d_{x_\lambda}$ for all $x_\lambda \in J(L^X)$ and $B \in L^Y$.

The category of $L$-topological derived remotehood spaces and $L$-topological derived remotehood preserving mappings is denoted by $L$-**TDRNS**. We have the following result.

**Remark 6.11.** (1) Let $(X, \overline{\sqsubset}^d)$ be an $L$-topological derived remotehood relation space. We define

$$\forall x_\lambda \in J(L^X), \ (\mathcal{R}^d_{\overline{\sqsubset}^d})_{x_\lambda} = \{A \in L^X : x_\lambda \overline{\sqsubset}^d A\}.$$

Then $\mathcal{R}^d_{\overline{\sqsubset}^d} = \{(\mathcal{R}^d_{\overline{\sqsubset}^d})_{x_\lambda} : x_\lambda \in J(L^X)\}$ is an $L$-topological derived remotehood systems on $L^X$.

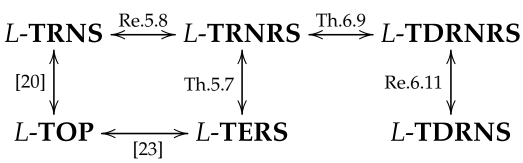
(2) Let $(X, \mathcal{R}^d)$ be an $L$-topological derived remotehood space. Define a binary relation $\overline{\sqsubset}^d_{\mathcal{R}^d}$ by

$$\forall x_\lambda \in J(L^X), \forall A \in L^X, \ x_\lambda \overline{\sqsubset}^d_{\mathcal{R}^d} A \ \Leftrightarrow \ A \in \mathcal{R}^d_{x_\lambda}.$$

Then $\overline{\sqsubset}^d_{\mathcal{R}^d}$ is an $L$-topological derived remotehood relation on $L^X$.

(3) The category $L$-**TDRNRS** is isomorphic to the category $L$-**TDRNS**.

Isomorphisms among the categories mentioned in Sections 5 and 6 are presented by as follows.



## 7. Conclusions

(1) In this paper, we introduce notions of $L$-topological neighborhood relation space, $L$-topological derived neighborhood relation space, $L$-topological remotehood relation space and $L$-topological derived remotehood space. We find that all of these spaces are either isomorphic to $L$-topological internal relation space or $L$-topological enclosed relation space. Thus they are all categorically isomorphic to $L$-topological space. Specifically, these isomorphisms are presented by the diagrams in Sections 4 and 6.

(2) In the introduction section, we are looking for some fuzzy relations that can be used to characterize $L$-topological neighborhood space and $L$-topological remotehood space. Actually, in Remark 3.9, we established a direct connection between $L$-topological neighborhood space and $L$-topological neighborhood relation space. Similarly, in Remark 5.8, we established a direct connection between $L$-topological remotehood space and $L$-topological remotehood relation space.

Also, we are seeking some $L$-topological derived neighborhood space and $L$-topological derived remotehood space that can be used to characterize $L$-topological neighborhood space and $L$-topological remote neighborhood space. Indeed, in Sections 4 and 6, we respectively introduced them and obtain the desired characterizations in Remarks 4.11 and 6.11.

(3) We present the following example to show the fuzzy relations mentioned in this paper.

Let $X = \{x\}$ and $L = \{\bot, a, b, \top\}$ be a diamond lattice, where $a$ and $b$ are incomparable.

| | $x_\bot$ | $x_a$ | $x_b$ | $x_\top$ |
|---|---|---|---|---|
| $x_\bot$ | $\preccurlyeq$ | $\preccurlyeq$ | $\preccurlyeq$ | $\preccurlyeq$ |
| $x_a$ | | | | $\preccurlyeq$ |
| $x_b$ | | | $\preccurlyeq$ | $\preccurlyeq$ |
| $x_\top$ | | | | $\preccurlyeq$ |

Table 1: An $L$-topological internal relation.

| | $x_\bot$ | $x_a$ | $x_b$ | $x_\top$ |
|---|---|---|---|---|
| $x_\bot$ | $\curlyeqprec$ | $\curlyeqprec$ | $\curlyeqprec$ | $\curlyeqprec$ |
| $x_a$ | | $\curlyeqprec$ | | $\curlyeqprec$ |
| $x_b$ | | | | $\curlyeqprec$ |
| $x_\top$ | | | | $\curlyeqprec$ |

Table 2: An $L$-topological enclosed relation.

| | $x_\bot$ | $x_a$ | $x_b$ | $x_\top$ |
|---|---|---|---|---|
| $x_a$ | | | | $\sqsubseteq$ |
| $x_b$ | | | $\sqsubseteq$ | $\sqsubseteq$ |

Table 3: An $L$-topological neighborhood relation.

| | $x_\bot$ | $x_a$ | $x_b$ | $x_\top$ |
|---|---|---|---|---|
| $x_a$ | $\overline{\sqsubset}$ | | | |
| $x_b$ | $\overline{\sqsubset}$ | $\overline{\sqsubset}$ | | |

Table 4: An $L$-topological remotehood relation.

| | $x_\bot$ | $x_a$ | $x_b$ | $x_\top$ |
|---|---|---|---|---|
| $x_a$ | | | $\sqsubseteq^d$ | $\sqsubseteq^d$ |
| $x_b$ | | $\sqsubseteq^d$ | $\sqsubseteq^d$ | $\sqsubseteq^d$ |

Table 5: An $L$-topological derived neighborhood relation.

| | $x_\bot$ | $x_a$ | $x_b$ | $x_\top$ |
|---|---|---|---|---|
| $x_a$ | $\overline{\sqsubset}^d$ | $\overline{\sqsubset}^d$ | | |
| $x_b$ | $\overline{\sqsubset}^d$ | | $\overline{\sqsubset}^d$ | $\overline{\sqsubset}^d$ |

Table 6: An $L$-topological derived remotehood relation.

Notions defined in the tables from (1) to (6) are mutually induced. In addition, they are all isomorphic to the $L$-topology $\mathcal{T} = \{x_\bot, x_b, x_\top\}$.

(4) Relations among $L$-topological space, $L$-topological neighborhood space, $L$-topological remotehood space, $L$-topological neighborhood relation space, $L$-topological derived neighborhood relation space, $L$-topological remotehood relation space and $L$-topological derived remotehood relation space may provide some alternative ways in discussing relations among $L$-topological space, $L$-matroid, $L$-convex space and $L$-convergence space.

## Acknowledgment

The authors deeply thank the editor for handling the paper and the reviewers for their precious comments and suggestions.

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