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Numerical Radius Inequalities for Products and Sums of Semi-Hilbertian Space Operators

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Abstract. New inequalities for the A-numerical radius of the products and sums of operators acting on a semi-Hilbert space, i.e. a space generated by a positive semidefinite operator A, are established. In particular, for every operators T and S which admit A-adjoints, it is proved that

$$\omega_A(TS) \leq \frac{1}{2}\omega_A(ST) + \frac{1}{4}\Big(\|T\|_A\|S\|_A + \|TS\|_A\Big),$$

where $\omega_A(T)$ and $\|T\|_A$ denote the A-numerical radius and the A-operator seminorm of an operator T respectively.

1. Introduction and Preliminaries

Let $\mathbb{B}(\mathcal{H})$ stand for the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. Let I be the identity operator. For $T \in \mathbb{B}(\mathcal{H})$, we denote by $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* the range, the kernel and the adjoint of T, respectively. For a given linear subspace $\mathcal{M}$ of $\mathcal{H}$, its closure in the norm topology of $\mathcal{H}$ will be denoted by $\overline{\mathcal{M}}$. Further, let $P_{\mathcal{S}}$ stand for the orthogonal projection onto a closed subspace $\mathcal{S}$ of $\mathcal{H}$. An operator $T \in \mathbb{B}(\mathcal{H})$ is called positive if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and we will write $T \geq 0$. Furthermore, if $T \geq 0$, then the square root of T is denoted by $T^{1/2}$. For $T \in \mathbb{B}(\mathcal{H})$, the absolute value of T, denoted by $|T|$, is defined as $|T| = (T^*T)^{1/2}$. Throughout this article, A denotes a non-zero positive operator on $\mathcal{H}$. The positive operator A induces the following semi-inner product

$$\langle\cdot,\cdot\rangle_A : \mathcal{H}\times\mathcal{H} \longrightarrow \mathbb{C},\ (x,y) \longmapsto \langle x,y\rangle_A := \langle Ax, y\rangle = \langle A^{1/2}x, A^{1/2}y\rangle.$$

The seminorm induced by $\langle\cdot,\cdot\rangle_A$ is given by $\|x\|_A = \|A^{1/2}x\|$ for all $x \in \mathcal{H}$. It is easy to check that $\|\cdot\|_A$ is a norm if and only if A is injective and that the semi-Hilbertian space $(\mathcal{H}, \|\cdot\|_A)$ is complete if and only if $\overline{\mathcal{R}(A)} = \mathcal{R}(A)$. It is well-known that the semi-inner product $\langle\cdot,\cdot\rangle_A$ induces an inner product on the quotient space $\mathcal{H}/\mathcal{N}(A)$ which is not complete unless $\mathcal{R}(A)$ is closed. However, a canonical construction due to de

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Branges and Rovnyak [12] (see also [19]) shows that the completion of $\mathcal{H}/\mathcal{N}(A)$ is isometrically isomorphic to the Hilbert space $\mathcal{R}(A^{1/2})$ with the inner product

$$\langle A^{1/2}x, A^{1/2}y\rangle_{\mathbf{R}(A^{1/2})} := \langle P_{\overline{\mathcal{R}(A)}}x, P_{\overline{\mathcal{R}(A)}}y\rangle, \quad \forall\, x, y \in \mathcal{H}.$$

For the sequel, the Hilbert space $\left(\mathcal{R}(A^{1/2}), \langle\cdot,\cdot\rangle_{\mathbf{R}(A^{1/2})}\right)$ will be simply denoted by $\mathbf{R}(A^{1/2})$. For an account of results related to $\mathbf{R}(A^{1/2})$, we refer the readers to [6] and the references therein.

Let $T \in \mathbb{B}(\mathcal{H})$. We recall that an operator $S \in \mathbb{B}(\mathcal{H})$ is called an A-adjoint of T if $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ for all $x, y \in \mathcal{H}$. One can observe that the existence of an A-adjoint of T is equivalent to the existence of a solution in $\mathbb{B}(\mathcal{H})$ of the equation $AX = T^*A$. Clearly, the existence of an A-adjoint operator is not guaranteed. If the set of all operators admitting A-adjoints is denoted by $\mathbb{B}_A(\mathcal{H})$, then by Douglas theorem [15], we have

$$\mathbb{B}_A(\mathcal{H}) = \{T \in \mathbb{B}(\mathcal{H})\,;\ \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\}.$$

If $T \in \mathbb{B}_A(\mathcal{H})$, then the reduced solution of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T, which will be denoted by $T^{\sharp_A}$ and satisfies $\mathcal{R}(T^{\sharp_A}) \subseteq \overline{\mathcal{R}(A)}$. Note that $T^{\sharp_A} = A^{\dagger}T^*A$, where $A^{\dagger}$ is the Moore-Penrose inverse of A (see [5]). If $T \in \mathbb{B}_A(\mathcal{H})$, then $T^{\sharp_A} \in \mathbb{B}_A(\mathcal{H})$, $(T^{\sharp_A})^{\sharp_A} = P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}$ and $((T^{\sharp_A})^{\sharp_A})^{\sharp_A} = T^{\sharp_A}$. Moreover, if $S \in \mathbb{B}_A(\mathcal{H})$ then $TS \in \mathbb{B}_A(\mathcal{H})$ and $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$. For more results concerning $T^{\sharp_A}$, we invite the readers to see [4, 5]. An operator T is called A-bounded if there exists $\lambda > 0$ such that $\|Tx\|_A \leq \lambda\|x\|_A$, for every $x \in \mathcal{H}$. In virtue of Douglas theorem, one can see that the set of all operators admitting $A^{1/2}$-adjoints, denoted by $\mathbb{B}_{A^{1/2}}(\mathcal{H})$, is same as the collection of all A-bounded operators, i.e.,

$$\mathbb{B}_{A^{1/2}}(\mathcal{H}) = \{T \in \mathbb{B}(\mathcal{H})\,;\ \exists\,\lambda > 0\,;\ \|Tx\|_A \leq \lambda\|x\|_A,\ \forall\, x \in \mathcal{H}\}.$$

It is well-known that $\mathbb{B}_A(\mathcal{H})$ and $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ are two subalgebras of $\mathbb{B}(\mathcal{H})$ which are, in general, neither closed nor dense in $\mathbb{B}(\mathcal{H})$. Further, we have $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H})$ (see [4, 17]). If $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, then the seminorm of T induced by $\langle\cdot,\cdot\rangle_A$ is given by

$$\|T\|_A := \sup_{\substack{x\in\overline{\mathcal{R}(A)},\\ x\neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup\left\{\|Tx\|_A\,;\ x \in \mathcal{H},\ \|x\|_A = 1\right\} < \infty.$$

It was shown in [5, Proposition 2.3] that, for every $T \in \mathbb{B}_A(\mathcal{H})$, we have

$$\|T^{\sharp_A}T\|_A = \|TT^{\sharp_A}\|_A = \|T\|_A^2 = \|T^{\sharp_A}\|_A^2.$$

Several generalizations for the notion of numerical radius of Hilbert space operators have recently been defined (see for example [8] and the reference therein). One of these generalizations is the A-numerical radius of an operator $T \in \mathbb{B}(\mathcal{H})$. This new concept was firstly introduced by Saddi in [28] as

$$\omega_A(T) := \sup\{|\langle Tx, x\rangle_A|\,;\ \ x \in \mathcal{H},\ \|x\|_A = 1\}.$$

We mention here that it may happen that $\|T\|_A$ and $\omega_A(T)$ are equal to $+\infty$ for some $T \in \mathbb{B}(\mathcal{H}) \setminus \mathbb{B}_{A^{1/2}}(\mathcal{H})$ (see [17]). However, it was shown in [7] that $\|\cdot\|_A$ and $\omega_A(\cdot)$ are equivalent seminorms on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$. More precisely, for every $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, the following inequalities hold

$$\frac{1}{2}\|T\|_A \leq \omega_A(T) \leq \|T\|_A.$$

Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then $\|T\|_A = 0$ if and only if $AT = 0$. Furthermore, $\|Tx\|_A \leq \|T\|_A\|x\|_A$, for every $x \in \mathcal{H}$. This implies that $\|TS\|_A \leq \|T\|_A\|S\|_A$ for all $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. An operator $T \in \mathbb{B}(\mathcal{H})$ is called A-selfadjoint if AT is selfadjoint and it is called A-positive if AT is a positive operator. For the sequel, if $A = I$ then $\|T\|$, $r(T)$ and $\omega(T)$ denote respectively the classical operator norm, the spectral radius and the numerical radius of an operator T.

For any operator $T \in \mathcal{B}_A(\mathcal{H})$, we write $\mathfrak{R}_A(T) := \frac{T+T^{\sharp_A}}{2}$. We keep into account from [31, Theorem 2.5] that for every $T \in \mathbb{B}_A(\mathcal{H})$, we have

$$\omega_A(T) = \sup_{\theta\in\mathbb{R}} \left\| \mathfrak{R}_A(e^{i\theta}T) \right\|_A. \tag{1}$$

The A-spectral radius of an operator $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ was defined by the second author in [17] as

$$r_A(T) := \inf_{n\geq 1} \|T^n\|_A^{\frac{1}{n}} = \lim_{n\to\infty} \|T^n\|_A^{\frac{1}{n}}. \tag{2}$$

The second equality in (2) is also proved in [17, Theorem 1]. In addition, it was shown in [17] that $r_A(\cdot)$ satisfies the commutativity property, i.e.,

$$r_A(TS) = r_A(ST), \quad \forall\ T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H}). \tag{3}$$

Also, the following relation between the A-spectral radius and the A-numerical radius of A-bounded operators is also proved in [17]:

$$r_A(T) \leq \omega_A(T), \quad \forall\ T \in \mathbb{B}_{A^{1/2}}(\mathcal{H}). \tag{4}$$

Recently, many mathematicians have obtained different A-numerical radius inequalities of semi-Hilbertian space operators, the interested readers are invited to see [9–11, 14, 18, 21, 26, 27, 31] and the references therein. Here, we obtain several new inequalities for the A-numerical radius of the products and the sums of semi-Hilbertian space operators. The bounds obtained here improve on the existing bounds.

2. On A-Numerical radius inequalities for products of operators

We begin this section with the following known lemma which can be found in [17].

Lemma 2.1. *Let $T \in \mathbb{B}(\mathcal{H})$ be an A-selfadjoint operator. Then,*

$$\|T\|_A = \omega_A(T) = r_A(T).$$

Our first result reads as:

Theorem 2.2. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_A(TS) \leq \|T\|_A\, \omega_A(S) + \frac{1}{2}\min\left\{\omega_A\left(TS + ST^{\sharp_A}\right), \omega_A\left(TS - ST^{\sharp_A}\right)\right\}.$$

Proof. Let $\theta \in \mathbb{R}$. Clearly, $\mathfrak{R}_A(e^{i\theta}TS)$ is an A-selfadjoint operator. Therefore, from Lemma 2.1, we get

$$\begin{aligned}
\left\|\mathfrak{R}_A(e^{i\theta}TS)\right\|_A &= \omega_A\left(\mathfrak{R}_A(e^{i\theta}TS)\right)\\
&= \omega_A\left(\frac{1}{2}(e^{i\theta}TS + e^{-i\theta}S^{\sharp_A}T^{\sharp_A})\right)\\
&= \omega_A\left(\frac{1}{2}(e^{i\theta}TS + e^{-i\theta}TS^{\sharp_A} + e^{-i\theta}S^{\sharp_A}T^{\sharp_A} - e^{-i\theta}TS^{\sharp_A})\right)\\
&= \omega_A\left(T\mathfrak{R}_A(e^{i\theta}S) + \frac{1}{2}e^{-i\theta}(S^{\sharp_A}T^{\sharp_A} - TS^{\sharp_A})\right)\\
&\leq \omega_A\left(T\mathfrak{R}_A(e^{i\theta}S)\right) + \omega_A\left(\frac{1}{2}e^{-i\theta}(S^{\sharp_A}T^{\sharp_A} - TS^{\sharp_A})\right)\\
&\leq \left\|T\mathfrak{R}_A(e^{i\theta}S)\right\|_A + \frac{1}{2}\omega_A\left(S^{\sharp_A}T^{\sharp_A} - TS^{\sharp_A}\right)\\
&\leq \|T\|_A \left\|\mathfrak{R}_A(e^{i\theta}S)\right\|_A + \frac{1}{2}\omega_A\left(S^{\sharp_A}T^{\sharp_A} - TS^{\sharp_A}\right)\\
&\leq \|T\|_A\omega_A(S) + \frac{1}{2}\omega_A\left(S^{\sharp_A}T^{\sharp_A} - TS^{\sharp_A}\right).
\end{aligned}$$

So, by taking the supremum over all $\theta \in \mathbb{R}$, we get

$$\omega_A(TS) \leq \|T\|_A \omega_A(S) + \frac{1}{2}\omega_A\left(S^{\sharp_A}T^{\sharp_A} - TS^{\sharp_A}\right). \tag{5}$$

On the other hand, for every $x \in \mathcal{H}$ we see that

$$\begin{aligned}\left|\langle (S^{\sharp_A}T^{\sharp_A} - TS^{\sharp_A})x, x\rangle_A\right| &= \left|\langle S^{\sharp_A}T^{\sharp_A}x, x\rangle_A - \langle TS^{\sharp_A}x, x\rangle_A\right| \\ &= \left|\langle S^{\sharp_A}T^{\sharp_A}x, x\rangle_A - \langle P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}S^{\sharp_A}x, x\rangle_A\right|,\end{aligned}$$

where the last equality follows from the fact that $AP_{\overline{\mathcal{R}(A)}} = A$. So, we get

$$\begin{aligned}\left|\langle (S^{\sharp_A}T^{\sharp_A} - TS^{\sharp_A})x, x\rangle_A\right| &= \left|\langle (S^{\sharp_A}T^{\sharp_A} - (T^{\sharp_A})^{\sharp_A}S^{\sharp_A})x, x\rangle_A\right| \\ &= \left|\langle \left(TS - ST^{\sharp_A}\right)^{\sharp_A} x, x\rangle_A\right| \\ &= \left|\langle \left(TS - ST^{\sharp_A}\right) x, x\rangle_A\right|.\end{aligned}$$

This implies that $\omega_A\left(S^{\sharp_A}T^{\sharp_A} - TS^{\sharp_A}\right) = \omega_A\left(TS - ST^{\sharp_A}\right)$. Thus, it follows from (5) that

$$\omega_A(TS) \leq \|T\|_A \omega_A(S) + \frac{1}{2}\omega_A\left(TS - ST^{\sharp_A}\right). \tag{6}$$

Also, by replacing T by iT in (6), we get

$$\omega_A(TS) \leq \|T\|_A \omega_A(S) + \frac{1}{2}\omega_A\left(TS + ST^{\sharp_A}\right). \tag{7}$$

Thus, the proof is finished by combining (6) together with (7). □

Remark 2.3. *It has been proved in [23, Theorem 2.13.] that*

$$\omega_A\left(TS \pm ST^{\sharp_A}\right) \leq 2\|T\|_A \omega_A(S), \quad \forall\, T, S \in \mathbb{B}_A(\mathcal{H}).$$

Therefore, $\|T\|_A \omega_A(S) + \frac{1}{2}\omega_A\left(TS \pm ST^{\sharp_A}\right) \leq 2\|T\|_A \omega_A(S)$. *Thus, the inequality obtained in Theorem 2.2 is stronger than the well-know inequality*

$$\omega_A(TS) \leq 2\|T\|_A \omega_A(S).$$

In order to obtain our next inequality that gives an upper bound for the A-numerical radius of product of two operators, we need the following lemmas. First we consider the 2×2 operator diagonal matrix $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Clearly, $\mathbb{A}$ is a positive operator on $\mathcal{H} \oplus \mathcal{H}$. So, $\mathbb{A}$ induces the following semi-inner product on $\mathcal{H} \oplus \mathcal{H}$ defined as

$$\langle x, y\rangle_{\mathbb{A}} = \langle \mathbb{A}x, y\rangle = \langle x_1, y_1\rangle_A + \langle x_2, y_2\rangle_A,$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{H} \oplus \mathcal{H}$.

Lemma 2.4. ([21]) *Let $T, S \in \mathbb{B}(\mathcal{H})$ be A-positive operators. Then,*

$$\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix}\right] = \frac{1}{2}\|T + S\|_A .$$

Lemma 2.5. ([20]) *Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then,* *(a)* $\omega_{\mathbb{A}}\left[\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}\right] = \max\{\omega_A(T), \omega_A(S)\}$. *In particular,*

$$\omega_{\mathbb{A}}\left[\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}\right] = \omega_{\mathbb{A}}\left[\begin{pmatrix} T & 0 \\ 0 & T^{\sharp_A} \end{pmatrix}\right] = \omega_A(T). \tag{8}$$

(b) $\left\|\begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix}\right\|_{\mathbb{A}} = \left\|\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}\right\|_{\mathbb{A}} = \max\left\{\|T\|_A, \|S\|_A\right\}.$

Now we are in a position to obtain the following inequality.

Theorem 2.6. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_A(TS) \leq \frac{1}{2}\omega_A(ST) + \frac{1}{4}\Big(\|T\|_A\|S\|_A + \|TS\|_A\Big). \tag{9}$$

Proof. Let $\theta \in \mathbb{R}$. Since $\Re_A(e^{i\theta}TS)$ is an A-selfadjoint operator, so by Lemma 2.1, we have

$$\begin{aligned} \|\Re_A(e^{i\theta}TS)\|_A &= r_A[\Re_A(e^{i\theta}TS)] \\ &= \frac{1}{2}r_A(e^{i\theta}TS + e^{-i\theta}S^{\sharp_A}T^{\sharp_A}). \end{aligned} \tag{10}$$

On the other hand, we have

$$\begin{aligned} r_A(e^{i\theta}TS + e^{-i\theta}S^{\sharp_A}T^{\sharp_A}) &= r_{\mathbb{A}}\left[\begin{pmatrix} e^{i\theta}TS + e^{-i\theta}S^{\sharp_A}T^{\sharp_A} & 0 \\ 0 & 0 \end{pmatrix}\right] \\ &= r_{\mathbb{A}}\left[\begin{pmatrix} e^{i\theta}T & S^{\sharp_A} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S & 0 \\ e^{-i\theta}T^{\sharp_A} & 0 \end{pmatrix}\right] \\ &= r_{\mathbb{A}}\left[\begin{pmatrix} S & 0 \\ e^{-i\theta}T^{\sharp_A} & 0 \end{pmatrix}\begin{pmatrix} e^{i\theta}T & S^{\sharp_A} \\ 0 & 0 \end{pmatrix}\right] \quad \text{(by (3))} \\ &= r_{\mathbb{A}}\left[\begin{pmatrix} e^{i\theta}ST & SS^{\sharp_A} \\ T^{\sharp_A}T & e^{-i\theta}T^{\sharp_A}S^{\sharp_A} \end{pmatrix}\right]. \end{aligned}$$

By applying (4), we get

$$\begin{aligned} r_A(e^{i\theta}TS + e^{-i\theta}S^{\sharp_A}T^{\sharp_A}) &\leq \omega_{\mathbb{A}}\left[\begin{pmatrix} e^{i\theta}ST & SS^{\sharp_A} \\ T^{\sharp_A}T & e^{-i\theta}T^{\sharp_A}S^{\sharp_A} \end{pmatrix}\right] \\ &\leq \omega_{\mathbb{A}}\left[\begin{pmatrix} e^{i\theta}ST & 0 \\ 0 & e^{-i\theta}T^{\sharp_A}S^{\sharp_A} \end{pmatrix}\right] + \omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & SS^{\sharp_A} \\ T^{\sharp_A}T & 0 \end{pmatrix}\right] \\ &= \omega_A(ST) + \frac{1}{2}\|SS^{\sharp_A} + T^{\sharp_A}T\|_A, \end{aligned}$$

where the last equality follows from Lemma 2.4 together with (8). Therefore, from (10), we get

$$\|\Re_A(e^{i\theta}TS)\|_A \leq \frac{1}{2}\omega_A(ST) + \frac{1}{4}\|SS^{\sharp_A} + T^{\sharp_A}T\|_A.$$

Hence, by taking the supremum over all $\theta \in \mathbb{R}$ and then using (1), we get

$$\omega_A(TS) \leq \frac{1}{2}\omega_A(ST) + \frac{1}{4}\|SS^{\sharp_A} + T^{\sharp_A}T\|_A. \tag{11}$$

If $AT = 0$ or $AS = 0$, then the inequality (9) holds trivially. Assume that $AT \neq 0$ and $AS \neq 0$. By Replacing T and S by $\sqrt{\frac{\|S\|_A}{\|T\|_A}}T$ and $\sqrt{\frac{\|T\|_A}{\|S\|_A}}S$, respectively in (11), we obtain

$$\omega_A(TS) \leq \frac{1}{2}\omega_A(ST) + \frac{1}{4}\left\|\frac{\|S\|_A}{\|T\|_A}T^{\sharp_A}T + \frac{\|T\|_A}{\|S\|_A}SS^{\sharp_A}\right\|_A. \tag{12}$$

It is easy to see that the operator $\frac{\|S\|_A}{\|T\|_A}T^{\sharp_A}T + \frac{\|T\|_A}{\|S\|_A}SS^{\sharp_A}$ is A-positive. So, an application of Lemma 2.1 gives

$$\left\|\frac{\|S\|_A}{\|T\|_A}T^{\sharp_A}T + \frac{\|T\|_A}{\|S\|_A}SS^{\sharp_A}\right\|_A = r_A\left(\frac{\|S\|_A}{\|T\|_A}T^{\sharp_A}T + \frac{\|T\|_A}{\|S\|_A}SS^{\sharp_A}\right). \tag{13}$$

Next, ones observes that

$$\begin{aligned}
r_A\left(\frac{\|S\|_A}{\|T\|_A}T^{\sharp_A}T + \frac{\|T\|_A}{\|S\|_A}SS^{\sharp_A}\right) &= r_{\mathbb{A}}\left[\begin{pmatrix} \frac{\|S\|_A}{\|T\|_A}T^{\sharp_A}T + \frac{\|T\|_A}{\|S\|_A}SS^{\sharp_A} & 0 \\ 0 & 0\end{pmatrix}\right] \\
&= r_{\mathbb{A}}\left[\begin{pmatrix} \sqrt{\frac{\|S\|_A}{\|T\|_A}}T^{\sharp_A} & \sqrt{\frac{\|T\|_A}{\|S\|_A}}S \\ 0 & 0\end{pmatrix}\begin{pmatrix} \sqrt{\frac{\|S\|_A}{\|T\|_A}}T & 0 \\ \sqrt{\frac{\|T\|_A}{\|S\|_A}}S^{\sharp_A} & 0\end{pmatrix}\right].
\end{aligned}$$

Further, by applying (3), we get

$$\begin{aligned}
r_A\left(\frac{\|S\|_A}{\|T\|_A}T^{\sharp_A}T + \frac{\|T\|_A}{\|S\|_A}SS^{\sharp_A}\right) &= r_{\mathbb{A}}\left[\begin{pmatrix} \sqrt{\frac{\|S\|_A}{\|T\|_A}}T & 0 \\ \sqrt{\frac{\|T\|_A}{\|S\|_A}}S^{\sharp_A} & 0\end{pmatrix}\begin{pmatrix} \sqrt{\frac{\|S\|_A}{\|T\|_A}}T^{\sharp_A} & \sqrt{\frac{\|T\|_A}{\|S\|_A}}S \\ 0 & 0\end{pmatrix}\right] \\
&= r_{\mathbb{A}}\left[\begin{pmatrix} \frac{\|S\|_A}{\|T\|_A}TT^{\sharp_A} & TS \\ S^{\sharp_A}T^{\sharp_A} & \frac{\|T\|_A}{\|S\|_A}S^{\sharp_A}S\end{pmatrix}\right].
\end{aligned} \tag{14}$$

In addition, one can see that $\begin{pmatrix} \frac{\|S\|_A}{\|T\|_A}TT^{\sharp_A} & TS \\ S^{\sharp_A}T^{\sharp_A} & \frac{\|T\|_A}{\|S\|_A}S^{\sharp_A}S\end{pmatrix}$ is an $\mathbb{A}$-selfadjoint operator. Hence, in view of Lemma 2.1, we have

$$\left\|\begin{pmatrix} \frac{\|S\|_A}{\|T\|_A}TT^{\sharp_A} & TS \\ S^{\sharp_A}T^{\sharp_A} & \frac{\|T\|_A}{\|S\|_A}S^{\sharp_A}S\end{pmatrix}\right\|_{\mathbb{A}} = r_{\mathbb{A}}\left[\begin{pmatrix} \frac{\|S\|_A}{\|T\|_A}TT^{\sharp_A} & TS \\ S^{\sharp_A}T^{\sharp_A} & \frac{\|T\|_A}{\|S\|_A}S^{\sharp_A}S\end{pmatrix}\right]. \tag{15}$$

So, it follows from (13), (14) and (15) that

$$\left\|\frac{\|S\|_A}{\|T\|_A}T^{\sharp_A}T + \frac{\|T\|_A}{\|S\|_A}SS^{\sharp_A}\right\|_A = \left\|\begin{pmatrix} \frac{\|S\|_A}{\|T\|_A}TT^{\sharp_A} & TS \\ S^{\sharp_A}T^{\sharp_A} & \frac{\|T\|_A}{\|S\|_A}S^{\sharp_A}S\end{pmatrix}\right\|_{\mathbb{A}}.$$

Finally, by applying the triangle inequality and then using Lemma 2.5, we get

$$\begin{aligned}
\left\|\frac{\|S\|_A}{\|T\|_A}T^{\sharp_A}T + \frac{\|T\|_A}{\|S\|_A}SS^{\sharp_A}\right\|_A &\leq \left\|\begin{pmatrix} \frac{\|S\|_A}{\|T\|_A}TT^{\sharp_A} & 0 \\ 0 & \frac{\|T\|_A}{\|S\|_A}S^{\sharp_A}S\end{pmatrix}\right\|_{\mathbb{A}} + \left\|\begin{pmatrix} 0 & TS \\ S^{\sharp_A}T^{\sharp_A} & 0\end{pmatrix}\right\|_{\mathbb{A}} \\
&= \max\left\{\frac{\|S\|_A}{\|T\|_A}\|TT^{\sharp_A}\|_A, \frac{\|T\|_A}{\|S\|_A}\|S^{\sharp_A}S\|_A\right\} + \|TS\|_A \\
&= \|S\|_A\|T\|_A + \|TS\|_A.
\end{aligned}$$

Therefore, we get (9) as desired by taking (12) into account. □

Remark 2.7. *(i) By taking $A = I$ in Theorem 2.6 we get a recent result proved by Kittaneh et al. in [2].*
(ii) If we consider $T = I$, then it is easy to see that the inequality in Theorem 2.2 is stronger than that in Theorem 2.6. On the other hand, if we consider $S = I$, then the inequality in Theorem 2.6 is stronger than that in Theorem 2.2. Thus, we conclude that the inequalities in Theorems 2.2 and 2.6 are, in general, not comparable.

The following corollary is an immediate consequence of Theorem 2.6.

Corollary 2.8. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_A(TS) \leq \frac{1}{2}\left(\omega_A(ST) + \|T\|_A\|S\|_A\right).$$

Next, we obtain the following inequalities when T is assuming to be A-positive.

Theorem 2.9. *Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. If T is A-positive, then*

$$\omega_A(TS) \leq \|T\|_A\omega_A(S) \quad \text{and} \quad \omega_A(ST) \leq \|T\|_A\omega_A(S).$$

Proof. For all $\alpha \in [0, 1]$, we have

$$\begin{aligned}\omega_A(TS) &= \omega_A\left((T - \alpha\|T\|_A I)S + \alpha\|T\|_A S\right)\\ &\leq \omega_A\left((T - \alpha\|T\|_A I)S\right) + \alpha\|T\|_A\omega_A(S)\\ &\leq \|(T - \alpha\|T\|_A I)S\|_A + \alpha\|T\|_A\omega_A(S)\\ &\leq \|T - \alpha\|T\|_A I\|_A\,\|S\|_A + \alpha\|T\|_A\omega_A(S).\end{aligned}$$

Since T is A-positive, so we observe that $\|T - \alpha\|T\|_A I\|_A = (1 - \alpha)\|T\|_A$ for all $\alpha \in [0, 1]$. Therefore,

$$\omega_A(TS) \leq \|T\|_A\Big((1 - \alpha)\|S\|_A + \alpha\omega_A(S)\Big). \tag{16}$$

So, by considering $\alpha = 1$ in (16), we get

$$\omega_A(TS) \leq \|T\|_A\omega_A(S).$$

Similarly, we can prove that

$$\omega_A(ST) \leq \|T\|_A\omega_A(S).$$

Thus, we complete the proof. □

Considering $A = I$ in Theorem 2.9, we get the following numerical radius inequalities for the product of Hilbert space operators.

Corollary 2.10. *Let $T, S \in \mathbb{B}(\mathcal{H})$ with T positive. Then,*

$$\omega(TS) \leq \|T\|\omega(S) \quad \text{and} \quad \omega(ST) \leq \|T\|\omega(S).$$

Remark 2.11. *(1) We would like to note that the numerical radius $\omega(.)$ satisfies $\omega(TS) \leq \omega(T)\omega(S)$ if either T or S is positive.*

(2) Abu-Omar and Kittaneh proved in [3, Cor. 2.6] the following result: if $T, S \in \mathbb{B}(\mathcal{H})$ with T positive, then $\omega(TS) \leq \frac{3}{2}\|T\|\omega(S)$. Thus, Corollary 2.10 is stronger than this result.

3. On A-Numerical radius inequalities for sums of operators

Our starting point in this section is the following lemma.

Lemma 3.1. *For any $x, y, z \in \mathcal{H}$, we have*

$$|\langle x, y\rangle_A|^2 + |\langle x, z\rangle_A|^2 \le \|x\|_A^2\Big(\max\{\|y\|_A^2, \|z\|_A^2\} + |\langle y, z\rangle_A|\Big). \tag{17}$$

Proof. First note that, by the proof of [16, Th. 3] we have,

$$|\langle x, y\rangle|^2 + |\langle x, z\rangle|^2 \le \|x\|^2\Big(\max\{\|y\|^2, \|z\|^2\} + |\langle y, z\rangle|\Big), \tag{18}$$

for every $x, y, z \in \mathcal{H}$. Now,

$$|\langle x, y\rangle_A|^2 + |\langle x, z\rangle_A|^2 = |\langle A^{1/2}x, A^{1/2}y\rangle|^2 + |\langle A^{1/2}x, A^{1/2}z\rangle|^2.$$

So, by applying (18), we obtain

$$|\langle x, y\rangle_A|^2 + |\langle x, z\rangle_A|^2 \le \|A^{1/2}x\|^2\Big(\max\{\|A^{1/2}y\|^2, \|A^{1/2}z\|^2\} + |\langle A^{1/2}y, A^{1/2}z\rangle|\Big).$$

Hence, we get (17) as required. □

Now, we are in a position to prove the following theorem.

Theorem 3.2. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_A(T+S) \le \sqrt{\frac{1}{2}\Big(\left\|TT^{\sharp_A} + SS^{\sharp_A}\right\|_A + \left\|TT^{\sharp_A} - SS^{\sharp_A}\right\|_A\Big) + \omega_A\left(ST^{\sharp_A}\right) + 2\omega_A(T)\,\omega_A(S)}.$$

Proof. Recall first that for every $t, s \in \mathbb{R}$ it holds

$$\max\{t, s\} = \frac{1}{2}\Big(t + s + |t - s|\Big). \tag{19}$$

Now, let $x \in \mathcal{H}$ with $\|x\|_A = 1$. By using Lemma 3.1, we get

$$\begin{aligned}
&|\langle (T+S)x, x\rangle_A|^2 \\
&\le |\langle x, T^{\sharp_A}x\rangle_A|^2 + |\langle x, S^{\sharp_A}x\rangle_A|^2 + 2|\langle Tx, x\rangle_A|\,|\langle Sx, x\rangle_A| \\
&\le \max\left\{\|T^{\sharp_A}x\|_A^2, \|S^{\sharp_A}x\|_A^2\right\} + |\langle ST^{\sharp_A}x, x\rangle_A| + 2|\langle Tx, x\rangle_A|\,|\langle Sx, x\rangle_A| \\
&= \frac{1}{2}\Big(\|T^{\sharp_A}x\|_A^2 + \|S^{\sharp_A}x\|_A^2 + \left|\|T^{\sharp_A}x\|_A^2 - \|S^{\sharp_A}x\|_A^2\right|\Big) + |\langle ST^{\sharp_A}x, x\rangle_A| + 2|\langle Tx, x\rangle_A|\,|\langle Sx, x\rangle_A| \quad \text{(by (19))} \\
&= \frac{1}{2}\Big(\langle (TT^{\sharp_A} + SS^{\sharp_A})x, x\rangle_A + \left|\langle (TT^{\sharp_A} - SS^{\sharp_A})x, x\rangle_A\right|\Big) + |\langle ST^{\sharp_A}x, x\rangle_A| + 2|\langle Tx, x\rangle_A|\,|\langle Sx, x\rangle_A| \\
&\le \frac{1}{2}\Big(\omega_A(TT^{\sharp_A} + SS^{\sharp_A}) + \omega_A(TT^{\sharp_A} - SS^{\sharp_A})\Big) + \omega_A(ST^{\sharp_A}) + 2\omega_A(T)\,\omega_A(S) \\
&= \frac{1}{2}\Big(\|TT^{\sharp_A} + SS^{\sharp_A}\|_A + \|TT^{\sharp_A} - SS^{\sharp_A}\|_A\Big) + \omega_A(ST^{\sharp_A}) + 2\omega_A(T)\,\omega_A(S),
\end{aligned}$$

where the last equality follows from Lemma 2.1 since the operators $TT^{\sharp_A} \pm SS^{\sharp_A}$ are A-selfadjoint. So, we infer that

$$|\langle (T+S)x, x\rangle_A|^2 \le \frac{1}{2}\Big(\|TT^{\sharp_A} + SS^{\sharp_A}\|_A + \|TT^{\sharp_A} - SS^{\sharp_A}\|_A\Big) + \omega_A(ST^{\sharp_A}) + 2\omega_A(T)\,\omega_A(S).$$

Therefore, the desired result follows by taking supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the last inequality. □

Our next objective is to refine the triangle inequality related to $\omega_A(\cdot)$. To do this, we need to recall from [22] the following lemma.

Lemma 3.3. *Let $T_1, T_2, S_1, S_2 \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then,*

$$r_A\left(T_1S_1 + T_2S_2\right) \le \left\| \begin{pmatrix} \|S_1T_1\|_A & \sqrt{\|S_1T_2\|_A \|S_2T_1\|_A} \\ \sqrt{\|S_1T_2\|_A \|S_2T_1\|_A} & \|S_2T_2\|_A \end{pmatrix} \right\|.$$

Now, we are ready to prove the following theorem which covers and generalizes a recent result proved by Abu-Omar and Kittaneh in [1].

Theorem 3.4. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\begin{aligned} \omega_A(T+S) &\le \frac{1}{2}\left[\omega_A(T) + \omega_A(S) + \sqrt{(\omega_A(T) - \omega_A(S))^2 + 4\sup_{\theta\in\mathbb{R}} \left\|\Re_A(e^{i\theta}T)\Re_A(e^{i\theta}S)\right\|_A}\right] \\ &\le \omega_A(T) + \omega_A(S). \end{aligned}$$

Proof. Let $\theta \in \mathbb{R}$. It can be seen that $\Re_A[e^{i\theta}(T+S)]$ is an A-selfadjoint operator. So, by Lemma 2.1, we get

$$\left\|\Re_A[e^{i\theta}(T+S)]\right\|_A = r_A\Big(\Re_A[e^{i\theta}(T+S)]\Big).$$

By letting $T_1 = I$, $S_1 = \Re_A(e^{i\theta}T)$, $T_2 = \Re_A(e^{i\theta}S)$ and $S_2 = I$ in Lemma 3.3 and then using the norm monotonicity of matrices with nonnegative entries, we get

$$\begin{aligned} \left\|\Re_A[e^{i\theta}(T+S)]\right\|_A &= r_A\Big(\Re_A(e^{i\theta}T) + \Re_A(e^{i\theta}S)\Big) \\ &\le \left\| \begin{pmatrix} \|\Re_A(e^{i\theta}T)\|_A & \sqrt{\left\|\Re_A(e^{i\theta}T)\Re_A(e^{i\theta}S)\right\|_A} \\ \sqrt{\left\|\Re_A(e^{i\theta}T)\Re_A(e^{i\theta}S)\right\|_A} & \|\Re_A(e^{i\theta}S)\|_A \end{pmatrix} \right\| \\ &\le \left\| \begin{pmatrix} \omega_A(T) & \sqrt{\sup_{\theta\in\mathbb{R}}\left\|\Re_A(e^{i\theta}T)\Re_A(e^{i\theta}S)\right\|_A} \\ \sqrt{\sup_{\theta\in\mathbb{R}}\left\|\Re_A(e^{i\theta}T)\Re_A(e^{i\theta}S)\right\|_A} & \omega_A(S) \end{pmatrix} \right\| \\ &= \frac{1}{2}\left[\omega_A(T) + \omega_A(S) + \sqrt{\Big(\omega_A(T) - \omega_A(S)\Big)^2 + 4\sup_{\theta\in\mathbb{R}} \left\|\Re_A(e^{i\theta}T)\Re_A(e^{i\theta}S)\right\|_A}\right]. \end{aligned}$$

By taking supremum over all $\theta \in \mathbb{R}$, we get

$$\omega_A(T+S) \le \frac{1}{2}\left[\omega_A(T) + \omega_A(S) + \sqrt{(\omega_A(T) - \omega_A(S))^2 + 4\sup_{\theta\in\mathbb{R}} \left\|\Re_A(e^{i\theta}T)\Re_A(e^{i\theta}S)\right\|_A}\right]. \tag{20}$$

This proves the first inequality in the theorem. Moreover,

$$\begin{aligned} \sqrt{(\omega_A(T) - \omega_A(S))^2 + 4\sup_{\theta\in\mathbb{R}} \left\|\Re_A(e^{i\theta}T)\Re_A(e^{i\theta}S)\right\|_A} &\le \sqrt{\Big(\omega_A(T) - \omega_A(S)\Big)^2 + 4\omega_A(T)\,\omega_A(S)} \\ &= \sqrt{\Big(\omega_A(T) + \omega_A(S)\Big)^2} = \omega_A(T) + \omega_A(S). \end{aligned}$$

So, by using (20), we easily get the second inequality. □

The following lemma is an extension of Buzano's inequality (see [13]) and plays a crucial role in proving our next result.

Lemma 3.5. *([28]) Let $x, y, e \in \mathcal{H}$ with $\|e\|_A = 1$. Then,*

$$|\langle x, e\rangle_A \langle e, y\rangle_A| \leq \frac{1}{2}\Big(|\langle x, y\rangle_A| + \|x\|_A\|y\|_A\Big).$$

Now, we prove the following theorem.

Theorem 3.6. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_A(T+S) \leq \sqrt{\omega_A^2(T) + \omega_A^2(S) + \frac{1}{2}\left\|T^{\sharp_A}T + SS^{\sharp_A}\right\|_A + \omega_A(ST)}.$$

Proof. Let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. One can verify that

$$\begin{aligned}|\langle (T+S)x, x\rangle_A|^2 &\leq |\langle Tx, x\rangle_A|^2 + |\langle Sx, x\rangle_A|^2 + 2|\langle Tx, x\rangle_A|\,|\langle Sx, x\rangle_A| \\ &= |\langle Tx, x\rangle_A|^2 + |\langle Sx, x\rangle_A|^2 + 2|\langle Tx, x\rangle_A|\,|\langle x, S^{\sharp_A}x\rangle_A|.\end{aligned}$$

By using Lemma 3.5, we get

$$\begin{aligned}|\langle (T+S)x, x\rangle_A|^2 &\leq |\langle Tx, x\rangle_A|^2 + |\langle Sx, x\rangle_A|^2 + \|Tx\|_A\|S^{\sharp_A}x\|_A + |\langle Tx, S^{\sharp_A}x\rangle_A| \\ &= |\langle Tx, x\rangle_A|^2 + |\langle Sx, x\rangle_A|^2 + \sqrt{\langle T^{\sharp_A}Tx, x\rangle_A \langle SS^{\sharp_A}x, x\rangle_A} + |\langle STx, x\rangle_A|.\end{aligned}$$

By using the arithmetic-geometric mean inequality, we get

$$\begin{aligned}|\langle (T+S)x, x\rangle_A|^2 &\leq \omega_A^2(T) + \omega_A^2(S) + \frac{1}{2}\left(\langle T^{\sharp_A}Tx, x\rangle_A + \langle SS^{\sharp_A}x, x\rangle_A\right) + \omega_A(ST) \\ &= \omega_A^2(T) + \omega_A^2(S) + \frac{1}{2}\langle (T^{\sharp_A}T + SS^{\sharp_A})x, x\rangle_A + \omega_A(ST) \\ &\leq \omega_A^2(T) + \omega_A^2(S) + \frac{1}{2}\omega_A\left(T^{\sharp_A}T + SS^{\sharp_A}\right) + \omega_A(ST) \\ &= \omega_A^2(T) + \omega_A^2(S) + \frac{1}{2}\left\|T^{\sharp_A}T + SS^{\sharp_A}\right\|_A + \omega_A(ST),\end{aligned}$$

where the last equality follows from Lemma 2.1. So, we infer that

$$|\langle (T+S)x, x\rangle_A|^2 \leq \omega_A^2(T) + \omega_A^2(S) + \frac{1}{2}\left\|T^{\sharp_A}T + SS^{\sharp_A}\right\|_A + \omega_A(ST),$$

for all $x \in \mathcal{H}$ with $\|x\|_A = 1$. Thus, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$, we get

$$\omega_A^2(T+S) \leq \omega_A^2(T) + \omega_A^2(S) + \frac{1}{2}\left\|T^{\sharp_A}T + SS^{\sharp_A}\right\|_A + \omega_A(ST).$$

This proves the desired result. □

As an application of the above theorem, we get the following corollary.

Corollary 3.7. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_A(T) \leq \frac{1}{2}\sqrt{\left\|TT^{\sharp_A} + T^{\sharp_A}T\right\|_A + 2\omega_A(T^2)} \leq \frac{\sqrt{2}}{2}\sqrt{\|T^{\sharp_A}T + TT^{\sharp_A}\|_A}.$$

Proof. Clearly, the first inequality follows by taking $S = T$ in Theorem 3.6. Moreover, it is well-known that $\omega_A(T^2) \leq \omega_A^2(T)$ (see [17]) and $\omega_A^2(T) \leq \frac{1}{2}\left\|TT^{\sharp_A} + T^{\sharp_A}T\right\|_A$. So, we get that

$$\begin{aligned}\frac{1}{4}\left\|TT^{\sharp_A} + T^{\sharp_A}T\right\|_A + \frac{1}{2}\omega_A(T^2) &\leq \frac{1}{4}\left\|TT^{\sharp_A} + T^{\sharp_A}T\right\|_A + \frac{1}{2}\omega_A^2(T)\\ &\leq \frac{1}{4}\left\|TT^{\sharp_A} + T^{\sharp_A}T\right\|_A + \frac{1}{4}\left\|TT^{\sharp_A} + T^{\sharp_A}T\right\|_A\\ &= \frac{1}{2}\left\|TT^{\sharp_A} + T^{\sharp_A}T\right\|_A.\end{aligned}$$

This proves that the second inequality in Corollary 3.7. □

Remark 3.8. *Note that Corollary 3.7 has been recently proved in [31].*

Our next improvement reads as:

Theorem 3.9. *Let $T, S \in \mathbb{B}(\mathcal{H})$ be A-selfadjoint aperators. Then,*

$$\omega_A(T+S) \leq \sqrt{\omega_A^2(T+\mathrm{i}S) + \omega_A(ST) + \|T\|_A\|S\|_A} \leq \omega_A(T) + \omega_A(S).$$

Proof. Let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. Then, we have

$$\begin{aligned}|\langle (T+S)x, x\rangle_A|^2 &\leq (|\langle Tx, x\rangle_A| + |\langle Tx, x\rangle_A|)^2\\ &= |\langle Tx, x\rangle_A|^2 + |\langle Sx, x\rangle_A|^2 + 2|\langle Tx, x\rangle_A||\langle Sx, x\rangle_A|\\ &= |\langle Tx, x\rangle_A + \mathrm{i}\langle Sx, x\rangle_A|^2 + 2|\langle Tx, x\rangle_A\langle Sx, x\rangle_A|\\ &= |\langle (T+\mathrm{i}S)x, x\rangle_A|^2 + 2|\langle Tx, x\rangle_A\langle x, S^{\sharp_A}x\rangle_A|.\end{aligned}$$

So, an application of Lemma 3.5 gives

$$\begin{aligned}|\langle (T+S)x, x\rangle_A|^2 &\leq |\langle (T+\mathrm{i}S)x, x\rangle_A|^2 + \|Tx\|_A\|S^{\sharp_A}x\|_A + |\langle Tx, S^{\sharp_A}x\rangle_A|\\ &= |\langle (T+\mathrm{i}S)x, x\rangle_A|^2 + \|Tx\|_A\|S^{\sharp_A}x\|_A + |\langle STx, x\rangle_A|\\ &\leq \omega_A^2(T+\mathrm{i}S) + \|T\|_A\|S\|_A + \omega_A(ST).\end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ yields that

$$\omega_A^2(T+S) \leq \omega_A^2(T+\mathrm{i}S) + \|T\|_A\|S\|_A + \omega_A(ST).$$

Thus, we prove the first inequality of the theorem. On the other hand, it is not difficult to show that $\omega_A^2(T+\mathrm{i}S) \leq \|T\|_A^2 + \|S\|_A^2$. Also we have $\omega_A(ST) \leq \|T\|_A\|S\|_A$. So, $\omega_A^2(T+\mathrm{i}S) + \|T\|_A\|S\|_A + \omega_A(ST) \leq (\|T\|_A + \|S\|_A)^2$. Finally, since T and S are A-selfadjoint operators, so Lemma 2.1 implies that $\omega_A(T) = \|T\|_A$ and $\omega_A(S) = \|S\|_A$. So, we get the required second inequality of the theorem. This finishes the proof of our result. □

Our next objective is to establish some A-numerical radius inequalities for the sum of d operators. To achieve this goal, we shall need the following three lemmas. Note that the second assertion of the first lemma is known as McCarthy inequality [29, p. 20]. The second lemma is known as Bohr's inequality.

Lemma 3.10. *([24, pp. 75-76], [29, p. 20]) Let $T \in \mathbb{B}(\mathcal{H})$. Then, the following assertions hold:*

(i) $|\langle Tx, x\rangle|^2 \leq \langle |T|x, x\rangle\langle |T^*|x, x\rangle$ *for every* $x \in \mathcal{H}$.

(ii) If $T \geq 0$, *then* $\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle$ *for every* $x \in \mathcal{H}$ *with* $\|x\| = 1$ *and all* $r \geq 1$.

Lemma 3.11. *([30]) Let a_i be positive real numbers for all $i \in \{1, 2, \dots, d\}$. Then, for all $r \geq 1$ we have*

$$\left(\sum_{i=1}^{d} a_i\right)^r \leq d^{r-1} \sum_{i=1}^{d} a_i^r.$$

Lemma 3.12. *([6, 17, 25]) Let $T \in \mathbb{B}(\mathcal{H})$. Then, $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ if and only if there exists a unique $\widetilde{T} \in \mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_A T = \widetilde{T} Z_A$. Here, $Z_A : \mathcal{H} \to \mathbf{R}(A^{1/2})$ is defined by $Z_A x = Ax$. Further, the following properties hold*

(i) $\|T\|_A = \|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}$.

(ii) $\omega_A(T) = \omega(\widetilde{T})$.

(iii) If $T \in \mathbb{B}_A(\mathcal{H})$, *then* $\widetilde{T^{\sharp_A}} = (\widetilde{T})^*$.

On the basis of the above results we obtain the following theorems.

Theorem 3.13. *Let $S_i \in \mathbb{B}_A(\mathcal{H})$ for all $i \in \{1, 2, \dots, d\}$. Then,*

$$\omega_A^{4n}\left(\sum_{i=1}^{d} S_i\right) \leq \frac{d^{4n-1}}{4}\left[\left\|\sum_{i=1}^{d}\left(\left(S_i^{\sharp_A} S_i\right)^{2n} + \left(S_i S_i^{\sharp_A}\right)^{2n}\right)\right\|_A + 2\sum_{i=1}^{d} \omega_A\left(\left(S_i^{\sharp_A} S_i\right)^n \left(S_i S_i^{\sharp_A}\right)^n\right)\right],$$

for all $n = 1, 2, 3, \dots$.

Proof. Let $x \in \mathcal{H}$ be such that $\|x\| = 1$. Since $S_i \in \mathbb{B}_A(\mathcal{H})$, then $S_i \in \mathbb{B}(\mathcal{H})$. So, we see that

$$\begin{aligned}
\left|\left\langle \left(\sum_{i=1}^{d} S_i\right) x, x\right\rangle\right|^{4n} &\leq \left(\sum_{i=1}^{d} \left|\langle S_i x, x\rangle\right|\right)^{4n} \\
&\leq d^{4n-1} \sum_{i=1}^{d} \left|\langle S_i x, x\rangle\right|^{4n}, \quad \big(\text{by Lemma 3.11}\big) \\
&\leq d^{4n-1} \sum_{i=1}^{d} \langle |S_i| x, x\rangle^{2n} \langle |S_i^*| x, x\rangle^{2n}, \quad \big(\text{by Lemma 3.10 } (i)\big) \\
&\leq d^{4n-1} \sum_{i=1}^{d} \langle |S_i|^{2n} x, x\rangle \langle |S_i^*|^{2n} x, x\rangle, \quad \big(\text{by Lemma 3.10 } (ii)\big) \\
&= d^{4n-1} \sum_{i=1}^{d} \langle |S_i|^{2n} x, x\rangle \langle x, |S_i^*|^{2n} x\rangle.
\end{aligned}$$

Moreover, by applying Lemma 3.5 with $A = I$, we obtain

$$\begin{aligned}
\left|\left\langle \left(\sum_{i=1}^{d} S_i\right) x, x\right\rangle\right|^{4n} &\leq \frac{d^{4n-1}}{2} \sum_{i=1}^{d} \left(\left\| |S_i|^{2n} x \right\| \left\| |S_i^*|^{2n} x \right\| + |\langle |S_i|^{2n} x, |S_i^*|^{2n} x\rangle| \right) \\
&\leq \frac{d^{4n-1}}{2} \sum_{i=1}^{d} \left(\frac{1}{2}\left(\left\| |S_i|^{2n} x \right\|^2 + \left\| |S_i^*|^{2n} x \right\|^2 \right) + \left| \langle |S_i|^{2n} |S_i^*|^{2n} x, x\rangle \right| \right) \\
&= \frac{d^{4n-1}}{4} \sum_{i=1}^{d} \left\langle \left(|S_i|^{4n} + |S_i^*|^{4n} \right) x, x \right\rangle + \frac{d^{4n-1}}{2} \sum_{i=1}^{d} \left| \langle |S_i|^{2n} |S_i^*|^{2n} x, x\rangle \right| \\
&= \frac{d^{4n-1}}{4} \left\langle \left(\sum_{i=1}^{d} (|S_i|^{4n} + |S_i^*|^{4n}) \right) x, x \right\rangle + \frac{d^{4n-1}}{2} \sum_{i=1}^{d} \left| \langle |S_i|^{2n} |S_i^*|^{2n} x, x\rangle \right| \\
&\leq \frac{d^{4n-1}}{4} \left\| \sum_{i=1}^{d} (|S_i|^{4n} + |S_i^*|^{4n}) \right\| + \frac{d^{4n-1}}{2} \sum_{i=1}^{d} \omega\left(|S_i|^{2n} |S_i^*|^{2n} \right).
\end{aligned}$$

Taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$ yields that

$$\omega^{4n}\left(\sum_{i=1}^{d} S_i\right) \leq \frac{d^{4n-1}}{4} \left\| \sum_{i=1}^{d} \left(|S_i|^{4n} + |S_i^*|^{4n} \right) \right\| + \frac{d^{4n-1}}{2} \sum_{i=1}^{d} \omega\left(|S_i|^{2n} |S_i^*|^{2n} \right). \tag{21}$$

Now, since $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H})$, so $S_i \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ for each $i \in \{1, 2, \ldots, d\}$. Therefore, there exists unique $\widetilde{S_i}$ in $\mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_A S_i = \widetilde{S_i} Z_A$ for all i. By taking into consideration the fact that $\mathbf{R}(A^{1/2})$ is a complex Hilbert space, then (21) implies that

$$\omega^{4n}\left(\sum_{i=1}^{d} \widetilde{S_i}\right) \leq \frac{d^{4n-1}}{4} \left\| \sum_{i=1}^{d} \left(|\widetilde{S_i}|^{4n} + |\widetilde{S_i}^*|^{4n} \right) \right\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \frac{d^{4n-1}}{2} \sum_{i=1}^{d} \omega\left(|\widetilde{S_i}|^{2n} |\widetilde{S_i}^*|^{2n} \right).$$

We keep into account from [19] that for every $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have

$$\widetilde{TS} = \widetilde{S}\widetilde{T} \quad \text{and} \quad \widetilde{T + \lambda S} = \widetilde{T} + \lambda \widetilde{S} \ \ \forall \lambda \in \mathbb{C}. \tag{22}$$

So, in view of (22), we obtain

$$\omega^{4n}\left(\widetilde{\sum_{i=1}^{d} S_i}\right) \leq \frac{d^{4n-1}}{4} \left\| \sum_{i=1}^{d} \left(\left((\widetilde{S_i})^* \widetilde{S_i} \right)^{2n} + \left(\widetilde{S_i} (\widetilde{S_i})^* \right)^{2n} \right) \right\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \frac{d^{4n-1}}{2} \sum_{i=1}^{d} \omega\left(\left((\widetilde{S_i})^* \widetilde{S_i} \right)^{n} \left(\widetilde{S_i} (\widetilde{S_i})^* \right)^{n} \right).$$

Also, by Lemma 3.12 (iii), we have $(\widetilde{S_i})^* = \widetilde{S_i^{\sharp_A}}$ for all i. So,

$$\begin{aligned}
\omega^{4n}\left(\widetilde{\sum_{i=1}^{d} S_i}\right) &\leq \frac{d^{4n-1}}{4} \left\| \sum_{i=1}^{d} \left(\left(\widetilde{S_i^{\sharp_A}} \widetilde{S_i} \right)^{2n} + \left(\widetilde{S_i} \widetilde{S_i^{\sharp_A}} \right)^{2n} \right) \right\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \frac{d^{4n-1}}{2} \omega\left(\left(\widetilde{S_i^{\sharp_A}} \widetilde{S_i} \right)^{n} \left(\widetilde{S_i} \widetilde{S_i^{\sharp_A}} \right)^{n} \right) \\
&= \frac{d^{4n-1}}{4} \left\| \sum_{i=1}^{d} \widetilde{\left(\left(S_i^{\sharp_A} S_i \right)^{2n} + \left(S_i S_i^{\sharp_A} \right)^{2n} \right)} \right\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \frac{d^{4n-1}}{2} \sum_{i=1}^{d} \omega\left(\widetilde{\left(S_i^{\sharp_A} S_i \right)^{n} \left(S_i S_i^{\sharp_A} \right)^{n}} \right).
\end{aligned}$$

Hence, by applying Lemma 3.12 (i) and (ii), we get

$$\omega_A^{4n}\left(\sum_{i=1}^{d} S_i\right) \leq \frac{d^{4n-1}}{4} \left\| \sum_{i=1}^{d} \left(\left(S_i^{\sharp_A} S_i \right)^{2n} + \left(S_i S_i^{\sharp_A} \right)^{2n} \right) \right\|_A + \frac{d^{4n-1}}{2} \sum_{i=1}^{d} \omega_A\left(\left(S_i^{\sharp_A} S_i \right)^{n} \left(S_i S_i^{\sharp_A} \right)^{n} \right).$$

Thus, we complete the proof. □

In particular, by considering $d = n = 1$ in Theorem 3.13, we get the following result.

Corollary 3.14. *Let $S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_A^4(S) \leq \frac{1}{4}\left\|\left(S^{\sharp_A}S\right)^2 + \left(SS^{\sharp_A}\right)^2\right\|_A + \frac{1}{2}\omega_A\left(S^{\sharp_A}S^2S^{\sharp_A}\right).$$

Theorem 3.15. *Let $S_i \in \mathbb{B}_A(\mathcal{H})$ for all $i \in \{1, 2, \dots, d\}$. Then*

$$\omega_A^{2n}\left(\sum_{i=1}^d S_i\right) \leq \frac{d^{2n-1}}{2}\left\|\sum_{i=1}^d \left(\left(S_i^{\sharp_A}S_i\right)^n + \left(S_iS_i^{\sharp_A}\right)^n\right)\right\|_A,$$

for all $n = 1, 2, 3, \dots$.

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. $S_i \in \mathbb{B}_A(\mathcal{H})$ implies $S_i \in \mathbb{B}(\mathcal{H})$. So, we have

$$\begin{aligned}
\left|\left\langle\left(\sum_{i=1}^d S_i\right)x, x\right\rangle\right|^{2n} &\leq \left(\sum_{i=1}^d \left|\langle S_ix, x\rangle\right|\right)^{2n} \\
&\leq d^{2n-1}\sum_{i=1}^d |\langle S_ix, x\rangle|^{2n}, \quad \left(\text{by Lemma 3.11}\right) \\
&\leq d^{2n-1}\sum_{i=1}^d \langle |S_i|x, x\rangle^n \langle |S_i^*|x, x\rangle^n, \quad \left(\text{by Lemma 3.10 } (i)\right) \\
&\leq d^{2n-1}\sum_{i=1}^d \langle |S_i|^n x, x\rangle \langle |S_i^*|^n x, x\rangle, \quad \left(\text{by Lemma 3.10 } (ii)\right) \\
&\leq \frac{d^{2n-1}}{2}\sum_{i=1}^d \left(\langle |S_i|^n x, x\rangle^2 + \langle |S_i^*|^n x, x\rangle^2\right) \\
&\leq \frac{d^{2n-1}}{2}\sum_{i=1}^d \left(\langle |S_i|^{2n} x, x\rangle + \langle |S_i^*|^{2n} x, x\rangle\right) \\
&= \frac{d^{2n-1}}{2}\sum_{i=1}^d \left\langle\left(|S_i|^{2n} + |S_i^*|^{2n}\right)x, x\right\rangle \\
&= \frac{d^{2n-1}}{2}\left\langle\sum_{i=1}^d \left(|S_i|^{2n} + |S_i^*|^{2n}\right)x, x\right\rangle \\
&\leq \frac{d^{2n-1}}{2}\left\|\sum_{i=1}^d \left(|S_i|^{2n} + |S_i^*|^{2n}\right)\right\|.
\end{aligned}$$

By taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$\omega^{2n}\left(\sum_{i=1}^d S_i\right) \leq \frac{d^{2n-1}}{2}\left\|\sum_{i=1}^d \left(|S_i|^{2n} + |S_i^*|^{2n}\right)\right\|. \tag{23}$$

On the other hand, since $S_i \in \mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H})$ for all i, then by Lemma 3.12, for each $i \in \{1, 2, \dots, d\}$ there exists a unique $\widetilde{S_i}$ in $\mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_AS_i = \widetilde{S_i}Z_A$. By taking into account the fact that $\mathbf{R}(A^{1/2})$ is a complex Hilbert space, then (23) implies that

$$\omega^{2n}\left(\sum_{i=1}^d \widetilde{S_i}\right) \quad \leq \quad \frac{d^{2n-1}}{2}\left\|\sum_{i=1}^d \left(|\widetilde{S_i}|^{2n} + |\widetilde{S_i}^*|^{2n}\right)\right\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}.$$

Moreover, by using (22) together with Lemma 3.12 (iii), we get

$$\begin{aligned}\omega^{2n}\left(\widetilde{\sum_{i=1}^{d} S_i}\right) &\leq \frac{d^{2n-1}}{2}\left\|\sum_{i=1}^{d}\left(\left((\widetilde{S_i})^*\widetilde{S_i}\right)^n+\left(\widetilde{S_i}(\widetilde{S_i})^*\right)^n\right)\right\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\\ &\leq \frac{d^{2n-1}}{2}\left\|\sum_{i=1}^{d}\left(\left(\widetilde{S_i^{\sharp_A}}\widetilde{S_i}\right)^n+\left(\widetilde{S_i}\widetilde{S_i^{\sharp_A}}\right)^n\right)\right\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\\ &= \frac{d^{2n-1}}{2}\left\|\widetilde{\sum_{i=1}^{d}\left(\left(S_i^{\sharp_A}S_i\right)^n+\left(S_iS_i^{\sharp_A}\right)^n\right)}\right\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}.\end{aligned}$$

Hence,

$$\omega_A^{2n}\left(\sum_{i=1}^{d} S_i\right) \leq \frac{d^{2n-1}}{2}\left\|\sum_{i=1}^{d}\left(\left(S_i^{\sharp_A}S_i\right)^n+\left(S_iS_i^{\sharp_A}\right)^n\right)\right\|_A.$$

Thus, we complete the proof. □

Remark 3.16. *The bounds obtained in Theorem 3.13 and Theorem 3.15 are not comparable, in general. Note that if* $S^{\sharp_A}S^2S=0$, *then Theorem 3.13 (*$d=n=1$*) gives,* $\omega_A^4(S)\leq\frac{1}{4}\left\|\left(S^{\sharp_A}S\right)^2+\left(SS^{\sharp_A}\right)^2\right\|_A$, *whereas Theorem 3.15 (*$d=1$, $n=2$*) gives* $\omega_A^4(S)\leq\frac{1}{2}\left\|\left(S^{\sharp_A}S\right)^2+\left(SS^{\sharp_A}\right)^2\right\|_A$. *Hence, if* $S^{\sharp_A}S^2S=0$, *then the inequality obtained in Theorem 3.13 (*$d=n=1$*) is a refinement of that obtained in Theorem 3.15 (*$d=1$, $n=2$*). On the other hand, consider* $S_1=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ *and* $S_2=\begin{pmatrix}0&2\\0&0\end{pmatrix}$. *Then Theorem 3.13 (*$n=1$, $d=2$, $A=I$*) gives* $w_A^2(S_1+S_2)\leq\sqrt{34}$, *whereas Theorem 3.15 (*$n=1$, $d=2$, $A=I$*) gives* $w_A^2(S_1+S_2)\leq 5$. *Thus for this example, the bound obtained in Theorem 3.15 is better than that obtained in Theorem 3.13.*

Finally, we obtain the following result.

Theorem 3.17. *Let* $S_i\in\mathbb{B}_A(\mathcal{H})$ *for all* $i\in\{1,2,\ldots,d\}$. *Then,*

$$\omega_A^{2n}\left(\sum_{i=1}^{d} S_i\right)\leq\frac{d^{2n-1}}{\sqrt{2}}\sum_{i=1}^{d}\omega_A\left(\left(S_i^{\sharp_A}S_i\right)^n+\mathrm{i}\left(S_iS_i^{\sharp_A}\right)^n\right),$$

for all $n=1,2,3,\ldots$.

Proof. Let $x\in\mathcal{H}$ with $\|x\|=1$. By using similar arguments to that used in proof of Theorem 3.15, one observes that

$$\left|\left\langle\left(\sum_{i=1}^{d} S_i\right)x,x\right\rangle\right|^{2n}\leq\frac{d^{2n-1}}{2}\sum_{i=1}^{d}\left(\langle|S_i|^{2n}x,x\rangle+\langle|S_i^*|^{2n}x,x\rangle\right).$$

Further, we observe that $|a+b|\leq\sqrt{2}|a+\mathrm{i}b|$ for all $a,b\in\mathbb{R}$. By using this inequality, we see that

$$\begin{aligned}\left|\left\langle\left(\sum_{i=1}^{d} S_i\right)x,x\right\rangle\right|^{2n}&\leq\frac{d^{2n-1}}{\sqrt{2}}\sum_{i=1}^{d}\left|\langle|S_i|^{2n}x,x\rangle+\mathrm{i}\langle|S_i^*|^{2n}x,x\rangle\right|\\&\leq\frac{d^{2n-1}}{\sqrt{2}}\sum_{i=1}^{d}\omega\left(|S_i|^{2n}+\mathrm{i}|S_i^*|^{2n}\right).\end{aligned}$$

Taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$ gives

$$\omega^{2n}\left(\sum_{i=1}^{d} S_i\right) \leq \frac{d^{2n-1}}{\sqrt{2}} \sum_{i=1}^{d} \omega\left(|S_i|^{2n} + \mathrm{i}|S_i^*|^{2n}\right). \tag{24}$$

Now, since $S_i \in \mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H})$ for all i, then by Lemma 3.12, for each $i \in \{1, 2, \dots, d\}$ there exists a unique $\widetilde{S_i}$ in $\mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_A S_i = \widetilde{S_i} Z_A$. Due to the fact that $\mathbf{R}(A^{1/2})$ is a complex Hilbert space, then an application of (24) gives

$$\omega^{2n}\left(\sum_{i=1}^{d} \widetilde{S_i}\right) \leq \frac{d^{2n-1}}{\sqrt{2}} \sum_{i=1}^{d} \omega\left(|\widetilde{S_i}|^{2n} + \mathrm{i}|\widetilde{S_i}^*|^{2n}\right). \tag{25}$$

So, by using (22) together with Lemma 3.12 (iii), we obtain

$$\omega^{2n}\left(\widetilde{\sum_{i=1}^{d} \widetilde{S_i}}\right) \leq \frac{d^{2n-1}}{\sqrt{2}} \sum_{i=1}^{d} \omega\left(\widetilde{(S_i^{\sharp_A} S_i)^n + \mathrm{i}(S_i S_i^{\sharp_A})^n}\right).$$

Hence, by Lemma 3.12 (ii), we get

$$\omega_A^{2n}\left(\sum_{i=1}^{d} S_i\right) \leq \frac{d^{2n-1}}{\sqrt{2}} \sum_{i=1}^{d} \omega_A\left(\left(S_i^{\sharp_A} S_i\right)^n + \mathrm{i}\left(S_i S_i^{\sharp_A}\right)^n\right),$$

as required. $\square$

The following corollary is an easy consequence of Theorem 3.17.

Corollary 3.18. *Let $S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_A^2(S) \leq \frac{1}{\sqrt{2}} \omega_A\left(S^{\sharp_A} S + \mathrm{i}\ S S^{\sharp_A}\right).$$

Remark 3.19. *Following the proofs of Theorems 3.15 and 3.17, we conclude that, in general, the inequality in Theorem 3.17 is weaker than that in Theorem 3.15. In particular, let $n = d = 1$, $A = I$ and $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then, Theorem 3.15 gives, $\omega_A^2(S) \leq \frac{1}{2}\|S^{\sharp_A} S + S S^{\sharp_A}\|_A = \frac{1}{2}$, whereas Theorem 3.17 gives $\omega_A^2(S) \leq \frac{1}{\sqrt{2}} \omega_A\left(S^{\sharp_A} S + \mathrm{i}\ S S^{\sharp_A}\right) = \frac{1}{\sqrt{2}}$. This example substantiates the fact that the inequality in Theorem 3.15 is better than that in Theorem 3.17.*

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