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Recurrence Relations Arising from Confluent Hypergeometric Functions

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Abstract. The aim of this paper is to present some recurrence relations arising from confluent hypergeometric functions. In addition, an explicit closed-form expression for a sequence associated to the hypergeometric series in terms of Bell partition polynomials is proposed. Several examples are given to illustrate our results.

1. Introduction

As usual, $(\lambda)_n$ (for $\lambda \in \mathbb{C}$) denotes the Pochhammer symbol defined by

$$(\lambda)_n = \lambda (\lambda + 1) \cdots (\lambda + n - 1)$$

with $(\lambda)_0 = 1$. The confluent hypergeometric function $M(a, c, z)$ is defined as [1]

$$M(a, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}, \tag{1}$$

which converges for any $z \in \mathbb{C}$, and is defined for any $a \in \mathbb{C}$, $c \in \mathbb{C} - \{0, -1, -2, \cdots\}$.
It is well-known that $M(a, c, z)$ is the simplest solution of Kummer's differential equation

$$zy'' + (c - z)y' - ay = 0. \tag{2}$$

A second solution of Kummer's differential equation (2) is the Tricomi confluent hypergeometric function $U(a, c, z)$ given by

$$U(a, c, z) = \frac{\Gamma(1 - c)}{\Gamma(a - c + 1)} M(a, c, z) + \frac{\Gamma(c - 1)}{\Gamma(a)} z^{1-c} M(a - c + 1, 2 - c, z), \tag{3}$$

where $\Gamma(z)$ is the Euler gamma function.

If $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$, the confluent hypergeometric function $M(a, c, z)$ can be represented as an integral

$$M(a, c, z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a))} \int_0^1 e^{zt} t^{a-1} (1 - t)^{c-a-1} dt \tag{4}$$

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and, if $\text{Re}(a) > 0$, $U(a,c,z)$ can be obtained by the Laplace integral

$$U(a,c,z) = \frac{1}{\Gamma(a)} \int_0^{+\infty} e^{-zt} t^{a-1} (1+t)^{c-a-1} \, dt. \tag{5}$$

The (exponential) partial Bell partition polynomials $B_{n,k}(x_1, x_2, \ldots)$ in an infinite number of variables x_j, $(j \geq 1)$, were introduced as a mathematical tool [2, 5, 6] for representing the n-th derivative of composite function. They are defined by their generating function

$$\sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots) \frac{z^n}{n!} = \frac{1}{k!} \left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!} \right)^k \tag{6}$$

and are given explicitly by the formula

$$B_{n,k}(x_1, x_2, \ldots, x_n) = \sum_{\pi(n,k)} \frac{n!}{k_1! \cdots k_n!} \left(\frac{x_1}{1!} \right)^{k_1} \left(\frac{x_2}{2!} \right)^{k_2} \cdots \left(\frac{x_n}{n!} \right)^{k_n}, \tag{7}$$

where

$$\pi(n,k) = \left\{ (k_1, \ldots, k_n) \in \mathbb{N}^n : \sum_{i=1}^{n} k_i = k, \ \sum_{i=1}^{n} i k_i = n \right\}.$$

An interesting identity is obtained from (6):

$$B_{n,k}(abx_1, ab^2 x_2, \ldots, ab^n x_n) = a^k b^n B_{n,k}(x_1, x_2, \ldots, x_n). \tag{8}$$

Now, for appropriate choices of the variables x_j, the (exponential) partial Bell partition polynomials can be reduced to some special combinatorial sequences. We will mention the following special cases:

$$s(n,k) = B_{n,k}(0!, -1!, 2!, -3!, \ldots), \text{ (signed) Stirling numbers of the first kind,} \tag{9}$$
$$S(n,k) = B_{n,k}(1,1,1,\ldots), \text{ Stirling numbers of the second kind.} \tag{10}$$

Over the years, generating functions have demonstrated to be a fundamental tool for dealing with mathematical problems, such as in special functions, probability theory and enumerative combinatorics. Recently, many research articles have been devoted to the closed-form expression for the classical sequences (generalized Bernoulli and Euler polynomials [4], Frobenius-Euler polynomials [13, 15], Truncated-exponential-based Frobenius–Euler polynomials [9], Frobenius-type Eulerian polynomials [14]).

The aim of this paper is to present an explicit closed-form expression for a sequence associated to the hypergeometric series in terms of Bell partition polynomials and to demonstrate that the sequence $(A_n)_{n\geq 0}$ associated to the confluent hypergeometric function

$$M(a,c,\alpha(z)) = \sum_{n=0}^{\infty} A_n \frac{z^n}{n!}, \tag{11}$$

satisfies the following recurrence relation:

$$\mathcal{A}_{0,m} = 1, \quad \mathcal{A}_{n+1,m} = \sum_{k=0}^{n} \binom{n}{k} v_{n-k+1} \left(\mathcal{A}_{k,m} - \frac{c+m-a}{c+m} \mathcal{A}_{k,m+1} \right), \tag{12}$$

with

$$\alpha(z) = \sum_{n=0}^{\infty} v_n \frac{z^n}{n!}, \quad v_0 = 0.$$

More precisely, if we construct an infinite matrix $(\mathcal{A})_{n,m\geq 0}$ with the initial sequence given by $\mathcal{A}_{0,m} = 1$, and each entry is given by (12). Then the first column of the matrix is $\mathcal{A}_{n,0} = A_n$.

2. Recurrence relation for $M(a, c, \alpha(z))$

First, we have obtained the following result.

Theorem 2.1. *The sequence $(A_n)_{n\geq 0}$ associated to the confluent hypergeometric $M(a, c, \alpha(z))$ is given explicitly by*

$$A_0 = 1,\ A_n = \sum_{k=1}^{n} \frac{(a)_k}{(c)_k} B_{n,k}(v_1, v_2, \ldots, v_{n-k+1}) \tag{13}$$

with

$$\alpha(z) = \sum_{n=0}^{\infty} v_n \frac{z^n}{n!}, \quad v_0 = 0.$$

Proof. It is easily derived directly from the Faà di Bruno formula [6, Theorem A, pp. 137]. □

As consequence of the last result, we give alternative proofs to some explicit sequences arising from confluent hypergeometric functions.

Example 2.2. *The exponential polynomials $\phi_n(x)$ are defined by means of the following generating function*

$$\exp(x(e^z - 1)) = \sum_{n=0}^{\infty} \phi_n(x) \frac{z^n}{n!},$$

and, can be represented as $M(a, a, x(e^z - 1))$.

From (13), (8) *and* (10), *we obtain the well-known explicit formula for $\phi_n(x)$*

$$\begin{aligned}\phi_n(x) &= \sum_{k=0}^{n} B_{n,k}(x, x, \ldots, x)\\ &= \sum_{k=0}^{n} B_{n,k}(1, 1, \ldots, 1) x^k\\ &= \sum_{k=0}^{n} S(n, k) x^k.\end{aligned}$$

Example 2.3. *The H-Cauchy numbers C_n^k are defined by the following generating function [8, 10]*

$$\frac{1}{k!} M(1, k+1, \ln(1+z)) = \sum_{n=0}^{\infty} C_n^{(k)} \frac{z^n}{n!},$$

or, equivalently,

$$C_n^{(k)} = n! \int_0^1 dx_k \int_0^{x_k} dx_{k-1} \cdots \int_0^{x_2} \binom{x_1}{n} dx_1.$$

Since

$$\ln(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)! \frac{z^n}{n!},$$

we get

$$C_n^{(k)} = \frac{1}{k!} \sum_{l=0}^{n} \frac{(1)_l}{(k+1)_l} B_{n,l}(0!, -1!, 2!, \ldots).$$

Now, from (9)*, we have obtained the explicit formula for* $C_n^{(k)}$

$$C_n^{(k)} = \sum_{l=0}^{n} \frac{l!}{(l+k)!} s(n,l).$$

Example 2.4. *The Gould-Hopper generalized Hermite polynomials* $g_n^m(x,h), (m > 0)$ *are defined by the following generating function (see [7, 12])*

$$M(1,1,xz+hz^m) = \sum_{n=0}^{\infty} g_n^m(x,h) \frac{z^n}{n!}.$$

From (13)*, we get*

$$g_n^m(x,h) = 1 + \sum_{k=1}^{n} B_{n,k}(x,0,\ldots,m!h,0,\ldots,0).$$

Using (7)*, we get*

$$\begin{aligned} g_n^m(x,h) &= 1 + \sum_{k=1}^{n} \left(\sum_{k_1+k_2=k, k_1+mk_2=n} \frac{n!}{k_1!k_2!} \left(\frac{x}{1!}\right)^{k_1} \left(\frac{m!h}{m!}\right)^{k_2} \right) \\ &= 1 + \sum_{k_1+mk_2=n} \frac{n!}{k_1!k_2!} x^{k_1} h^{k_2} \\ &= \sum_{k=0}^{[n/m]} \frac{n!}{(n-mk)!k!} x^{n-mk} h^k. \end{aligned}$$

By setting $m = 2, h = -1$ *and* $x := 2x$ *in the above formula, we obtain the explicit formula for the classical Hermite polynomials.*

In order to derive the recurrence relations for $M(a, c, \alpha(z))$, we suppose that

$$f_m(z) := \sum_{n=0}^{\infty} \mathcal{A}_{n,m} \frac{z^n}{n!} = M(a, c+m, \alpha(z)), \tag{14}$$

where m is any non-negative integer and

$$\alpha(z) = \sum_{n=0}^{\infty} v_n \frac{z^n}{n!}, \tag{15}$$

with $v_0 = 0$.
By differentiation (4) with respect to z, we obtain

$$\frac{d}{dz} f_m(z) = \frac{\Gamma(c+m)}{\Gamma(a)\Gamma(c-a+m)} \left(\frac{d}{dz}\alpha(z)\right) \int_0^1 t e^{\alpha(z)t} (1-t)^{c+m-a-1} t^{a-1} dt.$$

Thus,

$$\begin{aligned} \frac{d}{dz} f_m(z) = \frac{d}{dz}\alpha(z) \frac{\Gamma(c+m)}{\Gamma(a)\Gamma(c-a+m)} \int_0^1 e^{\alpha(z)t} (1-t)^{c+m-a-1} t^{a-1} dt \\ - \frac{d}{dz}\alpha(z) \frac{\Gamma(c+m)}{\Gamma(a)\Gamma(c-a+m)} \int_0^1 e^{\alpha(z)t} (1-t)^{c+m-a} t^{a-1} dt, \end{aligned}$$

and, we get

$$\frac{d}{dz}f_m(z) = \frac{d}{dz}\alpha(z)\left(f_m(z) - \frac{c+m-a}{c+m}f_{m+1}(z)\right).$$

This in turn leads to

$$\sum_{n=0}^{\infty}\mathcal{A}_{n+1,m}\frac{z^n}{n!} = \left(\sum_{n=0}^{\infty}v_{n+1}\frac{z^n}{n!}\right)\sum_{n=0}^{\infty}\left(\mathcal{A}_{n,m} - \frac{c+m-a}{c+m}\mathcal{A}_{n,m+1}\right)\frac{z^n}{n!}.$$

Applying the Cauchy product, we get

$$\sum_{n=0}^{\infty}\mathcal{A}_{n+1,m}\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}v_{n-k+1}\left(\mathcal{A}_{k,m} - \frac{c+m-a}{c+m}\mathcal{A}_{k,m+1}\right)\right)\frac{z^n}{n!}.$$

Equating the coefficients of $\frac{z^n}{n!}$ in both sides of the last expression, we may therefore state:

Theorem 2.5. *The sequence $(A_n)_{n\geq 0}$ associates to the exponential generating function $M(a,c,\alpha(z))$ satisfies the following recurrence relation*

$$\mathcal{A}_{0,m} = 1,$$
$$\mathcal{A}_{n+1,m} = \sum_{k=0}^{n}\binom{n}{k}v_{n-k+1}\left(\mathcal{A}_{k,m} - \frac{c+m-a}{c+m}\mathcal{A}_{k,m+1}\right), \tag{16}$$

with

$$\mathcal{A}_{n,0} := A_n.$$

In particular, for $\alpha(z) = z$, we derive the following recurrence relations for the solution of Kummer's differential equation.

Corollary 2.6. *The sequence associates to the exponential generating function $M(a,c,z)$ satisfies the following recurrence relation*

$$\mathcal{A}_{0,m} = 1,\ \ \mathcal{A}_{n+1,m} = \mathcal{A}_{n,m} - \frac{c+m-a}{c+m}\mathcal{A}_{n,m+1}, \tag{17}$$

with

$$\mathcal{A}_{n,0} := \frac{(a)_n}{(c)_n}.$$

Remark 2.7. *If $v_0 \neq 0$ in (15), then, the sequence $(A_n)_{n\geq 0}$ satisfies (16), with the initial sequence is given by $\mathcal{A}_{0,m} = M(a, c+m, \alpha(0))$.*

Remark 2.8. *The confluent hypergeometric function $M(a,c,\alpha(z_0))$ can be computed as power series. We use the following procedure : define*

$$S_N = \sum_{i=0}^{N}A_i\frac{z_0^i}{i!},$$

where A_i was computed using (16). For $n \geq 0$, let

$$Z_0 = 1,\ \ Z_{n+1} = \frac{z_0}{n+1}Z_n$$
$$T_n = A_n Z_n$$

Then $S_0 = A_0$ and, for $n > 0$, use the recurrence relationship to compute

$$S_{n+1} = S_n + T_{n+1}.$$

The process stop with $\left|\frac{T_{m+1}}{S_m}\right| < \varepsilon$ and return S_m.

Example 2.9. *The generating function of Hermite polynomials $H_n(x)$ can be expressed as*

$$M\left(1,1,2xz-z^2\right)=\sum_{n=0}^{\infty}H_n(x)\frac{z^n}{n!}.$$

In view of (16), *we present the following algorithm for $H_n(x)$: we start with the sequence $\mathcal{H}_{0,m}=1$ as the first row of the matrix $(\mathcal{H}_{n,m})_{n,m\geq 0}$. Each entry is determined recursively by*

$$\mathcal{H}_{n+1,m}=2x\left(\mathcal{H}_{n,m}-\frac{m}{m+1}\mathcal{H}_{n,m+1}\right)-2n\left(\mathcal{H}_{n-1,m}-\frac{m}{m+1}\mathcal{H}_{n-1,m+1}\right).$$

Then

$$H_n(x):=\mathcal{H}_{n,0}$$

where $\mathcal{H}_{n,0}$ are the first column of the matrix $(\mathcal{H}_{n,m})_{n,m\geq 0}$.

Example 2.10. *The generating function of exponential polynomials $\phi_n(x)$ can be expressed as*

$$M\left(1,1,x\left(e^z-1\right)\right)=\sum_{n=0}^{\infty}\phi_n(x)\frac{z^n}{n!}.$$

In view of (16), *we obtain*

$$\mathcal{A}_{0,m}=1,\ \ \mathcal{A}_{n+1,m}=x\sum_{k=0}^{n}\binom{n}{k}\left(\mathcal{A}_{k,m}-\frac{m}{m+1}\mathcal{A}_{k,m+1}\right).$$

Then

$$\phi_n(x):=\mathcal{A}_{n,0}.$$

3. Recurrence relation for $U(a,c,\alpha(z))$

In the present section, we derive a similar recurrence formula for $U(a,c,\alpha(z))$. Unlike Kummer's function which is an entire function of z, $U(a,c,\alpha(z))$ usually has a singularity at zero. If $a=-N$ with $N\in\mathbb{N}$, $U(a,c,\alpha(z))$ is a polynomial in z. In this case, letting

$$g_m(z):=\sum_{n=0}^{\infty}\mathcal{B}_{n,m}\frac{z^n}{n!}=U(a,c+m,\alpha(z)), \tag{18}$$

By differentiation (5) with respect to z, we get

$$\frac{d}{dz}g_m(z)=-\frac{1}{\Gamma(a)}\frac{d}{dz}\alpha(z)\int_0^{+\infty}e^{-\alpha(z)t}t^{a-1}(1+t)^{c-a+m}\,dt+\frac{1}{\Gamma(a)}\frac{d}{dz}\alpha(z)\int_0^{+\infty}e^{-\alpha(z)t}t^{a-1}(1+t)^{c-a-1+m}\,dt.$$

And so, we obtain

$$\frac{d}{dz}g_m(z)=\frac{d}{dz}\alpha(z)\left(g_m(z)-g_{m+1}(z)\right).$$

Applying some series manipulations, we get

$$\sum_{n=0}^{\infty}\mathcal{B}_{n+1,m}\frac{z^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}v_{n-k+1}\left(\mathcal{B}_{k,m}-\mathcal{B}_{k,m+1}\right)\right)\frac{z^n}{n!}.$$

Upon equating the coefficients of $\frac{z^n}{n!}$, we get the following Theorem.

Theorem 3.1. *The sequence $(B_n)_{n\geq 0}$ associates to the exponential generating function $U(a,c,\alpha(z))$ satisfies the following recurrence relation*

$$\mathcal{B}_{0,m} = U(a, c+m, \alpha(0)),$$

$$\mathcal{B}_{n+1,m} = \sum_{k=0}^{n} \binom{n}{k} v_{n-k+1} \left(\mathcal{B}_{k,m} - \mathcal{B}_{k,m+1}\right),$$

with $\mathcal{B}_{n,0} = B_n$.

Example 3.2. *The generalized Laguerre polynomials $L_N^{\gamma}(z)$ can be written as*

$$L_N^{\gamma}(x) = \frac{(-1)^N}{N!} U(-N, \gamma+1, z).$$

In view of Theorem 3.1, we obtain

$$\mathcal{L}_{0,m} = (-1)^N (\gamma+1+m)_N, \quad \mathcal{L}_{n+1,m} = \mathcal{L}_{n,m} - \mathcal{L}_{n,m+1}.$$

Then

$$L_N^{\gamma}(z) = \frac{(-1)^N}{N!} \sum_{k=0}^{N} \mathcal{L}_{k,0} \frac{z^k}{k!}.$$

Remark 3.3. *If $v_0 = 0$ and $\operatorname{Re}(c+m) < 1$ then $\mathcal{B}_{0,m} = \frac{\Gamma(1-(c+m))}{\Gamma(a-(c+m)+1)}$.*

4. Generalized hypergeometric function

It is well-known that a generalized hypergeometric series is a power series of the form

$$_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots \left(a_p\right)_n}{(b_1)_n \cdots \left(b_q\right)_n} \frac{z^n}{n!},$$

where p and q are nonnegative integers.

In this notation, the confluent hypergeometric function $M(a,c,z)$ and Tricomi confluent hypergeometric function $U(a,c,z)$ are

$$M(a,c,z) = {}_1F_1(a;c;z),$$

$$U(a,c,z) = z^{-a} {}_2F_0\left(a, 1+a-c; -; -\frac{1}{z}\right).$$

Theorem 2.1, can be extended as follows:

Theorem 4.1. *The sequence $(G_n)_{n\geq 0}$ associated to the generalized hypergeometric series*

$$_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;\alpha(z)\right) = \sum_{n=0}^{\infty} G_n \frac{z^n}{n!},$$

is given explicitly by

$$G_0 = 1,\ G_n = \sum_{k=1}^{n} \frac{(a_1)_n \cdots \left(a_p\right)_n}{(b_1)_n \cdots \left(b_q\right)_n} B_{n,k}\left(v_1, v_2, \ldots, v_{n-k+1}\right) \tag{19}$$

with

$$\alpha(z) = \sum_{n=0}^{\infty} v_n \frac{z^n}{n!}, \quad v_0 = 0.$$

Example 4.2. *The Bernoulli polynomials* $\mathfrak{B}_n(x)$ *are defined by the following generating function*

$$\frac{ze^{xz}}{e^z-1}=\sum_{n=0}^{\infty}\mathfrak{B}_n(x)\frac{z^n}{n!}$$

and can be expressed as [11]

$$\sum_{n=0}^{\infty}\mathfrak{B}_n(x)\frac{z^n}{n!}={}_2F_1(1,1;2;1-e^z)\,e^{xz}.$$

It follows from (19), (8) *and* (10) *that*

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_n(x)\frac{z^n}{n!}&=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{(1)_i(1)_i}{(2)_i}B_{n,i}(-1,-1,\ldots,-1)\right)\frac{z^n}{n!}\sum_{n=0}^{\infty}x^n\frac{z^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}(-1)^i\frac{i!}{i+1}S(n,i)\right)\frac{z^n}{n!}\sum_{n=0}^{\infty}x^n\frac{z^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{k}(-1)^i\frac{i!}{i+1}S(k,i)\,x^{n-k}\right)\frac{z^n}{n!}.\end{aligned}$$

Comparing coefficients, we obtain

$$\begin{aligned}\mathfrak{B}_n(x)&=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{k}(-1)^i\frac{i!}{i+1}S(k,i)\,x^{n-k}\\&=\sum_{i=0}^{n}(-1)^i\frac{i!}{i+1}\sum_{k=0}^{n}\binom{n}{k}S(k,i)\,x^{n-k}.\end{aligned}$$

Since

$$\begin{aligned}S_n^k(x)&=\frac{1}{k!}\Delta^k x^n\\&=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+j)^n\\&=\sum_{j=0}^{n}\binom{n}{j}S(j,k)\,x^{n-j},\end{aligned}$$

this reduces to

$$\mathfrak{B}_n(x)=\sum_{i=0}^{n}(-1)^i\frac{i!}{i+1}S_n^i(x).$$

Example 4.3. *For* $\lambda\in\mathbb{C}$ *with* $\lambda\neq 1$*, the Frobenius-Euler numbers* $H_n^{(\alpha)}(\lambda)$ *of order* $\alpha\in\mathbb{C}$ *are defined by the following generating function [13]*

$$\left(\frac{1-\lambda}{e^z-\lambda}\right)^{\alpha}=\sum_{n=0}^{\infty}H_n^{(\alpha)}(\lambda)\frac{z^n}{n!}.$$

It is not difficult to verify that

$$\left(\frac{1-\lambda}{e^z-\lambda}\right)^{\alpha}={}_1F_0\left(\alpha;-;\frac{e^z-1}{\lambda-1}\right).$$

It follows that

$$H_0^{(\alpha)}(\lambda) = 1,\ \ H_n^{(\alpha)}(\lambda) = \sum_{k=1}^{n} (\alpha)_k \, B_{n,k}\left(\frac{1}{\lambda-1}, \ldots, \frac{1}{\lambda-1}\right).$$

Using (8) *and* (10), *we get*

$$H_n^{(\alpha)}(\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda-1)^k} S(n,k). \tag{20}$$

By substituting $\lambda = -1$ *into (20), we obtain a known result for Euler numbers of order* α

$$E_n^{(\alpha)} = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} (\alpha)_k \, S(n,k).$$

The results obtained above can be generalized for the polynomials case

$$H_n^{(\alpha)}(x \mid \lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda-1)^k} S_n^k(x),$$

where $H_n(x \mid \lambda)$ *are defined by*

$$\left(\frac{1-\lambda}{e^z-\lambda}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} H_n(x \mid \lambda) \frac{z^n}{n!}.$$

Example 4.4. *The Lerch polynomials* $\Phi_n^{(\lambda)}(x)$ *of order* λ *are defined by the following ordinary generating function [3]*

$$\frac{1}{(1 - x\ln(1+z))^{\lambda}} = \sum_{n=0}^{\infty} \Phi_n^{(\lambda)}(x) \, z^n.$$

Since

$$\frac{1}{(1 - x\ln(1+z))^{\lambda}} = {}_1F_0(\lambda; -; x\ln(1+z)),$$

we have

$$\begin{aligned} n!\Phi_n^{(\lambda)}(x) &= 1 + \sum_{k=1}^{n} (\lambda)_k \, B_{n,k}(-0!x, 1!x, -2!x, \ldots) \\ &= \sum_{k=0}^{n} (\lambda)_k \, s(n,k) \, x^k. \end{aligned}$$

It follows that

$$\Phi_n^{(\lambda)}(x) = \sum_{k=0}^{n} \frac{(\lambda)_k}{n!} s(n,k) \, x^k.$$

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