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Optimal Control Problem for Fractional Stochastic Nonlocal Semilinear System

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Abstract. This article deals with the optimal control of the fractional stochastic nonlocal semilinear system in Hilbert space. The existence and uniqueness results for the mild solution are derived using Banach fixed point theorem. The optimal control is proved using minimizing sequence approach and Mazur's lemma. For better understanding of theory, we have included one example.

1. Introduction

Optimal Control theory deals with finding the control law for a period of time for a dynamical system such that an optimized objective function is obtained. It has vast applications in operation research, science, and engineering. Control problems consist of a function called cost function which is dependent on both state and control variables. Optimal control is a combination of differential equations that describes the path for the control variables which optimize the cost function.

Balakrishnan [8] considered the control problems where both state as well as control variables belongs to Banach space. He presented results on optimal control for the infinite-dimensional semilinear system using the bounded resolvent method. In [9] authors considered the linear term as an infinitesimal generator of strongly continuous cosine family and used various conditions on the non-linear term to obtain the results. They also derived the conditions in which the considered second-order semilinear system can be converted to a first-order system and also studied the case in which it is beneficial to study the system in the given form, that is, not converting it to a first-order system.

In [14] park et al. studied the optimality conditions for the semilinear control problems having bounded delay. They used the concept of a penalty function and Lipschitz continuity for non-linear terms to obtain the results. In [15] author derived necessary and sufficient conditions for strong-weak lower semicontinuity of cost function. In [17] Frankowska et al. studied some necessary conditions of the first and second order for local minimizers of optimal control problems. Nonconvexity was handled with the help of variational analysis and derived weak maximum principle using separation theorem. The authors also proposed

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stochastic inward pointing condition which is a sufficient condition for normality of weak maximum principle.

In ([33],[35]) authors discussed the existence of mild solution for fractional and stochastic semilinear Cauchy Problems. In [36] authors discussed some sufficient conditions for the existence of mild solution for the impulsive stochastic control system in abstract space. They derived the results using Leray-Schauder fixed point theorem, the uniform continuity of the resolvent and analytic resolvent operators. In [19] Klamka discussed certain sufficient conditions for controllability of non-linear control system in finite as well as infinite dimension. He used linear controllability and Schauder's fixed point theorem to obtain the results. In ([1]-[2],[10]-[13],[21], [28],[34],[42]-[43]) authors studied approximate controllability of delay systems, stochastic systems, fractional systems, etc. using basics of functional analysis and fixed point theorems.

In [22] authors studied optimal control problems in Hilbert space for an abstract semilinear control system with delay. They considered different set of cost functions, like observation of terminal value and averaging observation control. In [23] authors derived the existence results for the optimal control and maximal principle for parabolic type semilinear control system in which the non-linear term is Lipschitz continuous. They derived optimality conditions for a system relaxing the condition of differentiability on the non-linear term. If the principal operator is unbounded, Jeong et al. [24] obtained the results for time-optimal control for abstract semilinear control systems having time delay. They derived the results with the help of real interpolation space and construction of fundamental solution.

In [25] authors derived the necessary optimality condition for the retarded abstract control system in Hilbert space. In [26] Wang et al. studied results for optimal controls and solvability for a fractional integrodifferential control system in abstract spaces having infinite delay. With the help of Priori estimate, they derived the results for the existence of mild solutions and also the extension of the mild solution for a global interval. In [27] authors derived certain conditions for optimal feedback controls. They prove the existence of feasible pairs by using the Fillippove theorem and Cesari property.

In [30]-[32] Papageorgiou proved several results on the existence of optimal controls. He used Gronwall's inequality, penalty method for obtaining results for nonlinear evolution systems having non monotone nonlinearities. In [4] authors discussed the existence of mild solution and optimal control for second order semilinear control system in Hilbert space. The results are derived with the help of sine and cosine family theory and Banach fixed point theorem. In [37] authors obtained the existence and optimal control results using Krasnoselskii's fixed point theorem and minimizing sequence concept for the second-order SDE having mixed-fractional Brownian motion. In [38] Patel et al. discussed the existing result for the mild solution and optimal control for fractional-order $\alpha \in (1, 2]$ semilinear control system in Hilbert space. The results are derived with the help of α-order sine and cosine family theory, Banach fixed point theorem, and certain assumptions on nonlinearity. In [40]-[41] author discussed optimal control for fractional semilinear system. Using Weissinger's fixed point theorem, some assumptions on nonlinear function and contraction mapping some sufficient results for existence are derived. Recently [1] discussed the asymptotic stability of semilinear fractional stochastic system of order $(1, 2]$ using fixed point theorem approach.

Byszewski et al. [29] described the initial value problem with nonlocal conditions and explain the more accuracy of the system as due to nonlocal conditions there is less chance of ill-effects which are occurred due to a single initial measurement. Motivated by the above works and best of our knowledge there is no article dealing with optimal control problems for fractional-order $\alpha \in (1, 2]$ semilinear stochastic system in Hilbert space having nonlocal conditions. Using the basic ideas from [5]-[7] and [40]-[41] with suitable modifications, we have obtained our results. The results are advanced and will help researchers in the field of control theory. In theorem (3.1) we have studied the existence and uniqueness of mild solution for fractional stochastic system with nonlocal conditions using fixed point theory. This theorem is extension of [38] deterministic work in stochastic settings with suitable modifications. In theorem (4.1) we have studied the optimal control problem of Lagrange's problem of proposed system using Minimizing sequence approach, basics of Mazur's Lemma, and functional analysis. This theorem is extension of [38] deterministic work with addition of nonlocal conditions in stochastic settings.

2. Preliminaries

Some basic results, notations and definitions are considered in this section.These are helpful for obtaining results in section 3 and section 4 of the article. Throughout the article, we cosider the notations as below. Consider two separable Hilbert space which are denoted by G and K. For simplicity,$\|\cdot\|$ represent norm and $<,>$ denotes inner products . Let complete probability space be denoted by $(\Omega, F, \mathbf{P})$. It has complete family of sub σ-algebras $F_\tau, 0 \leq \tau \leq b$ which are right continuous increasing and $F_\tau \subset F$. The complete orthonormal system in K be denoted by $\{e_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ denotes sequence of Brownian motions which are independent and satisfying the below condition.

$$W(\tau) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n e_n,\ 0 \leq \tau \leq b$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence which is bounded and $\lambda_n \in \mathbb{R}^+ \cup 0$ for $n \in \mathbb{N}$ with the condition $Qe_n = \lambda_n e_n,\ n \in \mathbb{N}$ with $tr(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$. Then $W(\tau)$ which is K-valued stochastic process is called a Wiener Process. The sigma algebra generated by $\{W(s) : 0 \leq s \leq \tau\}$ is called normal filtration and is denoted by F_τ. Also $F_b = F$. Let the space of operators defined from K to G which are bounded be denoted by $L(K, G)$ where the norm is usual operator norm. For $\psi \in L(K, G)$, we define

$$\|\psi\|_Q^2 = tr(\psi Q \psi^*) = \sum_{n=1}^{\infty} \| \sqrt{\lambda_n}\psi e_n\|^2$$

ψ is called a Q-Hilbert Schmidt operator if $\|\psi\|_Q^2 < \infty$.The space of operators $\psi : K \to G$ where ψ is Q-Hilbert Schmidt operator be denoted by $L_Q(K, G)$. $L_Q(K, G)$ is a completion of $L(K, G)$ *w.r.to* the topology induced by the norm $\|\cdot\|_Q$ is a Hilbert space with respect to norm topology.
$L_2(\Omega, G)$ is a space of G-valued integrable,strongly measurable random variables,is a Banach space with respect to norm topology $\|s(\cdot)\| = (\mathbb{E}\|s(\tau)\|^2)^{1/2}$, where $\mathbb{E}(\cdot)$ denotes the expectation *w.r.to* the measure $\mathbb{P}$.
Let space of continuous maps defined from $[0, b]$ into $L_2(\Omega, G)$ which is Banach space and satisfying $\sup_{0\leq\tau\leq b} \mathbb{E}\|s(\tau\|^2 < \infty$ be denoted by $C([0, b], L_2(\Omega, G))$ .Consider G_2 as the subspace of $C([0, b], L_2(\Omega, G))$ which is closed and having F_τ-adapted, measurable, G-valued processes $s \in C([0, b], L_2(\Omega, G))$ endowed with the norm

$$\|s\|_{G_2} = \left(\sup_{0\leq\tau\leq b} \mathbb{E}\|s(\tau)\|_G^2\right)^{\frac{1}{2}}.$$

Consider the cost function as

$$J(s, v) := \mathbb{E}\left\{\int_0^b L(\tau, s^v(\tau), v(\tau))d\tau\right\}, \tag{1}$$

with respect to

$$\begin{aligned} {}^C D_\tau^\alpha s(\tau) &= As(\tau) + B(\tau)v(\tau) + \eta(\tau, s(\tau)) + \sigma(\tau, s(\tau))\frac{dW(\tau)}{d\tau},\ 0 < \tau \leq b; \\ s(0) &= s_0 + n_0(s) \in G, \\ s'(0) &= s_1 + n_1(s) \in G. \end{aligned} \tag{2}$$

where the integrand L is defined in section 4, for $1 < \alpha \leq 2$, ${}^C D_\tau^\alpha$ denotes the Caputo fractional derivative, b is time constant and $0 < b < \infty$. The control function $v(\cdot)$ has domain U which is a separable reflexive Hilbert space, the state $s(\cdot)$ is G-valued stochastic process. The infinitesimal generator for solution operator $C_\alpha(\tau),\ 0 \leq \tau < \infty$ be denoted by A and is defined as $A : D(A) \subseteq G \to G$ .A family of operators defined from U to G which are linear is represented as $\{B(\tau) : \tau \geq 0\}$; the functions $\eta : [0,b] \times G \to G$ and $\sigma : [0,b] \times G \to L(K,G)$ are nonlinear, $s(0)$, $s^{'}$ are G-valued F_0-measurable random variables which are independent of W. The functions $n_0(s)$ and $n_1(s)$ are continuous and defined as $C([0,b],G) \to G$.

Define the admissible set U_{ad}, the set of all $v(\cdot) : [0,b] \times \Omega \to U$ such that v is F_τ adapted stochastic process and $\mathbb{E} \int_0^b \|v(\tau)\|^p d\tau < \infty$. Clearly $U_{ad} \neq \emptyset$ and $U_{ad} \subset L_p([0,b];U)(1 < p < \infty)$ is closed, bounded and convex.

Let $\mathcal{A}_{ad}$ consist of pairs (s,v) where s denotes mild solution of system (2) with the control $v \in U_{ad}$, which are admissible. The major target of the paper is to find a pair $(s^0, v^0) \in \mathcal{A}_{ad}$ such that

$$J(s^0, v^0) := \inf J(s,v) : (s,v) \in \mathcal{A}_{ad} = \delta$$

We now recall some of the definition which are used as basic for fractional calculus.
Let V be a Hilbert space having norm $\|\cdot\|$.Consider the space of all V-valued Bochner integrable functions as $L_2([0,b];V)$.The norm for the function $f \in L_2([0,b];V)$ is defined as

$$\|f\| = \left(\int_0^b \|f(\tau)\|^2 d\tau \right)^{1/2} \tag{3}$$

Set of all the operators which are bounded and linear on V be denoted by $\mathbb{L}(V)$.
The space of functions which are continuous is represented by $C([0,b];V)$ and space of functions which has first order continuous derivative is represented by $C^1([0,b];V)$.

Definition 2.1. *[28] "If $s(\tau) \in C([0,b];V)$, then the Riemann-Liouville integral of fractional order $\alpha > 0$ can be given as*

$$I_t^\alpha s(\tau) = \frac{1}{\Gamma(\alpha)} \int_{\tau_0}^{\tau} (\tau - \varsigma)^{\alpha-1} s(\varsigma) d\varsigma"$$

Definition 2.2. *[28] "The Riemann-Liouville fractional derivative of a function $s(\tau) \in C([0,b];V)$ of order $\alpha \in (1,2]$ is defined by*

$$\begin{aligned} D_\tau^\alpha s(\tau) &= D^2 I^{2-\alpha} s(\tau) \\ &= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{d\tau^2} \int_{\tau_0}^{\tau} (\tau - \varsigma)^{1-\alpha} s(\varsigma) d\varsigma". \end{aligned}$$

Definition 2.3. *[28] "The Caputo fractional derivative of order $\alpha \in (1,2]$ is defined by*

$$\begin{aligned} {}^C D_\tau^\alpha s(\tau) &= I^{2-\alpha} D^2 s(\tau) \\ &= \frac{1}{\Gamma(2-\alpha)} \int_{\tau_0}^{\tau} (\tau - \varsigma)^{1-\alpha} \left[\frac{d^2}{d\varsigma^2} s(\varsigma) \right] d\varsigma, \end{aligned}$$

where $s(\tau) \in C^1([0,b];V)$".

Consider the linear system of fractional order as :

$${}^C D_\tau^\alpha s(\tau) = A s(\tau), \quad s(0) = \kappa, \quad s'(0) = 0, \tag{4}$$

where $\alpha \in (1,2]$ $A : D(A) \subset V \to V$ is a operator defined in V which is dense and closed.
Now using Riemann-Liouville integral of fractional order α on (2)

$$s(\tau) = \kappa + \frac{1}{\Gamma(\alpha)} \int_0^{\tau} (\tau - \varsigma)^{\alpha-1} A s(\varsigma) d\varsigma \tag{5}$$

Definition 2.4. *[28] "Let $\alpha \in (1,2]$. A family $\{C_\alpha(\tau)\}_{\tau\geq 0} \subset \mathbb{L}(V)$ is called the solution operator (or strongly continuous α-order fractional cosine family) for* (2) *and A is called the infinitesimal generator of $C_\alpha(\tau)$, if the following conditions are satisfied:*

1. *$C_\alpha(\tau)$ is strongly continuous for $\tau \geq 0$ and $C_\alpha(0) = I$. There exists a constant $M \geq 1$ such that $\|C_\alpha(\tau)\| \leq M$.*
2. *$C_\alpha(\tau)D(A) \subset D(A)$ and $AC_\alpha(\tau)\kappa = C_\alpha(\tau)A\kappa$ for all $\kappa \in D(A)$, $\tau \geq 0$;*
3. *$C_\alpha(\tau)\kappa$ is a solution of* (2) *for all $\kappa \in D(A)$".*

Definition 2.5. *[28] "The fractional sine family $S_\alpha : [0,\infty) \to \mathbb{L}(V)$ associated with C_α is defined by*

$$S_\alpha(\tau) = \int_0^\tau C_\alpha(\varsigma)d\varsigma, \quad \tau \geq 0". \tag{6}$$

Definition 2.6. *[28] "The fractional Riemann-Liouville family $P_\alpha : [0,\infty) \to \mathbb{L}(V)$ associated with C_α is defined by*

$$P_\alpha(\tau) = I^{\alpha-1}C_\alpha(\tau)" \tag{7}$$

With the help of definition (1) *, for $0 \leq \tau \leq b$*

$$\begin{aligned}
\|P_\alpha(\tau)\| &= \|I^{\alpha-1}C_\alpha(\tau)\| \\
&= \|\int_0^\tau \frac{(\tau-\varsigma)^{(\alpha-2)}}{\Gamma(\alpha-1)}C_\alpha(\varsigma)d\varsigma\| \\
&\leq \frac{\|C_\alpha(\varsigma)\|}{\Gamma(\alpha-1)}\|\int_0^\tau (\tau-\varsigma)^{\alpha-2}d\varsigma\| \\
&\leq \frac{M}{\Gamma(\alpha)}b^{\alpha-1} = M_P \ (let).
\end{aligned}$$

Definition 2.7. *"An F_τ-adapted stochastic process $s(\tau) \in C([0,b]; L_2(\Omega, G))$ is called a mild solution of system* (2) *if for each $v(\cdot) \in L_p([0,b]; U)$, $s(\tau)$ is measurable and the following stochastic integral equation is satisfied:*

$$\begin{aligned}
s(\tau) &= C_\alpha(\tau)(s_0 + n_0(s)) + S_\alpha(\tau)(s_1 + n_1(s)) + \int_0^\tau P_\alpha(\tau-\varsigma)Bv(\varsigma)d\varsigma \\
&+ \int_0^\tau P_\alpha(\tau-\varsigma)\eta(\varsigma, s(\varsigma))d\varsigma + \int_0^\tau P_\alpha(\tau-\varsigma)\sigma(\varsigma, s(\varsigma))dW(\varsigma)."
\end{aligned} \tag{8}$$

3. Existence and Uniqueness of Mild Solution

Existence and uniqueness results for the mild solution of semilinear control system (2) are obtained in this section. To obtain the results, certain conditions on nonlinear functions are imposed.

[$H1$] The functions $\tau \to \eta(\tau, s(\tau))$ and $\tau \to \sigma(\tau, s(\tau))$ for any $s \in G$ are measurable.

[$H2$] $\eta : [0,b] \times G \to G$, $\sigma : [0,b] \times G \to L(K,G)$ are functions satisfying Lipschitz conditions and linear growth condition. Also the functions η, σ are continuous. In general, we consider that there are positive constants L_η and L_σ such that

$$\begin{aligned}
\|\eta(t,s) - \eta(t,w)\|^2 &\leq K_1\|s-w\|^2, \quad \|\eta(t,s)\|^2 \leq K_2(1 + \|s\|^2), \\
\|\sigma(t,s) - \sigma(t,w)\|^2 &\leq N_1\|s-w\|^2, \quad \|\sigma(t,s)\|^2 \leq N_2(1 + \|s\|^2).
\end{aligned}$$

[$H3$] The functions $n_0(s)$ and $n_1(s)$ are continuous and there exists some positive constants M_{n_0} and M_{n_1} such that

$$\begin{aligned}
\|n_0(s) - n_0(w)\|^2 &\leq M_{n_0}\|s-w\|^2, \quad \|n_0(s)\|^2 \leq M_{n_0}(1 + \|s\|^2), \\
\|n_1(s) - n_1(w)\|^2 &\leq M_{n_1}\|s-w\|^2, \quad \|n_1(s)\|^2 \leq M_{n_1}(1 + \|s\|^2)
\end{aligned}$$

for all $s,\ w \in C([0,b], H)$.

[$H4$] The operator $B \in L_\infty([0,b]; L(U,G))$ and $\|B\|_\infty$ stands for the norm of operator B in the Banach space

$L_\infty([0,b];L(U,G))$.All operators which are bounded and linear and defined from U to V be denoted by $L(U,G)$.
[$H5$] The multivalued map $U(\cdot):[0,b]\rightrightarrows 2^U\setminus\{\emptyset\}$ has closed ,convex and bounded values. $U(\cdot)$ is graph measurable and $U(\cdot)\subseteq\aleph$, where $\aleph$ is a subset of U which is bounded.

Theorem 3.1. *For every control function $v(\cdot)\in U_{ad}$ and if assumptions $[H1]-[H5]$ holds then the system (2) has a unique mild solution in $[0,b]$.*
Proof: *Define an operator $\Phi:G_2\to G_2$ such that*

$$\begin{aligned}(\Phi s)(\tau) &= C_\alpha(\tau)(s_0+n_0(s))+S_\alpha(\tau)(s_1+n_1(s))+\int_0^\tau P_\alpha(\tau-\varsigma)[B(\varsigma)v(\varsigma)+\eta(\varsigma,s(\varsigma))]d\varsigma\\ &+ \int_0^\tau P_\alpha(\tau-\varsigma)\sigma(\varsigma,s(\varsigma))dW(\varsigma).\end{aligned}$$

Now we will establish the result which shows that the system (2) has mild solution as (8) on $[0,b]$. For this we will prove that in space G_2 Φ has a fixed point. Classical fixed point theorem for contractions is used to prove the result. First we will prove $\Phi(G_2)\subset G_2$. Let $s\in G_2$, then we have

$$\mathbb{E}\|(\Phi s)(\tau)\|^2\le 5[T_1+T_2+T_3+T_4+T_5] \tag{9}$$

Now

$$\begin{aligned}T_1 &= \mathbb{E}\|C_\alpha(\tau)(s_0+n_0(s))\|^2\\ &\le 2M^2(\|s_0\|^2+M_{n_0}(1+\|s\|^2_{H_2}))\end{aligned}$$

Similarly $T_2\le 2M^2(\|s_1\|^2+M_{n_1}(1+\|s\|^2_{H_2}))$. Next, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}T_3 &= \mathbb{E}\Big\|\int_0^\tau P_\alpha(\tau-\varsigma)B(\varsigma)v(\varsigma)d\varsigma\Big\|^2\\ &= \Big(\int_0^\tau \|P_\alpha(\tau-\varsigma)\|\ \|B(\varsigma)\|\mathbb{E}\|v(\varsigma)\|d\varsigma\Big)^2\\ &\le M_P^2\|B\|^2_\infty\Big[\int_0^\tau \mathbb{E}\|v(\varsigma)\|^2d\varsigma\Big]^2\\ &\le M_P^2\|B\|^2_\infty\Big[\Big(\int_0^\tau d\varsigma\Big)^{\frac{p-1}{p}}\Big(\int_0^\tau \|v(\varsigma)\|^p_U d\varsigma\Big)^{\frac{1}{p}}\Big]^2\\ &\le M_P^2\|B\|^2_\infty\|v\|^2_{L_p([0,b];U)}b^{\frac{2(p-1)}{p}}\end{aligned}$$

Using cauchy-Schwartz inequality and hypothesis [H2] implies that

$$\begin{aligned}T_4 &= \mathbb{E}\Big\|\int_0^\tau P_\alpha(\tau-\varsigma)\eta(\varsigma,s(\varsigma))d\varsigma\Big\|^2_G\\ &\le \mathbb{E}\Big(\int_0^\tau \|P_\alpha(\tau-\varsigma)\eta(\varsigma,s(\varsigma))\|_G d\varsigma\Big)^2\\ &\le M_p^2\mathbb{E}\Big(\int_0^\tau \|\eta(\varsigma,s(\varsigma))\|_G d\varsigma\Big)^2\end{aligned}$$

$$\begin{aligned}
T_4 &\le M_P^2 b \int_0^\tau \mathbb{E}\|\eta(\varsigma, s(\varsigma))\|_G^2 d\varsigma \\
&\le M_P^2 b \int_0^\tau K_2(1 + \mathbb{E}\|s(\varsigma)\|_G^2) d\varsigma \\
&\le M_P^2 b K_2 \int_0^\tau \Big(1 + \sup_{\varsigma\in[0,b]} \mathbb{E}\|s(\varsigma)\|_G^2\Big) d\varsigma \\
&\le M_P^2 b K_2 b (1 + \|s\|_{G_2}^2) \\
&= M_P^2 b^2 K_2 (1 + \|s\|_{G_2}^2)
\end{aligned}$$

and

$$\begin{aligned}
T_5 &= \mathbb{E}\Big\| \int_0^\tau P_\alpha(\tau-\varsigma)\sigma(\varsigma, s(\varsigma)) dW(\varsigma)\Big\|^2 \\
&\le \mathbb{E}\Big(\int_0^\tau \|P_\alpha(\tau-\varsigma)\sigma(\varsigma, s(\varsigma))\| d\varsigma\Big)^2 \\
&\le M_p^2 tr(Q) b \Big(\int_0^\tau \mathbb{E}\|\sigma(\varsigma, s(\varsigma))\|_Q^2 d\varsigma\Big) \\
&\le M_P^2 tr(Q) b N_2 b (1 + \|s\|_{G_2}^2) \\
&= M_P^2 tr(Q) b^2 K_2 (1 + \|s\|_{G_2}^2)
\end{aligned}$$

Thus (9) becomes

$$\mathbb{E}\|\Phi s)(\tau)\|^2 \le a + b\|s\|_{G_2}^2$$

where $a > 0$ and $b > 0$ are preferable constants. This leads to the result that Φ map G_2 into itself.
Next, we show that Φ is a contraction map. For $s, w \in G_2$, hypothesis (H2) and the Cauchy-Schwartz inequality yield that

$$\begin{aligned}
\|(\Phi s)(\tau) - (\Phi w)(\tau)|^2 &\le 4\mathbb{E}\|C_\alpha(\tau)(n_0(s) - n_0(w))\|^2 + 4\mathbb{E}\|S_\alpha(\tau)(n_1(s) - n_1(w))\|^2 \\
&+ 4\mathbb{E}\Big\| \int_0^\tau P_\alpha(\tau-\varsigma)[\eta(\varsigma, s(\varsigma)) - \eta(\varsigma, w(\varsigma)) d\varsigma\Big\|^2 \\
&+ 4\mathbb{E}\Big\| \int_0^\tau P_\alpha(\tau-\varsigma)[\sigma(\varsigma, s(\varsigma)) - \sigma(\varsigma, w(\varsigma)) dW(\varsigma)\Big\|^2 \\
&\le 4\Big(M^2(M_{n_0} + M_{n_1}) + M_P^2(K_1 + N_1 tr(Q))b^2\Big)\|s - w\|_{G_2}^2
\end{aligned}$$

Consequently if

$$4\Big(M^2(M_{n_0} + M_{n_1}) + M_P^2(K_1 + N_1 tr(Q))b^2\Big) < 1 \tag{10}$$

then it is clear that in G_2 , Φ has a fixed point which is unique and is a solution of (2). Continuing the above process on interval $[0, b^*]$, $[b^*, 2b^*], \ldots$ such that b^* satisfies (10), we can easily remove the extra condition on b.

To obtain the main results, we derive a priori estimate of mild solution for the system (2).

Lemma 3.2. *(A priori estimate). Consider that corresponding to control v, the mild solution of system (2) is system (8) on $[0, b]$. Then there exist a constant $C = C(v) > 0$ such that*

$$\mathbb{E}\|s(\tau)\|^2 \le C, \ \forall\ \tau \in [0, b].$$

Proof: *Using conditions H2 and Hölder's inequality, we obtain*

$$\begin{aligned}
\mathbb{E}\|s(\tau)\|^2 &\leq 5\mathbb{E}\|C_\alpha(\tau)(s_0+n_0(s))\|^2 + 5\mathbb{E}\|S_\alpha(\tau)(s_1+n_1(s))\|^2 + 5\mathbb{E}\left\|\int_0^\tau P_\alpha(\tau-\varsigma)Bv(\varsigma)d\varsigma\right\|^2 \\
&+ 5\mathbb{E}\left\|\int_0^\tau P_\alpha(\tau-\varsigma)\eta(\varsigma,s(\varsigma))d\varsigma\right\|^2 + 5\mathbb{E}\left\|\int_0^\tau P_\alpha(\tau-\varsigma)\sigma(\varsigma,s(\varsigma))dW(\varsigma)\right\|^2 \\
&\leq 5(2M^2(\|s_0\|^2 + M_{n_0}(1+\|s\|^2_{H_2})) + 5(2M^2(\|s_1\|^2 + M_{n_1}(1+\|s\|^2_{H_2})) \\
&+ 5M_P^2\|B\|^2_\infty\Big[\int_0^\tau \|v(\varsigma)\|d\varsigma\Big]^2 + 5M_P^2(K_2 + tr(Q)N_2)b\int_0^\tau \{1+\mathbb{E}\|s(\varsigma)\|^2\}d\varsigma \\
&\leq 5(2M^2(\|s_0\|^2 + M_{n_0}(1+\|s\|^2_{H_2})) + 5(2M^2(\|s_1\|^2 + M_{n_1}(1+\|s\|^2_{H_2})) \\
&+ 5M_P^2\|B\|^2_\infty\Big[\Big(\int_0^\tau d\varsigma\Big)^{\frac{p-1}{p}}\Big(\int_0^\tau \|v(\varsigma)\|^p_U d\varsigma\Big)^{\frac{1}{p}}\Big]^2 \\
&+ 5M_P^2(K_2 + tr(Q)N_2)b\int_0^\tau \{1+\mathbb{E}\|s(\varsigma)\|^2\}d\varsigma \\
&\leq 5(2M^2(\|s_0\|^2 + M_{n_0}(1+\|s\|^2_{H_2})) + 5(2M^2(\|s_1\|^2 + M_{n_1}(1+\|s\|^2_{H_2})) \\
&+ 5M_P^2\|B\|^2_\infty\|v\|^2_{L_p([0,b];U)}b^{\frac{2(p-1)}{p}} \\
&+ 5M_P^2(K_2 + tr(Q)N_2)b^2 + 5M_P^2(K_2 + tr(Q)N_2)b\int_0^\tau \mathbb{E}\|s(\varsigma)\|^2 d\varsigma.
\end{aligned}$$

Now using Gronwall's inequality, one can easily obtain the boundedness of s in G_2.

4. Existence of Fractional Optimal Control

Existence results for fractional stochastic optimal control are discussed in this section under certain assumptions:
Let the integrand be defined as:

$$L : [0,b] \times G \times U \to \mathbb{R} \cup \{\infty\}$$

Then the integrand L satisfies the following conditions:

(M1) The integrand $L : [0,b] \times G \times U \to \mathbb{R} \cup \{\infty\}$ is F_τ-measurable.
(M2) For $s \in G$, $\tau \in [0,b]$ the integrand $L(\tau,s,\cdot)$ is convex on U.
(M3) For almost all $\tau \in [0,b]$, the integrand $L(\tau,\cdot,\cdot)$ is sequentially lower semicontinuous on $G \times U$.
(M4) There exist l, j, α constants such that $l \in [0,\infty)$, $j \in (0,\infty)$, $\alpha \in [0,\infty)$ and $\alpha \in L_1([0,b];\mathbb{R})$ such that

$$L(\tau,s,v) \geq \alpha(\tau) + l\mathbb{E}\|s\|^2 + j\mathbb{E}\|v\|^p_U$$

Theorem 4.1. *Suppose* $(M1)-(M4)$ *holds and hypothesis of Theorem 3.1 is true, then there is a pair* $(s^0,v^0) \in \mathcal{A}_{ad}$, *i.e. atleast one optimal control exist corresponding to Lagrange problem (1) with*

$$J(s^0,v^0) := \mathbb{E}\Big\{\int_0^b L(\tau,s^0(\tau),v^0(\tau))d\tau\Big\} \leq J(s,v), \forall (s,v) \in \mathcal{A}_{ad}$$

Proof: *If greatest lower bound of* $\{J(s,v)|(s,v) \in \mathcal{A}_{ad}\}$ *is* $+\infty$, *then obviously we obtain the result. So, we will assume that greatest lower bound of* $\{J(s,v)|(s,v) \in \mathcal{A}_{ad}\}$ *as* $\delta < +\infty$. *From above* $(M1)-(M4)$ *condition, it is clear that* $\delta > -\infty$. *With the help of greatest lower bound there exist* $(s^m,v^m) \in \mathcal{A}_{ad}\}$ *a sequence of state-control pair such that* $J(s^m,v^m) \to \delta$ *as* $m \to +\infty$ *assuming* $(s^m,v^m) \in \mathcal{A}_{ad}\}$ *as minimizing sequence. We know* $L_p([0,b];U)$ *is a reflexive separable Banach space and* $\{v^m\}$ *is a bounded subset of* $L_p([0,b];U)$ *and also* $\{v^m\} \subseteq U_{ad} :\ m \in \mathbb{N}$, *so there is relabeled sequence* $\{v^m\}$ *and* $v^0 \in L_p([0,b];U)$ *such that* $v^m \to v^0$ *(weakly converges as* $m \to +\infty$*) in* $L_p([0,b];U)$. *As we know*

that the admissible set $U_{ad} \subset L_p([0,b];U)$ is bounded, closed and convex, so Mazur's lemma forces us to conclude that $v^0 \in U_{ad}$.
Now, let us assume that corresponding to sequence of controls $\{v^m\}$,the sequence of solutions of the system (2) be given by $\{s^m\}$, that is

$$\begin{aligned} s^m(\tau) &= C_\alpha(\tau)(s_0 + n_0(s^m)) + S_\alpha(\tau)(s_1 + n_1(s^m)) + \int_0^\tau P_\alpha(\tau-\varsigma)\{Bv^m(\varsigma) + \eta(\varsigma, s^m(\varsigma))\}d\varsigma \\ &+ \int_0^\tau P_\alpha(\tau-\varsigma)\sigma(\varsigma, s^m(\varsigma))dW(\varsigma). \end{aligned}$$

By Lemma 1, it is easy to see that there exists $\delta > 0$ such that

$$\mathbb{E}\|s^m\|^2 \le \delta, \quad m = 0, 1, 2, ...,$$

Let corresponding to the control $v^0 \in U_{ad}$ i, the mild solution for the system (2) be given as s^0, such that

$$\begin{aligned} s^0(\tau) &= C_\alpha(\tau)(s_0 + n_0(s^0)) + S_\alpha(\tau)(s_1 + n_1(s^0)) + \int_0^\tau P_\alpha(\tau-\varsigma)\{Bv^0(\varsigma) + \eta(\varsigma, s^0(\varsigma))\}d\varsigma \\ &+ \int_0^\tau P_\alpha(\tau-\varsigma)\sigma(\varsigma, s^0(\varsigma))dW(\varsigma). \end{aligned}$$

For all $\tau \in [0,b]$, using Hölder inequality,Cauchy-Schwarz inequality and condition$(M3)$, we get

$$\begin{aligned} \mathbb{E}\|s^m(\tau) - s^0(\tau)\|^2 &\le 5\mathbb{E}\|C_\alpha(\tau)[n_0(s^m) - n_0(s^0)]\|^2 \\ &+ 5\mathbb{E}\|S_\alpha(\tau)[n_1(s^m) - n_1(s^0)]\|^2 \\ &+ 5\mathbb{E}\Big\|\int_0^\tau P_\alpha(\tau-\varsigma)[B(\varsigma)v^m(\varsigma) - B(\varsigma)v^0(\varsigma)]d\varsigma\Big\|^2 \\ &+ 5\mathbb{E}\Big\|\int_0^\tau P_\alpha(\tau-\varsigma)[\eta(\varsigma, s^m(\varsigma)) - \eta(\varsigma, s^0(\varsigma))]d\varsigma\Big\|^2 \\ &+ 5\mathbb{E}\Big\|\int_0^\tau P_\alpha(\tau-\varsigma)[\sigma(\varsigma, s^m(\varsigma)) - \sigma(\varsigma, s^0(\varsigma))]dW(\varsigma)\Big\|^2 \\ &\le 5M^2(M_{n_0}^2 + M_{n_1}^2)\mathbb{E}\|s^m(\varsigma) - s^0(\varsigma)\|^2 \\ &+ 5M_P^2 b\Big(\int_0^\tau \|B(\varsigma)v^m(\varsigma) - B(\varsigma)v^0(\varsigma)\|^p d\varsigma\Big)^{\frac{2}{p}} \\ &+ 5M_P^2 b(K_1 + tr(Q)N_1)\int_0^\tau \mathbb{E}\|s^m(\varsigma)) - s^0(\varsigma))\|^2 d\varsigma. \end{aligned}$$

With the help of singular Gronwall's inequality, there exists a constant $K^*(\alpha)$ independent of v, m and τ such that

$$\begin{aligned} \mathbb{E}\|s^m(\tau) - s^0(\tau)\| &\le K^*(\alpha)\Big(\int_0^\tau \|B(\varsigma)v^m(\varsigma) - B(\varsigma)v^0(\varsigma)\|^p d\varsigma\Big)^{\frac{2}{p}} \\ &\le K^*(\alpha)\|Bv^m - Bv^0\|^2_{L_p([0,b];U)}. \end{aligned} \tag{11}$$

As B is strongly continuous, so

$$\|Bv^m - Bv^0\|_{L_p([0,b];U)} \to 0 \ \ as \ \ m \to \infty \tag{12}$$

From equation (11) and (12) , we conclude that

$$\mathbb{E}\|s^m_\tau - s^0_\tau\|^2 \to 0 \ \ as \ \ m \to \infty.$$

This implies that $\mathbb{E}\|s^m - s^0\|^2 \to 0$ *in* $C([0,b]; L_2(\Omega, G)]$ *as* $m \to \infty$.
It is clear that the assumptions $(M1) - (M4)$ *give the result of Balder (Theorem 2.1,[15]).Therefore, using Balder's theorem, we get*

$$(s,v) \to \mathbb{E}\int_0^{\tau} L(\tau, s(\tau), v(\tau))d\tau \tag{13}$$

in the strong topology of $L_1([0,b];G)$ *and weak topology of* $L_p([0,b];U) \subset L_1([0,b],U)$ *is sequentially lower semicontinuous. Therefore on* $L_p([0,b];U)$*,* J *is weakly lower semicontinuous and using condition* $(M4)$*, it is clear that* J *is greater than* $-\infty$*. This implies that* J *has its greatest lower bound at* $v^0 \in U_{ad}$*, that is,*

$$\begin{aligned} \epsilon &:= \lim_{m\to\infty} \mathbb{E}L(\tau, s^m(\tau), v^m(\tau))d\tau \\ &\geq \int_0^{\tau} \mathbb{E}L(\tau, s^0(\tau), v^0(\tau))d\tau = J(s^0, v^0) \geq \epsilon. \end{aligned}$$

This completes the proof.

5. Examples

Let $\Omega_1 \in \mathbb{R}^3$ be a bounded domain and $\partial\Omega \in \mathbb{C}^3$. Further let $G = U := L_2(\Omega_1)$, On a stochastic process $(\Omega, F, \mathbb{P})$ the standard cylindrical Wiener process in G is denoted by $W(\tau)$. Suppose $D(A) := G^2(\Omega_1) \cap G_0^1(\Omega_1)$ and for $x \in D(A)$, $Ax := \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)x$. The admissible control set $U_{ad} := \{v \in U : \|v\|_{L_p([0,1];U) \leq 1}\}$.

Let the control stochastic fractional order pde be given by:

$$\begin{aligned} {}^C D_\tau^{\frac{3}{2}} s(\tau, x) &= s_{xx}(\tau, x) + \int_0^1 \varrho(x,\nu)v(\nu,\tau)d\nu + \int_0^1 \kappa(x,\nu)\sin(s,\nu)d\nu \\ &+ \frac{(s(\tau,x))^2}{1 + (s(\tau,x))^2} dW(\tau), \\ s(\tau,x)|_{x\in\partial\Omega} &= 0, \ \tau > 0, \\ s(\tau,x) &= s_0(x) + \beta_1 s(\tau,x), \quad x \in \Omega_1, \\ \frac{\partial s}{\partial \tau}(0,x) &= s_1(x) + \beta_2 s(\tau,x), \ x \in \Omega_1. \end{aligned} \tag{14}$$

Define
$s(\tau)(x) = s(\tau,x)$, $(Bv)(\tau)(x) = \int_0^1 \varrho(x,\nu)v(\nu,\tau)d\nu$, $g(\tau,s(\tau))(x) = g(\tau,s(\tau,x)) = \int_0^1 \kappa(x,\nu)\sin(s,\nu)d\nu$, $\sigma(\tau,s(\tau))(x) = \sigma(\tau,s(\tau,x)) = \frac{(s(\tau,x))^2}{1+(s(\tau,x))^2}$, $s(0)(x) = s(0,x) = s_0(x)$ and $s'(0)(x) = s'(0,x) = s_1(x)$. β_1 and β_2 are finite constants. Moreover, we assume that $\varrho : \Omega_1 \times [0,1] \to \mathbb{R}$ is continuous. The function κ is measurable and $\int_{\Omega_1}\int_0^1 \kappa(x,\nu)d\nu dx < \infty$. The one-dimensional standard Brownian motion is denoted by $W(\tau)$. Thus for $\alpha = 3/2$ the problem (14) can be written as the abstract form of system (2) with the cost function

$$J(s,v) := \mathbb{E}\left\{\int_0^1 L(\tau, s(\tau), v(\tau))d\tau\right\},$$

where $L(\tau, s(\tau), v(\tau))(s) = \int_{\Omega_1} |s(\tau,x)|^2 dx + \int_{\Omega_1} |v(\tau,x)|^2 dx$. It is clear that the assumptions $(M1) - (M4)$ are satisfied. So,there exists an optimal control pair $(s^0, v^0) \in L_2([0,1] \times \Omega_1) \times L_2([0,1] \times \Omega_1)$ such that $J(s^0, v^0) \leq J(s,v)$ for all $(s,v) \in L_2([0,1] \times \Omega_1) \times L_2([0,1] \times \Omega_1)$.

6. Conclusion

In this paper fractional optimal control for stochastic semilinear equations in Hilbert space is considered. Under certain set of conditions, it is proved that lagrange's problem has atleast one optimal state-control pair. Using Banach fixed point theorem and priori estimate , existence and uniqueness conditions for mild solution are derived.

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