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Existence and Controllability Results for Integrodifferential Equations with State-Dependent Delay and Random Effects

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Abstract. The goal of this study is to investigate the existence and uniqueness of mild solutions, as well as controllability outcomes, for random integrodifferential equations with state-dependent delay. We prove the existence and uniqueness of mild solutions in the case where the nonlinear term is of the Carathéodory type and meets various weakly compactness conditions. Our research is based on Dardo's fixed point theorem, Mönch's fixed point theorem, a random fixed point with a stochastic domain, and Grimmer's resolvent operator theory. Finally, an example is provided to demonstrate the outcomes that were obtained.

1. Introduction

In this work, we deal with the random partial integro-differential equations with state dependent delay of the form :

$$\begin{cases} \vartheta'(t,\xi) = \mathcal{A}\vartheta(t,\xi) + \displaystyle\int_0^t \mathbf{\Gamma}(t-s)\vartheta(s,\xi)\mathsf{d}s + \mathsf{F}\Big(t, \vartheta_{\rho(t,\vartheta_t)}(\cdot,\xi),\xi\Big), \\ \qquad\qquad\qquad \text{a.e} \quad (t,\xi) \in [0,b]\times\Omega, \\ \vartheta(t,\xi) = \varphi(t,\xi), \qquad t\in(-\infty,0], \end{cases} \tag{1}$$

where the state $\vartheta(\cdot,\cdot)$ takes values in a separable Banach space Υ with norm $\|\cdot\|$, $(\Omega,\mathcal{F},\mathbb{P})$ is a complete probability space, $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \Upsilon \to \Upsilon$ is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ in Υ. Here $\mathbf{\Gamma}(t)$ is a closed linear operator on Υ, with domain $\mathcal{D}(\mathbf{\Gamma}(t)) \supset \mathcal{D}(\mathcal{A})$, which is independent of t. The time history $\vartheta_t(\cdot,\xi) : (-\infty,0] \to \Upsilon$ given by $\vartheta_t(r,\xi) = \vartheta(t+r,\xi)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically. The random nonlinear function $\mathsf{F} : [0,b]\times\mathcal{B}\times\Omega \to \Upsilon$, and $\rho : [0,b]\times\mathcal{B} \to (-\infty,0]$, are given functions to be specified later.

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The theory of functional differential equations has evolved as an important branch of nonlinear analysis. Differential delay equations, and more generally functional differential equations, have been employed in the modeling of scientific phenomena for a number of years. It is also assumed that the delay is either a fixed constant or an integral one; in the second case, the delay is referred to as the distributed delay [19, 21, 28, 32]. For evolution equations, a full theory has been constructed; for example, see [1, 14]. The outcomes of existence as well as uniqueness have recently been defined in the Benchohra and Baghli publications for infinite and finite delays for a variety of evolution issues (see [3, 4]). The nature of a complex system in engineering or natural science is determined by the precision with which the system's parameters are represented in the information. If the dynamic system information is accurate, a dynamic system information can arise. Unfortunately, much of the information used to describe and evaluate dynamic system parameters is unreliable, erroneous, or unclear. To put it another way, the determination of parameters in a complex system is not without its difficulties. For a system in which we have statistical knowledge of the parameters (information that is probabilistic), the standard approach for mathematical modeling of such systems is to use random differential equations or stochastic differential equations, which are both types of stochastic differential equations. Numerous applications of random differential equations, as fundamentally deterministic extensions, have been researched by a large number of writers; the readers are directed to monographs [6, 29, 30], papers [9, 12, 23, 24] and the relevant references for further information. There are some real-world phenomena that have anomalous dynamics, such as signal transmissions through strong magnetic fields, air emission diffusion, network traffic, the effect of betting on the profitability of stocks on financial markets, and so on, for which the classical models are insufficient to account for these characteristics.

Integrodifferential equations have become an active area of study due to their various applications in the fields like electrical engineering, mechanics, medical biolology, economical systems etc. During the last decades, many authors have investigated the existence, uniqueness, stability, controllability and others qualitative and quantitative properties for solutions of these equations by using fixed point technique and the theory of resolvent operator, which plays an important role in solving integrodifferential equations; see for example [10, 15, 16].

On the other hand, controllability is one of the fundamental qualitative features of a dynamical system, which means that it is possible to lead a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. As a result of its widespread application, the controllability of such systems has attracted an increasing amount of attention; for additional information, see [16, 25, 27]. Recently, Vijayn et al. [31] investigated the existence of approximate controllability of a random impulsive semilinear control system under sufficient conditions with a nondensely defined system. Aoued et al. [2] investigated the controllability of mild solutions for evolution equations with infinite state-dependent delay in the presence of infinite state-dependent delay. Chalishajar et al. [7] demonstrated the controllability of impulsive neutral evolution integrodifferential equations with state-dependent delays in Banach spaces by introducing a state-dependent delay into the equation. According to Kailasavalli et al. [22], the exact controllability of fractional neutral integro-differential systems with state-dependent delay in Banach spaces can be achieved by the use of state-dependent delay. In a recent paper, Diop et al. [11] investigated the existence, uniqueness, and controllability of solutions for stochastic partial integrodifferential equations with nonlocal conditions.

The primary purpose of this research, which is motivated by the previously listed publications, is to investigate the controllability of random nonlinear control systems. Currently, the study of the controllability of integrodifferential equations with state-dependent delay, as specified in the abstract form (1), is an unexplored issue in the literature, according to our knowledge. The following are the most significant contributions made by this work :

1. Integrodifferential system with random effects is formulated.
2. Resolvent operator theory is effectively used to derive sufficient conditions for the existence and controllability results by means of Darbo fixed-point Theorem and a random fixed point theorem with stochastic domain via the noncompactness measure.
3. Our work expands the usefulness of integrodifferential equations, since the literature shows results

for existence and controllability for such random equations in the case of semigroup only.

4. An example is provided to illustrate the obtained results.

The remainder of this paper is organized as follows: Section 2 provides preliminary details such as certain fundamental points, lemmas, and definitions. Section 3 ensures the existence of random mild solutions by using Darbo's fixed point Theorem and Grimmer's resolvent operator theory to guarantee the existence of sufficient conditions for their existence. Section 4 establishes the outcomes that are within control. In the final section, a model is offered in order to explain the theoretical conclusions that have been proposed.

2. Preliminaries

This section is concerned with some basic concepts, notations, definitions, lemmas, and preliminary facts, which are used through this work. Let $C([0,b],\Upsilon)$ be the Banach space of all continuous functions ϑ mapping from $[0,b]$ into Υ with the norm $\|\vartheta\|_C = \sup_{t\in[0,b]} \|\vartheta(t)\|$. Let $L^1([0,b],\Upsilon)$ be the space of Υ-valued Bochner integrable functions on $[0,b]$ with the norm $\|\vartheta\|_{L^1} = \int_0^b \|\vartheta(t)\|dt$. We will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [19] and follow the terminology used in [21]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty,0]$ into X, and satisfying the following axioms :

(A1) If $x : (-\infty,b) \to X$, $b > 0$, is continuous on $[0,b]$ and $x_0 \in \mathcal{B}$, then for every $t \in [0,b)$ the following conditions hold :

- (i) $x_t \in \mathcal{B}$;
- (ii) There exists a positive constant H such that $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$;
- (iii) There exist two functions $\gamma(\cdot), \lambda(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of x with γ continuous and λ locally bounded such that :

$$\|x_t\|_{\mathcal{B}} \leq \gamma(t)\sup\{\|x(s)\| : 0 \leq s \leq t\} + \lambda(t)\|x_0\|_{\mathcal{B}}.$$

(A2) For the function x in (A1), x_t is a $\mathcal{B}-$valued continuous function on $[0,b]$.

(A3) The space $\mathcal{B}$ is complete.

Remark 2.1. *In the sequel we assume that γ and λ are bounded on J and*

$$\kappa := \max\left\{\sup_{t\in\mathbb{R}_+}\{\gamma(t)\},\ \sup_{t\in\mathbb{R}_+}\{\lambda(t)\}\right\}.$$

For more details, we refer the reader to [21].

Definition 2.2. *A map $\mathsf{F} : I \times \mathcal{B} \times \Omega \to \Upsilon$ is said to be random Carathéodory if:*

1. $t \to \mathsf{F}\Big(t,z,\xi\Big)$ *is jointly measurable with respect to $(t,\xi) \in I \times \Omega$ for all $z \in \mathcal{B}$;*
2. $z \to \mathsf{F}\Big(t,z,\xi\Big)$ *is continuous for almost each $t \in I$ and all $\xi \in \Omega$.*

Definition 2.3 ([13]). *Let Υ be a separable Banach space with Borel σ-algebra B_Υ. A mapping $y : \Omega \to \Upsilon$ is said to be a random variable with values in Υ if for each $C \in \mathsf{B}_\Upsilon$, $y^{-1}(C) \in \mathcal{F}$.*

Definition 2.4 ([13]). *A mapping $\Upsilon : \Omega \times \Upsilon \to \Upsilon$ is called a random operator if $\Upsilon(\cdot,y)$ is measurable for each $y \in \Upsilon$ and is generally expressed as $\Upsilon(\xi,y) = \Upsilon(\xi)y$; we will use these two expressions interchangeably.*

Next, we provide a very helpful random fixed point Theorem with a stochastic domain.

Definition 2.5 ([13]). *Let* $\mathsf{D} : \Omega \to 2^{\mathbb{Y}}$*. A mapping* $\Upsilon : \{(\xi, y) : \xi \in \Omega$ *and* $y \in \mathsf{D}(\xi)\} \to \mathbb{Y}$ *is called a random operator with stochatic domain* D *if* D *is measurable (i.e, for all* $A \subseteq \mathbb{Y}$, $\{\xi \in \Omega : \mathsf{D}(\xi) \cap A \neq \emptyset\} \in \mathcal{F}$*) and for every open set* $O \subseteq \mathbb{Y}$ *and all* $y \in \mathbb{Y}$,

$$\left\{\xi \in \Omega : y \in \mathsf{D}(\xi) \text{ and } \Upsilon(\xi, y) \in O\right\} \in \mathcal{F}.$$

We say that Υ *is continuous if every* $\Upsilon(\xi)$ *is continuous.*

Definition 2.6 ([13]). *For a random operator* Υ*, a mapping* $y : \Omega \to \mathbb{Y}$ *is called a random (stochastic) fixed point of* Υ *if for* $\mathbb{P}$*-almost all* $\xi \in \Omega$*, we have* $y(\xi) \in \mathsf{D}(\xi)$, $\quad \Upsilon(\xi)y(\xi) = y(\xi)$*, and*

$$\{\xi \in \Omega : y(\xi) \in O\} \in \mathcal{F}$$

for every open set $O \subseteq \mathbb{Y}$ *(i.e, y is measurable).*

Lemma 2.7 ([13]). *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be complete,* $y_0 : \Omega \to \mathbb{Y}$ *and* $r : \Omega \to \mathbb{R}_+^*$ *be measurable. Then* $\mathsf{D} : \Omega \to 2^{\mathbb{Y}}$ *defined by*

$$\mathsf{D}(\xi) = \{y \in \mathbb{Y} : \|y - y_0(\xi)\| \leq r(\xi)\}$$

is a measurable multivalued mapping.

Lemma 2.8 ([13]). *Let* $\mathsf{D} : \Omega \to 2^{\mathbb{Y}}$ *be measurable with* $\mathsf{D}(\xi)$ *closed, convex and solid (i.e* $int(\mathsf{D}(\xi)) \neq \emptyset$*) for all* $\xi \in \Omega$*. Assume there exists a measurable random variable* $y_0 : \Omega \to \mathbb{Y}$ *with* $y_0(\xi) \in int(\mathsf{D}(\xi))$ *for all* $\xi \in \Omega$*. Let* Υ *be a continuous random operator with stochastic domain* D *such that for every* $\xi \in \Omega$,

$$\{y \in \mathsf{D}(\xi) : \Upsilon(\xi)y = y\} \neq \emptyset.$$

Then Υ *has a stochastic fixed point.*

Let x be a mapping of $[0, b] \times \Omega$ into $\mathbb{Y}$. x is said to be a stochastic process if for each $t \in [0, b]$, the function $x(t, \cdot)$ is measurable.

2.1. Noncompactness measure

We recall some fundamental definitions and lemmas on the measure of noncompactness. We first introduce the concept of Kuratowski's measure of noncompactness and its properties.

Definition 2.9 ([5]). *The Kuratowski measure of noncompactness* $\alpha(\cdot)$ *defined on bounded subset E of Banach space* $\mathbb{Y}$ *is*

$$\alpha(E) = \inf\left\{\epsilon > 0 \, : E = \cup_{i=1}^{k} E_i \text{ and } diam(E_i) \leq \epsilon\right\}.$$

Theorem 2.10 ([5, 8]). *Let* α *denote Kuratowski measure of noncompactness on the real Banach spaces* $\mathbb{Y}$ *and* $B, C \subseteq \mathbb{Y}$ *be bounded. The following properties are satisfied:*

(i) $C \subseteq B \implies \alpha(B) \leq \alpha(C)$ *(Monotonicity).*

(ii) $\alpha(B) = \alpha(\bar{B}) = \alpha(convB)$*, where* $\bar{B}$ *and convB are the closure and convex hull of B, respectively.*

(iii) *B is pre-compact if and only if* $\alpha(B) = 0$ *(Regularity).*

(iv) $\alpha(\lambda B) = |\lambda|\alpha(B)$ *for any* $\lambda \in \mathbb{R}$.

(v) $\alpha(B \cup V) \leq \max\{\alpha(B), \alpha(V)\}$.

(vi) $\alpha(B + \vartheta) = \alpha(B)$ *for all* $\vartheta \in \mathbb{Y}$.

(vii) $\alpha(B + C) \leq \alpha(B) + \alpha(C)$ *where* $B + C = \{x + y : \, x \in B, \, y \in C\}$.

(viii) *If the map* $Q : D(Q) \subseteq \Upsilon \to \mathbb{U}$ *is Lipschitz continuous with constant* κ*, then*

$$\alpha(QB) \leq \kappa\, \alpha(B)$$

for any bounded subset $B \subseteq D(Q)$.

More details on the Kuratowski measure of noncompacness can be found in Goebel [5] and Deimling [8].

The notation $\alpha(\cdot)$ and α_C stand for the Kuratowski measure of noncompactness on the bounded set of Υ and $C([0,b],\Upsilon)$, respectively. For any $V \subset C([0,b],\Upsilon)$ and $t \in [0,b]$, set $V(t) = \{\vartheta(t) : \vartheta \in V\}$ then $V(t) \subset \Upsilon$.

The next results play an important role in demonstrating our key findings.

Lemma 2.11 ([18]). *If* $B \subset C([0,b],\Upsilon)$ *is bounded and equicontinuous, then* $\alpha(B(t))$ *is continuous on* $[0,b]$ *and*

$$\alpha\left(\left\{\int_0^b y(s)\mathrm{d}s \;:\; y \in B\right\}\right) \leq \int_0^b \alpha(B(s))\mathrm{d}s,$$

where $B(s) = \{y(s) \;:\; y \in B\}$, $s \in [0,b]$.

Lemma 2.12 ([5]). *Let* $E \subset C([0,b],\Upsilon)$ *be bounded and equicontinuous. Then* $\alpha(E(t))$ *is continuous on* $[0,b]$*, and* $\alpha_C\Big(E\Big) = \max_{t\in[0,b]} \alpha(E(t))$.

To prove our existence results, we shall use the following famous Dardo's fixed point Theorem.

Lemma 2.13 ([18]). *Let* V *be a closed and convex subset of a real Banach space* Υ*. Suppose* $\mathrm{S} :\; V \to V$ *is a continuous operator and* $\mathrm{S}(V)$ *is bounded. If there exists a constant* $\delta \in [0,1)$ *such that for each bounded subset* $V_0 \subset V$*,*

$$\alpha(\mathrm{S}(V_0)) \leq \delta\; \alpha(V_0), \tag{2}$$

then S *has at least one fixed point in* V*.*

The following nonlinear fixed point Theorem-type alternative of Mönch plays an significant role in proving the main results of this work.

Lemma 2.14 ([26]). *Let* $\mathcal{O}$ *be a closed convex subset of a Banach space* $\mathcal{B}$ *and* $0 \in \mathcal{O}$*. Assume that* $\Psi : \mathcal{O} \to \mathcal{B}$ *is a continuous map which satisfies Mönch's condition, that is,*

$$\mathsf{N} \subseteq \mathcal{O} \text{ is countable, } \mathsf{N} \subseteq \overline{conv}(\{0\} \cup \Psi(\mathsf{N})) \Longrightarrow \overline{\mathsf{N}} \text{ is compact.}$$

Then Ψ *has a fixed point in* $\mathcal{O}$*.*

2.2. Integrodifferential equations in Banach spaces

We recall some knowledge on partial integrodifferential equations and the related resolvent operators. Let $\mathcal{D}$ be the Banach space $D(\mathcal{A})$ equipped with the graph norm defined by

$$\|\vartheta\|_{\mathcal{D}} := \|\mathcal{A}\vartheta\| + \|\vartheta\| \text{ for } \vartheta \in \mathcal{D}.$$

We denote by $C(\mathbb{R}^+,\mathcal{D})$, the space of all functions from $\mathbb{R}^+$ into $\mathcal{D}$ which are continuous. Let us consider the following system for further purposes :

$$\begin{cases} \vartheta'(t) &=& \mathcal{A}\vartheta(t) + \displaystyle\int_0^t \Gamma(t-s)\vartheta(s)\mathrm{d}s \quad \text{for } \; t \in [0,b], \\ \vartheta(0) &=& \vartheta_0 \in \Upsilon. \end{cases} \tag{3}$$

Definition 2.15 ([17]). *A resolvent operator for Eq.* (3) *is a bounded linear operator valued function* $\mathrm{R}(t) \in \mathcal{L}(\Upsilon)$ *for* $t \in [0,b]$*, having the following properties :*

(i) $\mathrm{R}(0) = I$ *(the identity map of* Υ*) and* $\|\mathrm{R}(t)\| \leq Ne^{\beta t}$ *for some constants* $N > 0$ *and* $\beta \in \mathbb{R}$.

(ii) *For each* $\vartheta \in \Upsilon$, $\mathrm{R}(t)\vartheta$ *is strongly continuous for* $t \in [0, b]$.

(iii) *For* $\vartheta \in \Upsilon$, $\mathrm{R}(\cdot)\vartheta \in C^1(\mathbb{R}^+; \Upsilon) \cap C(\mathbb{R}^+; \mathcal{D})$ *and*

$$\begin{aligned} \mathrm{R}'(t)\vartheta &= \mathcal{A}\mathrm{R}(t)\vartheta + \int_0^t \mathbf{\Gamma}(t-s)\mathrm{R}(s)\vartheta \mathrm{ds} \\ &= \mathrm{R}(t)\mathcal{A}\vartheta + \int_0^t \mathrm{R}(t-s)\mathbf{\Gamma}(s)\vartheta \mathrm{ds}, \quad \text{for } t \in [0, b]. \end{aligned}$$

In what follows, we make the following assumptions.

($\mathbf{R_1}$) The operator $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $(\mathsf{T}(t))_{t\in[0,b]}$ on Υ.

($\mathbf{R_2}$) For all $t \in [0, b]$, the operator $\mathbf{\Gamma}(t)$ is closed and linear from $D(\mathcal{A})$ to Υ and $\mathbf{\Gamma}(t) \in \mathcal{L}(\mathcal{D}, \Upsilon)$. For any $\vartheta \in \Upsilon$, the map $t \mapsto \mathbf{\Gamma}(t)\vartheta$ is bounded, differentiable and the derivative $t \mapsto \mathbf{\Gamma}'(t)\vartheta$ is bounded and uniformly continuous for $t \geq 0$. In addition, there is a function $\mu : [0, b] \to \mathbb{R}^+$ which is integrable such that for each $\vartheta \in \Upsilon$, the map $t \mapsto \mathbf{\Gamma}(t)\vartheta$ belongs to $\mathsf{W}^{1,1}(J, \Upsilon)$ and $\left\|\dfrac{d\mathbf{\Gamma}(t)\vartheta}{dt}\right\| \leq \mu(t)\|\vartheta\|$, $\vartheta \in \Upsilon, t \in [0, b]$.

Theorem 2.16. [17] *Assume that* ($\mathbf{R_1}$)-($\mathbf{R_2}$) *hold. Then there exists a unique resolvent operator to the Cauchy problem* (3)*.*

The following theorem gives the equivalence between the operator-norm continuity of the C_0-semigroup and the resolvent operator for integral equations.

Theorem 2.17 (Theorem 6, [16]). *Let* $\mathcal{A}$ *be the infinitesimal generator of a* C_0*-semigroup* $(T(t))_{t\geq 0}$ *and let* $(\mathbf{\Gamma}(t))_{t\geq 0}$ *satisfy* ($\mathbf{R_2}$)*. Then the resolvent operator* $(\mathrm{R}(t))_{t\geq 0}$ *for Eq.* (3) *is operator-norm continuous (or continuous in the uniform operator topology) for* $t > 0$ *if and only if* $(T(t))_{t\geq 0}$ *is operator-norm continuous for* $t > 0$.

3. Existence of mild solutions

Now we give our main existence result for problem (1). Before starting and proving this result, we give the definition of the random mild solution.

Definition 3.1. *A* Υ*-valued stochastic process* $\vartheta : [0, b] \times \Omega \to \Upsilon$ *is said to be a random mild solution of the random problem* (1)*, if*

1. $\vartheta(t, \xi) = \phi(t, \xi), \quad t \in (-\infty, 0]$;
2. *The restriction of* $\vartheta(\cdot, \xi)$ *to the interval* $[0, b]$ *is continuous and satisfies the following integral equation:*

$$\vartheta(t, \xi) = \mathrm{R}(t)\phi(0, \xi) + \int_0^t \mathrm{R}(t-s)\, \mathsf{F}(s, \vartheta_{\rho(s,\vartheta_s)}(\cdot, \xi), \xi)\mathrm{ds}. \tag{4}$$

Remark 3.2. *The random mild solution* $\vartheta(\cdot, \xi)$ *of the random problem* (1) *belongs to* $C([0, b], \Upsilon)$ *and is measurable with respect to the random parameter* $\xi \in \Omega$.

Set

$$\mathcal{R}_\rho^- := \{\rho(s,\ \psi) : \quad (s, \psi) \in J \times \mathcal{B}, \rho(t, \psi) \leq 0\}$$

We always assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous. Additionally, we introduce following hypothesis

($\mathbf{C}_\phi$) The function $t \to \phi_t$ is continuous from $\mathcal{R}_\rho^-$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}_\rho^- \to (0, \infty)$ such that $\|\phi_t\| \leq L^\phi(t)\|\phi\|_{\mathcal{B}}$ for all $t \in \mathcal{R}_\rho^-$.

Remark 3.3. *For more detailed information concerning* $(\mathbf{C}_\phi)$ *see ([21]).*

Lemma 3.4. [20] *Let* $\vartheta : (-\infty,\infty) \to X$ *continuous and bounded and* $\vartheta_0 = \phi$. *If* $(\mathbf{C}_\phi)$ *holds, then*

$$\|\vartheta_s\|_{\mathcal{B}} \leq (\lambda_b + L^\phi)\|\phi\|_{\mathcal{B}} + \gamma_b \sup\{\|\vartheta(\theta)\|;\ \theta \in [0, max\{0,s\}]\},\ s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup_{t\in\mathcal{R}(\rho^-)} L^\phi(t)$.

Now we introduce the following hypotheses used in our discussions:

$(\mathbf{C_1})$ The semigroup $(S(t))_{t\geq 0}$ is norm continuous for $t > 0$.

$(\mathbf{C_2})$ For each $\xi \in \Omega$, $\phi(\cdot,\xi)$ is continous and for each t, $\phi(t,\cdot)$ is measurable.

$(\mathbf{C_3})$ **(i)** The nonlinear function F is random Carathéodory.

(ii) There exist functions $\widetilde{K_1} : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ and $\mathsf{f} : [0,b] \times \Omega \to \mathbb{R}_+$ such that for each $\xi \in \Omega$, $\widetilde{K_1}(\cdot,\xi)$ is a continuous nondecreasing function and $\mathsf{f}(\cdot,\xi) \in L^1([0,b],\mathbb{R}_+)$ such that

$$\|\mathsf{F}(t,\psi,\xi)\| \leq \mathsf{f}(t,\xi)\,\widetilde{K_1}\,(\|\psi\|_{\mathcal{B}},\xi)\ \text{ for a.e } t \in [0,b] \text{ and each } \xi \in \Omega.$$

(iii) There exists a function $\widetilde{K_2} : [0,b] \times \Omega \to \mathbb{R}_+$ with $K_2(\cdot,\xi) \in L^1([0,b],\mathbb{R}_+)$ for each $\xi \in \Omega$ such that for any bounded $V \subset \Upsilon$,

$$\alpha\,(\mathsf{F}(t,V,\xi)) \leq \widetilde{K_2}(t,\xi)\ \alpha(V),\ t \in \mathrm{R}_\rho^-.$$

$(\mathbf{C_4})$ There exists a random function $\mathsf{q} : \Omega \to \mathbb{R}_+\backslash\{0\}$ such that:

$$\sigma\|\phi\|_{\mathcal{B}} + \sigma\widetilde{K_1}\Big([\kappa + L^\phi]\,\|\phi\|_{\mathcal{B}} + \kappa\,\mathsf{q}(\xi),\ \xi\Big)\int_0^b \mathsf{f}(s,\xi)\mathrm{d}s \leq \mathsf{q}(\xi).$$

Theorem 3.5. *Assume that* $(\mathbf{R_1}), (\mathbf{R_2}), (\mathbf{C}_\phi)$ *and* $(\mathbf{C_1}) - (\mathbf{C_4})$ *hold. If the resolvent operator* $(\mathrm{R}(t))_{t\geq 0}$ *is operator norm-continuous for* $t > 0$, *then the random problem* (1) *has at least one random mild solution.*

Proof. Let $\chi_0 = \{\vartheta(\cdot,\xi) \in C([0,b],\Upsilon)\ :\ \vartheta(0,\xi) = \phi(0,\xi) = 0\}$ endowed with the norm

$$\|\vartheta\|_{\chi_0} = \sup_{t\in[0,b]} \|\vartheta(t,\xi)\| + \|\vartheta(0,\xi)\|_{\mathcal{B}} = \sup_{t\in[0,b]} \|\vartheta(t,\xi)\|,$$

$\overline{\vartheta}\ :\ (-\infty,b] \times \Omega \to \Upsilon$ defined by

$$\overline{\vartheta}(t,\xi) = \begin{cases} \overline{\vartheta}_0(t,\xi) = \phi(t,\xi) & \text{if} \quad t \in (-\infty,0], \\ \vartheta(t,\xi) & \text{if} \quad t \in [0,b], \end{cases}$$

$\overline{\phi}\ :\ (-\infty,b] \times \Omega \to \Upsilon$ defined by

$$\overline{\phi}(t,\xi) = \begin{cases} \phi(t,\xi) & \text{if} \quad t \in (-\infty,0], \\ 0 & \text{if} \quad t \in [0,b], \end{cases}$$

and the operator $\Upsilon : \Omega \times \chi_0 \to \chi_0$ defined by

$$\Big(\Upsilon(\xi)\vartheta\Big)(t) = \mathrm{R}(t)\phi(0,\xi) + \int_0^t \mathrm{R}(t-s)\ \mathsf{F}(s,\overline{\vartheta}_{\rho(t,\overline{\vartheta}_t)},\xi)\mathrm{d}s, \qquad t \in [0,b]. \tag{5}$$

Step 1. Υ is a random variable with stochastic domain.

We only need to prove that for any $\vartheta \in \chi_0$, $\Upsilon(\cdot)(\vartheta) : \Omega \to \chi_0$ is a random variable. From $(\mathbf{C_3}) - (ii)$, we know that $\mathsf{F}(t, \vartheta, \cdot)$, $t \in [0, b]$, $\vartheta \in \chi_0$ is measurable and from $(\mathbf{C_2})$, $\phi(t, \cdot)$ is measurable, then we deduce that $\Upsilon(\cdot)(\vartheta)$ is measurable. Let $\mathsf{D} : \Omega \to 2^{\chi_0}$ be defined by $\mathsf{D}(\xi) = \{\vartheta \in \chi_0 : \quad \|\vartheta\| \leq \mathsf{q}(\xi)\}$.

The set $\mathsf{D}(\xi)$ is bounded, closed, convex and solid for all $\xi \in \Omega$. Using Lemma 2.7, we deduce that D is measurable. For each $\vartheta \in \mathsf{D}(\xi)$, using Lemma 3.4 and hypotheses $(\mathbf{C_3})$ and $(\mathbf{C_4})$, we get for each $t \in [0, b]$

$$\begin{aligned}
\|(\Upsilon(\xi)\vartheta)(t)\| &\leq \sigma\|\phi\|_{\mathcal{B}} + \sigma \int_0^t \|\mathsf{F}\Big(s, \overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}, \xi\Big)\mathrm{d}s \\
&\leq \sigma\|\phi\|_{\mathcal{B}} + \sigma \int_0^t \mathsf{f}(s,\xi)\widetilde{K_1}\Big(\|\overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}\|_{\mathcal{B}}, \xi\Big)\mathrm{d}s \\
&\leq \sigma\|\phi\|_{\mathcal{B}} + \sigma \int_0^t \mathsf{f}(s,\xi)\widetilde{K_1}\Big([\kappa + L^{\phi}]\,\|\phi\|_{\mathcal{B}} + \kappa\,\mathsf{q}(\xi), \xi\Big)\mathrm{d}s \\
&\leq \sigma\|\phi\|_{\mathcal{B}} + \sigma\widetilde{K_1}\Big((\kappa + L^{\phi})\,\|\phi\|_{\mathcal{B}} + \kappa\,\mathsf{q}(\xi), \xi\Big) \int_0^b \mathsf{f}(s,\xi)\mathrm{d}s \\
&\leq \mathsf{q}(\xi),
\end{aligned}$$

which implies that Υ is a random operator with stochastic domain D and $\Upsilon(\xi) : \mathsf{D}(\xi) \to \mathsf{D}(\xi)$ for each $\xi \in \Omega$.

Step 2. Υ is continuous.

Let $(\vartheta^{(n)})_{n\in\mathbb{N}}$ be a sequence in χ_0 such that $\vartheta^{(n)} \to \vartheta$ in χ_0. We have that

$$\|(\Upsilon(\xi)\vartheta^{(n)})(t) - (\Upsilon(\xi)\vartheta)(t)\| \;\leq\; \sigma \int_0^t \|\mathsf{F}(s, \overline{\vartheta^{(n)}}_{\rho(s,\overline{\vartheta_s^{(n)}})}, \xi) - \mathsf{F}(s, \overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}, \xi)\|\mathrm{d}s. \tag{6}$$

Hence, since the function F is Carathédory, the Lebesgue dominated convergence Theorem implies that $\|\Upsilon(\xi)\vartheta^{(n)} - \Upsilon(\xi)\vartheta\| \to 0$ as $n \to +\infty$. This implies that Υ is continuous.

Step 3. For every $\xi \in \Omega$, $\{\vartheta \in \mathsf{D}(\xi) \,:\, \Upsilon(\xi)\vartheta = \vartheta\} \neq \emptyset$.

Consider the measure of noncompactness $\overline{\alpha}(\cdot)$ defined on the family of bounded subsets of the space $C([0, b], \mathbb{Y})$ by

$$\overline{\alpha}(B) = \sup_{t\in[0,b]} e^{-n_0\,E(t,\xi)}\alpha(B(t)),$$

where $E(t, \xi) = \sigma \int_0^t \widetilde{K_2}(s, \xi)\gamma(s)\mathrm{d}s$. First, let prove that $\Upsilon(\mathsf{D}(\xi))$ is equicontinuous. Let $t_1, t_2 \in [0, b]$ with $t_1 < t_2$ and $\vartheta \in \mathsf{D}(\xi)$. Then, we have

$$\begin{aligned}
&\|(\Upsilon(\xi)\vartheta)(t_2) - (\Upsilon(\xi)\vartheta)(t_1)\| \\
\leq\;& \|\mathrm{R}(t_2)\phi(0,\xi) - \mathrm{R}(t_1)\phi(0,\xi)\| \\
&+\Big\| \int_0^{t_2} \mathrm{R}(t_2 - s)\mathsf{F}(s, \overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}, \xi)\mathrm{d}s - \int_0^{t_1} \mathrm{R}(t_1 - s)\mathsf{F}(s, \overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}, \xi)\mathrm{d}s\Big\| \\
\leq\;& \|\mathrm{R}(t_2)\phi(0,\xi) - \mathrm{R}(t_1)\phi(0,\xi)\| \\
&+ \int_0^{t_1} \|\mathrm{R}(t_2 - s) - \mathrm{R}(t_1 - s)\|\,\|\mathsf{F}(s, \overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}, \xi)\|\mathrm{d}s \\
&+ \int_{t_1}^{t_2} \|\mathrm{R}(t_2 - s)\|\,\|\mathsf{F}(s, \overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}, \xi)\|\mathrm{d}s \\
\leq\;& \|\mathrm{R}(t_2)\phi(0,\xi) - \mathrm{R}(t_1)\phi(0,\xi)\| \\
&+\widetilde{K_1}\Big((\kappa + L^{\phi})\,\|\phi\|_{\mathcal{B}} + \kappa\,\mathsf{q}(\xi), \xi\Big) \int_0^{t_1} \|\mathrm{R}(t_2 - s) - \mathrm{R}(t_1 - s)\|\,\mathsf{f}(s,\xi)\mathrm{d}s \\
&+\widetilde{K_1}\Big((\kappa + L^{\phi})\,\|\phi\|_{\mathcal{B}} + \kappa\,\mathsf{q}(\xi), \xi\Big) \int_{t_1}^{t_2} \|\mathrm{R}(t_2 - s)\|\,\mathsf{f}(s,\xi)\mathrm{d}s.
\end{aligned} \tag{7}$$

By the continuity of $(\mathsf{R}(t))_{t\geq 0}$ in the operator-norm topology and the dominated convergence theorem, we conclude that the right hand side of the above inequality tends to zero and independent of ϑ as $t_1 \to t_2$. Hence $\Upsilon(\mathsf{D}(\xi))$ is equicontinuous.
Secondly, let show that there exists a constant $\delta \in [0,1[$ such that $\overline{\alpha}(\Upsilon B) \leq \delta\,\overline{\alpha}(B)$ for $B \subset \mathsf{D}(\xi)$. Since (ΥB) is equicontinuous, using the properties of $\alpha(\cdot)$, Lemma 2.11 and hypothesis $(\mathbf{C_3})$, we get that

$$
\begin{aligned}
\alpha((\Upsilon B)(t)) &\leq \alpha\left(\int_0^t \mathsf{R}(t-s)\mathsf{F}(s,\overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)},\xi)\mathrm{d}s \,:\, \overline{\vartheta}\in B\right)\\
&\leq \sigma\int_0^t \alpha\left(\mathsf{F}(s,\overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)},\xi) \,:\, \overline{\vartheta}\in B\right)\mathrm{d}s\\
&\leq \sigma\int_0^t \widetilde{K_2}(s,\xi)\alpha\left(\overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)} \,:\, \overline{\vartheta}\in B\right)\mathrm{d}s\\
&\leq \sigma\int_0^t \widetilde{K_2}(s,\xi)\gamma(s)\sup_{\tau\in[0,s]}\alpha\left(B(\tau)\right)\mathrm{d}s\\
&\leq \int_0^t e^{-n_0\,E(s,\xi)}\,e^{n_0\,E(s,\xi)}\,[\sigma\,\widetilde{K_2}(s,\xi)\gamma(s)]\sup_{s\in[0,t]}\alpha\left(B(s)\right)\mathrm{d}s\\
&\leq \int_0^t e^{-n_0\,E(s,\xi)}\,e^{n_0\,E(s,\xi)}\,[\sigma\,\widetilde{K_2}(s,\xi)\gamma(s)]\sup_{s\in[0,t]}\alpha\left(B(s)\right)\mathrm{d}s\\
&\leq \int_0^t e^{n_0\,E(s,\xi)}\,[\sigma\,\widetilde{K_2}(s,\xi)\gamma(s)]\sup_{t\in[0,b]}e^{-n_0\,E(t,\xi)}\alpha\left(B(t)\right)\mathrm{d}s\\
&\leq \sup_{t\in[0,b]}e^{-n_0\,E(t,\xi)}\alpha\left(B(t)\right)\int_0^t\left(\frac{e^{n_0\,E(s,\xi)}}{n_0}\right)'\mathrm{d}s\\
&\leq \overline{\alpha}(B)\frac{e^{n_0\,E(t,\xi)}}{n_0}.
\end{aligned}
\tag{8}
$$

Therefore, we have

$$e^{-n_0\,E(t,\xi)}\alpha((\Upsilon B)(t)) \leq \frac{1}{n_0}\overline{\alpha}(B), \tag{9}$$

then

$$\overline{\alpha}((\Upsilon B)) \leq \frac{1}{n_0}\overline{\alpha}(B). \tag{10}$$

This conclude that $\Upsilon(\xi)$ is a set contraction. Consequently, by Darbo fixed point theorem, we deduce that $\Upsilon(\xi)$ has at least a fixed point $\vartheta \in \mathsf{D}(\xi)$. Thus for every $\xi \in \Omega$, $\{\vartheta \in \mathsf{D}(\xi) \,:\, \Upsilon(\xi)\vartheta = \vartheta\} \neq \emptyset$. Since $\cap_{\xi\in\Omega}\mathsf{D}(\xi) \neq \emptyset$, then $\mathrm{int}(\mathsf{D}(\xi)) \neq \emptyset$ and there exists a measurable random variable $\vartheta_0 : \Omega \to \mathrm{Y}$ with $\vartheta_0(\xi) \in \mathrm{int}(\mathsf{D}(\xi))$. By Lemma 2.8, we deduce that the random operator Υ has a stochastic fixed point $\vartheta^*(\xi)$ which is a mild solution of Eq. (1).

□

4. Controllability results

In this section, we study the controllability results for a class of nonlinear random systems with state-dependent delay. We consider the following controlled system of the form

$$\begin{cases} \vartheta'(t,\xi) &= \mathcal{A}\vartheta(t,\xi) + \displaystyle\int_0^t \Gamma(t-s)\vartheta(s,\xi)\mathrm{d}s + \mathsf{F}\Big(t, \vartheta_{\rho(t,\vartheta_t)}(\cdot,\xi),\xi\Big) + \Xi \mathrm{u}(t,\xi), \\ & \qquad\qquad\qquad \text{a.e} \quad (t,\xi)\in[0,b]\times\Omega, \\ \vartheta(t,\xi) &= \varphi(t,\xi), \qquad t\in(-\infty,0], \end{cases} \tag{11}$$

where $\mathcal{A}$, Γ, F and ρ are defined as in Eq. (1). Here $\mathrm{u}(\cdot,\xi)$ is the control function which takes values in $L^2([0,b],\mathbb{U})$, a Banach space of admissible control functions with $\mathbb{U}$ as a Banach space. $\Xi:\mathbb{U}\to\Upsilon$ is a bounded linear operator.

Definition 4.1. *A Υ-valued stochastic process $\vartheta:[0,b]\times\Omega\to\Upsilon$ is called a random mild solution of the random problem* (11)*, if*

1. $\vartheta(t,\xi)=\phi(t,\xi), \quad t\in(-\infty,0]$;
2. *The restriction of $\vartheta(\cdot,\xi)$ to the interval $[0,b]$ is continuous and satisfies the following integral equation:*

$$\vartheta(t,\xi) = \mathrm{R}(t)\phi(0,\xi) + \int_0^t \mathrm{R}(t-s)\mathsf{F}(s,\vartheta_{\rho(s,\vartheta_s)}(\cdot,\xi),\xi)\mathrm{d}s + \int_0^t \mathrm{R}(t-s)\Xi\mathrm{u}(s,\xi)\mathrm{d}s. \tag{12}$$

Definition 4.2. *The system* (11) *is controllable on the intervalle $[0,b]$, if there exists a random control $\mathrm{u}(\cdot,\xi)\in L^2([0,b],\mathbb{U})$ such that the solution $\vartheta(\cdot,\cdot)$ of* (11) *satisfies $\vartheta(b,\xi)=\vartheta^{(\mathbf{b})}$ where $\vartheta^{(\mathbf{b})}$ and b are preassigned terminal state and time respectively.*

Furthermore, we assume the following conditions:

($\mathbf{C_5}$) The linear operator $W: L^2([0,b])\to\Upsilon$ defined by

$$W\mathrm{u} = \int_0^b \mathrm{R}(b-s)\Xi\mathrm{u}(s,\xi)\mathrm{d}s,$$

has an inverse operator W^{-1} which takes values in $L^2(J,\mathbb{U})/\mathrm{Ker}W$ and there exists a positive constant M_Ξ such that $\|\Xi W^{-1}\|\le M_\Xi$

($\mathbf{C_6}$) There exists a random function $\widetilde{q}:\Omega\to\mathbb{R}_+\backslash\{0\}$ such that

$$b\sigma M_\Xi\|\vartheta^{(\mathbf{b})}\| + \sigma(1+b\sigma M_\Xi)\left(\|\phi\|_{\mathcal{B}} + \widetilde{K_1}\Big([\kappa+L^\phi]\|\phi\|_{\mathcal{B}} + \kappa\widetilde{q}(\xi),\ \xi\Big)\int_0^b \mathsf{f}(s,\xi)\mathrm{d}s\right) \le \widetilde{q}(\xi).$$

Theorem 4.3. *Assume that* $(\mathbf{R_1})-(\mathbf{R_2})$,$(\mathbf{C_\phi})$,$(\mathbf{C_1})-(\mathbf{C_3})$, $(\mathbf{C_5})$, *and* $(\mathbf{C_6})$ *hold and the resolvant operator* $(\mathrm{R}(t))_{t\ge0}$ *is continuous in the operator-norm topology for $t>0$. Then, the random problem* (11) *is controllable on* $[0,b]$ *provided that*

$$p_0 = \sigma(1+b\sigma M_\Xi)\int_0^b \widetilde{K_2}(s,\xi)\gamma(s)\mathrm{d}s < 1\ . \tag{13}$$

Proof. Using the hypothesis ($\mathbf{C_5}$), for an arbitrary function $\vartheta(\cdot,\cdot)$, we define the following control

$$\mathrm{u}_\vartheta(t,\xi) = W^{-1}\left(\vartheta^{(\mathbf{b})}(\xi) + \mathrm{R}(b)\phi(0,\xi) + \int_0^b \mathrm{R}(b-s)\mathsf{F}(s,\vartheta_{\rho(s,\vartheta_s)}(\cdot,\xi),\xi)\mathrm{d}s\right)(t,\xi)\ . \tag{14}$$

Using this control, we shall show that the operator Υ defined by

$$\begin{aligned} &(\Upsilon(\xi)\vartheta)(t)\\ =\;& \mathrm{R}(t)\phi(0,\xi) + \int_0^t \mathrm{R}(t-s)\mathsf{F}(s,\vartheta_{\rho(s,\vartheta_s)}(\cdot,\xi),\xi)\mathrm{d}s\\ &+\int_0^t \mathrm{R}(t-s)\Xi\, W^{-1}\left(\vartheta^{(\mathbf{b})}(\xi)+\mathrm{R}(b)\phi(0,\xi)+\int_0^b \mathrm{R}(b-r)\mathsf{F}(r,\vartheta_{\rho(r,\vartheta_r)}(\cdot,\xi),\xi)\mathrm{d}r\right)(s,\xi)\mathrm{d}s \end{aligned} \tag{15}$$

has a fixed point $\vartheta(\cdot,\cdot)$, which means that system (11) is controllable on $[0,b]$.
Let $\chi_0=\{\vartheta\in C([0,b],\Upsilon)\;:\;\vartheta(0,\xi)=\phi(0,\xi)\}$ endowed with the sup-norm and $\Lambda:\Omega\times\chi_0\to\chi_0$ be the random operator defined by :

$$\begin{aligned} &(\Lambda(\xi)\vartheta)(t)\\ =\;& \mathrm{R}(t)\phi(0,\xi) + \int_0^t \mathrm{R}(t-s)\mathsf{F}(s,\overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}(\cdot,\xi),\xi)\mathrm{d}s\\ &+\int_0^t \mathrm{R}(t-s)\Xi\, W^{-1}\left(\vartheta^{(\mathbf{b})}(\xi)+\mathrm{R}(b)\phi(0,\xi)+\int_0^b \mathrm{R}(b-r)\mathsf{F}(r,\overline{\vartheta}_{\rho(r,\overline{\vartheta}_r)}(\cdot,\xi),\xi)\mathrm{d}r\right)(s)\mathrm{d}s \end{aligned} \tag{16}$$

where $\overline{\vartheta}:(-\infty,b]\times\Omega\to\Upsilon$ is such that $\overline{\vartheta_0}(\cdot,\xi)=\phi(\cdot,\xi)$ and $\overline{\vartheta}(\cdot,\xi)=\vartheta(\cdot,\xi)$ on $[0,b]$. Let $\overline{\phi}:(-\infty,b]\times\Omega\to\Upsilon$ be the extention of ϕ to $(-\infty,0]$ such that $\overline{\phi}(t,\xi)=\phi(0,\xi)$ on $[0,b]$. We split the proof in several steps :

Step 1. Λ is a random variable with stochastic domain.

By the measurability of mappings $\mathcal{F}(t,\vartheta,\cdot)$, $t\in[0,b]$, $\vartheta\in\chi_0$ and $\phi(t,\cdot)$, $t\in[0,b]$, we obtain that for any $\vartheta\in\chi_0$, $\Lambda(\cdot)(\vartheta):\Omega\to\chi_0$ is a random variable. Let $\mathsf{D}:\Omega\to 2^{\chi_0}$ be defined by $\mathsf{D}(\xi)=\{\vartheta\in\chi_0\;:\;\|\vartheta\|\le\widetilde{q}(\xi)\}$. The set $\mathsf{D}(\xi)$ is bounded, closed, convex and solid for all $\xi\in\Omega$. Then by Lemma 2.7, D is measurable. Let $\xi\in\Omega$ be fixed, for each $\vartheta\in\mathsf{D}(\xi)$ and $t\in[0,b]$, we get

$$\begin{aligned} &\|(\Lambda(\xi)\vartheta)(t)\|\\ \le\;& \|\mathrm{R}(t)\phi(0,\xi)\| + \int_0^t \|\mathrm{R}(t-s)\mathsf{F}(s,\overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}(\cdot,\xi),\xi)\|\mathrm{d}s\\ &+\int_0^t \left\|\mathrm{R}(t-s)\Xi\, W^{-1}\left(\vartheta^{(\mathbf{b})}(\xi)+\mathrm{R}(b)\phi(0,\xi)+\int_0^b \mathrm{R}(b-r)\mathsf{F}(r,\overline{\vartheta}_{\rho(r,\overline{\vartheta}_r)}(\cdot,\xi),\xi)\mathrm{d}r\right)(s)\right\|\mathrm{d}s\\ \le\;& \sigma\|\phi\|_{\mathcal{B}}+\sigma\int_0^t \|\mathsf{F}(s,\overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}(\cdot,\xi),\xi)\|\mathrm{d}s\\ &+\sigma M_{\Xi}\int_0^t \|\left(\|\vartheta^{(\mathbf{b})}(\xi)\|+\sigma\|\phi\|_{\mathcal{B}}+\sigma\int_0^b \|\mathsf{F}(r,\overline{\vartheta}_{\rho(r,\overline{\vartheta}_r)}(\cdot,\xi),\xi)\|\mathrm{d}r\right)\mathrm{d}s\\ \le\;& \sigma\|\phi\|_{\mathcal{B}}+\sigma\int_0^t \mathsf{f}(s,\xi)\widetilde{K_1}(\|\overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}\|_{\mathcal{B}},\xi)\|\mathrm{d}s\\ &+\sigma M_{\Xi}\int_0^t \left(\|\vartheta^{(\mathbf{b})}(\xi)\|+\sigma\|\phi\|_{\mathcal{B}}+\sigma\int_0^b \mathsf{f}(r,\xi)\widetilde{K_1}(\|\overline{\vartheta}_{\rho(r,\overline{\vartheta}_r)}\|_{\mathcal{B}},\xi)\mathrm{d}r\right)\mathrm{d}s\\ \le\;& \sigma\|\phi\|_{\mathcal{B}}+\sigma\int_0^t \mathsf{f}(s,\xi)\widetilde{K_1}\Big([\kappa+L^{\phi}]\,\|\phi\|_{\mathcal{B}}+\kappa\,\widetilde{\mathsf{q}}(\xi),\,\xi\Big)\mathrm{d}s\\ &+\sigma M_{\Xi}\int_0^t \left(\|\vartheta^{(\mathbf{b})}(\xi)\|+\sigma\|\phi\|_{\mathcal{B}}+\sigma\int_0^b \mathsf{f}(r,\xi)\widetilde{K_1}\Big([\kappa+L^{\phi}]\,\|\phi\|_{\mathcal{B}}+\kappa\,\widetilde{\mathsf{q}}(\xi),\,\xi\Big)\mathrm{d}r\right)\mathrm{d}s\\ \le\;& \sigma\|\phi\|_{\mathcal{B}}+\sigma\widetilde{K_1}\Big([\kappa+L^{\phi}]\,\|\phi\|_{\mathcal{B}}+\kappa\,\widetilde{\mathsf{q}}(\xi),\,\xi\Big)\int_0^b \mathsf{f}(s,\xi)\mathrm{d}s+\sigma M_{\Xi}\, b\,\|\vartheta^{(\mathbf{b})}(\xi)\|\\ &+\sigma^2 M_{\Xi}\, b\,\|\phi\|_{\mathcal{B}}\;+\sigma^2 M_{\Xi}\, b\widetilde{K_1}\Big([\kappa+L^{\phi}]\,\|\phi\|_{\mathcal{B}}+\kappa\,\widetilde{\mathsf{q}}(\xi),\,\xi\Big)\int_0^b \mathsf{f}(r,\xi)\mathrm{d}r \end{aligned} \tag{17}$$

$$
\begin{aligned}
&\le \sigma(1+\sigma M_{\Xi}\, b)\widetilde{K_1}\Big([\kappa+L^{\phi}]\,\|\phi\|_{\mathcal{B}}+\kappa\,\widetilde{\mathsf{q}}(\xi),\ \xi\Big)\int_0^b \mathsf{f}(s,\xi)\mathrm{d}s \\
&\quad +\sigma M_{\Xi}\, b\,\|\vartheta^{(\mathbf{b})}(\xi)\|+\sigma(1+\sigma M_{\Xi}\, b)\|\phi\|_{\mathcal{B}} \\
&\le \sigma(1+\sigma M_{\Xi}\, b)\left[\|\phi\|_{\mathcal{B}}+\widetilde{K_1}\Big([\kappa+L^{\phi}]\,\|\phi\|_{\mathcal{B}}+\kappa\,\widetilde{\mathsf{q}}(\xi),\ \xi\Big)\int_0^b \mathsf{f}(s,\xi)\mathrm{d}s\right]+\sigma M_{\Xi}\, b\,\|\vartheta^{(\mathbf{b})}(\xi)\| \\
&\le \widetilde{\mathsf{q}}(\xi).
\end{aligned}
$$

This implies that Λ is a random operator with stochastic domain D and $\Gamma(\xi):\mathsf{D}(\xi)\to\mathsf{D}(\xi)$ for each $\xi\in\Omega$.

Step 2. Λ is continous.

Let $(\vartheta^{(n)})_{n\in\mathbb{N}}$ be a sequence in $\mathsf{D}(\xi)$ such that $\vartheta^{(n)}\to\vartheta$ in $\mathsf{D}(\xi)$. Then by using the hypothesis $(\mathbf{C_2})$, we obtain

$$
\begin{aligned}
&\lim_{n\to+\infty}\mathsf{F}(s,\overline{\vartheta^{(n)}}_{\rho(s,\overline{\vartheta^{(n)}}_s)}(\cdot,\xi),\xi)=\mathsf{F}(s,\overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}(\cdot,\xi),\xi) \text{ and} \\
&\|\mathsf{F}(s,\overline{\vartheta^{(n)}}_{\rho(s,\overline{\vartheta^{(n)}}_s)}(\cdot,\xi),\xi)-\mathsf{F}(s,\overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}(\cdot,\xi),\xi)\| \\
&\qquad\qquad \le 2\widetilde{K_1}\Big([\kappa+L^{\phi}]\,\|\phi\|_{\mathcal{B}}+\kappa\,\widetilde{\mathsf{q}}(\xi),\ \xi\Big)\int_0^b \mathsf{f}(r,\xi)\mathrm{d}r.
\end{aligned}
\tag{18}
$$

Moreover, we have that

$$
\begin{aligned}
&\|(\Lambda(\xi)\vartheta^{(n)})(t)-(\Lambda(\xi)\vartheta)(t)\| \\
\le\ &\sigma\int_0^t\|\mathsf{F}(s,\overline{\vartheta^{(n)}}_{\rho(s,\overline{\vartheta^{(n)}}_s)}(\cdot,\xi),\xi)-\mathsf{F}(s,\overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}(\cdot,\xi),\xi)\|\mathrm{d}s+\sigma\int_0^t\|\mathrm{u}_{\overline{\vartheta^{(n)}}}(s,\xi)-\mathrm{u}_{\overline{\vartheta}}(s,\xi)\|\mathrm{d}s.
\end{aligned}
\tag{19}
$$

Now,

$$
\begin{aligned}
&\|\mathrm{u}_{\overline{\vartheta^{(n)}}}(s,\xi)-\mathrm{u}_{\overline{\vartheta}}(s,\xi)\| \\
&\qquad \le \sigma M_{\Xi}\int_0^b\|\mathsf{F}(s,\overline{\vartheta^{(n)}}_{\rho(s,\overline{\vartheta^{(n)}}_s)}(\cdot,\xi),\xi)-\mathsf{F}(s,\overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}(\cdot,\xi),\xi)\|\mathrm{d}s.
\end{aligned}
\tag{20}
$$

Substituting this into (19), by (18), the fact that the function $s\to 2\widetilde{K_1}\Big([\kappa+L^{\phi}]\,\|\phi\|_{\mathcal{B}}+\kappa\,\widetilde{\mathsf{q}}(\xi),\ \xi\Big)\int_0^b \mathsf{f}(r,\xi)\mathrm{d}r$ is Lebesgue integrable for each $s\in[0,t]$ and Lebesgue dominated convergence Theorem, we get that

$$
\|(\Lambda(\xi)\vartheta^{(n)})(t)-(\Lambda(\xi)\vartheta)(t)\|\to 0 \text{ as } n\to+\infty.
$$

Hence $\Lambda(\xi)$ is continuous in $\mathsf{D}(\xi)$.

Step 3. For every $\xi\in\Omega$, $\{\vartheta\in\mathsf{D}(\xi)\ :\ \Lambda(\xi)\vartheta=\vartheta\}\neq\emptyset$.

We first show that the operator $\Lambda(\xi)$ is equicontinuous on $[0,b]$. For any $t_1<t_2\in[0,b]$, $\vartheta\in\mathsf{D}(\xi)$, we

have

$$\begin{aligned}
& \|(\Lambda(\xi)\vartheta)(t_2) - (\Lambda(\xi)\vartheta)(t_1)\| \\
\leq\ & \|\mathrm{R}(t_2)\phi(0,\xi) + \int_0^{t_2} \mathrm{R}(t_2 - s)\mathsf{F}(s, \overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}(\cdot,\xi),\xi)\mathrm{ds} \\
& + \int_0^{t_2} \mathrm{R}(t_2 - s)\Xi\, W^{-1}\left(\vartheta^{(\mathbf{b})}(\xi) + \mathrm{R}(b)\phi(0,\xi) + \int_0^b \mathrm{R}(b-r)\mathsf{F}(r, \overline{\vartheta}_{\rho(r,\overline{\vartheta}_r)}(\cdot,\xi),\xi)\mathrm{dr}\right)(s,\xi)\mathrm{ds} \\
& - \mathrm{R}(t_1)\phi(0,\xi) + \int_0^{t_1} \mathrm{R}(t_1 - s)\mathsf{F}(s, \overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}(\cdot,\xi),\xi)\mathrm{ds} \\
& - \int_0^{t_1} \mathrm{R}(t_1 - s)\Xi\, W^{-1}\left(\vartheta^{(\mathbf{b})}(\xi) + \mathrm{R}(b)\phi(0,\xi) + \int_0^b \mathrm{R}(b-r)\mathsf{F}(r, \overline{\vartheta}_{\rho(r,\overline{\vartheta}_r)}(\cdot,\xi),\xi)\mathrm{dr}\right)(s,\xi)\mathrm{ds}\| \\
\leq\ & \|\mathrm{R}(t_2)\phi(0,\xi) - \mathrm{R}(t_1)\phi(0,\xi)\| \\
& + \int_0^{t_1} \|\mathrm{R}(t_2 - s) - \mathrm{R}(t_1 - s)\|\,\|\mathsf{F}(s, \overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}(\cdot,\xi),\xi)\|\mathrm{ds} \\
& + \int_{t_1}^{t_2} \|\mathrm{R}(t_2 - s)\|\,\|\mathsf{F}(s, \overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}(\cdot,\xi),\xi)\|\mathrm{ds} \\
& + \int_0^{t_1} \|\mathrm{R}(t_2 - s) - \mathrm{R}(t_1 - s)\|\,\|\Xi\, W^{-1}\left(\vartheta^{(\mathbf{b})}(\xi) + \mathrm{R}(b)\phi(0,\xi) + \int_0^b \mathrm{R}(b-r)\mathsf{F}(r, \overline{\vartheta}_{\rho(r,\overline{\vartheta}_r)}(\cdot,\xi),\xi)\mathrm{dr}\right)(s,\xi)\|\mathrm{ds} \\
& + \int_{t_1}^{t_2} \|\mathrm{R}(t_2 - s)\|\,\|\Xi\, W^{-1}\left(\vartheta^{(\mathbf{b})}(\xi) + \mathrm{R}(b)\phi(0,\xi) + \int_0^b \mathrm{R}(b-r)\mathsf{F}(r, \overline{\vartheta}_{\rho(r,\overline{\vartheta}_r)}(\cdot,\xi),\xi)\mathrm{dr}\right)(s,\xi)\|\mathrm{ds} \\
\leq\ & \|\mathrm{R}(t_2)\phi(0,\xi) - \mathrm{R}(t_1)\phi(0,\xi)\| \\
& + \widetilde{K_1}\Big([\kappa + L^{\phi}]\,\|\phi\|_{\mathcal{B}} + \kappa\,\mathsf{q}(\xi),\ \xi\Big) \int_0^{t_1} \|\mathrm{R}(t_2 - s) - \mathrm{R}(t_1 - s)\|\,\mathsf{f}(s,\xi)\mathrm{ds} \\
& + \widetilde{K_1}\Big([\kappa + L^{\phi}]\,\|\phi\|_{\mathcal{B}} + \kappa\,\mathsf{q}(\xi),\ \xi\Big) \int_{t_1}^{t_2} \|\mathrm{R}(t_2 - s)\|\mathsf{f}(s,\xi)\mathrm{ds} \\
& + \int_0^{t_1} \|\mathrm{R}(t_2 - s) - \mathrm{R}(t_1 - s)\|\, M_{\Xi}\left(\|\vartheta^{(\mathbf{b})}(\xi)\| + \sigma\|\phi\|_{\mathcal{B}} + \sigma\widetilde{K_1}\Big([\kappa + L^{\phi}]\,\|\phi\|_{\mathcal{B}} + \kappa\,\widetilde{\mathsf{q}}(\xi),\ \xi\Big)\int_0^b \mathsf{f}(r,\xi)\mathrm{dr}\right)\mathrm{ds} \\
& + \int_{t_1}^{t_2} \|\mathrm{R}(t_2 - s)\|\, M_{\Xi}\left(\|\vartheta^{(\mathbf{b})}(\xi)\| + \sigma\|\phi\|_{\mathcal{B}} + \sigma\widetilde{K_1}\Big([\kappa + L^{\phi}]\,\|\phi\|_{\mathcal{B}} + \kappa\,\widetilde{\mathsf{q}}(\xi),\ \xi\Big)\int_0^b \mathsf{f}(r,\xi)\mathrm{dr}\right)\mathrm{ds}.
\end{aligned} \tag{21}$$

Using similar argument as in the **Step 3** of the proof of Theorem 3.5, we see that the right hand side of the above inequality tends to zero independently of $\vartheta \in \mathsf{D}(\xi)$ as $t_2 - t_1 \to 0$. Hence, $\Lambda(\xi)$ is equicontinuous on $[0, b]$.

Secondly, let show that the condition of Mönch holds. Let $\xi \in \Omega$, $\mathsf{N} = \{\vartheta^{(k)} : k \in \mathbb{N}\}$ be a subset of $\mathsf{D}(\xi)$ such that $\mathsf{N} \subset \overline{conv}(\Lambda(\xi)(\mathsf{N}) \cup \{0\})$. Knowing that $\Lambda(\xi)(\mathsf{D}(\xi))$ is bounded and equicontinuous on $[0, b]$, we can deduce that $\Lambda(\xi)(\mathsf{N})$ is bounded and equicontinuous. We have

$$\begin{aligned}
& \alpha(\{\Xi \mathrm{u}_{\overline{\vartheta^{(k)}}}\}_{k\geq 1}(t)) \\
=\ & \alpha\Big(\Xi\, W^{-1}\left\{\vartheta^{(\mathbf{b})}(\xi) + \mathrm{R}(b)\phi(0,\xi) + \int_0^b \mathrm{R}(b-r)\mathsf{F}(r, \overline{\vartheta^{(k)}}_{\rho(r,\overline{\vartheta^{(k)}}_r)}(\cdot,\xi),\xi)\mathrm{dr}\right\}_{k\geq 1}(t)\Big) \\
\leq\ & M_{\Xi}\alpha(\vartheta^{(\mathbf{b})}(\xi) + \mathrm{R}(b)\phi(0,\xi)) + M_{\Xi}\sigma\int_0^b \widetilde{K_2}(r,\xi)\alpha\left(\{\overline{\vartheta^{(k)}}_{\rho(r,\overline{\vartheta^{(n)}_r})}(\cdot,\xi)\}_{n\geq 1}\right)\mathrm{dr} \\
\leq\ & M_{\Xi}\sigma\int_0^b \widetilde{K_2}(r,\xi)\gamma(r)\sup_{t\in[0,b]}\alpha\left(\{\vartheta^{(n)}(t)\}_{k\geq 1}\right)\mathrm{dr} \\
\leq\ & M_{\Xi}\sigma\int_0^b \widetilde{K_2}(r,\xi)\gamma(r)\alpha_C\left(\{\vartheta^{(n)}\}_{n\geq 1}\right)\mathrm{dr}.
\end{aligned} \tag{22}$$

It follows that

$$\begin{aligned}
\alpha(\{(\Lambda(\xi)\vartheta^{(n)})(t)\}_{n\geq 1}) &\leq \alpha\Big(\Big\{\mathrm{R}(t)\phi(0,\xi) + \int_0^t \mathrm{R}(t-s)\mathsf{F}(s, \overline{\vartheta^{(n)}}_{\rho(s,\overline{\vartheta}_s^{(n)})}(\cdot,\xi),\xi)\mathrm{ds} \\
&\quad + \int_0^t \mathrm{R}(t-s)\Xi \mathrm{u}_{\overline{\vartheta^{(n)}}}\mathrm{ds}\Big\}_{n\geq 1}\Big) \\
&\leq \alpha\Big(\Big\{\int_0^t \mathrm{R}(t-s)\mathsf{F}(s,\overline{\vartheta}_{\rho(s,\overline{\vartheta}_s)}(\cdot,\xi),\xi)\mathrm{ds}\Big\}_{n\geq 1}\Big) \\
&\quad +\alpha\Big(\Big\{\int_0^t \mathrm{R}(t-s)\Xi \mathrm{u}_{\overline{\vartheta^{(n)}}}\mathrm{ds}.\Big\}_{n\geq 1}\Big) \\
&\leq \sigma \int_0^t \widetilde{K_2}(s,\xi)\gamma(s) \sup_{s\in[0,t]} \alpha\left(\{\vartheta^{(n)}(s)\}_{n\geq 1}\right)\mathrm{ds} \\
&\quad +\sigma \int_0^t \alpha\Big(\Big\{\Xi \mathrm{u}_{\overline{\vartheta^{(n)}}}(s,\xi)\Big\}_{n\geq 1}\Big)\mathrm{ds} \\
&\leq \sigma \int_0^t \widetilde{K_2}(s,\xi)\gamma(s) \sup_{s\in[0,t]} \alpha\left(\{\vartheta^{(n)}(s)\}_{n\geq 1}\right)\mathrm{ds} \\
&\quad +\sigma \int_0^t M_\Xi \sigma \int_0^b \widetilde{K_2}(r,\xi)\gamma(r) \sup_{t\in[0,b]} \alpha\left(\{\vartheta^{(k)}(t)\}_{k\geq 1}\right)\mathrm{drds}\Big) \\
&\leq \sigma \int_0^b \widetilde{K_2}(s,\xi)\gamma(s)\alpha_C\left(\{\vartheta^{(n)}\}_{n\geq 1}\right)\mathrm{ds} \\
&\quad +\sigma^2\, b\, M_\Xi \int_0^b \widetilde{K_2}(r,\xi)\gamma(r)\alpha_C\left(\{\vartheta^{(k)}\}_{k\geq 1}\right)\mathrm{dr} \\
&\leq \left[\sigma(1+\sigma\, b\, M_\Xi)\int_0^b \widetilde{K_2}(s,\xi)\gamma(s)\mathrm{ds}\right]\alpha_C\left(\{\vartheta^{(n)}\}_{n\geq 1}\right).
\end{aligned} \tag{23}$$

Using Lemma 2.12 and inequalities (13), (23), it follows that

$$\alpha_C(\Lambda(\xi)\mathsf{N}) \leq \left[\sigma(1+\sigma\, b\, M_\Xi)\int_0^b \widetilde{K_2}(s,\xi)\gamma(s)\mathrm{ds}\right]\alpha_C(\mathsf{N}) = p_0\alpha_C(\mathsf{N}). \tag{24}$$

Thus,

$$\alpha_C(\mathsf{N}) \leq \alpha_C(\overline{conv}(\{0\}\cup\Lambda(\xi)\mathsf{N})) = \alpha_C(\Lambda(\xi)(\mathsf{N})) \leq p_0\alpha_C(\mathsf{N}). \tag{25}$$

We obtain that $\alpha_C(\mathsf{N}) = 0$, since $p_0 < 1$, proving that $\overline{\mathsf{N}}$ is compact on χ_0 and the Mönch condition holds. Then by Mönch fixed point Theorem $\{\vartheta \in \mathsf{D}(\xi) : \Lambda(\xi)\vartheta = \vartheta\} \neq \emptyset$.

Since $\cap_{\xi\in\Omega}\mathsf{D}(\xi) \neq \emptyset$, then $\mathrm{int}(\mathsf{D}(\xi)) \neq \emptyset$ and there exists a measurable random variable $\vartheta_0 : \Omega \to \Upsilon$ with $\vartheta_0(\xi) \in \mathrm{int}(\mathsf{D}(\xi))$. By Lemma 2.8, we deduce that the random operator Λ has a stochastic fixed point ϑ^* which is a mild solution of problem (11) and satisfying $\vartheta^*(b,\xi) = \vartheta^{(\mathbf{b})}(\xi)$. Hence, system (11) is controllable on $[0,b]$. □

5. Example

We consider the following random control system:

$$\begin{cases} \frac{\partial}{\partial t}x(t,y,\xi) &= [\frac{\partial^2 x(t,y,\xi)}{\partial y^2} + \tilde{b}\frac{\partial x(t,y,\xi)}{\partial y} + \tilde{c}x(t,y,\xi)] \\ & + \int_0^t E(t-s)[\frac{\partial^2 x(s,y,\xi)}{\partial y^2} + \tilde{b}\frac{\partial x(s,y,\xi)}{\partial y} + \tilde{c}x(s,y,\xi)]ds \\ & + \mathcal{M}\beta(t,\xi,) + d_1(\xi)d_2(t)\int_{-\infty}^0 h(x(t+\rho_1(t,x(t+s,y,\xi)),y,\xi))\mathrm{d}s, \\ & \text{for } t\in J=[0,b],\ y\in[0,\pi],\ \text{and } \xi\in\Omega, \\ x(t,0,\xi) &= x(t,\pi,\xi)=0, \quad \text{for } t\in[0,b],\ \xi\in\Omega, \\ x(s,y,\xi) &= x_0(s,y,\xi), \quad s\in(-\infty,0], \quad y\in[0,\pi], \quad \xi\in\Omega, \end{cases} \tag{26}$$

where $E:\mathbb{R}_+\mapsto\mathbb{R}_+$ is a C^1 function with a derivative $|E'(t)|\le E(t)$ for all $t\ge 0$, $\tilde{b},\ \tilde{c}\in\mathbb{R}$, $\mathcal{M}>0$, $\beta\ ;\ [0,b]\times\Omega\ \to\mathbb{R}$ is continuous in t, d_1 is a real-valued random variable, $d_2\in L^1(J;\mathbb{R}_+)$, $x_0:(-\infty,0]\times[0,\pi]\times\Omega\to\mathbb{R}$ and $\rho_1: J\times\mathbb{R}\to\mathbb{R}$ are given functions.

Let $\Upsilon=L^2[0,\pi]$, $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space and $\mathcal{B}=BCU(\mathbb{R}^-;\Upsilon)$ be the space of uniformly bounded continuous functions endowed with the following norm $\|\phi\|=\sup_{s\le 0}\|\phi(s)\|$ for $\phi\in\mathcal{B}$. We define the operator $\mathcal{A}$ induced on $\mathbb{H}$ as follows:

$$\begin{aligned} &D(\mathcal{A}) = H^2(0,\pi)\cap H_0^1(0,\pi), \\ &\mathcal{A}z = z'' + \tilde{b}z' + \tilde{c}z, \quad \tilde{b},\tilde{c}\in\mathbb{R}. \end{aligned} \tag{27}$$

From [14, p. 173], we know that $\mathcal{A}$ is the infinitesimal generator of an analytic C_0-semigroup $(S(t))_{t\ge 0}$ on Υ. Since the semigroup generated by $\mathcal{A}$ is analytic, then it is norm continuous for $t>0$. Thus by Theorem 2.17 the corresponding resolvent operator is operator-norm continuous for $t>0$.
We define the operator $\mathbf{\Gamma}(t):\mathcal{B}\mapsto\Upsilon$ as follows: $\mathbf{\Gamma}(t)z=E(t)\mathcal{A}z$ for $t\ge 0$ and $z\in D(\mathcal{A})$.

Furthermore we set

$$\begin{aligned} \vartheta(t,\xi)(y) &= x(t,y,\xi,) \ \text{ for } \ t\in[0,b],\ \ y\in[0,\pi] \text{ and } \xi\in\Omega, \\ \phi(s,\xi)(y) &= x_0(s,y,\xi) \ \text{ for } \ s\in(-\infty,b]\ \ ,y\in[0,\pi] \text{ and } \ \xi\in\Omega, \end{aligned}$$

and define for every $t\in J=[0,b]$, $y\in[0,\pi]$ and $\xi\in\Omega$,

$$\mathsf{F}(t,\vartheta_t,\xi)(y) = d_1(\xi)d_2(t)\int_{-\infty}^0 h\Big(\vartheta(t+\rho(t,\vartheta_t),y,\xi)\Big)\mathrm{d}s,$$

where $\rho: J\times\mathcal{B}\to\mathbb{R}_+$ is defined by

$$\rho(t,\vartheta_t)(y) = \rho_1(t,x(t+s,y,\xi)).$$

Let $\Xi\ :\ \mathbb{U}\to\ \Upsilon$ be defined by

$$(\Xi u(t,\xi))(y) = \mathcal{M}\beta(t,y,\xi,),\ \ \xi\in\Omega,\ y\in[0,\pi] \text{ and } u(\cdot,\xi)\in L^2([0,1],\mathbb{U}).$$

Using these definitions we can represent the system (26) in the following abstract form

$$\begin{cases} \vartheta'(t,\xi) = \mathcal{A}\vartheta(t,\xi) + \int_0^t \Gamma(t-s)\vartheta(s,\xi)ds + F\Big(t, \vartheta_{\rho(t,\vartheta_t)}(\cdot,\xi),\xi\Big) + \Xi u(t,\xi), \\ \qquad\qquad \text{a.e} \quad (t,\xi) \in [0,b]\times\Omega, \\ \vartheta(t,\xi) = \varphi(t,\xi), \qquad t\in(-\infty,0]. \end{cases} \tag{28}$$

Moreover, let $\phi \in \mathcal{B}$ be such that $(\mathbf{C_4})$ holds, $t \to \phi_t$ be continous on $\mathcal{R}(\rho^-)$. Suppose that the function F satisfies $(\mathbf{C_3})$ and $(\mathbf{C}_\phi)$, $(\mathbf{C_4}) - (\mathbf{C_6})$ hold.

Now for $\xi \in \Omega$, the operator $\mathcal{W}$ is given by

$$\mathcal{W}u = \mathcal{M}\int_0^b \mathrm{R}(b-s)u(s,\xi)ds.$$

Assuming that $\mathcal{W}$ satisfies $(\mathbf{C_5})$. Then all the conditions of Theorem 4.3 are satisfied. Hence, the random problem (26) is controllable on $[0,b]$.

6. Conclusion

In this paper, we have studied the existence and controllability of a random integrodifferential equation with state-dependent delay by using the theory of resolvent operator in the sense of Grimmer. By applying the measure of noncompactness and a random fixed point theorem with stochastic domain, we have proved some existence and controllability results for random nonlinear systems with state-dependent delay. However, we can extend the obtained results to some random integrodifferential equations with random impulses.

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