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On the Boundary and Global Behavior of Mappings of Riemannian Surfaces

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Abstract. In this article, we study non-homeomorphic mappings of Riemannian surfaces of the Sobolev class. We have established estimates for the distortion of the modulus of families of paths, and as a consequence, we obtained results on the boundary behavior of such mappings between domains of Riemannian surfaces.

1. Introduction

One of the main problems of modern analysis is the extension of mappings to the boundary of a domain. There are a number of results on this topic related, in particular, to the theory of quasiconformal mappings and their generalizations, see [1]–[12]. Among the listed papers, we note the fundamental assertion of Näkki, see [1, Theorem 2.4] (see also [2, Theorem 17.15])

Theorem (on the extension of quasiconformal mappings to the boundary). *Let D, D' be domains in $\mathbb{R}^n$, $n \geqslant 2$, and let f be a quasiconformal mapping of D onto D'. Suppose that D is locally connected on its boundary, and $\partial D'$ is quasiconformally accessible. Then f has a continuous extension $\overline{f} : \overline{D} \to \overline{\mathbb{R}^n}$.*

This result was developed in a number of papers by other authors. First of all, Srebro and Vuorinen extended Näkki's theorem to quasiregular mappings, see [3, Theorem 4.2] and [4, Theorem 4.10.II]. More recently, Martio, Ryazanov, Srebro and Yakubov, as well as Ignat'ev and Ryazanov obtained results on the boundary extension of homeomorphisms with unbounded characteristic, see [5, Lemma 5.16] and [6, Lemma 2.1, Corollary 2.1]. Later they were carried over to Riemannian manifolds and metric measure spaces (see, e.g. [7], [9], [10] and [11]).

Let us now dwell on the recent results of Ryazanov and Volkov [12]. Here the authors proposed an approach that allows one to study the boundary behavior of Sobolev classes acting between two Riemannian surfaces. It should be noted that the paper [12] contains important results in this direction, however, they all concern only homeomorphisms. In our opinion, it would be important to describe the boundary behavior of similar mappings with branching, and this problem is largely solved in this article. As in [12], the main research tool is the modulus method. To a large extent, our publication is devoted to the development of the modulus method and the identification of fundamental opportunities of this method in this context.

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Conventionally, the paper may be divided into three parts: establishing an estimate for the distortion of the modulus of families of curves, §§ 2 and 3; boundary behavior of mappings, § 4; local the behavior of the mappings (at the inner points of the domain) and the global behavior of the mappings (at the inner and boundary points of the domain), §§ 5–6. In the last sections, homeomorphisms and mappings with branching are studied separately, since the corresponding results have been proved under various conditions and are new in both cases.

Here are some definitions. A *Riemannian surface* is a two-dimensional manifold with a countable base in which transition maps between corresponding maps are conformal, see, e.g., [12]. The Riemannian surface $\mathbb{S}$ considered below will be assumed to be a *of hyperbolic type*, that is, a surface conformally equivalent to the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with "glued" points (see [13, item 6, Section 1]). In other words, we consider those and only those Riemannian surfaces that are conformally equivalent to the factor space $\mathbb{D}/G$, where G is some group of linear fractional automorphisms of the unit disk that has no fixed points and acts discontinuously in $\mathbb{D}$. Recall that each element p_0 of the factor space $\mathbb{D}/G$ is an *orbit* of the point $z_0 \in \mathbb{D}$, that is, $p_0 = \{z \in \mathbb{D} : z = g(z_0), g \in G\}$. In what follows we identify the Riemannian surface $\mathbb{S}$ with its factor representation $\mathbb{D}/G$ in terms of some group G of linear fractional mappings $g : \mathbb{D} \to \mathbb{D}$.

In what follows, we use the *hyperbolic metric* on the unit disk $\mathbb{D}$ defined by the equalities

$$h(z_1, z_2) = \log \frac{1+t}{1-t}, \quad t = \frac{|z_1 - z_2|}{|1 - z_1\overline{z_2}|}, \tag{1}$$

as well as the *hyperbolic area* of the set $S \subset \mathbb{D}$ and the *length* of the path $\gamma : [a, b] \to \mathbb{D}$, which are given, respectively, by the relations

$$h(S) = \int\limits_S \frac{4\, dm(z)}{(1-|z|^2)^2}, \quad s_h(\gamma) = \int\limits_\gamma \frac{2\,|dz|}{1-|z|^2}, \quad z = x + iy \tag{2}$$

(see [12, (2.4), (2.5)]). It is easy to verify by direct calculations that the hyperbolic metric, length, and area are invariant under linear fractional mappings of the unit disk onto itself.

In what follows, for a point $y_0 \in \mathbb{D}$ and a number $r \geqslant 0$, we define the *hyperbolic disk* $B_h(y_0, r)$ and the *hyperbolic circle* $S_h(y_0, r)$ by means of equalities

$$B_h(y_0, r) := \{y \in \mathbb{D} : h(y_0, y) < r\}, S_h(y_0, r) := \{y \in \mathbb{D} : h(y_0, y) = r\}. \tag{3}$$

We perform the metrization of the surface $\mathbb{D}/G$ as follows. If $p_1, p_2 \in \mathbb{D}/G$, we put

$$\widetilde{h}(p_1, p_2) := \inf_{g_1, g_2 \in G} h(g_1(z_1), g_2(z_2)), \tag{4}$$

where $p_i = G_{z_i} = \{\xi \in \mathbb{D} : \exists\, g \in G : \xi = g(z_i)\}$, $i = 1, 2$. In the latter case, the set G_{z_i} will be called the *orbit* of the point z_i, and p_1 and p_2 will be called the *orbits* of the points z_1 and z_2, respectively. Note that $\widetilde{h}$ is a metric on $\mathbb{D}/G$ (see Section 2 in [12]).

Everywhere below, $\mathbb{S} = \mathbb{D}/G$ and $\mathbb{S}_* = \mathbb{D}/G_*$ are two different Riemannian surfaces of a hyperbolic type. In what follows, we do not distinguish between the original Riemannian surface $\mathbb{S}$ and its factor representation $\mathbb{D}/G$. A continuous extension of the mapping $f : D \to D_*$ to a point $f : D \to D_*$ $p_0 \in \overline{D}$, as well as other concepts related to limit, continuity, topology, etc., are understood in the sense of the metric spaces $(\mathbb{D}/G, \widetilde{h})$ and $(\mathbb{D}/G_*, \widetilde{h}_*)$, where $\widetilde{h}$ and $\widetilde{h}_*$ are metrics defined in (4). The elements of length and area on the surfaces $\mathbb{S}$ and $\mathbb{S}_*$ are denoted $ds_{\widetilde{h}}$, $d\widetilde{h}$ and $ds_{\widetilde{h}_*}$, $d\widetilde{h}_*$, respectively.

Let D and D_* be domains on Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively. A mapping $f : D \to D_*$ is called *discrete* if the preimage $f^{-1}(y)$ of any point $y \in D_*$ consists of isolated points only. A mapping $f : D \to D_*$ is called *open* if the image of any open set $U \subset D$ is an open set in D_*. The definition of mappings of the Sobolev class $W^{1,1}_{\text{loc}}$ on a Riemannian surface can be found, for example, in [12]. In further,

for mappings $f : D \to D_*$ of a class $W^{1,1}_{\rm loc}$ in local coordinates $f_{\overline{z}} = \left(f_x + if_y\right)/2$ and $f_z = \left(f_x - if_y\right)/2$, $z = x+iy$. In addition, the *norm* and the *Jacobian* of the mapping of f in local coordinates are expressed, respectively, by the equalities $\|f'(z)\| = |f_z| + |f_{\overline{z}}|$ and $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2$. A *dilatation* of the mapping f at a point z is defined by the relation

$$K_f(z) = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \tag{5}$$

for $J_f(z) \neq 0$, $K_f(z) = 1$ for $\|f'(z)\| = 0$ and $K_f(z) = \infty$ otherwise. It is easy to verify by direct calculations that $K_f(z)$ does not depend on local coordinates. A mapping $f : D \to D_*$ is called a *mapping with finite distortion*, if $f \in W^{1,1}_{\rm loc}(D)$ and, in addition, there is almost everywhere a finite function $K(z)$ such that $\|f'(z)\|^2 \leqslant K(z)\cdot J_f(z)$ for almost all $z \in D$.

As usual, a path γ on the Riemannian surface $\mathbb{S}$ is defined as a continuous mapping $\gamma : I \to \mathbb{S}$, where I is a finite segment, an interval or a half-interval of a real axis. Let Γ be a family of paths in $\mathbb{S}$. A Borel function $\rho : \mathbb{S} \to [0,\infty]$ is called *admissible* for the family Γ of paths γ, if $\int\limits_{\gamma} \rho(p)\, ds_{\widetilde{h}}(p) \geqslant 1$ for any path $\gamma \in \Gamma$. The latter is briefly written in the form: $\rho \in \operatorname{adm}\Gamma$. A *modulus* of the family Γ is a real-valued function

$$M(\Gamma) := \inf_{\rho\in\operatorname{adm}\Gamma} \int\limits_{\mathbb{S}} \rho^2(p)\, d\widetilde{h}(p)\,.$$

To avoid confusion, we introduce separately the notation for the modulus of the family Γ in the Euclidean sense, namely, put

$$M_e(\Gamma) := \inf_{\rho\in\operatorname{adm}_e\Gamma} \int\limits_{\mathbb{D}} \rho^2(z)\, dm(z)\,,$$

where $\rho \in \operatorname{adm}_e\Gamma$ if and only if $\int\limits_{\gamma} \rho(z)\,|dz| \geqslant 1$ for any (locally rectifiable) path $\gamma \in \Gamma$.

Let D and D_* be given domains lying in the Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively. Given a mapping $f : D \to D_*$, a set $E \subset D$ and $y \in D_*$, we define a *multiplicity function* $N(y, f, E)$ as the number of preimages of the point y in E, that is

$$N(y,f,E) = \operatorname{card}\{p \in E : f(p) = y\}\,, \quad N(f,E) = \sup_{y\in\mathbb{S}_*} N(y,f,E)\,. \tag{6}$$

We say that the function $\rho\colon \mathbb{S} \to [0,\infty]$ measurable with respect to the area $\widetilde{h}$ is *extensively admissible* for the family Γ, abbr. $\rho \in \operatorname{ext}\operatorname{adm}\Gamma$, if the inequality $\int\limits_{\gamma} \rho\, ds_{\widetilde{h}}(p) \geqslant 1$ is satisfied for all locally rectifiable paths $\gamma \in \Gamma \setminus \Gamma_0$, where $M(\Gamma_0) = 0$.

The next class of mappings is related to the Gehring ring definition of quasiconformality (see [14]) and is necessary from a technical point of view. Its definition includes a distortion of the modulus of families of paths in such a way as is necessary to solve the corresponding problems of interest to us. Let D and D_* be domains lying in the Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively, and $Q\colon D \to (0,\infty)$ be a measurable function with respect to the measure $\widetilde{h}$ on $\mathbb{S}$. We say that $f\colon D \to D_*$ is a *lower Q-mapping* at a point $p_0 \in \overline{D}$, if for some $\varepsilon_0 = \varepsilon_0(p_0) > 0$, $\varepsilon_0 < d_0 = \sup\limits_{p\in D} \widetilde{h}(p,p_0)$, and any ring $\widetilde{A}(p_0,\varepsilon,\varepsilon_0) = \{p \in \mathbb{S} : \varepsilon < \widetilde{h}(p,p_0) < \varepsilon_0\}$ the inequality

$$M(f(\Sigma_\varepsilon)) \geqslant \inf_{\rho\in\operatorname{ext}\operatorname{adm}\Sigma_\varepsilon} \int\limits_{D\cap\widetilde{A}(p_0,\varepsilon,\varepsilon_0)} \frac{\rho^2(p)}{Q(p)}\, d\widetilde{h}(p) \tag{7}$$

holds, where Σ_ε denotes the family of all intersections of circles $\widetilde{S}(p_0, r) = \{p \in \mathbb{S} : \widetilde{h}(p, p_0) = r\}$ with D, $r \in (\varepsilon, \varepsilon_0)$ (see [8, Chapter 9]).

The next assertion contains a fundamental estimate for the distortion of families of paths in Sobolev classes (see also [12, Lemma 3.1] and [15, Lemma 3.1] in this regard). Further studies related to the boundary behavior and equicontinuity of mappings are based on estimates of such a plan (see paragraphs 4–6).

Theorem 1.1. *Let D and D_* be domains of Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively, $\overline{D_*}$ is compact in $\mathbb{S}_*$ and $p_0 \in \overline{D}$. Then any open discrete mapping $f : D \to D_*$ of finite distortion of the class $W^{1,1}_{\rm loc}$ such that $N(f, D) < \infty$ satisfies (7) at p_0 for $Q(p) = c \cdot N(f, D) \cdot K_f(p)$, where $K_f(p)$ is defined by the relation (5), the function $N(f, D)$ is given in (6), and $c > 0$ is some constant depending only on the point p_0 and the domain D_*.*

Let us now formulate the main result of the article concerning the continuous boundary extension of mappings. Let $E, F \subset \mathbb{S}$ be arbitrary sets. In the future, everywhere by $\Gamma(E, F, D)$ we denote the family of all paths $\gamma : [a, b] \to D$, which join E and F in D, that is, $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a, b)$. Let us agree to say that the boundary ∂G of the domain G is *strongly accessible at the point* $p_0 \in \partial G$ if for each neighborhood U of p_0 there is a compactum $E \subset G$, a neighborhood $V \subset U$ of the same point and a number $\delta > 0$ such that $M(\Gamma(E, F, G)) \geqslant \delta$ for any continua E and F intersecting both ∂U, and ∂V. We will also say that a boundary ∂G is *strongly accessible* if it is strongly accessible at each of its points. Note that, for an open closed mapping $f : D \to D_*$, the following condition holds: $N(f, D) < \infty$ (see [16, Theorem 5.5]).

A mapping $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$, is called *quasiconformal* if f is a homeomorphism and, moreover, there is a constant $K \geqslant 1$ such that $M_e(f(\Gamma)) \leqslant K \cdot M_e(\Gamma)$ for any family of paths Γ in D. We say that the boundary of a domain D in $\mathbb{S}$ is *locally quasiconformal* if each point $p_0 \in \partial D$ has a neighborhood U in $\mathbb{S}$, which can be mapped by a quasiconformal mapping ψ onto the unit disk $\mathbb{D} \subset \mathbb{C}$ so that $\psi(\partial D \cap U)$ is the intersection of $\mathbb{D}$ with the straight line $x = 0$, where $z = x + iy \in \mathbb{D}$. The most important result of this article can be formulated as follows.

Theorem 1.2. *Let D and D_* be domains on the Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively, let $\overline{D_*}$ be a compactum in $\mathbb{S}_*$, let $p_0 \in \partial D$ and let $Q : \mathbb{S} \to (0, \infty)$ be a measurable function with respect to the measure $\widetilde{h}$, $Q(p) \equiv 0$ in $\mathbb{S} \setminus D$. Let also $f : D \to D_*$ be an open discrete closed $W^{1,1}_{\rm loc}$-mapping with a finite distortion of D onto D_* such that $K_f(p) \leqslant Q(p)$ for almost all $p \in D$. Suppose that the domain D has a locally quasiconformal boundary, and the boundary of the domain D_* is strongly accessible. If the relations*

$$\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dt}{\|Q\|(t)} < \infty\,, \qquad \int\limits_{0}^{\varepsilon_0} \frac{dt}{\|Q\|(t)} = \infty\,, \tag{8}$$

hold for some $\varepsilon_0 > 0$ and any $\varepsilon \in (0, \varepsilon_0)$, then f has a continuous extension to p_0. Here

$$\|Q\|(r) := \int\limits_{\widetilde{S}(p_0, r)} Q(p)\, ds_{\widetilde{h}}(p)$$

denotes L_1-norm of the function Q over the circle $\widetilde{S}(p_0, r)$.

2. Preliminaries

Let us start the section with the next most important Remark.

Remark 2.1. *Following [17, item 7.2], the hyperbolic distance $h(z_1, z_2)$ in the unit disk [17, item 7.2], $\mathbb{D}$ can be equivalently is defined as* inf *of hyperbolic lengths of all piecewise smooth paths joining the points z_1, z_2. Note that the indicated* inf *is calculated and is exactly equal to the right-hand side in (1) (see [17, theorem 7.2.1, relation (7.2.5)]).*

If we define the length $l(\gamma)$ of the path $\gamma : [a, b] \to \mathbb{D}$ by the equality

$$l(\gamma) := \sup_{\pi} \sum_{k=0}^{n-1} h(\gamma(t_k), \gamma(t_{k+1}))\,, \tag{9}$$

where h is from (1), and $\pi = \{a = t_0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n = b\}$, then $l(\gamma) = s_h(\gamma)$ for absolutely continuous paths.

The proof of this fact can be carried out similarly to [2, Theorem 1.3], and therefore goes down. Further, for the rectifiable path $\gamma : [a, b] \to \mathbb{D}$ there is a unique normal representation $\gamma_0 : [0, l(\gamma)] \to \mathbb{D}$ such that $\gamma_0(s(t)) = \gamma(t)$ for any $t \in [a, b]$, where $s(t)$ is the length of the path γ on the segment $[a, t]$ (see, for example, [18, Sec. 7.1] or [2, Sec. 2]). If the path is only locally rectifiable, then $l(\gamma) = \infty$ and, accordingly, $\gamma_0 : [0, \infty) \to \mathbb{D}$, $\gamma_0(s(t)) = \gamma(t)$ for any $t \in [a, b]$. Let $\rho : \mathbb{D} \to \mathbb{R}$ be a nonnegative Borel function. Then the integral from ρ over the (locally rectifiable) path γ can be defined by equality

$$\int\limits_{\gamma} \rho(x)\, ds_h(x) = \int\limits_{0}^{l(\gamma)} \rho(\gamma_0(s))\, ds\,. \tag{10}$$

Observe that, the integral in (10) coincides with

$$\int\limits_{\gamma} \rho(z) \frac{2\,|dz|}{1-|z|^2} = \int\limits_{a}^{b} \frac{2\rho(\gamma(t))\,|\gamma'(t)|}{1-|\gamma(t)|^2}\, dt$$

for absolutely continuous path γ (see [19, Corollary 2.1]).

It should be noted that the normal representation of $\gamma_0(s)$ by an arbitrary locally rectifiable path $\gamma(t)$ is trivial locally absolutely continuous with respect to its natural parameter $s \in [0, l(\gamma)]$ ($s \in [0, l(\gamma))$, if $l(\gamma) = \infty$). In particular,

$$\int\limits_{0}^{l(\gamma)} \rho(\gamma_0(s))\, ds = \int\limits_{0}^{l(\gamma)} \frac{2\rho(\gamma_0(s))\,|\gamma_0'(s)|}{1-|\gamma_0(s)|^2}\, ds\,.$$

In view of the above, we will not distinguish between the length of the path (integral over it) in (9)–(10) and in (2), respectively.

In order to simplify research, we introduce into consideration the so-called *fundamental set F*. We define it as a subset of $\mathbb{D}$, containing one and only one point of the orbit $z \in G_{z_0}$ (see [17, item 9.1, Ch. 9]). A *fundamental domain* D_0 is a domain in $\mathbb{D}$ with the property $D_0 \subset F \subset \overline{D_0}$ such that $h(\partial D_0) = 0$ (see ibid). The existence of fundamental sets and fundamental domains is justified by the presence of their examples, the most important of which is *Dirichlet polygon*,

$$D_{\zeta} = \bigcap_{g \in G, g \neq I} H_g(\zeta)\,, \tag{11}$$

where $H_g(\zeta) = \{z \in \mathbb{D} : h(z, \zeta) < h(z, g(\zeta))\}$ (see [12, relation (2.6)]). Let π be the natural projection of $\mathbb{D}$ onto $\mathbb{D}/G$, then π is an analytic function conformal on D_0 (see also [17, Proposition 9.2.2] and comments after (2.11) in [12]). Note, in addition, that there is a one-to-one correspondence between the points of F and $\mathbb{D}/G$. For $z_1, z_2 \in F$ we put

$$d(z_1, z_2) := \widetilde{h}(\pi(z_1), \pi(z_2))\,, \tag{12}$$

where $\widetilde{h}$ is defined in (4). Observe that $d(z_1, z_2) \leqslant h(z_1, z_2)$ and, moreover, for any compactum $A \subset \mathbb{D}$ there is $\delta = \delta(A) > 0$ such that

$$d(z_1, z_2) = h(z_1, z_2), \quad \forall\ z_1, z_2 \in A : h(z_1, z_2) < \delta\,, \tag{13}$$

see e.g. [19, Lemma 2.3].

Note that the metric space (F, d) is homeomorphic to $(\mathbb{D}/G, \widetilde{h})$. Define the elements of length and volume on (F, d) according to the relations (2), in addition, we also define the elements $ds_{\widetilde{h}}$ and $d\widetilde{h}$ of length and area on the surface $\mathbb{S}$, respectively as the elements ds_h and dh in their respective local coordinates. These local coordinates can be, in particular, fundamental domains D_0 in $\mathbb{D}$. Due to $d(z_1, z_2) \leqslant h(z_1, z_2)$ and (13), the lengths of the paths in the metrics h and d of the domain D_0 coincide.

Here and below, $B(z_0, r)$ and $S(z_0, r)$ denote the Euclidean disk and a circle on a plane centered at the point $z_0 \in \mathbb{C}$ and of a radius $r > 0$, respectively. Let $p_0 \in \mathbb{S}$ and $z_0 \in \mathbb{D}$ be such that $\pi(z_0) = p_0$, where π is the natural projection of $\mathbb{D}$ onto $\mathbb{D}/G$. Denote by D_0 the Dirichlet polygon centered at the point z_0, and put $\varphi := \pi^{-1}$. Note that the mapping φ is a homeomorphism of $(\mathbb{S}, \widetilde{h})$ onto (F, d), where $\widetilde{h}$ is a metric on the surface $\mathbb{S}$, a d is the above-defined metric on the fundamental set F, $D_0 \subset F \subset \overline{D_0}$. Without loss of generality, we may also assume that $z_0 = 0$. Indeed, otherwise consider an auxiliary mapping $g_0(z) = (z - z_0)/(1 - z\overline{z_0})$, having no fixed points inside the unit disk. Then, if G is a group of linear fractional maps corresponding to the surface $\mathbb{S}$, then $G' = \{g_0 \circ g, g \in G\}$, obviously also corresponds to $\mathbb{S}$ in the sense that the surface $\mathbb{S}$ is again conformally equivalent to the factor space $\mathbb{D}/G'$. Choose a compact neighborhood $V \subset \mathbb{D}$ of the point $0 \in F \subset \mathbb{D}$, such that $d(x, z) = h(x, z)$ for all $x, z \in V$, which is possible due to condition (13). In addition, we choose V so that $V \subset B(0, r_0)$ for some $0 < r_0 < 1$. Put $U := \pi(V)$. In this case, the neighborhood U is called *a normal neighborhood of the point* p_0. Note that the ball $\widetilde{B}(p_0, r) \subset U$ corresponds to the set $B_h(0, r) \subset \mathbb{C}$, more precisely,

$$\widetilde{B}(p_0, r) := \{p \in \mathbb{S} : \widetilde{h}(p, p_0) < r\} = \{p \in \mathbb{S} : h(\varphi(p), 0) < r\} =$$
$$= \left\{p \in \mathbb{S} : |\varphi(p)| < \frac{e^r - 1}{e^r + 1}\right\} = \pi\left(B\left(0, \frac{e^r - 1}{e^r + 1}\right)\right), \tag{14}$$

where h is a hyperbolic metric, see (1). Similarly,

$$\widetilde{S}(p_0, r) := \{p \in \mathbb{S} : \widetilde{h}(p, p_0) = r\} =$$
$$= \left\{p \in \mathbb{S} : |\varphi(p)| = \frac{e^r - 1}{e^r + 1}\right\} = \pi\left(S\left(0, \frac{e^r - 1}{e^r + 1}\right)\right). \tag{15}$$

Throughout what follows, the normal neighborhood U of the point p_0, as well as the mapping φ and the fundamental set F, we will considered selected and fixed. The following analogue of Fubini's theorem for Riemannian surfaces holds.

Lemma 2.2. *Let U be some normal neighborhood of the point $p_0 \in \mathbb{S}$, and let $Q : U \to [0, \infty]$ be a function measurable with respect to the measure $\widetilde{h}$, and $d_0 := \operatorname{dist}(p_0, \partial U) := \inf\limits_{p \in \partial U} \widetilde{h}(p_0, p)$. Then, for any $0 < r_0 \leqslant d_0$*

$$\int\limits_{\widetilde{B}(p_0, r_0)} Q(p)\, d\widetilde{h}(p) = \int\limits_0^{r_0} \int\limits_{\widetilde{S}(p_0, r)} Q(p)\, ds_{\widetilde{h}}(p)\, dr\,, \tag{16}$$

where $d\widetilde{h}(p)$ and $ds_{\widetilde{h}}$ are area and length elements on $\mathbb{S}$, respectively, see (2), and the disk $\widetilde{B}(p_0, r_0)$ and the circle $\widetilde{S}(p_0, r)$ are defined in (14) and (15).

The assertion of Lemma 2.2 includes the measurability of $\psi(r) := \int\limits_{\widetilde{S}(p_0, r)} Q(p)\, ds_{\widetilde{h}}(p)$ with respect to r on the right-hand side of the integral in (16).

Proof. According to the definition of a normal neighborhood, the ball $\widetilde{B}(p_0, r_0) \subset \mathbb{S}$ corresponds to the ball $B_h(0, r_0) \subset F$ in the hyperbolic metric h. Taking into account the relation (14), $B_h(0, r_0) = B\left(0, \frac{e^{r_0}-1}{e^{r_0}+1}\right)$. By definition

$$\int\limits_{\widetilde{B}(p_0,r_0)} Q(p)\ d\widetilde{h}(p) = 4 \int\limits_{B\left(0,\frac{e^{r_0}-1}{e^{r_0}+1}\right)} \frac{Q(\pi(z))}{(1-|z|^2)^2}\ dm(z)\,. \tag{17}$$

We use the classical Fubini theorem on the plane (see, for example, [20, Theorem 2.6.2] or [21, Theorem 8.1.III]). Using polar coordinates and applying this theorem, we will have that

$$I = 4 \int\limits_0^{\frac{e^{r_0}-1}{e^{r_0}+1}} \int\limits_{S(0,r)} \frac{Q(\pi(z))}{(1-|z|^2)^2}\,|dz|\,dr =$$

$$= 2 \int\limits_0^{\frac{e^{r_0}-1}{e^{r_0}+1}} \frac{1}{1-r^2} \int\limits_{S(0,r)} \frac{2\,Q(\pi(z))}{1-|z|^2}\,|dz|\,dr\,. \tag{18}$$

The last relation takes into account that the function $\int\limits_{S(0,r)} \frac{2Q(\pi(z))}{1-|z|^2}\,|dz|$ is measurable by r (which is also part of the statement of the classical Fubini theorem). Let us make the change $t = \log\frac{1+r}{1-r}$ in the last integral in accordance with [20, Theorem 3.2.6]. Since $dt = \frac{2\,dr}{1-r^2}$, we get:

$$2 \int\limits_0^{\frac{e^{r_0}-1}{e^{r_0}+1}} \frac{1}{1-r^2} \int\limits_{S(0,r)} \frac{2Q(\pi(z))}{1-|z|^2}\,|dz|\,dr = \int\limits_0^{r_0} \int\limits_{S\left(0,\frac{e^r-1}{e^r+1}\right)} \frac{2Q(\pi(z))}{1-|z|^2}\,|dz|\,dr\ =$$

$$= \int\limits_0^{r_0} \int\limits_{S_h(0,r)} Q(\pi(z))\,ds_h(z)\,dr = \int\limits_0^{r_0} \int\limits_{\widetilde{S}(p_0,r)} Q(p) ds_{\widetilde{h}}(p)\,dr = \int\limits_0^{r_0} \psi(r)\,dr\,. \tag{19}$$

In particular, by [20, Theorem 3.2.6] the function $\psi(r)$ is measurable by r. Now combining (17), (18) and (19), we obtain that

$$\int\limits_{\widetilde{B}(p_0,r_0)} Q(p)\,d\widetilde{h}(p) = \int\limits_0^{r_0} \int\limits_{\widetilde{S}(p_0,r)} Q(p)\,ds_{\widetilde{h}}(p)\,dr\,,$$

as required to prove. □

Before proceeding directly to the study of mappings on Riemannian surfaces (including mappings with (7)), we formulate the following statement, which relates the concept of "almost all" with respect to the modulus of families of paths and the Lebesgue sense (its proof is similar to [22, Lemma 4.1]).

Lemma 2.3. *Let D be a domain of the Riemannian surface $\mathbb{S}$, $p_0 \in \overline{D}$ and let U be some normal neighborhood of the point p_0. If some property P holds for almost all intersections $D(p_0, r) := \widetilde{S}(p_0, r) \cap D$ of circles $\widetilde{S}(p_0, r)$ with a domain D, lying in U, where "almost all" is understood in the sense of the modulus of families of paths and the set*

$$E = \{r \in \mathbb{R} : P \ \ \textit{holds for}\ \ \widetilde{S}(p_0, r) \cap D\}$$

is Lebesgue measurable, then P also holds for almost all $D(p_0, r)$ in U with respect to the linear Lebesgue measure by a parameter $r \in \mathbb{R}$. Conversely, if P holds for almost all $D(p_0, r) := \widetilde{S}(p_0, r) \cap D$ with respect to

the linear Lebesgue measure in $r \in \mathbb{R}$, then P also holds for almost all $D(p_0, r) := \widetilde{S}(p_0, r) \cap D$ in the sense of a modulus.

The proof of the following statement is similar to [8, Theorem 9.2] (see also [22, Lemma 4.2]), and is therefore omitted.

Lemma 2.4. *Let D and D_* be domains in $\mathbb{S}$ and $\mathbb{S}_*$, respectively, let $p_0 \in \overline{D}$ and let $Q\colon D \to (0, \infty)$ be a measurable function. Then $f\colon D \to D_*$ satisfies estimate (7) at the point p_0 if and only if there is $0 < d_0 < \sup\limits_{p \in D} \widetilde{h}(p, p_0)$ such that*

$$M(f(\Sigma_\varepsilon)) \geqslant \int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{\|Q\|(r)} \quad \forall\, \varepsilon \in (0, \varepsilon_0)\,,\ \varepsilon_0 \in (0, d_0)\,, \tag{20}$$

where, as above, Σ_ε denotes the family of all intersections of the circles $\widetilde{S}(p_0, r)$ with the domain D, $r \in (\varepsilon, \varepsilon_0)$,

$$\|Q\|(r) = \int\limits_{D(p_0, r)} Q(p)\, ds_{\widetilde{h}}(p)$$

is L_1-norm of the function Q over the intersection $D \cap \widetilde{S}(p_0, r) = D(p_0, r) = \{p \in D : \widetilde{h}(p, p_0) = r\}$.

3. Basic bound for distortion of the modulus of families of paths

We say that a set $A \subset \mathbb{S}$ has *Lebesgue measure zero* if A can be covered by at most countable the number of normal neighborhoods $U_k \subset \mathbb{S}$, $k = 1, 2, \ldots$, such that $\varphi_k(U_k) \to \mathbb{D}$, where φ_k is some homeomorphisms related to each other conformal transformation, in this case, $m(\varphi_k(U_k \cap A)) = 0$ for any $k = 1, 2, \ldots$, m is the Lebesgue measure in $\mathbb{C}$. The following statements are true.

Lemma 3.1. *Suppose that $B_0 \subset \mathbb{S}$ has a Lebesgue measure zero, $p_0 \in \mathbb{S}$, U is a normal neighborhood of the point p_0, $\overline{U} \neq \mathbb{S}$ and $0 < \varepsilon_0 < \operatorname{dist}(p_0, \partial U)$. Then*

$$\mathcal{H}^1(\varphi(B_0 \cap S_r)) = 0 \tag{21}$$

for almost all circles $S_r := \widetilde{S}(p_0, r)$ centered at a point p_0, where $\varphi = \pi^{-1}$ is a homeomorphism of U into $\mathbb{D}$, corresponding to the definition of a normal neighborhood U, $\mathcal{H}^1$ is a 1-dimensional Hausdorff measure in $\mathbb{C}$, and "almost all" should be understood with respect to the parameter $r \in (0, \varepsilon_0)$.

Proof. Indeed, since the Lebesgue measure is regular, there is a Borel set $B \subset U$ such that $\varphi(B_0) \subset \varphi(B)$ and $m(\varphi(B_0)) = m(\varphi(B)) = 0$, where m is, as usual, the Lebesgue measure in $\mathbb{C}$. Let g be the characteristic function of the set $\varphi(B)$. According to [20, Theorem 3.2.5] for $m = 1$, we have that

$$\int\limits_{\varphi(\gamma)} g(z)|dz| = \mathcal{H}^1(\varphi(B \cap |\gamma|))\,, \tag{22}$$

where $\gamma : [a, b] \to U$ is any locally rectifiable path, $|\gamma|$ us a locus of γ in U, and $|dz|$ is an element of the Euclidean measure. Arguing similarly to the proof of [2, Theorem 33.1], we put

$$\rho(p) = \begin{cases} \infty, & p \in B, \\ 0, & p \notin B\,. \end{cases}$$

Observe that ρ is a Borel function. Let Γ be a family of all circles $S_r := \widetilde{S}(p_0, r)$ centered at the point p_0, for which $\mathcal{H}^1(\varphi(B \cap S_r)) > 0$. By (22), for any $S_r \in \Gamma$ we obtain that

$$\int\limits_{S_r} \rho(p)\, ds_{\widetilde{h}}(p) = \int\limits_{S_h(0,r)} \rho(\varphi^{-1}(y))\, ds_h(y) = 2 \int\limits_{S\left(0, \frac{e^r-1}{e^r+1}\right)} \frac{\rho(\varphi^{-1}(y))}{1-|y|^2}|dy| =$$

$$= 2 \int\limits_{S\left(0, \frac{e^r-1}{e^r+1}\right)} \frac{g(y)\rho(\varphi^{-1}(y))}{1-|y|^2}|dy| = \infty\,.$$

Now $\rho \in \operatorname{adm}\Gamma$. Thus, $M(\Gamma) \leqslant \int\limits_{\mathbb{S}} \rho^2(p)\, d\widetilde{h}(p) = 0$. Let Γ^* be a family of all circles $S_r := \widetilde{S}(p_0, r)$ centered at p_0 for which $\mathcal{H}^1(\varphi(B_0 \cap S_r)) > 0$. Observe that $\Gamma^* \subset \Gamma$, whence $M(\Gamma^*) = 0$. Finally, note that the function $\psi(r) := \mathcal{H}^1(\varphi(B_0 \cap S_r))$ is Lebesgue measurable by the classical Fubini theorem, so that (21) is true for almost all $r \in (0, \varepsilon_0)$ by Lemma 2.3. □

Let $\gamma : [a, b] \to \mathbb{S}$ be a (locally rectifiable) path on the Riemannian surface $\mathbb{S}$. Then we define the function $l_\gamma(t)$ as the length of the path $\gamma|_{[a,t]}$, $a \leqslant t \leqslant b$ (where "length" is understood in the sense of a Riemannian surface). For a set $B \subset \mathbb{S}$, put

$$l_\gamma(B) = \operatorname{mes}_1 \{s \in [0, l(\gamma)] : \gamma(s) \in B\}\,, \tag{23}$$

where, as usual, mes_1 denotes the linear Lebesgue measure in $\mathbb{R}$, and $l(\gamma)$ is the length of γ. Similarly, we may define the value $l_\gamma(B)$ for the dashed line γ, i.e. when $\gamma : \bigcup\limits_{i=1}^{\infty}(a_i, b_i) \to \mathbb{S}$, where $a_i < b_i$ for any $i \in \mathbb{N}$ and $(a_i, b_i) \cap (a_j, b_j) = \varnothing$ for any $i \neq j$.

Lemma 3.2. *Let D and D_* be domains in $\mathbb{S}$ and $\mathbb{S}_*$, respectively, and let $f : D \to D_*$ be a mapping of the Sobolev class $W^{1,1}_{\text{loc}}$. Let $p_0 \in \overline{D}$, let U be a normal neighborhood of p_0, $\overline{U} \neq \mathbb{S}$ and $0 < \varepsilon_0 < \operatorname{dist}(p_0, \partial U)$, and let $B_0 \subset D$ has a Lebesgue measure zero. Then $\mathcal{H}^1(f(B_0 \cap \widetilde{S}(p_0, r))) = 0$ for almost any $r \in (0, \varepsilon_0)$ in local coordinates and, in addition,*

$$l_{f(\widetilde{S}(p_0,r)\cap D)}(f(B_0)) = 0\,, \tag{24}$$

where l is defined in (23).

Proof. Since the mapping f is continuous, the domain $f(D)$ can be covered by at most a countable number of neighborhoods V_k, $k = 1, 2, \ldots$, in such a way, that V_k is conformally homeomorphic to some neighborhood $W_k \subset \mathbb{D}$, $\overline{W_k}$ is compact in $\mathbb{D}$ and, moreover, $f^{-1}(V_k) = U_k \subset D$, where U_k is an open set, $\bigcup\limits_{k=1}^{\infty} U_k = D$. We may also assume that the length and area in V_k are calculated in terms of the hyperbolic length and hyperbolic area in W_k. By what was said above, without loss of generality, we may assume that $f(D)$ is conformally homeomorphic to the set $W \subset \mathbb{D}$, whose closure is compact in $\mathbb{D}$. Let the indicated conformal homeomorphism be realized for using the mapping $\psi : f(D) \to W$, and let φ be a homeomorphism of U into $\mathbb{D}$, corresponding to the definition of normal neighborhood U, and let $\varphi(U) \subset B(0, r_0)$, where $\overline{B(0, r_0)}$ is a compact set in $\mathbb{D}$. Consider a partition of the set $B(0, r_0)$ into a countable number of pairwise disjoint ring segments

$$A_m = \{z \in \mathbb{C} : z = Re^{i\alpha}, R \in (r_{m-1}, r_m], \alpha \in (\psi_{m-1}, \psi_m]\}\,, m \in \mathbb{N}\,. \tag{25}$$

Let h_m be an auxiliary quasiisometry that maps A_m onto a rectangle B_m such that arcs of circles centered at zero are mapped to line segments, see Figure 1.

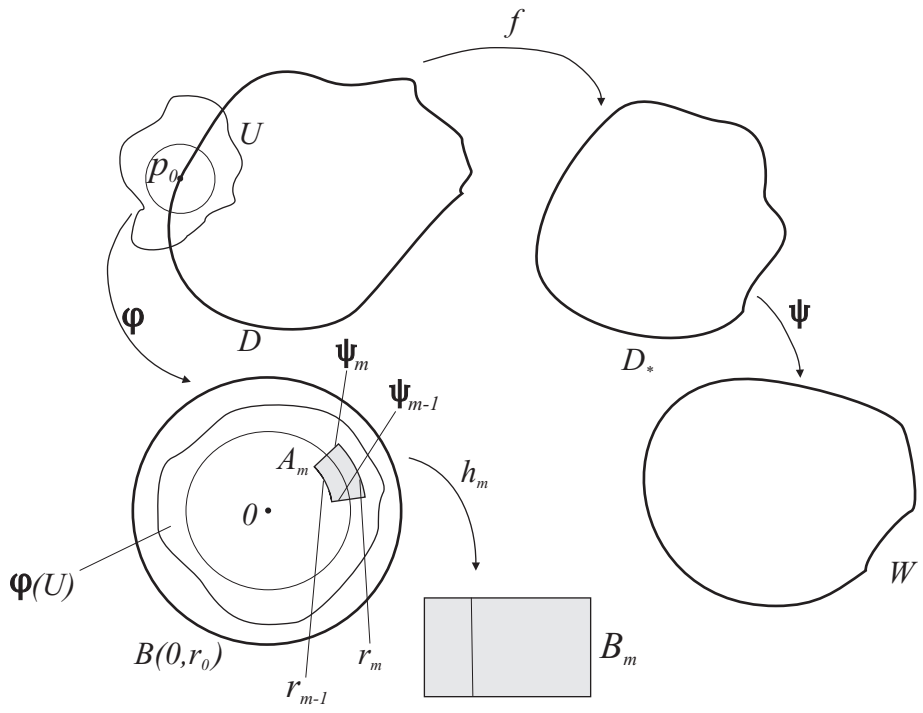

Figure 1: To the proof of Lemma 3.2.

More precisely, put $h_m(\omega) = \log \omega$, $\omega \in A_m$ and $\widetilde{A_m} := h_m(A_m \cap D)$. Then, for each $m \in \mathbb{N}$, consider the mapping

$$g_m := \psi \circ f \circ \varphi^{-1} \circ h_m^{-1}, \quad g_m : \widetilde{A_m} \to \mathbb{C}.$$

Observe that $g_m \in W^{1,1}_{\rm loc}(\widetilde{A_m})$ (see [23, Section 1.1.7]), whence, in particular, $g_m \in ACL$ (see [23, Theorems 1 and 2, Section 1.1.3, § 1.1, Section. I]). Set, as above, $S_r := \widetilde{S}(p_0, r)$. By Lemma 3.1 and in view of the smoothness of the mapping h_m, we obtain that

$$\mathcal{H}^1(\varphi(B_0 \cap S_r) \cap A_m) = \mathcal{H}^1(h_m(\varphi(B_0 \cap S_r) \cap A_m)) = 0$$

for any $r \in [0, \varepsilon_0] \setminus A_0$, where $\mathrm{mes}_1 A_0 = 0$. Set $D_r := D \cap S_r$. Then also for any $r \in [0, \varepsilon_0] \setminus A_0$

$$\mathcal{H}^1(\psi(f(B_0 \cap D_r \cap \varphi^{-1}(A_m)))) = \mathcal{H}^1(g_m(h_m(\varphi(B_0 \cap D_r) \cap A_m))) = 0, \tag{26}$$

since the absolute continuity of the map g_m on a fixed interval implies the N -property with respect to the linear Lebesgue measure (see [20, Section 2.10.13]). Observe that $U \subset \bigcup\limits_{m=1}^{\infty} \varphi^{-1}(A_m)$, so from (26), in view of the countable semi-additivity of the Hausdorff measure,

$$\mathcal{H}^1(\psi(f(B_0 \cap D_r))) = 0, \quad r \in [0, \varepsilon_0] \setminus A_0. \tag{27}$$

Let γ_i be an arbitrary arc of the dashed line $\psi(f(D_r))$. Then we parametrize $\gamma_i : [0, l(\gamma_i)] \to \mathbb{D}$, $\gamma_i = \gamma_i(s)$, where s is a natural parameter on γ_i in the sense of the Euclidean length. Setting $m = 1$ in [20, Theorem 3.2.5], by (27) we obtain that the set $B := \{s \in [0, l(\gamma_i)] : \gamma_i(s) \in \psi(f(B_0))\}$ has a linear measure zero. Let $\chi_{\psi(f(B_0))}(z)$ be a characteristic function of the set $\psi(f(B_0))$. Taking into account the Remark 2.1, we obtain that

$$l_{f(\widetilde{S}(p_0,r)\cap D)}(f(B_0)) = \sum_{i=1}^{\infty} 2 \int\limits_0^{l(\gamma_i)} \frac{\chi_{\psi(f(B_0))}(\gamma_i(s))\, ds}{1 - |\gamma_i(s)|^2} = 0$$

for almost any $r \in (0, \varepsilon_0)$, which was required to be established. $\Box$

Lemma 3.3. *Let D and D_* be domains in Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively, and let $f : D \to D_*$ be an open discrete $W^{1,1}_{\rm loc}$-mapping with a finite distortion. Let $p_0 \in \overline{D}$, let U be a normal neighborhood of the point p_0, let $0 < \varepsilon_0 < \operatorname{dist}(p_0, \partial U)$, and let $B_* \subset D$ be the set of such points of D, in which the mapping f is differentiable (in local coordinates), however, $J_f(p) = 0$. Then*

$$l_{f(\widetilde{S}(p_0,r)\cap D)}(f(B_*)) = 0 \tag{28}$$

for almost all $r \in (0, \varepsilon_0)$, where the function l is defined in (23).

Proof. Observe that f is differentiable almost everywhere in D in local coordinates (see the remarks made in the introduction in [12]). In particular, the set U can be split into a countable number sets B_k, $k = 0, 1, 2, \ldots$, such that $f|_{B_k}$ is a bilipschitz homeomorphism for $k = 1, 2, \ldots$, and B_0 has a measure zero (see [20, items 3.2.2, 3.1.4 and 3.1.8]). Let, as before, $S_r := \widetilde{S}(p_0, r)$ and $D_r := S_r \cap D$. By Lemma 3.2 $\mathcal{H}^1(f(B_0 \cap D_r)) = 0$ for almost of any $r \in (0, \varepsilon_0)$ in local coordinates, therefore, a 1-dimensional change of variables holds for almost all $r \in (0, \varepsilon_0)$, (see [20, Theorem 3.2.5]).

Repeating the reasoning given in the proof of Lemma 3.2 and using the notation of this proposition, we conclude that the mapping h_m maps $\varphi(S_r) \cap A_m$ to some part of the segment $I(m, R) = \{z \in \mathbb{C} : z = \log R + it, t \in (\psi_{m-1}, \psi_m), R = (e^r - 1)/(e^r + 1)\}$. Since f has a finite distortion, $g'_m(\log R + it) = 0$ for all $t \in (\psi_{m-1}, \psi_m)$ such that $\varphi^{-1}(h_m^{-1}(\log R + it)) \in B_*$. Then, by virtue of [20, Theorem 3.2.5] and in view of the above remarks

$$\mathcal{H}^1(\psi(f(B_* \cap D_r \cap \varphi^{-1}(A_m)))) = \mathcal{H}^1(g_m(h_m(\varphi(B_* \cap D_r) \cap A_m))) \leqslant$$

$$\leqslant \int\limits_{g_m(h_m(\varphi(B_*\cap D_r)\cap A_m))} N(y, g_m, h_m(\varphi(B_* \cap D_r) \cap A_m))\, d\mathcal{H}^1 y =$$

$$= \int\limits_{\psi_{m-1}}^{\psi_m} \chi_{h_m(\varphi(B_*)\cap A_m)}(\log R + it)|g'_m(\log R + it)|dt = 0\,,$$

where $\chi_{h_m(\varphi(B_*)\cap A_m)}$ is a characteristic function of the set $h_m(\varphi(B_*) \cap A_m)$. Semiadditivity with respect to m of the one-dimensional Hausdorff measure in the last chain of equalities gives us $\mathcal{H}^1(\psi(f(B_* \cap D_r))) = 0$ for almost all $r \in (0, \varepsilon_0)$. Let γ_i be an arbitrary dashed arc line $\psi(f(D_r))$. Parametrize γ_i as $\gamma_i : [0, l(\gamma_i)] \to \mathbb{D}$, $\gamma_i = \gamma_i(s)$, where $s \in [0, l(\gamma_i)]$ is a natural parameter. Setting $m = 1$ in [20, Theorem 3.2.5], we obtain that the set $B_i := \{s \in [0, l(\gamma_i)] : \gamma_i(s) \in \psi(f(B_*))\}$ has a linear measure zero. Let $\chi_{\psi(f(B_*))}(z)$ be the characteristic function of the set $\psi(f(B_*))$. Taking into account the Remark 2.1, we obtain that

$$l_{f(\widetilde{S}(p_0,r)\cap D)}(f(B_*)) = \sum_{i=1}^{\infty} 2 \int\limits_0^{l(\gamma_i)} \frac{\chi_{\psi(f(B_*))}(\gamma_i(s))\, ds}{1 - |\gamma_i(s)|^2} = 0$$

for almost any $r \in (0, \varepsilon_0)$, which was required to be established. $\Box$

Proof of Theorem 1.1. Since f is open, the mapping f is differentiable almost everywhere in D local coordinates (see the remarks made in the introduction to [12]; see also [24, Theorem III.3.1]). Let B be the Borel set of all points $p \in D$, where f has a total differential $f'(p)$ and $J_f(p) \neq 0$ in local coordinates. Note that B may be represented as at most countable unions of Borel sets B_l, $l = 1, 2, \ldots$, such that $f_l = f|_{B_l}$ are bilipschitz homeomorphisms (see [20, Sections 3.2.2, 3.1.4 and 3.1.8]). See Figure 2 for illustrations.

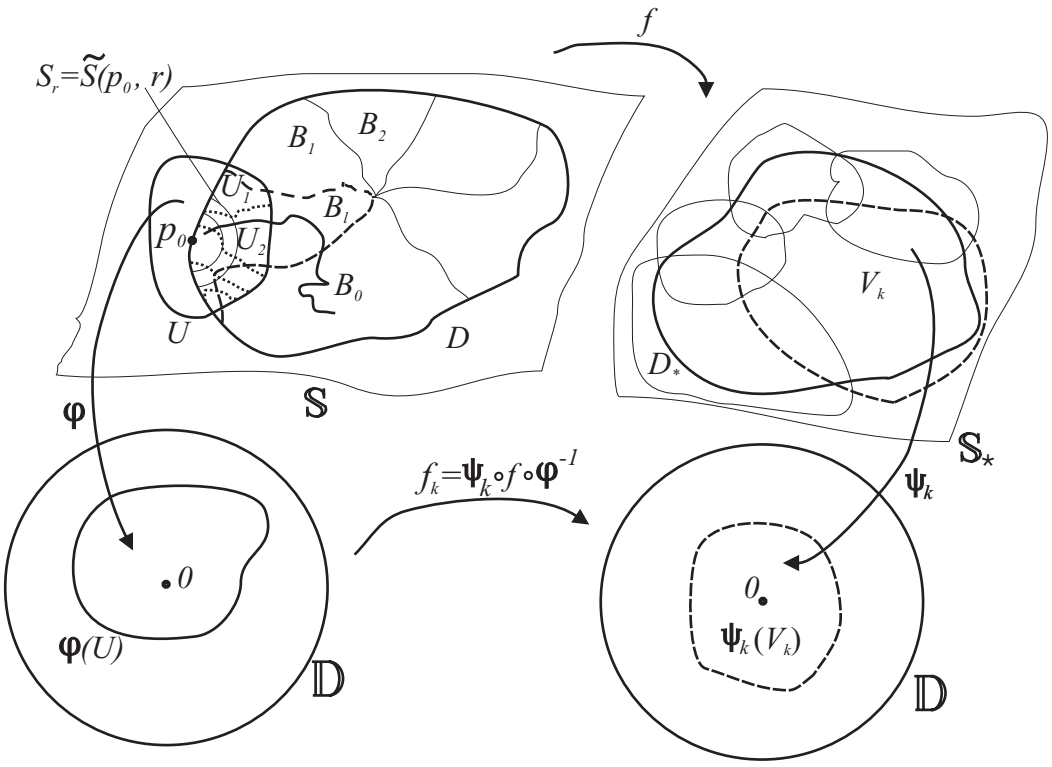

Figure 2: To the proof of Theorem 1.1

Without loss of generality, we may assume that the sets B_l are pairwise disjoint. We also denote by B_* the set of all points $p \in D$, where f has a total differential and $f'(p) = 0$.

Since f has a finite distortion, $f'(p) = 0$ for almost all points p, where $J_f(p) = 0$. Thus, by construction, the set $B_0 := D \setminus (B \bigcup B_*)$ has $\widetilde{h}$-measure zero. Let U be a normal neighborhood of the point p_0 and $\varphi : U \to \mathbb{D}$ be a mapping corresponding to this normal neighborhood. We may assume that $\varphi(U) \subset B(0, r_0)$, $0 < r_0 < 1$. Since $\overline{D_*}$ is compact in $\mathbb{S}_*$, we can cover $\overline{D_*}$ by a finite number of neighborhoods V_k, $k = 1, 2, \ldots, m$, such that $\psi_k : V_k \to B(0, R_k)$, $0 < R_k < 1$, and ψ_k are conformal mappings. Let $R_0 := \max\limits_{1\leqslant k\leqslant m} R_k$. Since the mapping f is continuous, the sets $U'_k := f^{-1}(V_k \cap D_*) \cap U$ are open in U and the mapping

$$f_k := \psi_k \circ f \circ \varphi^{-1}$$

is a mapping from $\varphi(U'_k) \subset B(0, r_0)$ into $\psi_k(V_k) \subset B(0, R_0)$.

Set $U_1 = U'_1$, $U_2 = U'_2 \setminus U'_1$, $U_3 = U'_3 \setminus (U'_1 \cup U'_2)$, $\ldots$, $U_m = U'_m \setminus (U'_1 \cup U'_2 \ldots U'_{m-1})$. Observe that, by the definition, $U_m \subset U'_m$ for $m \geqslant 1$ and $U_s \cap U_k = \varnothing$ for $s \neq k$. Let Γ be a family $D_r := D \cap S_r$ of all intersections of circles $S_r = \widetilde{S}(p_0, r)$, $r \in (\varepsilon, r_0)$, with D. We fix an admissible function $\rho_* \in \operatorname{adm} f(\Gamma)$, $\rho_* \equiv 0$ outside $f(D)$, and put $\rho \equiv 0$ outside U and on B_0, and

$$\rho(p) := \rho_*(f(p))\|f'_k(\varphi(p))\| \qquad \text{for } p \in U_k \setminus B_0\,,$$

where the matrix norm of the derivative $\|g'(z)\|$ of a given function $g : \mathbb{D} \to \mathbb{C}$, $\mathbb{D} \subset \mathbb{C}$, as usual, is defined as $\|g'(z)\| = |g_z| + |g_{\bar z}|$, $g_z = (g_x - ig_y)/2$, $g_{\bar z} = (g_x + ig_y)/2$, $z = x + iy \in \mathbb{C}$. Observe that

$$D_r := D \cap S_r = \left(\bigcup\limits_{\substack{1\leqslant k\leqslant m\\ 1\leqslant l<\infty}} D^r_{kl}\right) \bigcup \left(\bigcup\limits_{k=1}^{m} S_r \cap U_k \cap B_*\right) \bigcup \left(\bigcup\limits_{k=1}^{m} S_r \cap U_k \cap B_0\right),$$

where $D_{kl}^r = S_r \cap U_k \cap B_l$. By Lemmas 3.2 and 3.3 $l_{f(D_r)}(f(U_k \cap B_0)) = 0$ and $l_{f(D_r)}(f(U_k \cap B_*)) = 0$ for any $1 \leqslant k \leqslant m$ and almost any $r \in (0, r_0)$. Thus,

$$1 \leqslant \int\limits_{D_r^*} \rho_*(p_*)\, ds_{\widetilde{h}_*}(p_*) \leqslant 2 \sum_{k=1}^{m} \sum_{l=1}^{\infty} \int\limits_{\psi_k(f(D_{kl}^r))} \frac{\rho_*(\psi_k^{-1}(z))}{1-|z|^2}\, d\mathcal{H}^1(z) \leqslant$$

$$\leqslant \frac{2}{1-R_0^2} \sum_{k=1}^{m} \sum_{l=1}^{\infty} \int\limits_{\psi_k(f(D_{kl}^r))} \rho_*(\psi_k^{-1}(z))\, d\mathcal{H}^1(z) \tag{29}$$

for almost any $r \in (0, r_0)$, where $\mathcal{H}^1$ denotes 1-dimensional Hausdorff measure. On the other hand, arguing at each set D_{kl}^r separately and using [20, item 1.7.6, theorem 2.10.43 and theorem 3.2.6], we obtain that

$$\int\limits_{D_{kl}^r} \rho(p)\, ds_{\widetilde{h}}(p) = 2 \int\limits_{\varphi(D_{kl}^r)} \frac{\rho_*(\psi_k^{-1}(f_k(y)))\|f_k'(y)\|}{1-|y|^2}\, |dy| \geqslant$$

$$2 \int\limits_{\varphi(D_{kl}^r)} \rho_*(\psi_k^{-1}(f_k(y)))\|f_k'(y)\|\, |dy| \geqslant 2 \int\limits_{\psi_k(f(D_{kl}^r))} \rho_*(\psi_k^{-1}(z))\, d\mathcal{H}^1(z)\,. \tag{30}$$

Summing (30) over all $1 \leqslant k \leqslant m$ and $1 \leqslant l < \infty$, and considering (29) and Lemma 2.3, we conclude that $\rho/(1-R_0^2) \in \operatorname{ext}\operatorname{adm}\Gamma$.

Using the change of variables on each B_l, $l = 1, 2, \ldots$ (see, e.g. [20, Theorem 3.2.5]), countable additivity of the Lebesgue integral, and also taking into account (2), we obtain the estimate

$$\frac{1}{(1-R_0^2)^2} \int\limits_{D} \frac{\rho^2(p)}{K_\mu(p)}\, d\widetilde{h}(p) =$$

$$= \frac{4}{(1-R_0^2)^2} \sum_{k=1}^{m} \sum_{l=1}^{\infty} \int\limits_{\varphi(U_k \cap B_l)} \frac{\rho_*^2((f\circ\varphi^{-1})(z))\|f_k'(z)\|^2}{(1-|z|^2)^2 K_\mu(\varphi^{-1}(z))}\, dm(z) \leqslant$$

$$\leqslant \frac{4}{(1-r_0^2)^2(1-R_0^2)^2} \sum_{k=1}^{m} \sum_{l=0}^{\infty} \int\limits_{\mathbb{S}_*} \rho_*^2(\psi_k^{-1}(y)) N(y, f_k, \varphi(U_k \cap B_l))\, dm(y) \leqslant$$

$$\leqslant \frac{4}{(1-r_0^2)^2(1-R_0^2)^2} \sum_{k=1}^{m} \int\limits_{\mathbb{S}_*} \rho_*^2(\psi_k^{-1}(y)) N(y, f_k, \varphi(U_k))\, dm(y) \leqslant$$

$$\leqslant \frac{4}{(1-r_0^2)^2(1-R_0^2)^2} \sum_{k=1}^{m} \int\limits_{\mathbb{S}_*} \frac{\rho_*^2(\psi_k^{-1}(y)) N(\psi_k^{-1}(y), f, U_k)}{(1-|y|^2)^2}\, dm(y) =$$

$$= \frac{1}{(1-r_0^2)^2(1-R_0^2)^2} \sum_{k=1}^{m} \int\limits_{\mathbb{S}_*} \rho_*^2(p_*) N(p_*, f, U_k)\, d\widetilde{h}_*(p_*) \leqslant$$

$$\leqslant \frac{N(f, D)}{(1-r_0^2)^2(1-R_0^2)^2} \int\limits_{\mathbb{S}_*} \rho_*^2(p_*)\, d\widetilde{h}_*(p_*)\,.$$

To complete the proof, one should put $c := \frac{1}{(1-r_0^2)^2(1-R_0^2)^2}$. □

4. Boundary extension of mappings with lower modulus distortion

So, we have established the main modulus inequality for the Sobolev classes, with which we will work further. Now let us talk on the boundary extension of the Sobolev classes, for which we consider an auxiliary class of mappings with the condition (7).

We recall the following definitions. A mapping $f : D \to D_*$, $f(D) = D_*$, is called *boundary preserving*, if $C(f, \partial D) \subset \partial D_*$, where, as usual,

$$C(f, \partial D) = \{p_* \in \mathbb{S}_* : \exists\ p_k \in D, p \in \partial D : p_k \to p, f(p_k) \to p_*, k \to \infty\}\,.$$

The following statement is established in [4, Theorem 3.3] for the case of the space $\mathbb{R}^n$. In our case, its validity directly follows from [25, Proposition 2.1].

Proposition 4.1. *Let D and D_* be domains in Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively. Let $f : D \to D_*$ be open discrete and closed mapping in D. Then f is boundary preserving.*

A Borel function $\rho : \mathbb{D} \to [0, \infty]$ is called *admissible* for the family Γ of paths γ in the sense of hyperbolic length, write $\rho \in \mathrm{adm}_h\,\Gamma$, if $\int\limits_{\gamma} \rho(z)\,ds_h(z) \geqslant 1$ for any path $\gamma \in \Gamma$, where $ds_h(z)$ is an element of length corresponding to (2). *The modulus* of the family Γ in the sense of a hyperbolic measure is the quantity $M_h(\Gamma) := \inf\limits_{\rho \in \mathrm{adm}_h\,\Gamma} \int\limits_{\mathbb{D}} \rho^2(z)\,dh(z)$, where $dh(z)$ is an element of the hyperbolic area. The following result holds, see [25, Remark 5.2].

Proposition 4.2. *Let Γ be a family of paths in $\mathbb{D}$. Now*

$$M_h(\Gamma) = M_e(\Gamma)\,.$$

An analogue of the following statement is established for the space $\mathbb{R}^n$ in Väisälä's monograph [2, Theorem 7.5].

Proposition 4.3. *Let $\mathbb{S}$ be a Riemannian surface and $p_0 \in \mathbb{S}$. Let U be a neighborhood of the point p_0 such that $\widetilde{h}(p, p_0) = h(\varphi(p), \varphi(p_0))$, where $\varphi = \pi^{-1}$ and π is the natural projection of the fundamental Dirichlet polygon D_0 with center at the point $\varphi(p_0)$ on $\mathbb{S}$. Let $0 < r_1 < r_2 < \mathrm{dist}\,(p_0, \partial U)$, $\widetilde{S}_i = \widetilde{S}(p_0, r_i)$, $i = 1, 2$, $\widetilde{A}(p_0, r_1, r_2) = \{p \in \mathbb{S} : r_1 < \widetilde{h}(p_0, p) < r_2\}$. If $\Gamma = \Gamma(\widetilde{S}_1, \widetilde{S}_2, \widetilde{A})$ is family of paths joining $\widetilde{S}_1$ and $\widetilde{S}_2$ in $\widetilde{A}$, then*

$$M(\Gamma) = \frac{2\pi}{L(r_1, r_2)}\,,$$

where $L(r_1, r_2) = \left(\frac{e^{r_2}-1}{e^{r_2}+1}\right) : \left(\frac{e^{r_1}-1}{e^{r_1}+1}\right)$.

Proof. By the definition of the mapping φ and the neighborhood U, $M(\Gamma) = M_h(\Gamma^*)$, where $\Gamma^* = \Gamma(S_h(\varphi(p_0), r_1), S_h(\varphi(p_0), r_2), A_h)$, $A_h := \{z \in \mathbb{D} : r_1 < h(z, \varphi(p_0)) < r_2\}$. By Proposition 4.2 $M(\Gamma) = M_e(\Gamma^*)$. The required conclusion now follows from [26, Corollary 5.18]. □

Let Ω be a domain in $\mathbb{C}$, or a domain in $\mathbb{S}$. According to [27, item 3], the connected set $E \subset \Omega$ is called *cut* if E is closed in Ω, $\overline{E} \cap \partial\Omega \neq \varnothing$ and $\Omega \setminus E$ consists of two components, the boundary of each of which intersects $\partial\Omega$. A sequence of cuts $E_1, E_2, \ldots, E_k, \ldots$ is called a *chain* if E_k separates E_{k-1} from E_{k+1} in Ω, that is, E_{k-1} and E_{k+1} belong to different components of $\Omega \setminus E_k$. It follows from the above definitions that one of the subdomains $\Omega \setminus E_k$ contains all E_m for $m > k$. This subdomain will be denoted by d_k. Two chains of cuts $\{\sigma_m\}$ and $\{\sigma'_k\}$ are called *equivalent* if for each $m = 1, 2, \ldots$ the domain d_m contains all domains d'_k except for a finite number, and for each $k = 1, 2, \ldots$, the domain d'_k also contains all domains d_m for excluding a finite number.

The following statement contains some information on important properties of domains with locally quasiconformal boundaries.

Lemma 4.4. *Suppose that a domain D in $\mathbb{S}$ has a locally quasiconformal boundary, $p_0 \in \partial D$ and $r_m > 0$ is an arbitrary sequence such that $r_m \to 0$ as $m \to \infty$. Let U be a neighborhood of the point p_0, for which there is a quasiconformal mapping $\psi : U \to \mathbb{D}$, $\psi(\partial D \cap U) = \mathbb{D} \cap I$, $I := (-1, 1) = \{z \in \mathbb{D} : y = 0, z = x + iy\}$. Then:*

1) there is a sequence of neighborhoods U_k, $k = 1, 2, \ldots$, of the point p_0, contracting to this point, such that $\psi(U_k) = B(0, 1/2^k)$, $\psi(U_k \cap D) = \{z = x + iy \in B(0, 1/2^k) : x > 0\}$;

2) the sets $\sigma_k := \partial U_k \cap D$ are cuts of the domain D, in this case, $\widetilde{h}(\sigma_k) \to 0$ as $k \to \infty$, $\widetilde{h}(\sigma_k) := \sup\limits_{x,y \in \sigma_k} \widetilde{h}(x, y)$, and the corresponding domain d_k is $U_k \cap D$;

3) there is a subsequence $r_{m_l} \to 0$, $l \to \infty$, and the corresponding sequence of arcs $\gamma_l \subset \widetilde{S}(p_0, r_{m_l}) \cap D$, $l = 1, 2, \ldots$, forming a chain of cuts equivalent to σ_k, $k = 1, 2, \ldots$.

Proof. Arguing similarly to the proof of Theorem 17.10 in [2], we show that the neighborhood U in the definition of a locally quasiconformal boundary can be chosen arbitrarily small. Indeed, by definition, the point $p_0 \in \partial D$ has a neighborhood U, which can be mapped by a quasiconformal mapping ψ onto the unit disk $\mathbb{D} \subset \mathbb{C}$ so that $\psi(\partial D \cap U) = I$, where $I := (-1, 1) = \{z \in \mathbb{D} : x = 0, z = x + iy\}$. Since ψ is a homeomorphism, then either $\psi(U \cap D) = \mathbb{D}_+$, or $\psi(U \cap D) = \mathbb{D} \setminus \overline{\mathbb{D}_+}$, where $\mathbb{D}_+ := \{z \in \mathbb{D} : x > 0, z = x + iy\}$. Thus, without loss of generality, we may assume that $\psi(U \cap D) = \mathbb{D}_+$. Choose a neighborhood $V \subset U$ containing the point p_0. If $r < 1 - |\psi(p_0)|$, then, by the triangle inequality, the ball $B(\psi(p_0), r)$ lies strictly inside $\mathbb{D}$. Since ψ is a homeomorphism in U, then, in particular, ψ^{-1} is a continuous mapping. In this case, there is $r < 1 - |\psi(p_0)|$ with the following property: the condition $|\psi(p) - \psi(p_0)| < r$ implies that $p \in V$. In addition, if $\psi(p) \in \mathbb{D}_+ \cup I$ and $|\psi(p) - \psi(p_0)| < r$, then $p \in V \cap \overline{D}$. Setting $U_1 := \psi^{-1}(B(\psi(p_0), r))$, we note that $U_1 \subset V$ and U_1 is a neighborhood of the point p_0. In this case, $U_1 \cap \overline{D} = \psi^{-1}(B(\psi(p_0), r) \cap (\mathbb{D}_+ \cup I))$. Setting $H(p) = (\psi(p) - \psi(p_0))/r$, we obtain the mapping H of the neighborhood U_1 on $\mathbb{D}$ such that $H(U_1 \cap \overline{D}) = \mathbb{D}_+ \cup I$ and $H(p_0) = 0$. Since H is a homeomorphism, it follows that $H(U_1 \cap \partial D) = I$. It is also clear that if the original mapping ψ is quasiconformal, then the same is the mapping H. Thus, the neighborhood U_1 satisfies all the same conditions as the original neighborhood U. In what follows, we use the notation ψ instead of H, and we assume that $\psi(p_0) = 0$.

From the above reasoning it follows that there is a decreasing sequence of neighborhoods U_k of the point p_0, for which $p_0 = \bigcap\limits_{k=1}^{\infty} \overline{U_k \cap D}$, $\psi(U_k) = B(0, 1/2^k)$, $\psi(\partial U_k \cap D) = S(0, 1/2^k) \cap \mathbb{D}_+$. By direct calculations it is easy to see that that the sequence $\sigma_k := \partial U_k \cap D$ forms a chain of cuts of the domain D. From the equality $\bigcap\limits_{k=1}^{\infty} \overline{U_k \cap D} = p_0$ it follows that $\widetilde{h}(\sigma_k) \leqslant \widetilde{h}(\overline{U_k \cap D}) \to 0$ as $k \to \infty$, where we use the notation $\widetilde{h}(A) := \sup\limits_{x,y \in A} \widetilde{h}(x, y)$. Items 1) and 2) of Lemma 4.4 are established. It remains to establish item 3). Consider the segment $\beta(t) = it$, $t \in (0, 1)$, in $\mathbb{D}_+$. Put $\alpha(t) := \psi^{-1}(\beta(t))$. Then α is a path in $U \cap D$ with origin at the point p_0. Let $m_1 \in \mathbb{N}$ be that $r_{m_1} < \widetilde{h}(p_0, \sigma_1)$. By [28, Theorem 1.I.5, § 46] $\widetilde{S}(p_0, r_{m_1}) \cap \alpha \neq \varnothing$. Let γ_1 be an arc of the set $\widetilde{S}(p_0, r_{m_1}) \cap D$, such that $\alpha(t_1) \in \gamma_1$, where $t_1 := \min\{t \in (0, 1) : \alpha(t) \in \widetilde{S}(p_0, r_{m_1})\}$ (see Figure 3).

By construction, the ends of the path γ_1 lie on ∂D, therefore $\psi(\gamma_1)$ is a path whose ends lie on the segment $I \subset \mathbb{D}$. Obviously, $\psi(\gamma_1)$ splits $\mathbb{D}_+$ into two domains. Therefore, γ_1 divides the domain D into two domains, as well. Let d_1' be the component of $D \setminus \gamma_1$, containing the path $\alpha_1 := \alpha|_{(0, t_1)}$. Note that there is $k_1 \in \mathbb{N}$ such that $\overline{U_{k_1}} \cap D \subset d_1'$. Indeed, by construction $\overline{U_{k_1}} \subset \widetilde{B}(p_0, r_{m_1})$ for sufficiently large $k_1 \in \mathbb{N}$ and the point $\alpha_1(t)$ belongs to $U_{k_1} \cap D$ for sufficiently small t, since U_{k_1} is a neighborhood of the point p_0. Hence, $\overline{U_{k_1}} \cap D$ belongs to some component of $D \setminus \gamma_1$ containing α_1, that is, $\overline{U_{k_1}} \cap D \subset d_1'$. Note that $\sigma_{k_1} \subset d_1'$, because $\sigma_{k_1} = \partial U_{k_1} \cap D \subset d_1'$.

Observe also that $d_1' \subset D \setminus \sigma_1$. This follows from the fact that d_1' is bounded by γ_1 and some part of the boundary of the domain $U \cap D$, and σ_1 lies in the domain bounded by γ_1 and another part of the boundary $U \cap D$. Thus, γ_1 separates σ_1 from $U_{k_1} \cap D$ in D; in particular, γ_1 separates σ_1 from σ_{k_1} in D.

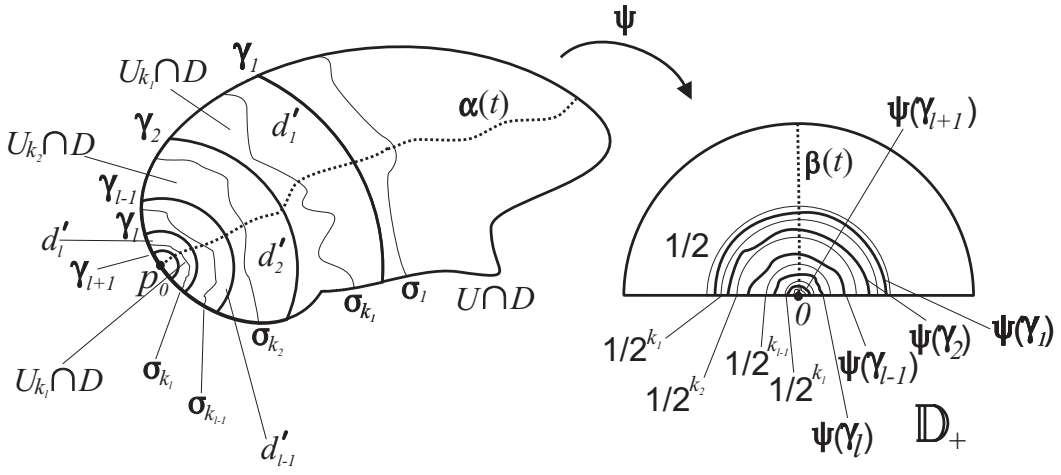

Figure 3: On the proof of Lemma 4.4

Let $m_2 \in \mathbb{N}$, $m_2 > m_1$, be such that $r_{m_2} < \widetilde{h}(p_0, \sigma_{k_1})$. By [28, Theorem 1.I.5, § 46] $\widetilde{S}(p_0, r_{m_2}) \cap \alpha \neq \varnothing$. Let γ_2 be an arc of the set $\widetilde{S}(p_0, r_{m_2}) \cap D$ such that $\alpha(t_2) \in \gamma_2$, where $t_2 := \min\{t \in (0,1) : \alpha(t) \in \widetilde{S}(p_0, r_{m_2})\}$. By construction, the ends of the path γ_2 lie on ∂D, therefore $\psi(\gamma_2)$ is a path whose ends lie on the segment $I \subset \mathbb{D}$. Obviously, $\psi(\gamma_2)$ splits $\mathbb{D}_+$ into two domains, therefore, γ_2 divides the domain D into two domains. Let d_2' be a component of $D \setminus \gamma_2$, containing a path $\alpha_2 := \alpha|_{(0,t_2)}$. Observe that there is $k_2 \in \mathbb{N}$ such that $\overline{U_{k_2}} \cap D \subset d_2'$. Indeed, by construction $\overline{U_{k_2}} \subset \widetilde{B}(p_0, r_{m_2})$ for large enough $k_2 \in \mathbb{N}$ and $\alpha_2(t)$ belongs to $U_{k_2} \cap D$ for sufficiently small t, since U_{k_2} is a neighborhood of the point p_0. Hence, $\overline{U_{k_2}} \cap D$ belongs to some component of $D \setminus \gamma_2$, containing α_2, that is, $\overline{U_{k_2}} \cap D \subset d_2'$. Observe that $\sigma_{k_2} = \partial U_{k_2} \cap D \subset d_2'$.

Observe also that, $d_2' \subset D \setminus \sigma_{k_1}$. Thus γ_2 separates σ_{k_1} from $U_{k_2} \cap D$ in D; in particular, γ_2 separates σ_{k_1} from σ_{k_2} in D.

For the same reason, $\gamma_1 \subset D \setminus d_2'$. Indeed, d_2' is one of the components of $D \setminus \gamma_1$, not the same as d_1', in particular, $d_2' \subset D \setminus \gamma_1$. Thus, γ_2 also separates γ_1 from $U_{k_2} \cap D$ in D; in particular, γ_2 separates γ_1 from σ_{k_2} in D.

Etc. As a result of the endless process, we will have that some sequence of arcs $\gamma_l \subset \widetilde{S}(p_0, r_{m_l})$, $l = 1, 2, \ldots$, separating σ_{k_l} from $\sigma_{k_{l-1}}$ in D, in this case,

(1) $U_{k_l} \cap D \subset d_l'$ and, in addition,

(2) γ_l separates γ_{l-1} from σ_{k_l} in D,

(3) $\sigma_{k_l} \in d_l'$;

(4) $\gamma_{l+1} \subset U_{k_l} \cap D$.

Let us show that γ_l separates γ_{l+1} from γ_{l-1} for any $l \in \mathbb{N}$, more precisely, show that $\gamma_{l+1} \subset d_l'$ and $\gamma_{l-1} \subset D \setminus d_l'$. Indeed, step by step **(1)** and **(4)** $\gamma_{l+1} \subset U_{k_l} \cap D \subset d_l'$. Since as proved, γ_l separates γ_{l-1} from σ_{k_l} in D and $\sigma_{k_l} \subset d_l'$ by **(3)**, by [28, Theorem 1.I.5, § 46] $\gamma_{l-1} \subset D \setminus d_l'$, as required to establish.

It follows from the above that the sequence of cuts γ_l, $l = 1, 2, \ldots$, forms a chain. Note that the sequences of cuts γ_l and σ_k are equivalent. Indeed, given $l \in \mathbb{N}$, by the property **(1)** $U_{k_l} \cap D \subset d_l'$, therefore also $U_k \cap D \subset d_l'$ for $k \geqslant k_l$. Conversely, fix $k \in \mathbb{N}$ and consider the corresponding number $l = l(k) \in \mathbb{N}$ such that $k_l \geqslant k$, where k_l, $l = 1, 2, \ldots$ is the subsequence constructed above. Notice, that d_{l+1}' belongs to exactly one of the components of $D \setminus \sigma_{k_l}$, namely, either $d_{l+1}' \subset U_{k_l} \cap D$, or $d_{l+1}' \subset D \setminus U_{k_l}$. On the other hand, by the condition **(3)** $\sigma_{k_{l+1}} \subset d_{l+1}'$, in addition, $\sigma_{k_{l+1}} \subset U_{k_{l+1}-1} \cap D \subset U_{k_l} \cap D$, because σ_k, $k = 1, 2, \ldots$ is a chain of cuts. Moreover, $k_{l+1} - 1 \geqslant k_l$. In this case, d_{l+1}' belongs to the component of $D \setminus \sigma_{k_l}$, containing $\sigma_{k_{l+1}}$, that is,

$d_l' \subset U_{k_l} \cap D \subset U_k \cap D$. Then also $d_l' \subset U_k \cap D$ for any $l \geqslant l(k)$. Equivalence of chains γ_l and σ_k is established. Lemma 4.4 is completely proved. $\square$

The following lemma is technically necessary to establish the main result on the boundary behavior of mappings. We have specially highlighted it into a separate statement, emphasizing that it refers to mappings satisfying rather general topological conditions.

Lemma 4.5. *Let D and D_* be domains on Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively, let $\overline{D_*}$ be a compactum in $\mathbb{S}_*$, let $p_0 \in \partial D$ and let $Q : D \to (0, \infty)$ be a given function measurable with respect to measure $\widetilde{h}$. Let also $f : D \to D_*$ be an open discrete closed mapping of the domain D onto the domain $f(D) = D_*$. Suppose the domain D has a locally quasiconformal boundary , and the boundary of the domain D_* is strongly accessible.*

Suppose that $p_0 \in \partial D$, and that there are at least two sequences $p_i, p_i' \in D$, $i = 1, 2, \ldots$, such that $p_i \to p_0$, $p_i' \to p_0$ as $i \to \infty$, $f(p_i) \to y$, $f(p_i') \to y'$ as $i \to \infty$ and $y' \neq y$.

Then there are $0 < \delta_0'$ and $l_0 > 0$ such that the inequality

$$l(f(\widetilde{S}(p_0, r) \cap D)) \geqslant l_0, \quad \forall\ r \in (0, \delta_0')\,, \tag{31}$$

where l denotes the length of the path (dashed line) on the Riemannian surface $\mathbb{S}_*$.

Proof. By the definition of a strongly accessible boundary at the point $y \in \partial D_*$, for the neighborhood U of the point y, not containing the point y', there is a compact set $C_0' \subset D_*$, a neighborhood V of the point y, $V \subset U$, and a number $\delta > 0$ such that

$$M(\Gamma(C_0', F, D_*)) \geq \delta > 0 \tag{32}$$

for an any continuum F, intersecting ∂U and ∂V. By Lemma 4.4, there is a sequence of neighborhoods U_i, $i = 1, 2, \ldots$, of p_0, such that the set $d_i := U_i \cap D$ is connected. Without loss of generality, we may assume that p_i and p_i' belong to d_i. In this case, join the points p_i and p_i' by the path α_i, lying in d_i. Since $f(p_i) \in V$ and $f(p_i') \in D \setminus \overline{U}$ for sufficiently large $i \in \mathbb{N}$, there is a number $i_0 \in \mathbb{N}$, such that by (32)

$$M(\Gamma(C_0', f(|\alpha_i|), D_*)) \geq \delta > 0 \tag{33}$$

for any $i \geq i_0 \in \mathbb{N}$ (see Figure 4). Let us prove Lemma 4.5 by contradiction. Suppose that (31) does not hold. Then for any $k \in \mathbb{N}$ there is $r = r_k > 0$ such that $l(f(\widetilde{S}(p_0, r_k) \cap D)) < 1/k$, $r_k \to 0$, $k \to \infty$.

By Lemma 4.4, there is a subsequence r_{k_i} of the sequence r_k and some sequence of arcs $\gamma_i \subset \widetilde{S}(p_0, r_{k_i})$ such that $d_i \subset d_i'$ and d_i' is the corresponding component of the set $D \setminus |\gamma_i|$, $i = 1, 2, \ldots$. Let ζ_i, $i = 1, 2, \ldots$, is an arbitrary sequence of points from $f(|\gamma_i|)$. Since $\overline{D_*}$ is a compactum in $\mathbb{S}_*$, we may assume that $\zeta_i \to \zeta_0$ as $i \to \infty$, $\zeta_0 \in \overline{D_*}$. Observe that $\zeta_i = f(x_i)$, $x_i \in \widetilde{S}(p_0, r_{k_i}) \cap D$, so $\zeta_0 \in \partial D_*$ by the closeness of the mapping f and Proposition 4.1.

Note that, since the mapping f is closed, there is a number $i_0 \in \mathbb{N}$ such that

$$C_0' \subset D_* \setminus \overline{f(d_i')}\,. \tag{34}$$

for $i \geqslant i_0$. Indeed, if we assume that the inclusion (34) fails, then there is a sequence $i_m > 0$, $m = 1, 2, \ldots$, $i_m \to \infty$ as $m \to \infty$, and $y_m \in \overline{f(d_{i_m}')} \cap C_0'$. Since C_0' is a compactum in $f(D)$, we may assume that $y_m \to y_0 \in C_0'$ as $m \to \infty$. Since $y_m \in \overline{f(d_{i_m}')} \cap C_0'$, for any $m \in \mathbb{N}$ there is a sequence $y_{km} \in \overline{f(d_{i_m}')}$ such that $y_{km} \to y_m$ as $k \to \infty$. Observe that $y_{km} = f(q_{km})$, $q_{km} \in d_{i_m}'$.

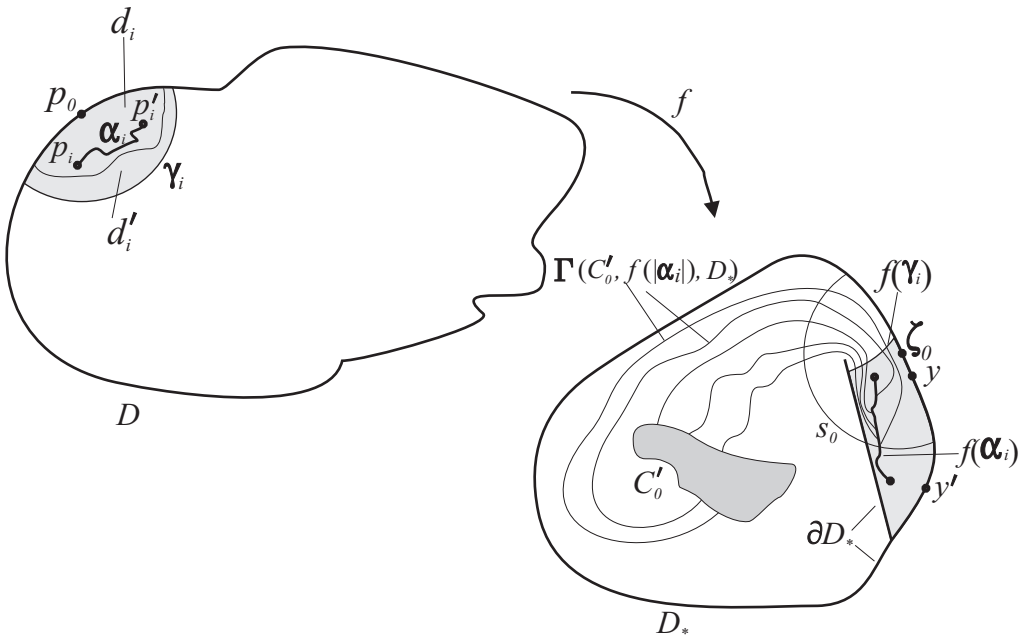

Figure 4: To the proof of Lemma 4.5

Due to the convergence of y_{k1} to y_1, for the number $1/2$, there is a number k_1 such that $\widetilde{h}_*(y_1, y_{k_1 1}) < 1/2$. Similarly, due to the convergence of y_{k1} to y_2, for the number $1/4$ there is a number k_2 such that $\widetilde{h}_*(y_2, y_{k_2 2}) < 1/4$. Generally, since y_{km} converges to y_m, for the number $1/2^m$ there is a number k_m such that $\widetilde{h}_*(y_m, y_{k_m m}) < 1/2^m$. But then, since by the construction of $y_m \to y_0$ as $m \to \infty$, for any fixed $\varepsilon > 0$ we have that

$$\widetilde{h}_*(y_0, y_{k_m m}) \leqslant \widetilde{h}_*(y_0, y_m) + \widetilde{h}_*(y_m, y_{k_m m}) \leqslant \varepsilon + 1/2^m$$

for any $m \geqslant M = M(\varepsilon)$, and thus $y_{k_m m} \to y_0$ as $m \to \infty$. But, on the other hand hand, $y_{k_m m} = f(q_{k_m m})$, $q_{k_m m} \in d'_{i_m}$, therefore $y_0 \in C(f, p_0)$, which contradicts the closeness of the mapping f. Indeed, by Lemma 4.4 there exists a sequence $t_i > 0$, $t_i \to 0$ as $i \to \infty$, such that $d_i' \subset \widetilde{B}(p_0, t_i)$. Then $y_0 \in C(f, \partial D) \subset \partial D_*$ (see Proposition 4.1). At the same time, $y_0 \in C_0'$, that is, y_0 is an inner point of the domain D_*. The resulting contradiction indicates the validity of the inclusion (34).

Let us now show that

$$\partial f(d_i') \cap f(D) \subset f(|\gamma_i|) \tag{35}$$

for any $i \in \mathbb{N}$.

Indeed, let $y_0 \in \partial f(d_i') \cap f(D)$, then there is a sequence $y_m \in f(d_i') \cap f(D)$ such that $y_m \in f(d_i') \cap f(D)$, $y_m \to y_0$ as $m \to \infty$, where $y_m = f(\xi_m)$, $\xi_m \in d_i'$. Without loss of generality, we may assume that $\xi_m \to \xi_0$ as $m \to \infty$. Note that the case $\xi_0 \in \partial D$ is impossible, since in this case $y_0 \in C(f, \partial D)$, which contradicts the closeness of the mapping f. Then $\xi_0 \in D$. Two situations are possible: 1) $\xi_0 \in d_i'$ and 2) $\xi_0 \in |\gamma_i|$. Note that case 1) is impossible, since, in this case, $f(\xi_0) = y_0$ and y_0 is an inner point of the set $f(d_i')$ by the openness of the mapping f, which contradicts the choice of y_0. Thus, the inclusion (35) is established.

By the assumption $l(f(\widetilde{S}(p_0, r_k) \cap D)) < 1/k$, $r_k \to 0$ as $k \to \infty$, we obtain that

$$\widetilde{h}_*(f(|\gamma_i|)) \to 0$$

as $i \to \infty$, $\widetilde{h}_*(f(|\gamma_i|)) := \sup\limits_{p_*, q_* \in f(|\gamma_i|)} \widetilde{h}_*(p_*, q_*)$. Then, for any $s \in \mathbb{N}$ there is a number $i_s \in \mathbb{N}$ such that $f(|\gamma_{i_s}|) \subset \widetilde{B}(\zeta_0, 1/s)$. Since C_0' is a compactum in D_*, there is $s_0 \geqslant 1$ such that $C_0' \cap \widetilde{B}(\zeta_0, 1/s_0) = \varnothing$.

Now, note that $\Gamma(f(|\gamma_{i_s}|), C_0', D_*) > \Gamma(\widetilde{S}(\zeta_0, 1/s_0), \widetilde{S}(\zeta_0, 1/s), D_*)$ for any $s > s_0$ (see [28, Theorem 1.I.5, § 46]). Hence, by the minorization of the modulus of families of paths and by Proposition 4.3

$$M(\Gamma(f(|\gamma_{i_s}|), C_0', D_*)) \leqslant M(\Gamma(\widetilde{S}(\zeta_0, 1/s_0), \widetilde{S}(\zeta_0, 1/s), D_*)) \to 0 \tag{36}$$

as $s \to \infty$. On the other hand, recall that $|\alpha_{i_s}| \subset d_{i_s} \subset d_{i_s}'$. Now, by (34)

$$|\beta| \cap f(d_{i_s}') \neq \varnothing \neq |\beta| \cap (D_* \setminus f(d_{i_s}')\,,$$

for any path $\beta \in \Gamma(f(|\alpha_{i_s}|), C_0', D_*)$. Thus, by [28, Theorem 1.I.5, § 46] and by (35),

$$\Gamma(f(|\alpha_{i_s}|), C_0', D_*) > \Gamma(f(|\gamma_{i_s}|), C_0', D_*).$$

From this and by (36), we obtain $M(\Gamma(f(|\alpha_{i_s}|), C_0', D_*)) \to 0$ as $s \to \infty$, which contradicts relation (33). The resulting contradiction indicates the validity of the inequality (31). □

Theorem 4.6. *Let D and D_* be domains on Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively, let $\overline{D_*}$ be a compactum in $\mathbb{S}_*$, let $p_0 \in \partial D$ and let $Q : D \to (0, \infty)$ be a given function measurable with respect to the measure $\widetilde{h}$. Let also $f : D \to D_*$ be an open discrete closed mapping of D onto $f(D) = D_*$ with the condition (7) at the point p_0. Suppose that the domain D has a locally quasiconformal boundary, and the boundary of the domain D_* is strongly accessible. If the relations*

$$\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dt}{\|Q\|(t)} < \infty\,, \qquad \int\limits_{0}^{\varepsilon_0} \frac{dt}{\|Q\|(t)} = \infty\,, \tag{37}$$

hold for some $0 < \varepsilon_0$ and any $\varepsilon \in (0, \varepsilon_0)$, where $\|Q\|(r) := \int\limits_{\widetilde{S}(p_0, r)} Q(p)\, ds_{\widetilde{h}}(p)$, then f has a continuous extension to p_0.

Proof. Suppose the opposite. Then, since $\overline{D_*}$ is compact in $\mathbb{S}_*$, there are at least two sequences p_i, $p_i' \in D$, $i = 1, 2, \ldots$, such that $p_i \to p_0$, $p_i' \to p_0$ as $i \to \infty$, $f(p_i) \to y$, $f(p_i') \to y'$ as $i \to \infty$ and $y' \neq y$. Let Γ_i^{δ} be a family of all dished lines $f(\widetilde{S}(p_0, r) \cap D)$, $r \in (2^{-i}, \delta)$. By Lemmas 2.4 and 2.2, there is $0 < d_0 < \sup\limits_{p \in D} \widetilde{h}(p, p_0)$ such that

$$M(\Gamma_i^{\delta_0}) \geqslant \int\limits_{2^{-i}}^{\delta_0} \frac{dr}{\|Q\|(r)} \quad \forall\ i \in \mathbb{N}\,, \tag{38}$$

for any $0 < \delta_0 < d_0$, where $\|Q\|(r) = \int\limits_{D(p_0, r)} Q(p)\, ds_{\widetilde{h}}(p)$ denotes L_1-norm of the function Q under the circle $D(p_0, r) := \widetilde{S}(p_0, r) \cap D$. By (38) and (37) we obtain that

$$M(\Gamma_i^{\delta_0}) \to \infty\,, \quad i \to \infty\,. \tag{39}$$

On the other hand, by Lemma 4.5 there are $0 < \delta_0' < d_0$ and $l_0 > 0$ such that

$$l(f(\widetilde{S}(p_0, r) \cap D)) \geqslant l_0, \quad \forall\ r \in (0, \delta_0')\,,$$

where l denotes the length of the dished line in $\mathbb{S}_*$. In particular, the function

$$\rho(p)=\begin{cases} 1/l_0\,, & p\in D_*\,,\\ 0\,, & p\notin D_*\end{cases}$$

is admissible for $\Gamma_i^{\delta_0'}$, $0<\delta_0'<d_0$. Since $\overline{D_*}$ is a compactum in $\mathbb{S}_*$, the $\widetilde{h}_*$-area of D_* is finite. Therefore,

$$M(\Gamma_i^{\delta_0'})\leqslant 1/l_0^2\cdot\widetilde{h}_*(D_*)<\infty\,.$$

The last condition contradicts the relation (39) for $\delta_0:=\delta_0'$, which refutes the assumption that the mapping f has no limit at the point p_0. □

Proof of Theorem 1.2 follows immediately from Theorems 1.1 and 4.6. □

Let $p_0\in\mathbb{S}$ and let $\varphi:\mathbb{S}\to\mathbb{R}$ be a function integrable in some neighborhood U of the point p_0 with respect to $\widetilde{h}$. Following [29, Section 2] (see also [8, Section 6.1, Ch. 6]), we say that a function $\varphi:\mathbb{S}\to\mathbb{R}$ has a *finite mean oscillation* at the point $p_0\in D$, we write $\varphi\in FMO(p_0)$, if

$$\limsup_{\varepsilon\to 0}\frac{1}{\widetilde{h}(\widetilde{B}(p_0,\varepsilon))}\int\limits_{\widetilde{B}(p_0,\varepsilon)}|\varphi(p)-\overline{\varphi}_\varepsilon|\,d\widetilde{h}(p)<\infty\,,$$

where $\overline{\varphi}_\varepsilon=\frac{1}{\widetilde{h}(\widetilde{B}(p_0,\varepsilon))}\int\limits_{\widetilde{B}(p_0,\varepsilon)}\varphi(p)\,d\widetilde{h}(p)$. In what follows, we will talk about results related to the function of the finite mean oscillation, therefore it is extremely important for us to use the following two most important facts related to these functions.

Let D be a domain in $\mathbb{S}$, and let $\varphi:\mathbb{S}\to\mathbb{R}$ be a nonnegative function with a finite mean oscillation at the point $p_0\in\overline{D}\subset\mathbb{S}$, $\varphi(x)=0$ for $x\notin D$. By [17, Theorem 7.2.2], the surface $\mathbb{S}$ is locally Ahlfors 2-regular, so that by [9, Lemma 3]

$$\int\limits_{\varepsilon<\widetilde{h}(p,p_0)<\varepsilon_0}\frac{\varphi(p)\,d\widetilde{h}(p)}{\left(\widetilde{h}(p,p_0)\log\frac{1}{\widetilde{h}(p,p_0)}\right)^2}=O\left(\log\log\frac{1}{\varepsilon}\right)\tag{40}$$

as $\varepsilon\to 0$ for some $0<\varepsilon_0<\operatorname{dist}(p_0,\partial U)$ and some normal neighborhood U of the point p_0. The following statement may be proved similarly to [8, Lemma 7.4, Ch. 7], cf. [30, Lemma 3.7] or [10, Lemma 4.2].

Proposition 4.7. *Let $p_0\in\mathbb{S}$, let U be some normal neighborhood of p_0, $0<r_1<r_2<\operatorname{dist}(p_0,\partial U)$, and let $Q:\mathbb{S}\to[0,\infty]$ be an integrable function in U with respect to the measure $\widetilde{h}$. Set $\widetilde{A}=\widetilde{A}(p_0,r_1,r_2)=\{p\in\mathbb{S}:r_1<\widetilde{h}(p,p_0)<r_2\}$, $\|Q\|(r)=\int\limits_{\widetilde{S}(p_0,r)}Q(p)\,ds_{\widetilde{h}}(p)$, $\eta_0(r):=\frac{1}{J\cdot\|Q\|(r)}$, where $J=J(p_0,r_1,r_2):=\int\limits_{r_1}^{r_2}\frac{dr}{\|Q\|(r)}$. Then*

$$J^{-1}=\int\limits_{\widetilde{A}(p_0,r_1,r_2)}Q(p)\cdot\eta_0^2(\widetilde{h}(p,p_0))\,d\widetilde{h}(p)\leqslant$$

$$\leqslant\int\limits_{\widetilde{A}(p_0,r_1,r_2)}Q(p)\cdot\eta^2(\widetilde{h}(p,p_0))\,d\widetilde{h}(p)\tag{41}$$

for any Lebesgue measurable function $\eta:(r_1,r_2)\to[0,\infty]$ such that $\int\limits_{r_1}^{r_2}\eta(r)\,dr=1$.

We now state and prove the following statement.

Theorem 4.8. *The conclusion of Theorem 1.2 holds, if instead of conditions (8) we require that* $Q \in FMO(p_0)$.

Proof. Set $\psi(t) = \frac{1}{\left(t \log \frac{1}{t}\right)}$. Observe that $I(\varepsilon, \varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt \geqslant \log \frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{\varepsilon_0}}$. Set $\eta(t) := \psi(t)/I(\varepsilon, \varepsilon_0)$. Then, by the relation (40), there is a constant $C > 0$ such that

$$\int\limits_{\widetilde{A}(p_0,\varepsilon,\varepsilon_0)} Q(p) \cdot \eta^2(\widetilde{h}(p, p_0))\, d\widetilde{h}(p) = \frac{1}{I^2(\varepsilon, \varepsilon_0)} \int\limits_{\varepsilon < \widetilde{h}(p,p_0) < \varepsilon_0} \frac{Q(p)\, d\widetilde{h}(p)}{\left(\widetilde{h}(p, p_0) \log \frac{1}{\widetilde{h}(p,p_0)}\right)^2} \leqslant$$

$$\leqslant C \cdot \left(\log \frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{\varepsilon_0}}\right)^{-1} \to 0 \tag{42}$$

as $\varepsilon \to 0$. Then by (41), taking into account (42), conditions (8) follow, in view of which the required conclusion follows directly from Theorem 1.2. □

5. Equicontinuity of families homeomorphisms

Our immediate goal is to prove the equicontinuity of the classes of mappings consisting of Sobolev homeomorphisms of finite distortion. First of all, let us clarify the question on the equicontinuity of these families at the inner points of the domain. Let us recall some definitions. Let (X, d) and (X', d') be metric spaces with distances d and d', respectively. A family $\mathfrak{F}$ of mappings $f : X \to X'$ is called *equicontinuous at the point* $x_0 \in X$, if for any $\varepsilon > 0$ there is $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ for all $x \in X$ such that $d(x, x_0) < \delta$ and for all $f \in \mathfrak{F}$. A family $\mathfrak{F}$ is *equicontinuous* if $\mathfrak{F}$ is equicontinuous at every point $x_0 \in X$. Everywhere below, unless otherwise stated, $(X, d) = (\mathbb{S}, \widetilde{h})$ and $(X', d') = (\mathbb{S}_*, \widetilde{h}_*)$ are Riemannian surfaces with metrics $\widetilde{h}$ and $\widetilde{h}_*$, respectively.

Let $Q : \mathbb{S} \to [0, \infty]$ be a function measurable with respect to the measure $\widetilde{h}$ function, $Q(x) \equiv 0$ for $x \notin D \subset \mathbb{S}$. We say that $f : D \to \mathbb{S}_*$ is a *ring Q-mapping at* $p_0 \in \overline{D}$, if the relation

$$M(f(\Gamma(E_1,\, E_2,\, D))) \leqslant \int\limits_{A} Q(p) \cdot \eta^2(\widetilde{h}(p, p_0))\, d\widetilde{h}(p)\,, \tag{43}$$

holds for some $r_0 = r(p_0) > 0$, any ring $\widetilde{A} = \widetilde{A}(p_0, r_1, r_2) = \{p \in \mathbb{S} : r_1 < \widetilde{h}(p, p_0) < r_2\}$, $0 < r_1 < r_2 < r_0$, and any continua $E_1 \subset \overline{\widetilde{B}(p_0, r_1)} \cap D$, $E_2 \subset \left(\mathbb{S} \setminus \widetilde{B}(p_0, r_2)\right) \cap D$, where $\eta : (r_1, r_2) \to [0, \infty]$ may be arbitrary nonnegative Lebesgue measurable function such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,. \tag{44}$$

The next definition can be found, e.g., in [31]. A domain $D \subset \mathbb{S}$ is called a *uniform* if for every $r > 0$ there is $\delta > 0$ such that $M(\Gamma(F, F^*, D)) \geqslant \delta$ for any continua F and F^* in D, satisfying the conditions $\widetilde{h}(F) \geqslant r$ and $\widetilde{h}(F^*) \geqslant r$. Domains D_i, $i \in I$, are called *equi-uniform* if for each $r > 0$ the above the inequality holds for every D_i with the same number δ.

For a given $\delta > 0$, $D \subset \mathbb{S}$ and a measurable function $Q : D \to [0, \infty]$ with respect to the measure $\widetilde{h}$, we denote by $\mathfrak{R}_{Q,\delta}(D)$ the family of all homeomorphisms $f : D \to \mathbb{S}_* \setminus G_f$ of class $W^{1,1}_{\rm loc}$ with finite a distortion, such that $K_f(p) \leqslant Q(p)$ for almost all $p \in D$, where G_f is some continuum in $\mathbb{S}_*$ and $\widetilde{h}_*(G_f) = \sup\limits_{x,y \in G_f} \widetilde{h}_*(x, y) \geqslant \delta$. The following statement holds.

Lemma 5.1. *The family $\mathfrak{R}_{Q,\delta}(D)$ is equicontinuous in D, if $\mathbb{S}_*$ is a uniform domain, $Q \in L^1_{\rm loc}$ and, for any $p_0 \in D$, one of the following conditions is satisfied: either (8), or $Q \in FMO(p_0)$.*

Proof. Since $\mathbb{S}$ is a manifold, $\mathbb{S}$ is locally compact and locally path connected. Besides that, $f(D)$ is a domain by Brower's theorem, see [32, Theorem VI 9 and Corollary].

Observe that the condition

$$\int\limits_{\varepsilon<\widetilde{h}(p,p_0)<\varepsilon_0} Q(x)\cdot\psi^2(\widetilde{h}(p,p_0))\,d\widetilde{h}(p) = o(I^2(\varepsilon,\varepsilon_0)) \tag{45}$$

holds as $\varepsilon\to 0$ for some nonnegative measurable function $\psi:(0,\infty)\to(0,\infty)$ such that $I(\varepsilon,\varepsilon_0):=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)dt<\infty$ for some $\varepsilon_0>0$ and any $\varepsilon\in(0,\varepsilon_0)$.

Indeed, if the relations (8) hold, then (45) holds by Proposition 4.7 by selecting a function $I(\varepsilon,\varepsilon_0):=\int\limits_{\varepsilon}^{\varepsilon_0}\frac{dr}{\|Q\|(r)}$. If $Q\in FMO(p_0)$ at any $p_0\in D$, then, by the reasoning given in the proof of Theorem 4.8, conditions (8) are satisfied, and therefore, by what was proved above, (45) holds.

Finally, by [12, Lemma 3.1], the mapping $f\in\mathfrak{R}_{Q,\delta}(D)$ satisfies (43) in D, therefore the desired conclusion follows from [25, Lemma 5.1]. □

Let us turn to the question on the equicontinuity of mappings in the closure of a domain. For this purpose, consider the following class mappings. Given $\delta>0$, $D\subset\mathbb{S}$, a continuum $A\subset D$ and a measurable function $Q:D\to[0,\infty]$ we denote $\mathfrak{F}_{Q,\delta,A}(D)$ the family of all Sobolev homeomorphisms $f:D\to\mathbb{S}_*\setminus G_f$ with a finite distortion such that $G_f\subset\mathbb{S}_*$ is some continuum satisfying the condition $\widetilde{h}_*(G_f)=\sup\limits_{x,y\in G_f}\widetilde{h}_*(x,y)\geqslant\delta$, moreover, $\widetilde{h}_*(f(A))\geqslant\delta$. An analogue of the following theorem was obtained in [31, Theorem 3.1] for quasiconformal of mappings of the Euclidean space.

Lemma 5.2. *Let D be a domain in $\mathbb{S}$ and $Q:\mathbb{S}\to(0,\infty)$ is a function locally integrable in D, $Q(x)\equiv 0$ for $x\in\mathbb{S}\setminus D$. Assume that, for any point $p_0\in\overline{D}$ there are $\varepsilon_0=\varepsilon_0(p_0)>0$ and a function $\psi:(0,\infty)\to(0,\infty)$ such that*

$$I(\varepsilon,\varepsilon_0):=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt<\infty\quad\forall\ \varepsilon\in(0,\varepsilon_0) \tag{46}$$

and, in addition,

$$\int\limits_{\varepsilon<\widetilde{h}(p,p_0)<\varepsilon_0} Q(x)\cdot\psi^2(\widetilde{h}(p,p_0))\,d\widetilde{h}(p) = o(I^2(\varepsilon,\varepsilon_0))\,,\quad \varepsilon\to 0\,. \tag{47}$$

Let $D_f=f(D)$. Assume also that D is locally connected on ∂D, $\overline{D_f}$ is a compact in $\mathbb{S}_*$ for any $f\in\mathfrak{F}_{Q,\delta,A}(D)$, besides that, domains D_f and $\mathbb{S}_*$ are equi-uniform over $f\in\mathfrak{F}_{Q,\delta,A}(D)$. Then any $f\in\mathfrak{F}_{Q,\delta,A}(D)$ has a continuous extension $\overline{f}:\overline{D}\to\overline{D_f}$ and, besides that, the family $\mathfrak{F}_{Q,\delta,A}(\overline{D})$ consisting of all extended mappings $\overline{f}:\overline{D}\to\overline{D_f}$ is equicontinuous in $\overline{D}$.

Proof. Observe that $\partial D_f=\partial f(D)$ is strongly accessible for any $f\in\mathfrak{F}_{Q,\delta,A}(D)$. Indeed, let $x_0\in\partial D_f$ and let U be an arbitrary neighborhood of x_0. Choose $\varepsilon_1>0$ such that $V:=\widetilde{B}(x_0,\varepsilon_1)$, $\overline{V}\subset U$. Let $\partial U\ne\varnothing$ and $\partial V\ne\varnothing$. Now $\varepsilon_2:=\widetilde{h}_*(\partial U,\partial V)>0$. Observe that, the inequalities $\widetilde{h}_*(F)\geqslant\varepsilon_2$ and $\widetilde{h}_*(G)\geqslant\varepsilon_2$ hold for any F and G in

D_f satisfying the conditions $F\cap\partial U\ne\varnothing\ne F\cap\partial V$ and $G\cap\partial U\ne\varnothing\ne G\cap\partial V$. Now, by the uniformity of D_f there is $\delta>0$ depending only on ε_2 such that

$$M(\Gamma(F,G,D_f))\geqslant\delta\,.$$

Thus, ∂D_f is strongly accessible. Now, by [12, Lemma 6.1] any $f\in\mathfrak{F}_{Q,\delta,A}(D)$ has a continuous extension $f:\overline{D}\to\overline{D_f}$.

Observe that $\mathfrak{F}_{Q,\delta,A}(D)\subset\mathfrak{R}_{Q,\delta}(D)$. Besides that, by (46)–(47) we obtain that the divergence conditions (8) hold. Indeed, in (41) we set $\eta(t)=\psi(t)/I(\varepsilon,\varepsilon_0)$ and let us use Proposition 4.7. Then the desired conclusion immediately follows from (41). In this case, the equicontinuity of the family $\mathfrak{F}_{Q,\delta,A}(D)$ in the inner points of the domain D follows directly from Lemma 5.1.

It remains to prove the equicontinuity of the family $\mathfrak{F}_{Q,\delta,A}(\overline{D})$ on ∂D. Suppose the opposite. Then there is $p_0\in\partial D$ and a number $a>0$ such that, for each $m=1,2,\ldots$ there is a point $p_m\in\overline{D}$ and an element $\overline{f}_m$ of the family $\mathfrak{F}_{Q,\delta,A}(\overline{D})$ such that $\widetilde{h}(p_0,p_m)<1/m$ and $\widetilde{h}_*(\overline{f}_m(p_m),\overline{f}_m(p_0))\geqslant a$. Since $f_m:=\overline{f}_m|_D$ has a continuous extension to the point p_0, we may assume that $p_m\in D$. In view of the same considerations, there is a sequence $f_m:=\overline{f}_m|_D p'_m\in D$, $p'_m\to p_0$ as $m\to\infty$ such that $\widetilde{h}_*(f_m(p'_m),\overline{f}_m(p_0))\leqslant 1/m$. Thus

$$\widetilde{h}_*(f_m(p_m),f_m(p'_m))\geqslant a/2\qquad\forall\ m\in\mathbb{N}\,. \tag{48}$$

Since D is locally connected at the point $p_0\in\partial D\subset\mathbb{S}$, and $\mathbb{S}$ is a smooth manifold, D is also locally path-connected at p_0 (see [8, Proposition 13.1]). In other words, for any neighborhood U of the point p_0 there is a neighborhood $V\subset U$ of the same point such that $V\cap D$ is a path-connected set. Then there is a sequence neighborhoods V_m of the point p_0 with $\widetilde{h}(V_m)\to 0$ as $m\to\infty$, such that the sets $D\cap V_m$ are domains and $D\cap V_m\subset\widetilde{B}(p_0,2^{-m})$. Without loss of generality, passing to a subsequence, if necessary, we may assume that $p_0\in\partial D\subset\mathbb{S}$, $p_m,p'_m\in D\cap V_m$. Join the points p_m and p'_m of the path $\gamma_m:[0,1]\to\mathbb{S}$ such that $\gamma_m(0)=p_m$, $\gamma_m(1)=p'_m$ and $\gamma_m(t)\in V_m\cap D$ for $t\in(0,1)$. Denote by C_m the image of the path $\gamma_m(t)$ under the mapping f_m. It follows from the relation (48) that

$$\widetilde{h}_*(C_m)\geqslant a/2\qquad\forall\, m\in\mathbb{N}\,, \tag{49}$$

where $\widetilde{h}_*(C_m)$ the diameter of the set C_m in the metrics $\widetilde{h}_*$ (see Figure 5).

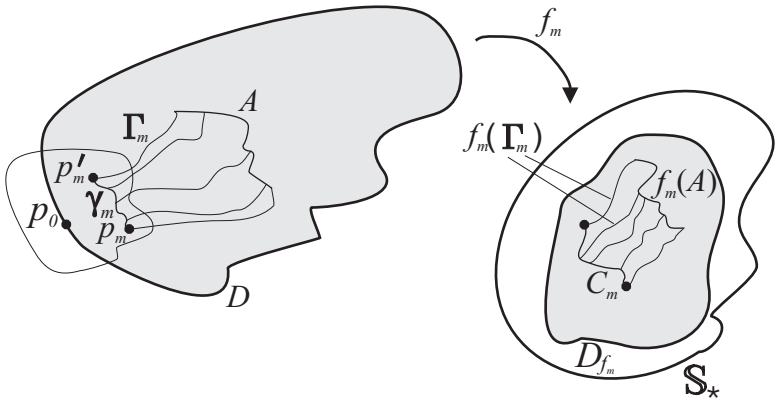

Figure 5: To the proof of Lemma 5.2

Without loss of generality, we may assume that the continuum A from the definition of the family $\mathfrak{F}_{Q,\delta,A}(D)$ is such that $\widetilde{B}(p_0,\varepsilon_0)\cap A=\varnothing$ and $B(p_0,2^{-m})\cap A=\varnothing$, $m=1,2,\ldots,$. Let Γ_m be family of paths joining $|\gamma_m|$ and A in D. By [12, Lemma 3.1] the mapping $f\in\mathfrak{R}_{Q,\delta}(D)$ satisfies the relation (43) in $\overline{D}$, so that

$$M(f_m(\Gamma_m))\leqslant\int\limits_{\widetilde{A}(p_0,2^{-m},\varepsilon_0)}Q(p)\cdot\eta^2(\widetilde{h}(p,p_0))\,d\widetilde{h}(p) \tag{50}$$

for any Lebesgue measurable function $\eta:(2^{-m},\varepsilon_0)\to[0,\infty]$ such that $\int\limits_{2^{-m}}^{\varepsilon_0}\eta(r)\,dr\geqslant 1$. Observe that, the function

$$\eta(t)=\begin{cases}\psi(t)/I(2^{-m},\varepsilon_0), & t\in(2^{-m},\varepsilon_0),\\ 0, & t\in\mathbb{R}\setminus(2^{-m},\varepsilon_0),\end{cases}$$

$I(\varepsilon,\varepsilon_0):=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)dt$, satisfies the condition (44) for $r_1:=2^{-m}$, $r_2:=\varepsilon_0$, therefore by (46)–(47) and (50) we obtain that

$$M(f_m(\Gamma_m))\leqslant\alpha(2^{-m})\to 0 \tag{51}$$

as $m\to\infty$, where $\alpha(\varepsilon)$ is some nonnegative function converging to zero as $\varepsilon\to 0$, which exists by (46)–(47).

On the other hand, observe that $f_m(\Gamma_m)=\Gamma(C_m,f_m(A),D_{f_m})$, moreover, $\widetilde{h}(f_m(A))\geqslant\delta$ for any $m\in\mathbb{N}$ by the definition of the class $\mathfrak{F}_{Q,\delta,A}(D)$. Taking into account the relation (49) and the definition of an equally uniform family of domains, we conclude that there exists $\sigma>0$ such that

$$M(f_m(\Gamma_m))=M(\Gamma(C_m,f_m(A),D_{f_m}))\geqslant\sigma\qquad\forall\ m\in\mathbb{N},$$

which contradicts the condition (51). The resulting contradiction proves the lemma. □

By Lemma 5.2 and Proposition 4.7 and also the reasoning used in the proof of Theorem 4.8, we obtain the following statement.

Theorem 5.3. *Let D be a domain in $\mathbb{S}$ and let $Q:\mathbb{S}\to(0,\infty)$ be a function locally integrable in D, $Q(x)\equiv 0$ for $x\in\mathbb{S}\setminus D$. Assume that, for any $p_0\in D$, one of the following conditions is satisfied: either (8), or $Q\in FMO(p_0)$. Let also D be locally path-connected on ∂D, $\overline{D_f}=\overline{f(D)}$ be a compactum in $\mathbb{S}_*$ for any $f\in\mathfrak{F}_{Q,\delta,A}(D)$. Assume that the domains D_f and $\mathbb{S}_*$ are equi-uniform over $f\in\mathfrak{F}_{Q,\delta,A}(D)$.*

Then any $f\in\mathfrak{F}_{Q,\delta,A}(D)$ has a continuous extension $\overline{f}:\overline{D}\to\overline{D_f}$ and, in addition, the family $\mathfrak{F}_{Q,\delta,A}(\overline{D})$, consisting of all extended maps $\overline{f}:\overline{D}\to\overline{D_f}$, is equicontinuous in $\overline{D}$.

6. Equicontinuity of Sobolev Classes with Branching

The question of the local and global behavior of mappings with a branching looks much more complicated, since for mappings of Riemannian surface estimates of the form (43) have not been established. Instead, we may only use the estimates (7) or (20), which are obtained in this paper. As in the previous section, let us start by investigation of mappings at interior points. Let us prove, first of all, the following auxiliary statement.

Proposition 6.1. *Let (X,d) be an arbitrary metric space with metric d and F_j, $j=1,2,\ldots,$ be a sequence of continua in X such that $d(F_j)=\sup\limits_{x,y\in F_j}d(x,y)\geqslant\delta\quad\forall\ j=1,2,\ldots$. Let $x_0\in X$ and $B(x_0,\delta/4)=\{x\in X: d(x,x_0)<\delta/4\}$. Then there is $\varepsilon_0>0$ and a sequence of continua C_j such that $C_j\subset F_j\setminus B(x_0,\delta/4)$ and $d(C_j)=\sup\limits_{x,y\in C_j}d(x,y)\geqslant\delta/4,\quad j=1,2,\ldots$.*

Proof. Fix $j \in \mathbb{N}$. If $F_j \cap B(x_0, \delta/4) = \varnothing$, there is nothing to prove. Let $F_j \cap B(x_0, \delta/4) \neq \varnothing$.

Since F_j is a continuum in X, there are $x_j, y_j \in F_j$ such that $d(F_j) = d(x_j, y_j)$. Since $d(F_j) \geqslant \delta$, at least one of the points x_j or y_j does not belong to $B(x_0, \delta/4)$, because, otherwise, by the triangle inequality $d(x_j, y_j) \leqslant d(x_j, x_0) + d(x_0, y_j) < \delta/2$. Let, for definiteness, $x_j \in D \setminus B(x_0, \delta/4)$. Then two situations are possible:

1) $y_j \in B(x_0, \delta/4)$. Let C_j be x_j-component of $F_j \setminus B(x_0, \delta/4)$. Since F_j is connected and $F_j \cap B(x_0, \delta/4) \neq \varnothing$, we obtain that $C_j \cap \overline{F_j \setminus C_j} \neq \varnothing$ (see, e.g., [28, Section I.5.46]). Observe that

$$F_j \setminus C_j = (F_j \cap B(x_0, \delta/4)) \cup \bigcup\limits_{\alpha \in A} K_\alpha \,, \tag{52}$$

where A is some set of indices α and $\bigcup\limits_{\alpha \in A} K_\alpha$ is a union of all components of $F_j \setminus B(x_0, \delta/4)$, except C_j. By [28, Theorem 1.III.46.5], K_α and C_j are closed disjoint sets in $F_j \setminus B(x_0, \delta/4)$, $\alpha \in A$. Then, by (52) the relation $C_j \cap \overline{F_j \setminus C_j} \neq \varnothing$ is possible if and only if $C_j \cap \overline{B(x_0, \delta/4)} \neq \varnothing$. Then there is $z_j \in C_j \cap S(x_0, \delta/4)$. By the triangle inequality

$$\delta \leqslant d(x_j, y_j) \leqslant d(x_j, z_j) + d(z_j, y_j) < d(C_j) + \delta/2 \,,$$

whence it follows that $d(C_j) > \delta/2$, as required. Consider the second situation:

2) $y_j \in D \setminus B(x_0, \delta/4)$. Let, as before, C_j be x_j-component of $F_j \setminus B(x_0, \delta/4)$, and let D_j be-y_j-$_j \setminus B(x_0, \delta/4)$. Reasoning similar to the above, we conclude that there are $z_j \in C_j \cap S(x_0, \delta/4)$ and $z_j' \in D_j \cap S(x_0, \delta/4)$. Then, by the triangle inequality

$$\delta \leqslant d(x_j, y_j) \leqslant d(x_j, z_j) + d(z_j, z_j') + d(z_j', y_j) \leqslant d(C_j) + d(D_j) + \delta/2 \,,$$

whence it follows that either $d(C_j) \geqslant \delta/4$, or $d(D_j) \geqslant \delta/4$. The proposition is proved. □

The next statement concerns the situation in which the images of two points under mappings are separated by a fixed nonzero number. It will be shown below that in this case the length of the images of circles centered at one of the points under these mappings is separated from zero from below.

Lemma 6.2. *Let D_* be a uniform domain in $\mathbb{S}_*$ such that $\overline{D_*}$ is a compactum. Let $f_k : D \to D_* \setminus G_k$, $k = 1, 2, \ldots$ be a family of mappings open in D such that $\widetilde{h}_*(G_k) = \sup\limits_{x, y \in G_k} \widetilde{h}_*(x, y) \geqslant \delta$, where $G_k \subset D_*$ is some continuum and the number δ does not depend on k.*

Suppose that $p_0 \in D$, $p_k \in D$, $k = 1, 2, \ldots$, and $\delta_0 > 0$ such that $p_k \to p_0$ as $k \to \infty$ and

$$\widetilde{h}_*(f_k(p_k), f_k(p_0)) \geqslant \delta_0 \quad k = 1, 2, \ldots \,. \tag{53}$$

Then there are $l_0 > 0$, $r_0 > 0$ and $k_0 \geqslant 1$ such that

$$l(f_k(\widetilde{S}(p_0, r)) \geqslant l_0, \quad \forall\ r \in (\widetilde{h}(p_0, p_k), r_0) \,, \quad \forall\ k \geqslant k_0 \,, \tag{54}$$

where l denotes the length of the path on the Riemannian surface $\mathbb{S}_*$.

Proof. Suppose the opposite. Then for each $i \in \mathbb{N}$ there are $k_i > i$ and $\widetilde{h}(p_0, p_{k_i}) < r_i < 1/i$ such that

$$l(f_{k_i}(\widetilde{S}(p_0, r_i))) < 1/i, \qquad i = 1, 2, \ldots, \qquad r_i \to 0 \tag{55}$$

as $i \to \infty$. Without loss of generality, we may assume that the sequence numbers k_i, $i = 1, 2, \ldots$ is increasing. Let ζ_i, $i = 1, 2, \ldots$, be an arbitrary sequence of points from $f_{k_i}(\widetilde{S}(p_0, r_i))$. Since $\overline{D_*}$ is a compactum in $\mathbb{S}_*$, we may assume that $\zeta_i \to \zeta_0$ as $i \to \infty$, $\zeta_0 \in \overline{D_*}$. Note that $\zeta_i = f_{k_i}(p_i')$, $p_i' \in \widetilde{S}(p_0, r_i)$, and that

$$G_{k_i} \subset D_* \setminus \overline{f_{k_i}(\widetilde{B}(p_0, r_i))} \,, \tag{56}$$

because by the condition $G_k \subset D_* \setminus f_k(D)$ for any $k \in \mathbb{N}$. Since f_{k_i} is open for any $i \in \mathbb{N}$, we obtain that

$$\partial f_{k_i}(\widetilde{B}(p_0, r_i)) \subset f_{k_i}(\widetilde{S}(p_0, r_i))\,. \tag{57}$$

The assumption (55) implies that

$$\widetilde{h}_*(f_{k_i}(\widetilde{S}(p_0, r_i))) \to 0$$

as $i \to \infty$, $\widetilde{h}_*(f_{k_i}(\widetilde{S}(p_0, r_i))) := \sup\limits_{p_*, q_* \in f_{k_i}(\widetilde{S}(p_0, r_i))} \widetilde{h}_*(p_*, q_*)$. Now, for any $s \in \mathbb{N}$ there exists a number $i_s \in \mathbb{N}$ such that

$$f_{k_i}(\widetilde{S}(p_0, r_i)) \subset \widetilde{B}(\zeta_0, 1/s)\,, \qquad i \geqslant i_s\,. \tag{58}$$

By Proposition 6.1, there is $s_0 \in \mathbb{N}$ and a sequence E_{k_i} of continua such that

$$E_{k_i} \subset G_{k_i} \setminus \widetilde{B}(\zeta_0, 1/s_0), \quad \widetilde{h}_*(E_{k_i}) \geqslant \delta/4\,, \quad i = 1, 2, \ldots\,. \tag{59}$$

We fix $s > s_0$ and consider the family $\Gamma(f_{k_i}(\widetilde{S}(p_0, r_i)), E_{k_i}, D_*)$ for $i \geqslant i_s$, see Figure 6.

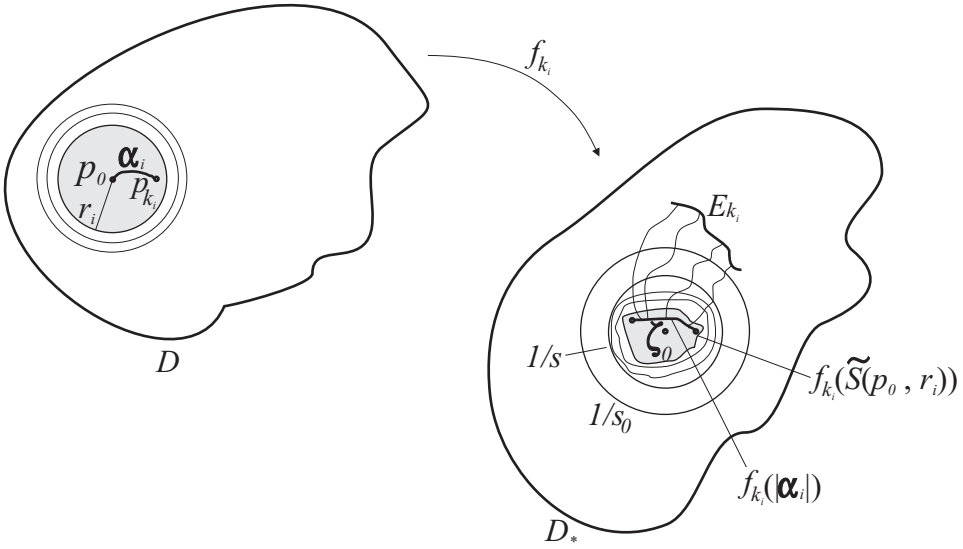

Figure 6: To the proof of Lemma 6.2

Let $\gamma \in \Gamma(f_{k_i}(\widetilde{S}(p_0, r_i)), E_{k_i}, D_*)$, i.e., $\gamma = \gamma(t)$, $t \in (0,1)$, $\gamma(0) \in f_{k_i}(\widetilde{S}(p_0, r_i))$, $\gamma(1) \in E_{k_i}$ and $\gamma(t) \in D_*$ for $t \in (0,1)$. By (58) and (59) $|\gamma| \cap \widetilde{B}(\zeta_0, 1/s) \neq \varnothing \neq |\gamma| \cap (D_* \setminus \widetilde{B}(\zeta_0, 1/s))$, therefore by [28, Theorem 1.I.5, § 46] there is $t_1 \in (0,1)$ such that $\gamma(t_1) \in \widetilde{S}(\zeta_0, 1/s)$. Without loss of generality we may assume that $\gamma(t) \in D_* \setminus \overline{\widetilde{B}(\zeta_0, 1/s)}$ for $t > t_1$. Set $\gamma_1 := \gamma|_{[t_1,1]}$. Again, by (58) and (59) $|\gamma_1| \cap \widetilde{B}(\zeta_0, 1/s_0) \neq \varnothing \neq |\gamma_1| \cap (D_* \setminus \widetilde{B}(\zeta_0, 1/s_0))$, therefore by [28, Theorem 1.I.5, § 46] there is $t_2 \in (t_1, 1) : \gamma_1(t_2) \in \widetilde{S}(\zeta_0, 1/s_0)$. Without loss of generality we may assume that $\gamma_1(t) \in \widetilde{B}(\zeta_0, 1/s_0)$ for $t \in (t_1, t_2)$. Set $\gamma_2 := \gamma_1|_{[t_1,t_2]}$, $\gamma_2 \in \Gamma(\widetilde{S}(\zeta_0, 1/s), \widetilde{S}(\zeta_0, 1/s_0), \widetilde{A}(\zeta_0, 1/s, 1/s_0))$, $\widetilde{A}(\zeta_0, 1/s, 1/s_0)) = \{p_* \in \mathbb{S}_* : 1/s < \widetilde{h}_*(p_*, \zeta_0) < 1/s_0\}$. From the above it follows that

$$\Gamma(f_{k_i}(\widetilde{S}(p_0, r_i)), E_{k_i}, D_*) > \Gamma(\widetilde{S}(\zeta_0, 1/s), \widetilde{S}(\zeta_0, 1/s_0), \widetilde{A}(\zeta_0, 1/s, 1/s_0))\,, \quad i \geqslant i_s\,,$$

and, therefore, in view of [33, Theorem 1(c)] and by Proposition 4.3

$$M(\Gamma(f_{k_i}(\widetilde{S}(p_0, r_i)), E_{k_i}, D_*)) \leqslant$$

$$\leqslant M(\Gamma(\widetilde{S}(\zeta_0, 1/s), \widetilde{S}(\zeta_0, 1/s_0), \widetilde{A}(\zeta_0, 1/s, 1/s_0))) \to 0 \tag{60}$$

as $s \to \infty$ for $i \geqslant i_s$. Let us fix $\varepsilon > 0$ and find for it a number $S = S(\varepsilon)$ such that

$$M(\Gamma(\widetilde{S}(\zeta_0, 1/s), \widetilde{S}(\zeta_0, 1/s_0), \widetilde{A}(\zeta_0, 1/s, 1/s_0))) < \varepsilon, \quad s > S(\varepsilon)\,.$$

Set $I_0 = I_0(\varepsilon) := i_{S(\varepsilon)}$. Now, it follows from (60) that

$$M(\Gamma(f_{k_i}(\widetilde{S}(p_0, r_i)), E_{k_i}, D_*)) < \varepsilon, \quad i > I_0 = I_0(\varepsilon)\,. \tag{61}$$

Since $\mathbb{S}$ is a smooth manifold, we may consider that the balls $\widetilde{B}(p_0, r_i)$ are path connected for any $i \in \mathbb{N}$. Let α_i be a path joining p_{k_i} and p_0 in $\widetilde{B}(p_0, r_i)$. By (53) $\widetilde{h}_*(f_{k_i}(|\alpha_i|)) \geqslant \delta_0$. Now, by the definition of a uniform domain

$$M(\Gamma(f_{k_i}(|\alpha_i|), E_{k_i}, D_*)) > \varepsilon_1\,, \quad \forall\ i \in \mathbb{N}\,. \tag{62}$$

On the other hand, by (56) and (57)

$$\Gamma(f_{k_i}(|\alpha_i|), E_{k_i}, D_*) > \Gamma(f_{k_i}(\widetilde{S}(p_0, r_i)), E_{k_i}, D_*)\,,$$

whence by [33, Theorem 1(c)] and also by (62)

$$\varepsilon_1 < M(\Gamma(f_{k_i}(|\alpha_i|), E_{k_i}, D_*)) \leqslant M(\Gamma(f_{k_i}(\widetilde{S}(p_0, r_i)), E_{k_i}, D_*))\,. \tag{63}$$

The inequalities (63) and (61) contradict each other, which proves 54). □

Let $D \subset \mathbb{S}$ and $D_* \subset \mathbb{S}_*$ be fixed domains. Given $\delta > 0$ and a measurable function $Q : \mathbb{S} \to (0, \infty)$ with a respect to the measure $\widetilde{h}$, $Q(x) \equiv 0$ for $x \in \mathbb{S} \setminus D$, we demote by $\mathfrak{G}_{Q,\delta}(D)$ the family of all open mappings $f : D \to D_* \setminus G_f$ satisfying the relation (7) in D such that $\widetilde{h}_*(G_f) = \sup\limits_{x,y \in G_f} \widetilde{h}_*(x, y) \geqslant \delta$, where $G_f \subset D_*$ is some continuum.

Lemma 6.3. *Assume that Q satisfies (8) in D, or $Q \in FMO(p_0)$ at any point $p_0 \in D$. If D_* is uniform and $\overline{D_*}$ is a compactum in $\mathbb{S}_*$, then the family $\mathfrak{G}_{Q,\delta}(D)$ is equicontinuous in D.*

Proof. In view of the reasoning used in the proof of Theorem 4.8, it suffices to establish Lemma 6.3 in the case when Q satisfies relations (8) in D.

Suppose that the conclusion of the lemma does not hold. Then there are $p_0 \in D$, $p_k \in D$, $k = 1, 2, \ldots$, $f_k \in \mathfrak{G}_{Q,\delta}(D)$ and δ_0 such that $p_k \to p_0$ as $k \to \infty$ and

$$\widetilde{h}_*(f_k(p_k), f_k(p_0)) \geqslant \delta_0\,. \tag{64}$$

By Lemma 6.2, there are $l_0 > 0$ and $r_0 > 0$ such that

$$l(f_k(\widetilde{S}(p_0, r)) \geqslant l_0 \tag{65}$$

for all $r \in (\widetilde{h}(p_0, p_k), r_0)$, for any $k \geqslant k_0$ and some $k_0 \geqslant 1$. Without loss of generality, we may assume that $r_0 < \varepsilon_0$, where ε_0 is a number from (8) that exists by the condition of the lemma. In this case, the function

$$\rho(p) = \begin{cases} 1/l_0\,, & p \in D_*\,, \\ 0\,, & p \notin D_* \end{cases}$$

is admissible for $\Gamma_k^{r_0}$, consisting of the union of the paths $f_k(\widetilde{S}(p_0, r))$ over all $r \in (\widetilde{h}(p_0, p_k), r_0)$, $k = 1, 2, \ldots$. In this case, by the definition of the modulus of families of paths,

$$M(\Gamma_k^{r_0}) \leqslant (1/l_0^2) \cdot \widetilde{h}_*(D_*) < \infty\,, \tag{66}$$

because $\overline{D_*}$ is a compactum in $\mathbb{S}_*$. On the other hand, by Lemma 2.4 and also by (8) we obtain that

$$M(\Gamma_k^{r_0}) \geqslant \int\limits_{\widetilde{h}(p_0,p_k)}^{r_0} \frac{dr}{\|Q\|(r)} \to \infty \tag{67}$$

as $k \to \infty$. The relations (66) and (67) contradict each other, which refutes the assumption made in (64). □

As usual, we formulate the main results of this section for Sobolev classes. For this purpose, consider the following definition of the family of mappings. Given numbers $\delta > 0$, $N \in \mathbb{N}$, a domain $D \subset \mathbb{S}$ and a function $Q : \mathbb{S} \to (0, \infty)$ that is measurable with respect to the measure $\widetilde{h}$, $Q(x) \equiv 0$ for $x \in \mathbb{S} \setminus D$, denote by $\mathfrak{Z}_{Q,\delta,N}(D)$ a family of all open discrete mappings $f : D \to D_* \setminus G_f$ of the class $W^{1,1}_{\rm loc}(D)$ with a finite distortion such that $K_f(p) \leqslant Q(p)$ for almost all $p \in D$, $N(f, D) \leqslant N$ and $\widetilde{h}_*(G_f) = \sup\limits_{x,y\in G_f} \widetilde{h}_*(x, y) \geqslant \delta$, where $G_f \subset D_*$ is some continuum. The following theorem holds.

Theorem 6.4. *Suppose that the function Q satisfies the relations (8) in D, or $Q \in FMO(p_0)$ at any point $p_0 \in D$. If the domain D_* is uniform and $\overline{D_*}$ is a compactum in $\mathbb{S}_*$, then the family $\mathfrak{Z}_{Q,\delta,N}(D)$ is equicontinuous at any point $p_0 \in D$.*

Proof immediately follows from Lemma 6.3 and Theorem 1.1. □

Let us turn to the study of equicontinuity at the points of the boundary. First of all, similarly to Lemma 6.2, we prove the following statement.

Lemma 6.5. *Let D and D_* be domains in $\mathbb{S}$ and $\mathbb{S}_*$, respectively, $p_0 \in \partial D$, $p_k \in D$, $k = 1, 2, \ldots$, $p_k \to p_0$ as $k \to \infty$, and let a domain D has a locally quasiconformal boundary. Let $f_k : \overline{D} \to \overline{D_*}$, $k = 1, 2, \ldots$ be a family of mappings such that $f_k|_D$ is open and closed in D. Suppose that*

1) the domains $D_k := f_k(D)$ and D_* are equi-uniformly over $k \in \mathbb{N}$, in addition, $\overline{D_*}$ is a compactum in $\mathbb{S}_*$;

2) there is a number $\delta > 0$ with the following property: for any $k \in \mathbb{N}$ there is a continuum $A_k \subset f_k(D)$, such that $\widetilde{h}_*(A_k) \geqslant \delta > 0$, moreover, $\widetilde{h}(f_k^{-1}(A_k), \partial D) \geqslant \delta > 0$;

3) there is $\delta_0 > 0$ such that

$$\widetilde{h}_*(f_k(p_k), f_k(p_0)) \geqslant \delta_0 \quad \forall\ k = 1, 2, \ldots . \tag{68}$$

Then there are $l_0 > 0$, $r_0 > 0$ and $k_0 \geqslant 1$ such that

$$l(f_k(\widetilde{S}(p_0, r) \cap D) \geqslant l_0, \quad \forall\ r \in (\widetilde{h}(p_0, p_k), r_0)\,, \quad \forall\ k \geqslant k_0\,, \tag{69}$$

where l denotes the length of the dashed line $\widetilde{S}(p_0, r) \cap D$ on the Riemannian surface $\mathbb{S}_*$.

Proof. Suppose the opposite. Then for each $i \in \mathbb{N}$ there are $k_i > i$ and $\widetilde{h}(p_0, p_{k_i}) < r_i < 1/i$, for which

$$l(g_i(\widetilde{S}(p_0, r_i) \cap D)) < 1/i, \qquad i = 1, 2, \ldots, \qquad r_i \to 0\,, \tag{70}$$

as $i \to \infty$, $g_i := f_{k_i}$. Without loss of generality, we may assume that the sequence numbers k_i, $i = 1, 2, \ldots$ is increasing. We also denote $p_{k_i} := q_i$.

Since D is a domain with a locally quasiconformal boundary, by Lemma 4.4 there is a sequence of neighborhoods U_m of the point p_0, $m = 1, 2, \ldots$, contracting to p_0 such that $U_m \cap D$ is connected and, moreover, for each $m = 1, 2, \ldots$, there is a cut γ_m of D such that $U_m \cap D \subset D \setminus |\gamma_m|$, $|\gamma_m| \subset \widetilde{S}(p_0, r_{i_m})$ for some

subsequence r_{i_m} of the sequence r_i, $i = 1, 2, \ldots$. Since f_k is continuous in $\overline{D}$, there is a sequence $p_k' \in D$ such that $\widetilde{h}_*(f_k(p_0), f_k(p_k')) < 1/k$. Put $q_i' := p_{k_i}'$. Then it follows from (68) that

$$\widetilde{h}_*(g_i(q_i), g_i(q_i')) \geqslant \delta_0/2 \tag{71}$$

for $i \geqslant i_0 \in \mathbb{N}$. For any $m \in \mathbb{N}$, we may find $j_m \in \mathbb{N}$, $j_m > i_0$, such that $q_{j_m}, q_{j_m}' \in U_m$. Now, by (71) we obtain that

$$\widetilde{h}_*(g_{j_m}(q_{j_m}), g_{j_m}(q_{j_m}')) \geqslant \delta_0/2 \quad \forall\ m = 1, 2, \ldots . \tag{72}$$

Let us show that

$$A_{j_m} \subset D_* \setminus g_{j_m}(U_m \cap \overline{D}) \qquad \forall\ m \geqslant m_0\,. \tag{73}$$

Indeed, if $y_l \in A_{j_{m_l}} \cap g_{j_{m_l}}(U_{m_l} \cap \overline{D})$ for arbitrarily large $l = 1, 2, \ldots$ and some increasing sequence m_l, $l = 1, 2, \ldots$, then $y_l = g_{j_{m_l}}(x_l)$, $x_l \in U_{m_l} \cap \overline{D}$ and, at the same time, $x_l \in g_{j_{m_l}}^{-1}(A_{j_{m_l}})$. Since U_{m_l} contracts to $p_0 \in \partial D$, then $x_l \rightarrow p_0$ as $l \rightarrow \infty$, which contradicts the condition $\widetilde{h}(f_k^{-1}(A_k), \partial D) \geqslant \delta > 0$, $g_{j_{m_l}} = f_{k_{j_{m_l}}}$. Thus, (73) holds.

Now let d_m be the component of $D \setminus |\gamma_m|$, containing $U_m \cap D$. Let us show that

$$\partial g_{j_m}(d_m) \cap g_{j_m}(D) \subset g_{j_m}(|\gamma_m|) \tag{74}$$

for any $m = 1, 2, \ldots$.

We fix $m \in \mathbb{N}$ and consider $y_m \in \partial g_{j_m}(d_m) \cap g_{j_m}(D)$. Then there is $y_{mk} \in g_{j_m}(d_m)$, $y_{mk} \rightarrow y_m$ as $k \rightarrow \infty$. Since f_k are open, $g_{j_m}(D)$ is a domain, so we may assume that $y_{mk} \in g_{j_m}(d_m) \cap g_{j_m}(D)$. Since $y_{mk} \in g_{j_m}(d_m)$, then $y_{mk} = g_{j_m}(\eta_{mk})$, $\eta_{mk} \in d_m$. By lemma 4.4 $d_n \subset U_m$ for all $n \geqslant n(M)$, therefore d_m also contract to the point p_0 as $m \rightarrow \infty$. Thus, we may assume that $\overline{d_m}$ is a compactum in $\mathbb{S}$, and that $\eta_{mk} \rightarrow \eta_0$ as $k \rightarrow \infty$. Observe that the case $\eta_0 \in \partial D$ is impossible, because now $y_m \in C(g_{j_m}, \partial D) \subset \partial g_{j_m}(D)$ by the closeness of the mapping $g_{j_m} = f_{k_{j_m}}$, which contradicts the choice of y_m. Then $\eta_0 \in D$. Two situations are possible: 1) $\eta_0 \in d_m$ and 2) $\eta_0 \in |\gamma_m|$. Observe that the case 1) is impossible, because now $g_{j_m}(\eta_0) = y_m$ and y_m is an inner point of the set $g_{j_m}(d_m)$ by the openness of the mapping g_{j_m}, which also contradicts the choice of y_m. Thus, the inclusion (74) is proved.

The further course of reasoning largely repeats the scheme of the proof of Lemma 6.2. Let ξ_m, $m = 1, 2, \ldots$, be an arbitrary sequence of points from $|\gamma_m|$. Since $\overline{D_*}$ is a compactum in $\mathbb{S}_*$, without loss of generality, we may assume that $\zeta_m := g_{j_m}(\xi_m) \rightarrow \zeta_0$ as $m \rightarrow \infty$, $\zeta_0 \in \overline{D_*}$. It follows from (70) that

$$\widetilde{h}_*(g_{j_m}(|\gamma_m|)) \rightarrow 0$$

as $m \rightarrow \infty$, $\widetilde{h}_*(g_{j_m}(|\gamma_m|)) := \sup\limits_{p_*, q_* \in g_{j_m}(|\gamma_m|)} \widetilde{h}_*(p_*, q_*)$. Now, for any $s \in \mathbb{N}$ there is a number $m_s \in \mathbb{N}$ such that

$$g_{j_m}(|\gamma_m|) \subset \widetilde{B}(\zeta_0, 1/s)\,, \qquad m \geqslant m_s\,. \tag{75}$$

By Proposition 6.1 there is $s_0 \in \mathbb{N}$ and a sequence of continua E_{j_m} such that

$$E_{j_m} \subset A_{j_m} \setminus \widetilde{B}(\zeta_0, 1/s_0), \quad \widetilde{h}_*(E_{j_m}) \geqslant \delta/4\,, \quad i = 1, 2, \ldots . \tag{76}$$

We fix $s > s_0$ and consider the family $\Gamma(g_{j_m}(|\gamma_m|), E_{j_m}, g_{j_m}(D))$ for $m \geqslant m_s$. Let $\gamma \in \Gamma(g_{j_m}(|\gamma_m|), E_{j_m}, g_{j_m}(D))$, i.e., $\gamma = \gamma(t)$, $t \in (0, 1)$, $\gamma(0) \in f_{k_i}(g_{j_m}(|\gamma_m|))$, $\gamma(1) \in E_{j_m}$ and $\gamma(t) \in g_{j_m}(D)$ for $t \in (0, 1)$. By (75) and (76) $|\gamma| \cap \widetilde{B}(\zeta_0, 1/s) \neq \varnothing \neq |\gamma| \cap (g_{j_m}(D) \setminus \widetilde{B}(\zeta_0, 1/s))$, therefore, according to [28, Theorem 1.I.5,§ 46] there is

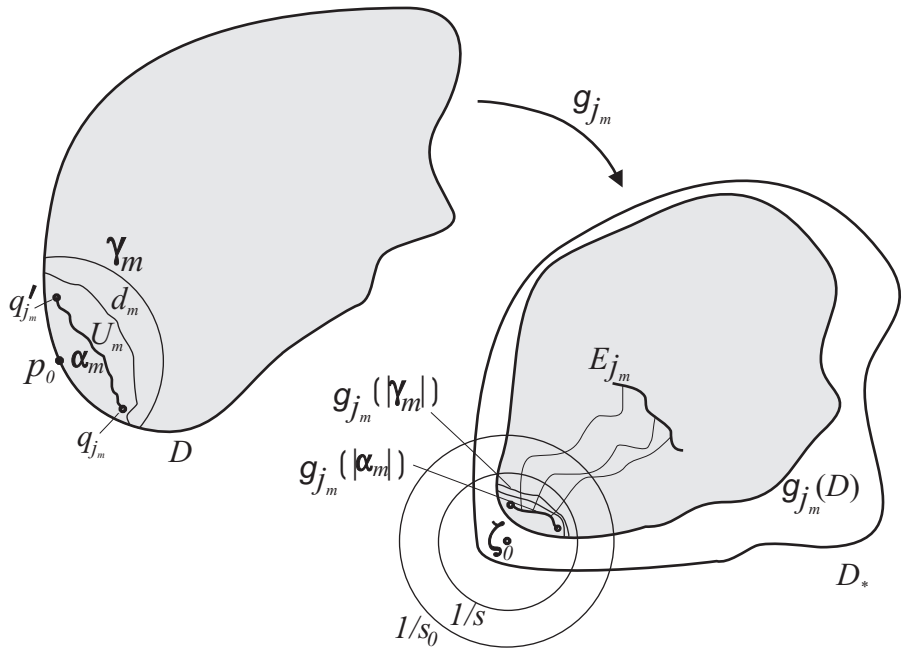

Figure 7: To the proof of Lemma 6.5

$t_1 \in (0,1)$ such that $\gamma(t_1) \in \widetilde{S}(\zeta_0, 1/s)$. Without loss of generality, we may assume that $\gamma(t) \in g_{j_m}(D) \setminus \overline{\widetilde{B}(\zeta_0, 1/s)}$ for $t > t_1$. Put $\gamma_1 := \gamma|_{[t_1,1]}$. Again, by (75) and (76) $|\gamma_1| \cap \widetilde{B}(\zeta_0, 1/s_0) \neq \varnothing \neq |\gamma_1| \cap (g_{j_m}(D) \setminus \widetilde{B}(\zeta_0, 1/s_0))$, therefore by [28, Theorem 1.I.5,§ 46] there is $t_2 \in (t_1, 1)$ such that $\gamma_1(t_2) \in \widetilde{S}(\zeta_0, 1/s_0)$. Without loss of generality, we may assume that $\gamma_1(t) \in \widetilde{B}(\zeta_0, 1/s_0)$ for $t \in (t_1, t_2)$. Put $\gamma_2 := \gamma_1|_{[t_1,t_2]}$, $\gamma_2 \in \Gamma(\widetilde{S}(\zeta_0, 1/s), \widetilde{S}(\zeta_0, 1/s_0), \widetilde{A}(\zeta_0, 1/s, 1/s_0))$, $\widetilde{A}(\zeta_0, 1/s, 1/s_0)) = \{p_* \in \mathbb{S}_* : 1/s < \widetilde{h}_*(p_*, \zeta_0) < 1/s_0\}$, see Figure 7.

Hence it follows that

$$\Gamma(g_{j_m}(|\gamma_m|), E_{j_m}, g_{j_m}(D)) > \Gamma(\widetilde{S}(\zeta_0, 1/s), \widetilde{S}(\zeta_0, 1/s_0), \widetilde{A}(\zeta_0, 1/s, 1/s_0))$$

for $m \geqslant m_s$. Thus, by [33, Theorem 1(c)] and by Proposition 4.3

$$M(\Gamma(g_{j_m}(|\gamma_m|), E_{j_m}, g_{j_m}(D))) \leqslant$$

$$\leqslant M(\Gamma(\widetilde{S}(\zeta_0, 1/s), \widetilde{S}(\zeta_0, 1/s_0), \widetilde{A}(\zeta_0, 1/s, 1/s_0)))\,, \tag{77}$$

$$M(\Gamma(\widetilde{S}(\zeta_0, 1/s), \widetilde{S}(\zeta_0, 1/s_0), \widetilde{A}(\zeta_0, 1/s, 1/s_0))) \to 0$$

as $s \to \infty$. Given $\varepsilon > 0$ we may find a number $S = S(\varepsilon)$ such that

$$M(\Gamma(\widetilde{S}(\zeta_0, 1/s), \widetilde{S}(\zeta_0, 1/s_0), \widetilde{A}(\zeta_0, 1/s, 1/s_0))) < \varepsilon, \quad s > S(\varepsilon)\,.$$

Set $M_0 = M_0(\varepsilon) := m_{S(\varepsilon)}$. It follows from (77) that

$$M(\Gamma(g_{j_m}(|\gamma_m|), E_{j_m}, g_{j_m}(D))) < \varepsilon, \quad m > M_0 = M_0(\varepsilon)\,. \tag{78}$$

Now let us join the points q_{j_m} and q'_{j_m} by the path $\alpha_m \subset U_m$. By assumption (71) $\widetilde{h}_*(g_{j_m}(|\alpha_m|)) \geqslant \delta_0/2$. Then by the definition of a uniform domain, there exists $\varepsilon_1 > 0$ such that

$$M(\Gamma(g_{j_m}(|\alpha_m|), E_{j_m}, g_{j_m}(D))) > \varepsilon_1\,, \qquad m \in \mathbb{N}\,. \tag{79}$$

On the other hand, by (73) and (74)

$$\Gamma(g_{j_m}(|\alpha_m|), E_{j_m}, g_{j_m}(D)) > \Gamma(g_{j_m}(|\gamma_m|), E_{j_m}, g_{j_m}(D))\,,$$

whence by [33, Theorem1(c)] and also by (79) we obtain that

$$\varepsilon_1 < M(\Gamma(g_{j_m}(|\alpha_m|), E_{j_m}, g_{j_m}(D))) \leqslant M(\Gamma(g_{j_m}(|\gamma_m|), E_{j_m}, g_{j_m}(D))) \tag{80}$$

for $m = 1, 2, \ldots$. The inequalities (80) and (78) contradict each other, which proves (69). □

Let$D$ and D_* be domains in $\mathbb{S}$ and $\mathbb{S}_*$, respectively. Given $\delta > 0$ and a measurable function $Q : \mathbb{S} \to (0, \infty)$ with respect the measure $\widetilde{h}$, $Q(x) \equiv 0$ for $x \in \mathbb{S} \setminus D$, denote by $\mathfrak{Q}_{Q,\delta}(D)$ the family of all open discrete mappings $f : D \to D_*$ with (7) for which: 1) there is a continuum $G_f \subset D_*$ such that $f : D \to D_* \setminus G_f$ and $\widetilde{h}_*(G_f) := \sup\limits_{x,y\in G_f} \widetilde{h}_*(x, y) \geqslant \delta$; 2) there exists a continuum $A_f \subset f(D)$ such that $\widetilde{h}_*(A_f) \geqslant \delta$ and $\widetilde{h}(f^{-1}(A_f), \partial D) \geqslant \delta$.
The following theorem holds.

Theorem 6.6. *Assume that the following conditions are satisfied:*

1) Q is locally integrable in D and either Q satisfies (8), or $Q \in FMO(p_0)$ at any point $p_0 \in \overline{D}$;

2) domains $D_f = f(D)$ and D_* are equi-uniform over $f \in \mathfrak{Q}_{Q,\delta}(D)$;

3) a domain D has a locally quasiconformal boundary;

4) the set $\overline{D_*}$ is a compactum in $\mathbb{S}_*$.

Then any mapping $f \in \mathfrak{Q}_{Q,\delta}(D)$ has a continuous extension $\overline{f} : \overline{D} \to \overline{D_*}$ and a family $\mathfrak{Q}_{Q,\delta}(\overline{D})$ of all extended mappings $\overline{f}$ is equicontinuous in $\overline{D}$.

Proof. Arguing similarly to the beginning of the proof of Lemma 5.2, we conclude that the domain $D_f = f(D)$, $f \in \mathfrak{Q}_{Q,\delta}(D)$, has a strongly accessible boundary. In this case, the possibility of continuous extension $\overline{f} : \overline{D} \to \overline{D_*}$ follows from Theorem 4.6, and the equicontinuity of $\mathfrak{Q}_{Q,\delta}(\overline{D})$ in D is from Lemma 6.3, since $\mathfrak{Q}_{Q,\delta}(D) \subset \mathfrak{G}_{Q,\delta}(D)$. It remains to show the equicontinuity of the family $\mathfrak{Q}_{Q,\delta}(\overline{D})$ in ∂D.

Suppose the opposite. Then there are $p_0 \in \partial D$, $p_k \in \overline{D}$, $k = 1, 2, \ldots$, $f_k \in \mathfrak{Q}_{Q,\delta}(\overline{D})$ and δ_0 such that $p_k \to p_0$ as $k \to \infty$ and

$$\widetilde{h}_*(f_k(p_k), f_k(p_0)) \geqslant \delta_0\,. \tag{81}$$

By Theorem 4.6 we may assume that $p_k \in D$, besides that, by Lemma 6.2 there are $l_0 > 0$ and $r_0 > 0$ such that for some $k_0 \geqslant 1$

$$l(f_k(\widetilde{S}(p_0, r)) \geqslant l_0, \quad \forall\ r \in (\widetilde{h}(p_0, p_k), r_0)\,, \quad \forall\ k \geqslant k_0\,, \tag{82}$$

where l denotes the length of a path on the Riemannian surface $\mathbb{S}_*$. Without loss of generality, we may assume that $r_0 < \varepsilon_0$, where ε_0 is a number from (8), existing by the conditions of the lemma. In this case, the function

$$\rho(p) = \begin{cases} 1/l_0\,, & p \in D_*\,, \\ 0\,, & p \notin D_* \end{cases}$$

is admissible for $\Gamma_k^{r_0}$, consisting from the union of all dished lines $f_k(\widetilde{S}(p_0, r)), k = 1, 2, \ldots$, over $r \in (\widetilde{h}(p_0, p_k), r_0)$. In this case, by the definition of the modulus of families of paths, we obtain that

$$M(\Gamma_k^{r_0}) \leqslant (1/l_0^2) \cdot \widetilde{h}_*(D_*) < \infty\,, \tag{83}$$

because $\overline{D_*}$ is a compactum in $\mathbb{S}_*$. On the other hand, by Lemma 2.4 and by the conditions (8) we have that

$$M(\Gamma_k^{r_0}) \geqslant \int\limits_{\widetilde{h}(p_0,p_k)}^{r_0} \frac{dr}{\|Q\|(r)} \to \infty \tag{84}$$

as $k \to \infty$. The relations (83) and (84) contradict each other,, which refutes the assumption made in (81). □

Let D and D_* be domains in $\mathbb{S}$ and $\mathbb{S}_*$, respectively. Given $\delta > 0$, a natural number $N \geqslant 1$ and a function $Q : \mathbb{S} \to (0, \infty)$, $Q(x) \equiv 0$ for $x \in \mathbb{S} \setminus D$, measurable with respect to the measure $\widetilde{h}$, denote by $\mathfrak{M}_{Q,\delta,N}(D)$ the family of all open discrete and closed mappings $f : D \to D_*$ of the class $W^{1,1}_{\rm loc}(D)$ with a finite distortion for which: 1) there is a continuum $G_f \subset D_*$ such that $f : D \to D_* \setminus G_f$ and $\widetilde{h}_*(G_f) := \sup\limits_{x,y\in G_f} \widetilde{h}_*(x, y) \geqslant \delta$; 2) there is a continuum $A_f \subset f(D)$ such that $\widetilde{h}_*(A_f) \geqslant \delta$ and $\widetilde{h}(f^{-1}(A_f), \partial D) \geqslant \delta$; 3) $K_f(p) \leqslant Q(p)$ for any $p \in D$; 4) $N(f, D) \leqslant N$.

The following theorem holds.

Theorem 6.7. *Assume that the following conditions are satisfied:*

1) Q is locally integrable in D and either Q satisfies (8), or $Q \in FMO(p_0)$ at any point $p_0 \in \overline{D}$;

2) domains $D_f = f(D)$ and D_* are equi-uniform over $f \in \mathfrak{M}_{Q,\delta,N}(D)$;

3) a domain D has a locally quasiconformal boundary;

4) the set $\overline{D_*}$ is a compactum in $\mathbb{S}_*$.

Then any mapping $f \in \mathfrak{M}_{Q,\delta,N}(D)$ has a continuous extension $\overline{f} : \overline{D} \to \overline{D_*}$ and the family $\mathfrak{M}_{Q,\delta,N}(\overline{D})$ of all extended mappings $\overline{f}$ is equicontinuous in $\overline{D}$.

Proof of Theorem 6.7 immediately follows from Theorems 6.6 and 1.1. □

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