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# A New Method of Constructing Weak Crossed Products

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**Abstract.** This paper is devoted to constructing a new class of weak crossed products.

## 1. Introduction

In [2], an associative product on $A \otimes V$ which was called weak crossed product, was defined, for an algebra $A$ and an object $V$ both living in a strict monoidal category $C$ where every idempotent splits. To obtain such weak crossed product, we must consider crossed product systems, that is, two morphisms $\psi_V^A : V \otimes A \to A \otimes V$ and $\sigma_V^A : V \otimes V \to A \otimes V$ satisfying some twisted-like and cocycle-like conditions. The theory presented in [2] contains the classical crossed product in [4], the weak smash products in [5], the weak wreath product in [12] and the weak crossed product in [11].

Recently some new types of crossed products were presented in different settings, for example, partial crossed product was introduced by Alves, Batista, Dokuchaev and Paques in order to characterize cleft extensions of algebras in the partial setting [1], and unified crossed product was introduced by Agore and Militaru in order to solve the restricted (H-C) extending structures problem [3]. In [10], Vilaboa, Rodríguez and Raposo proved that partial and unified crossed products are weak crossed products.

A natural question occurs to us: *How to construct a new class of weak crossed products*. In the paper, we shall present a new method to construct weak crossed products, the inspiration is from the partial crossed product and the unified crossed product.

The paper is organized as follows.

In Section 2, we shall recall some basic concepts in the strict monoidal category. In Section 3, we shall introduce a partial extending datum of a bialgebra, and a new algebra structure is presented.

## 2. Preliminaries

Throughout this paper, $C$ denotes a strict monoidal category with tensor product $\otimes$ and base object $\mathbb{K}$. Given an object $A$, we use $id_A$ to denote the identity map on $A$. Also we assume that idempotents split, i.e., for every morphism $\nabla_Y : Y \to Y$ such that $\nabla_Y = \nabla_Y \circ \nabla_Y$, there exist an object $Z$ and morphisms $i_Y : Z \to Y$(injection) and $p_Y : Y \to Z$(projection) satisfying $\nabla_Y = i_Y \circ p_Y$ and $p_Y \circ i_Y = id_Z$.

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An algebra in $C$ is a triple $A = (A, m_A, \eta_A)$, where $A$ is an object in $C$ and $\eta_A : \mathbb{K} \to A$(unit), $m_A : A \otimes A \to A$(product) are morphisms in $C$ such that

$$m_A \circ (id_A \otimes \eta_A) = id_A = m_A \circ (\eta_A \otimes id_A), \tag{1}$$

$$m_A \circ (id_A \otimes m_A) = m_A \circ (m_A \otimes id_A). \tag{2}$$

Given two algebras $A = (A, m_A, \eta_A)$ and $B = (B, m_B, \eta_B)$, $f : A \to B$ is an algebra morphism, if

$$m_B \circ (f \otimes f) = f \circ m_A, f \circ \eta_A = \eta_B. \tag{3}$$

Also, if $C$ is braided with braid $\mathfrak{c}$ and $A$, $B$ are algebras in $C$, the object $A \otimes B$ is also an algebra in $C$, where

$$\eta_{A\otimes B} = \eta_A \otimes \eta_B \text{ and } m_{A\otimes B} = (m_A \otimes m_B) \circ (id_A \otimes \mathfrak{c}_{B,A} \otimes id_B).$$

A coalgebra in $C$ is a triple $D = (D, \Delta_D, \varepsilon_D)$, where $D$ is an object in $C$ and $\varepsilon_D : D \to \mathbb{K}$(counit), $\Delta_D : D \to D \otimes D$(coproduct) are morphisms in $C$ such that

$$(\varepsilon_D \otimes id_D) \circ \Delta_D = id_D = (id_D \otimes \varepsilon_D) \circ \Delta_D, \tag{4}$$

$$(id_A \otimes \Delta_D) \circ \Delta_D = (\Delta_D \otimes id_D) \circ \Delta_D. \tag{5}$$

Given two coalgebras $D = (D, \Delta_D, \varepsilon_D)$ and $E = (E, \Delta_E, \varepsilon_E)$, $f : D \to E$ is a coalgebra morphism, if

$$(f \otimes f) \circ \Delta_D = \Delta_E \circ f, \varepsilon_E \circ f = \varepsilon_D. \tag{6}$$

Also, if $C$ is braided with the braid $\mathfrak{c}$ and $D$, $E$ are coalgebras in $C$, the object $D \otimes E$ is also an coalgebra in $C$, where

$$\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E \text{ and } \Delta_{D\otimes E} = (id_D \otimes \mathfrak{c}_{D,E} \otimes id_E) \circ (\Delta_D \otimes \Delta_E).$$

If $C$ is braided with the braid $\mathfrak{c}$, we say that $H$ is a bialgebra in $C$, if $(H, m_H, \eta_H)$ is an algebra, $(H, \Delta_H, \varepsilon_H)$ is a coalgebra and $\varepsilon_H$ and $\Delta_H$ are algebra morphisms (equivalently $\eta_H$ and $m_H$ are coalgebra morphisms). If moreover, there exists a morphism $S_H : H \to H$ satisfying the identities

$$m_H \circ (id_H \otimes S_H) \circ \Delta_H = \varepsilon_H \otimes \eta_H = m_H \circ (S_H \otimes id_H) \circ \Delta_H,$$

we say that $H$ is a Hopf algebra.

Let $A$ be an algebra. The pair $(M, \phi_M)$ is a right $A$-module, if $M$ is an object in $C$ and $\phi_M : M \otimes A \to M$ is a morphism in $C$ satisfying

$$\phi_M \circ (id_M \otimes \eta_A) = id_M, \phi_M \circ (\phi_M \otimes id_A) = \phi_M \circ (id_M \otimes m_A). \tag{7}$$

Given two right $A$-modules $(M, \phi_M)$ and $(N, \phi_N)$, $f : M \to N$ is a morphism of right $A$-modules, if $\phi_N \circ (f \otimes id_A) = f \circ \phi_M$. In a similar way, we can define the notions of left $A$-modules and morphism of left $A$-modules.

Assume that the monoidal category $C$ is braided with the braid $\mathfrak{c}$. We shall recall the theory of weak crossed products in a monoidal category $C$ introduced in [9]. Let $A$ be an algebra and $V$ be an object in $C$. Suppose that there exists a morphism

$$\psi_V^A : V \otimes A \to A \otimes V,$$

such that the following equality holds

$$(m_A \otimes id_V) \circ (id_A \otimes \psi_V^A) \circ (\psi_V^A \otimes id_A) = \psi_V^A \circ (id_V \otimes m_A). \tag{8}$$

As a consequence of (8), the morphism $\nabla_{A\otimes V} : A \otimes V \to A \otimes V$ defined by

$$\nabla_{A\otimes V} = (m_A \otimes id_V) \circ (id_A \otimes \psi_V^A) \circ (id_A \otimes id_V \otimes \eta_A) \tag{9}$$

is idempotent. With $A \times V$, $i_{A\otimes V} : A \times V \to A \otimes V$ and $p_{A\otimes V} : A \otimes V \to A \times V$ we denote the object, the injection and the projection associated to the factorization of $\nabla_{A\otimes V}$.

We consider the quadruples $(A, V, \psi_V^A, \sigma_V^A)$, where $A$ is an algebra, $V$ an object, $\psi_V^A : V \otimes A \to A \otimes V$ a morphism satisfying (8) and $\sigma_V^A : V \otimes V \to A \otimes V$ a morphism in $C$. If $(A, V, \psi_V^A, \sigma_V^A)$ satisfies the following conditions

$$(m_A \otimes id_V) \circ (id_A \otimes \psi_V^A) \circ (\sigma_V^A \otimes id_A) \\ = (m_A \otimes id_V) \circ (id_A \otimes \sigma_V^A) \circ (\psi_V^A \otimes id_V) \circ (id_V \otimes \psi_V^A), \tag{10}$$

and

$$(m_A \otimes id_V) \circ (id_A \otimes \sigma_V^A) \circ (\sigma_V^A \otimes id_V) \\ = (m_A \otimes id_V) \circ (id_A \otimes \sigma_V^A) \circ (\psi_V^A \otimes id_V) \circ (id_V \otimes \sigma_V^A). \tag{11}$$

For $(A, V, \psi_V^A, \sigma_V^A)$, we define the product

$$m_{A\otimes V} = (m_A \otimes id_V) \circ (m_A \otimes \sigma_V^A) \circ (id_A \otimes \psi_V^A \otimes id_V), \tag{12}$$

and let $m_{A\times V}$ be the product

$$m_{A\times V} = p_{A\otimes V} \circ m_{A\otimes V} \circ (i_{A\otimes V} \otimes i_{A\otimes V}). \tag{13}$$

If $(A, V, \psi_V^A, \sigma_V^A)$ satisfies (10) and (11), we say that $(A \otimes V, m_{A\otimes V})$ is a weak crossed product.

If $A$ is an algebra, $V$ an object in $C$ and $m_{A\otimes V}$ is an associative product defined in $A \otimes V$, a preunit $\nu : \mathbb{K} \to A \otimes V$ is a morphism satisfying

$$m_{A\otimes V} \circ (id_A \otimes id_V \otimes \nu) = m_{A\otimes V} \circ (\nu \otimes id_A \otimes id_V) \\ = m_{A\otimes V} \circ (id_A \otimes id_V \otimes ((m_{A\otimes V}) \circ (\nu \otimes \nu))). \tag{14}$$

## 3. Partial Extending Datums and Weak Crossed Products

**Definition 3.1.** *Let $A$ be a bialgebra in $C$. A partial extending datum of $A$ is a system*

$$\Omega(A) = (H, \phi_H : H \otimes A \to H, \varphi_A : H \otimes A \to A, \omega : H \otimes H \to A)$$

*where:*

*(i) There exist morphisms $\eta_H : K \to H$, $m_H : H \otimes H \to H$, $\varepsilon_H : H \to K$ and $\Delta_H : H \to H \otimes H$ such that*

*(i-1) $(H, \Delta_H, \varepsilon_H)$ is a coalgebra,*

*(i-2) $\Delta_H \circ \eta_H = \eta_H \otimes \eta_H$, $\varepsilon_H \circ \eta_H = id_K$,*

*(i-3) $m_H \circ (\eta_H \otimes id_H) = id_H = m_H \circ (id_H \otimes \eta_H)$,*

*(i-4) $\Delta_H \circ m_H = m_{H\otimes H} \circ (\Delta_H \otimes \Delta_H)$, $\varepsilon_H \circ m_H = \varepsilon_H \otimes \varepsilon_H$.*

*(ii) $\phi_H$ is a coalgebra morphism, i.e.,*

$$(\phi_H \otimes \phi_H) \circ \Delta_{H\otimes A} = \Delta_H \circ \phi_H, \varepsilon_H \circ \phi_H = \varepsilon_H \otimes \varepsilon_A.$$

*(iii) $\varphi_A$ is a coalgebra morphism, i.e.,*

$$(\varphi_A \otimes \varphi_A) \circ \Delta_{H\otimes A} = \Delta_A \circ \varphi_A, \varepsilon_A \circ \varphi_A = \varepsilon_H \otimes \varepsilon_A.$$

*(iv) $\omega$ is a coalgebra morphism, i.e.,*

$$(\omega \otimes \omega) \circ \Delta_{H\otimes H} = \Delta_A \circ \omega, \varepsilon_A \circ \omega = \varepsilon_H \otimes \varepsilon_H.$$

*(v) The following conditions hold*

*(v-1)* $\phi_H \circ (\eta_H \otimes id_A) = \eta_H \otimes \varepsilon_A$, $\phi_H \circ (id_H \otimes \eta_A) = id_H$,
*(v-2)* $\varphi_A \circ (\eta_H \otimes id_A) = id_A$,
*(v-3)* $\omega \circ (id_H \otimes \eta_H) = \omega \circ (\eta_H \otimes id_H) = \varphi_A \circ (id_H \otimes \eta_A)$,
*(v-4)* $\varphi_A \circ (id_H \otimes m_A) = m_A \circ (id_A \otimes \varphi_A) \circ ((\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A} \otimes id_A)$,
*(v-5)* $m_A \circ (id_A \otimes \omega) \circ ((\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A} \otimes id_H) \circ (id_H \otimes (\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A}) = m_A \circ (id_A \otimes \varphi_A) \circ ((\omega \otimes m_H) \circ \Delta_{H\otimes H} \otimes id_A)$,
*(v-6)* $\omega = m_A \circ (id_A \otimes \varphi_A) \circ ((\omega \otimes m_H) \circ \Delta_{H\otimes H} \otimes \eta_A)$,
*(v-7)* $c_{A,H} \circ (\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A} = (\phi_H \otimes \varphi_A) \circ \Delta_{H\otimes A}$,
*(v-8)* $c_{A,H} \circ (\omega \otimes m_H) \circ \Delta_{H\otimes H} = (m_H \otimes \omega) \circ \Delta_{H\otimes H}$.

**Lemma 3.2.** *If $\Omega(A)$ is a partial extending datum of $A$, then we have*

$$\varphi_A = m_A \circ (\varphi_A \otimes \varphi_A) \circ (id_H \otimes \eta_A \otimes id_H \otimes id_A) \circ (\Delta_H \otimes id_A), \tag{15}$$

$$\begin{aligned} & m_A \circ (id_A \otimes \omega) \circ ((\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A} \otimes id_H) \circ (id_H \otimes (\varphi_A \circ (id_H \otimes \eta_A)) \otimes id_H) \circ (id_H \otimes \Delta_H) \\ &= \omega \\ &= m_A \circ (\varphi_A \circ (id_H \otimes \eta_A) \otimes \omega) \circ (\Delta_H \otimes id_H). \end{aligned}$$

*Proof.* Since

$$\begin{aligned} \varphi_A &= \varphi_A \circ (id_H \otimes m_A) \circ (id_H \otimes \eta_A \otimes id_A) \\ &\overset{(v\text{-}4)}{=} m_A \circ (id_A \otimes \varphi_A) \circ ((\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A} \otimes id_A) \circ (id_H \otimes \eta_A \otimes id_A) \\ &= m_A \circ (id_A \otimes \varphi_A) \circ (\varphi_A \otimes \phi_H \otimes id_A) \\ &\quad \circ (id_H \otimes \eta_A \otimes id_H \otimes \eta_A \otimes id_A) \circ (\Delta_H \otimes id_A) \\ &= m_A \circ (\varphi_A \otimes \varphi_A) \circ (id_H \otimes \eta_A \otimes id_H \otimes id_A) \circ (\Delta_H \otimes id_A) \end{aligned}$$

as desired. By using (v-5) and (15), we can check that (16) holds in a straightforward way. □

Let $\Omega(A)$ be a partial extending datum of a bialgebra $A$. We define a product $m_{A\otimes H}$ on the object $A \otimes H$ as follows:

$$\begin{aligned} m_{A\otimes H} &= (m_A \otimes id_H) \circ (m_A \otimes (\omega \otimes m_H) \circ \Delta_{H\otimes H}) \\ &\quad \circ (id_A \otimes (\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A} \otimes id_H). \end{aligned} \tag{16}$$

**Lemma 3.3.** *Let $A$ be a bialgebra in $\mathcal{C}$ and $\Omega(A)$ a partial extending datum of $A$. The following cross-relations hold:*

$$\begin{aligned} m_{A\otimes H} \circ (id_A \otimes \eta_H \otimes id_A \otimes id_H) &= (m_A \otimes id_H) \circ (id_A \otimes \varphi_A \otimes id_H) \\ &\quad \circ (m_A \otimes (id_H \otimes \eta_A \otimes id_H) \circ \Delta_H), \end{aligned} \tag{17}$$

$$\begin{aligned} & m_{A\otimes H} \circ (id_A \otimes id_H \otimes \eta_A \otimes id_H) \\ &= (m_A \otimes id_H) \circ (m_A \otimes \omega \otimes m_H) \circ (id_A \otimes \varphi_A \circ (id_H \otimes \eta_A) \otimes \Delta_{H\otimes H}) \circ (id_A \otimes \Delta_H \otimes id_H), \end{aligned} \tag{18}$$

$$m_{A\otimes H} \circ (id_A \otimes id_H \otimes id_A \otimes \eta_H) = (m_A \otimes id_H) \circ (id_A \otimes (\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A}). \tag{19}$$

**Theorem 3.4.** *Let $A$ be a bialgebra in $\mathcal{C}$ and $\Omega(A)$ a partial extending datum of $A$. The product $m_{A\otimes H}$ on the object $A \otimes H$ is associative if and only if*

$$\phi_H \circ (\phi_H \otimes id_A) = \phi_H \circ (id_H \otimes m_A), \tag{20}$$

$$m_H \circ (\phi_H \otimes id_H) \circ (id_H \otimes \varphi_A \otimes \phi_H) \circ (id_H \otimes \Delta_{H\otimes A}) = \phi_H \circ (m_H \otimes id_A), \tag{21}$$

$$m_H \circ (m_H \otimes id_H) = m_H \circ (\phi_H \otimes id_H) \circ (id_H \otimes \omega \otimes m_H) \circ (id_H \otimes \Delta_{H\otimes H}), \tag{22}$$

$$\begin{aligned} & m_A \circ (id_A \otimes \omega) \circ (\omega \otimes m_H \otimes id_H) \circ (\Delta_{H\otimes H} \otimes id_H) \\ &= m_A \circ (id_A \otimes \omega) \circ (\varphi_A \otimes \phi_H \otimes id_H) \circ (\Delta_{H\otimes A} \otimes id_H) \circ (id_H \otimes \omega \otimes m_H) \circ (id_H \otimes \Delta_{H\otimes H}). \end{aligned} \tag{23}$$

*Proof.* Since

$$\begin{aligned}
&m_{A\otimes H}\circ(id_A\otimes id_H\otimes m_{A\otimes H})\circ(\eta_A\otimes id_H\otimes id_A\otimes\eta_H\otimes id_A\otimes\eta_H)\\
\overset{(19)}{=}\ & m_{A\otimes H}\circ(\eta_A\otimes id_H\otimes id_A\otimes\eta_H)\circ(id_H\otimes m_A)\\
\overset{(19)}{=}\ & (\varphi_A\otimes\phi_H)\circ\Delta_{H\otimes A}\circ(id_H\otimes m_A)
\end{aligned}$$

and

$$\begin{aligned}
&m_{A\otimes H}\circ(m_{A\otimes H}\otimes id_A\otimes id_H)\circ(\eta_A\otimes id_H\otimes id_A\otimes\eta_H\otimes id_A\otimes\eta_H)\\
\overset{(19)}{=}\ & (m_A\otimes id_H)\circ(id_A\otimes(\varphi_A\otimes\phi_H)\circ\Delta_{H\otimes A})\circ((\varphi_A\otimes\phi_H)\circ\Delta_{H\otimes A}\otimes id_A)\\
=\ & (m_A\otimes id_H)\circ(id_A\otimes\varphi_A\otimes\phi_H)\\
& \circ((\varphi_A\otimes\phi_H)\circ\Delta_{H\otimes A}\otimes(c_{H,A}\otimes id_A)\circ(\phi_H\otimes\Delta_A))\circ(\Delta_{H\otimes A}\otimes id_A)\\
\overset{(v\text{-}4)}{=}\ & (\varphi_A\circ(id_H\otimes m_A)\otimes\phi_H)\\
& \circ(id_H\otimes id_A\otimes(c_{H,A}\otimes id_A)\circ(\phi_H\otimes\Delta_A))\circ(\Delta_{H\otimes A}\otimes id_A).
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
&(\varphi_A\otimes\phi_H)\circ\Delta_{H\otimes A}\circ(id_H\otimes m_A)\\
&\quad=\ (\varphi_A\circ(id_H\otimes m_A)\otimes\phi_H)\\
&\quad\quad\circ(id_H\otimes id_A\otimes(c_{H,A}\otimes id_A)\circ(\phi_H\otimes\Delta_A))\circ(\Delta_{H\otimes A}\otimes id_A).
\end{aligned}$$

Applying $\varepsilon_A\otimes id_H$ to both sides of (24), we gain (20).

$$\begin{aligned}
&m_{A\otimes H}\circ(id_A\otimes id_H\otimes m_{A\otimes H})\circ(\eta_A\otimes id_H\otimes\eta_A\otimes id_H\otimes id_A\otimes\eta_H)\\
\overset{(19)}{=}\ & (m_A\otimes id_H)\circ(id_A\otimes\omega\otimes m_H)\circ(id_A\otimes\Delta_{H\otimes H})\\
& \circ(\varphi_A\otimes\phi_H\otimes id_H)\circ(\Delta_{H\otimes A}\otimes id_H)\circ(id_H\otimes(\varphi_A\otimes\phi_H)\circ\Delta_{H\otimes A})\\
\overset{(iii)}{=}\ & (m_A\otimes id_H)\circ(id_A\otimes\omega\otimes m_H)\circ(id_A\otimes id_H\otimes c_{H,H}\otimes id_H)\\
& \circ(\varphi_A\otimes\phi_H\otimes\phi_H\otimes\Delta_H)\circ(id_H\otimes id_A\otimes\Delta_{H\otimes A}\otimes id_H)\\
& \circ(\Delta_{H\otimes A}\otimes id_H)\circ(id_H\otimes(\varphi_A\otimes\phi_H)\circ\Delta_{H\otimes A})\\
=\ & (m_A\otimes id_H)\circ(id_A\otimes\omega\otimes m_H)\circ(id_A\otimes id_H\otimes c_{H,H}\otimes id_H)\\
& \circ(\varphi_A\otimes\phi_H\otimes\phi_H\otimes\Delta_H)\circ(\Delta_{H\otimes A}\otimes id_H\otimes id_A\otimes id_H)\\
& \circ(\Delta_{H\otimes A}\otimes id_H)\circ(id_H\otimes\varphi_A\otimes\phi_H)\circ(id_H\otimes\Delta_{H\otimes A})\\
\overset{(ii)}{=}\ & (m_A\otimes id_H)\circ(id_A\otimes\omega\otimes m_H)\circ(id_A\otimes id_H\otimes c_{H,H}\otimes id_H)\\
& \circ(\varphi_A\otimes\phi_H\otimes\phi_H\otimes id_H\otimes id_H)\circ(\Delta_{H\otimes A}\otimes id_H\otimes id_A\otimes id_H\otimes id_H)\\
& \circ(\Delta_{H\otimes A}\otimes id_H\otimes id_H)\circ(id_H\otimes\varphi_A\otimes\phi_H\otimes\phi_H)\\
& \circ(id_H\otimes id_H\otimes id_A\otimes\Delta_{H\otimes A})\circ(id_H\otimes\Delta_{H\otimes A})\\
=\ & (m_A\otimes id_H)\circ(id_A\otimes\omega\otimes m_H)\circ(\varphi_A\otimes\phi_H\otimes\underline{c_{H,H}\circ(\phi_H\otimes id_H)}\otimes id_H)\\
& \circ((\Delta_{H\otimes A}\otimes id_H\otimes id_A)\circ\Delta_{H\otimes A}\otimes id_H\otimes id_H)\circ(id_H\otimes\varphi_A\otimes\phi_H\otimes\phi_H)\\
& \circ(id_H\otimes(\Delta_{H\otimes A}\otimes id_H\otimes id_A)\circ\Delta_{H\otimes A})\\
=\ & (m_A\otimes id_H)\circ(id_A\otimes\omega\otimes m_H)\circ(\varphi_A\otimes\phi_H\otimes id_H\otimes\phi_H\otimes id_H)\\
& \circ(id_H\otimes id_A\otimes id_H\otimes id_A\otimes c_{H\otimes A,H}\otimes id_H)\\
& \circ((\Delta_{H\otimes A}\otimes id_H\otimes id_A)\circ\Delta_{H\otimes A}\otimes id_H\otimes id_H)\circ(id_H\otimes\varphi_A\otimes\phi_H\otimes\phi_H)\\
& \circ(id_H\otimes(\Delta_{H\otimes A}\otimes id_H\otimes id_A)\circ\Delta_{H\otimes A})
\end{aligned}$$

$$
\begin{aligned}
\overset{(iii),(v\text{-}7)}{=} & (m_A \otimes id_H) \circ (id_A \otimes \omega \otimes m_H) \circ (\varphi_A \otimes \phi_H \otimes id_H \otimes \phi_H \otimes id_H) \\
& \circ(\Delta_{H\otimes A} \otimes id_H \otimes id_H \otimes id_A \otimes id_H) \circ (id_H \otimes c_{H,A\otimes H} \otimes id_A \otimes id_H) \\
& \circ(id_H \otimes id_H \otimes \varphi_A \otimes \phi_H \otimes \varphi_A \otimes \phi_H) \\
& \circ(id_H \otimes id_H \otimes \Delta_{H\otimes A} \otimes id_H \otimes id_A \otimes id_H \otimes id_A) \\
& \circ(id_H \otimes id_H \otimes \Delta_{H\otimes A} \otimes id_H \otimes id_A) \circ (\Delta_H \otimes \Delta_{H\otimes A}) \\
= & (m_A \otimes id_H) \circ (id_A \otimes \omega \otimes m_H) \circ (\varphi_A \otimes \phi_H \otimes id_H \otimes \phi_H \otimes id_H) \\
& \circ(\Delta_{H\otimes A} \otimes id_H \otimes id_H \otimes id_A \otimes id_H) \circ (id_H \otimes \varphi_A \otimes \phi_H \otimes id_H \otimes id_A \otimes id_H) \\
& \circ(id_H \otimes \Delta_{H\otimes A} \otimes id_H \otimes id_A \otimes id_H) \circ (id_H \otimes c_{H,H\otimes A} \otimes \varphi_A \otimes \phi_H) \\
& \circ(id_H \otimes id_H \otimes \Delta_{H\otimes A} \otimes id_H \otimes id_A) \circ (\Delta_H \otimes \Delta_{H\otimes A}) \\
\overset{(v\text{-}5)}{=} & (m_A \otimes id_H) \circ (id_A \otimes \varphi_A \otimes m_H) \circ (\omega \otimes m_H \otimes id_A \otimes \phi_H \otimes id_H) \\
& \circ(\Delta_{H\otimes H} \otimes id_A \otimes id_H \otimes id_A \otimes id_H) \circ (id_H \otimes c_{H,H\otimes A} \otimes \varphi_A \otimes \phi_H) \\
& \circ(id_H \otimes id_H \otimes \Delta_{H\otimes A} \otimes id_H \otimes id_A) \circ (\Delta_H \otimes \Delta_{H\otimes A}).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& m_{A\otimes H} \circ (m_{A\otimes H} \otimes id_A \otimes id_H) \circ (\eta_A \otimes id_H \otimes \eta_A \otimes id_H \otimes id_A \otimes \eta_H) \\
= & (m_A \otimes id_H) \circ (m_A \otimes (\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A}) \circ (id_A \otimes \omega \otimes m_H \otimes id_A) \\
& \circ(\varphi_A \circ (id_H \otimes \eta_A) \otimes \Delta_{H\otimes H} \otimes id_A) \circ (\Delta_H \otimes id_H \otimes id_A) \\
= & (m_A \otimes id_H) \circ (m_A \otimes (\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A}) \\
& \circ((\varphi_A \circ (id_H \otimes \eta_A) \otimes \omega) \circ (\Delta_H \otimes id_H) \otimes m_H \otimes id_A) \circ (\Delta_{H\otimes H} \otimes id_A) \\
= & (m_A \otimes id_H) \circ (\omega \otimes ((\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A}) \circ (m_H \otimes id_A)) \circ (\Delta_{H\otimes H} \otimes id_A).
\end{aligned}
$$

Thus it follows that

$$
\begin{aligned}
& (m_A \otimes id_H) \circ (id_A \otimes \varphi_A \otimes m_H) \circ (\omega \otimes m_H \otimes id_A \otimes \phi_H \otimes id_H) \\
& \circ(\Delta_{H\otimes H} \otimes id_A \otimes id_H \otimes id_A \otimes id_H) \circ (id_H \otimes c_{H,H\otimes A} \otimes \varphi_A \otimes \phi_H) \\
& \circ(id_H \otimes id_H \otimes \Delta_{H\otimes A} \otimes id_H \otimes id_A) \circ (\Delta_H \otimes \Delta_{H\otimes A}) \\
= & (m_A \otimes id_H) \circ (\omega \otimes ((\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A}) \circ (m_H \otimes id_A)) \circ (\Delta_{H\otimes H} \otimes id_A).
\end{aligned}
$$

Applying $\varepsilon_A \otimes id_H$ to both sides of the above equation, we can obtain (21).

On one hand

$$
\begin{aligned}
& m_{A\otimes H} \circ (id_A \otimes id_H \otimes m_{A\otimes H}) \circ (\eta_A \otimes id_H \otimes \eta_A \otimes id_H \otimes \eta_A \otimes id_H) \\
= & (m_A \otimes id_H) \circ (id_A \otimes (\omega \otimes m_H) \circ \Delta_{H\otimes H}) \circ ((\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A} \otimes id_H) \\
& \circ(id_H \otimes m_A \otimes id_H) \circ (id_H \otimes \varphi_A \circ (id_H \otimes \eta_A) \otimes \omega \otimes m_H) \\
& \circ(id_H \otimes \Delta_H \otimes id_H \otimes id_H \otimes id_H) \circ (id_H \otimes \Delta_{H\otimes H}) \\
\overset{(16)}{=} & (m_A \otimes id_H) \circ (id_A \otimes (\omega \otimes m_H) \circ \Delta_{H\otimes H}) \circ ((\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A} \otimes id_H) \\
& \circ(id_H \otimes \omega \otimes m_H) \circ (id_H \otimes \Delta_{H\otimes H}).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& m_{A\otimes H} \circ (m_{A\otimes H} \otimes id_A \otimes id_H) \circ (\eta_A \otimes id_H \otimes \eta_A \otimes id_H \otimes \eta_A \otimes id_H) \\
= \; & (m_A \otimes id_H) \circ (m_A \otimes \omega \otimes m_H) \circ (id_A \otimes \varphi_A \circ (id_H \otimes \eta_A) \otimes \Delta_{H\otimes H}) \\
& \circ (id_A \otimes \Delta_H \otimes id_H) \circ (m_A \otimes id_H \otimes id_H) \circ (id_A \otimes \omega \otimes m_H \otimes id_H) \\
& \circ (\varphi_A \circ (id_H \otimes \eta_A) \otimes \Delta_{H\otimes H} \otimes id_H) \circ (\Delta_H \otimes id_H \otimes id_H) \\
\overset{(16)}{=} \; & (m_A \otimes id_H) \circ (m_A \otimes \omega \otimes m_H) \circ (id_A \otimes \varphi_A \circ (id_H \otimes \eta_A) \otimes \Delta_{H\otimes H}) \\
& \circ (id_A \otimes \Delta_H \otimes id_H) \circ (\omega \otimes m_H \otimes id_H) \circ (\Delta_{H\otimes H} \otimes id_H) \\
\overset{(16)}{=} \; & (m_A \otimes id_H) \circ (id_A \otimes \omega \otimes m_H) \circ (id_A \otimes \Delta_{H\otimes H}) \\
& \circ (\omega \otimes m_H \otimes id_H) \circ (\Delta_{H\otimes H} \otimes id_H).
\end{aligned}
$$

Thus it follows that

$$
\begin{aligned}
& (m_A \otimes id_H) \circ (id_A \otimes (\omega \otimes m_H) \circ \Delta_{H\otimes H}) \\
& \quad \circ ((\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A} \otimes id_H) \circ (id_H \otimes \omega \otimes m_H) \circ (id_H \otimes \Delta_{H\otimes H}) \\
= \; & (m_A \otimes id_H) \circ (id_A \otimes \omega \otimes m_H) \circ (id_A \otimes \Delta_{H\otimes H}) \\
& \quad \circ (\omega \otimes m_H \otimes id_H) \circ (\Delta_{H\otimes H} \otimes id_H).
\end{aligned}
$$

By applying $id_A \otimes \varepsilon_H$ and $\varepsilon_A \otimes id_H$ to both sides of the equality above respectively, we can gain (22) and (23).

The converse can be checked in a straightforward way. □

Let $C$ be the monoidal category of **vector spaces** over a fixed field $k$. If we write $m_{A\otimes H}$ using the Sweedler notation and denoting the product of two elements $a, b \in A$ by $ab$, $\phi_H(h \otimes a)$ by $h \triangleleft a$, $\varphi_A(h \otimes a)$ by $h \triangleright a$ and $m_H(h \otimes g) = h \cdot g$, we have the multiplication on the vector space $A \otimes H$ as follows:

$$(a \otimes h)(b \otimes g) = a(h_{(1)} \triangleright b_{(1)})\omega(h_{(2)} \triangleleft b_{(2)}, g_{(1)}) \otimes (h_{(3)} \triangleleft b_{(3)}) \cdot g_{(2)}. \tag{24}$$

By Theorem 3.4, we have the following result.

**Theorem 3.5.** *Let $A$ be an ordinary bialgebra and $\Omega(A) = (H, \triangleleft, \triangleright, \omega)$ a partial extending datum of $A$. Then the multiplication defined by (24) on $A \otimes H$ is associative if and only if the following conditions hold: for all $h, g, l \in H$ and $b, c \in A$,*

$$(h \cdot g) \cdot l = (h \triangleleft \omega(g_{(1)}, l_{(1)})) \cdot (g_{(2)} \cdot l_{(2)}), \tag{25}$$

$$(h \triangleleft b) \triangleleft c = h \triangleleft (bc), \tag{26}$$

$$\omega(h_{(1)}, g_{(1)})\omega(h_{(2)} \cdot g_{(2)}, l) = (h_{(1)} \triangleright \omega(g_{(1)}, l_{(1)}))\omega(h_{(2)} \triangleleft \omega(g_{(2)}, l_{(2)}), g_{(3)} \cdot l_{(3)}), \tag{27}$$

$$(h \cdot g) \triangleleft c = (h \triangleleft (g_{(1)} \triangleright c_{(1)})) \cdot (g_{(2)} \triangleleft c_{(2)}). \tag{28}$$

For $\Omega(A)$ a partial extending datum of $A$, we can define the morphisms

$$\psi_H^A : H \otimes A \to A \otimes H$$

by

$$\psi_H^A = (\varphi_A \otimes \phi_H) \circ \Delta_{H\otimes A},$$

and

$$\sigma_H^A : H \otimes H \to A \otimes H$$

by

$$\sigma_H^A = (\omega \otimes m_H) \circ \Delta_{H\otimes H}.$$

Using the morphisms $\psi_H^A$ and $\sigma_H^A$, we can rewrite the equalities (v-4)-(v-8) and (21)-(23) in the following form:

(A1) $m_H \circ (m_H \otimes id_H) = m_H \circ (\phi_H \otimes id_H) \circ (id_H \otimes \sigma_H^A)$,

(A2) $\varphi_A \circ (id_H \otimes m_A) = m_A \circ (id_A \otimes \varphi_A) \circ (\psi_H^A \otimes id_A)$,

(A3) $\phi_H \circ (m_H \otimes id_A) = m_H \circ (\phi_H \otimes id_H) \circ (id_H \otimes \psi_H^A)$,

(A4) $m_A \circ (id_A \otimes \omega) \circ (\psi_H^A \otimes id_H) \circ (id_H \otimes \psi_H^A) = m_A \circ (id_A \otimes \varphi_A) \circ (\sigma_H^A \otimes id_A)$,

(A5) $m_A \circ (id_A \otimes \omega) \circ (\psi_H^A \otimes id_H) \circ (id_H \otimes \sigma_H^A) = m_A \circ (id_A \otimes \omega) \circ (\sigma_H^A \otimes id_A)$,

(A6) $\tau_{A,H} \circ \psi_H^A = (\phi_H \otimes \varphi_A) \circ \Delta_{H \otimes A}$,

(A7) $\tau_{A,H} \circ \sigma_H^A = (m_H \otimes \omega) \circ \Delta_{H \otimes H}$.

By using Theorem 3.4 and [10], we have the following conclusion.

**Theorem 3.6.** *Let $\Omega(A)$ be a partial extending datum of a bialgebra $A$. Let $(A, H, \psi_H^A, \sigma_H^A)$ be a 4-tuple, where $\psi_H^A, \sigma_H^A$ are defined as above. Then the pair $(A \otimes H, m_{A \otimes H})$ where*

$$m_{A \otimes H} = (m_A \otimes id_H) \circ (m_A \otimes \sigma_H^A) \circ (id_A \otimes \psi_H^A \otimes id_H)$$

*is a weak crossed product with preunit $\nu = \eta_A \otimes \eta_H$ if and only if (A1), (A3), (A5) and (20).*

As the end of this paper, we shall present a concrete example as follows.

**Example 3.7.** *Let $G = \langle g | g^2 = e \rangle$ be a cycle group. Then we have a group Hopf algebra $H = k[G]$. Let $A = k\langle 1_A, a, b, c \rangle$ be an algebra with the following multiplication table:*

| $m_A$ | $1_A$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $1_A$ | $1_A$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $a$ | $c$ | $c$ |
| $b$ | $b$ | $c$ | $1_A$ | $a$ |
| $c$ | $c$ | $c$ | $a$ | $a$ |

*It is not hard to check that $A$ is a bialgebra with the coalgebra structures given as follows:*

$$\Delta_A(a) = a \otimes a, \Delta_A(b) = b \otimes b, \Delta_A(c) = c \otimes c,$$
$$\varepsilon_A(a) = \varepsilon_A(b) = \varepsilon_A(c) = 1.$$

*Now, we define the action $\triangleright$ of $H$ on $A$ given via*

$$g \triangleright 1_A = a, g \triangleright a = a, g \triangleright b = a, g \triangleright c = a, e \triangleright 1_A = 1_A, e \triangleright a = a, e \triangleright b = b, e \triangleright c = c.$$

*The action $\triangleleft$ of $A$ on $H$ is given by the trivial action and the linear map $\omega : H \otimes H \to A$ is defined as follows:*

$$\omega(e, e) = 1_A, \omega(g, e) = a = \omega(e, g) = \omega(g, g).$$

*with $\triangleright, \triangleleft, \omega$ defined as above, it is not hard to check that $(H, \triangleleft, \triangleright, \omega)$ is a partial extending datum of $A$ and the conditions (25)-(26) are satisfied. Thus we have the weak crossed product $A \otimes H$ with the multiplication table as follows:*

| $\cdot$ | $1_A \otimes e$ | $a \otimes e$ | $b \otimes e$ | $c \otimes e$ | $1_A \otimes g$ | $a \otimes g$ | $b \otimes g$ | $c \otimes g$ |
|---|---|---|---|---|---|---|---|---|
| $1_A \otimes e$ | $1_A \otimes e$ | $a \otimes e$ | $b \otimes e$ | $c \otimes e$ | $a \otimes g$ | $a \otimes g$ | $c \otimes g$ | $c \otimes g$ |
| $a \otimes e$ | $a \otimes e$ | $a \otimes e$ | $c \otimes e$ | $c \otimes e$ | $a \otimes g$ | $a \otimes g$ | $c \otimes g$ | $c \otimes g$ |
| $b \otimes e$ | $b \otimes e$ | $c \otimes e$ | $1_A \otimes e$ | $a \otimes e$ | $c \otimes g$ | $c \otimes g$ | $a \otimes g$ | $a \otimes g$ |
| $c \otimes e$ | $c \otimes e$ | $c \otimes e$ | $a \otimes e$ | $a \otimes e$ | $c \otimes g$ | $c \otimes g$ | $a \otimes g$ | $a \otimes g$ |
| $1_A \otimes g$ | $a \otimes g$ | $a \otimes g$ | $a \otimes g$ | $a \otimes g$ | $a \otimes e$ | $a \otimes e$ | $a \otimes e$ | $a \otimes e$ |
| $a \otimes g$ | $a \otimes g$ | $a \otimes g$ | $a \otimes g$ | $a \otimes g$ | $a \otimes e$ | $a \otimes e$ | $a \otimes e$ | $a \otimes e$ |
| $b \otimes g$ | $c \otimes g$ | $c \otimes g$ | $c \otimes g$ | $c \otimes g$ | $c \otimes e$ | $c \otimes e$ | $c \otimes e$ | $c \otimes e$ |
| $c \otimes g$ | $c \otimes g$ | $c \otimes g$ | $c \otimes g$ | $c \otimes g$ | $c \otimes e$ | $c \otimes e$ | $c \otimes e$ | $c \otimes e$ |

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