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Existence of Solutions for Non-Autonomous Second-Order Stochastic Inclusions with Clarke's Subdifferential and non Instantaneous Impulses

Anjali Upadhyay[a], Surendra Kumar[a]

[a] *Department of Mathematics, University of Delhi, Delhi 110007, India*

Abstract. This manuscript explores a new class of non-autonomous second-order stochastic inclusions of Clarke's subdifferential form with non-instantaneous impulses (NIIs), unbounded delay, and the Rosenblatt process in Hilbert spaces. The existence of a solution is deduced by employing a fixed point strategy for a set-valued map together with the evolution operator and stochastic analysis approach. An example is analyzed for theoretical developments.

1. Introduction

Due to its practical applications in several fields, for instance, finance, physics, electrical engineering, medicine, and telecommunication, among others, many scholars have addressed stochastic evolution equations, and have already gained several fruitful results. To explore more, we refer to the following books and articles and the references cited therein [1–4]. In numerous regions of research, there has been a growing revenue in the evaluation of the frameworks fusing memory, i.e., there is the impact of postponement on state conditions. In this way, there is a genuine desire to talk about stochastic differential systems with delay.

In abstract spaces, Henríquez [23] assessed the presence of mild solutions, as well as classical solutions for a non-autonomous second-order (NASO) delayed functional differential equation with unbounded delay. Henríquez et al. [29] considered NASO differential structure with nonlocal initial data and developed the existence of solutions by applying the principle of the Hausdorff measure of non-compactness. Benchohra et al. [37] used a fixed point theorem developed by Darbo with the Kuratowski measure of non-compactness to build certain adequate conditions that guarantee the presence of a solution for a NASO non-instantaneous integro-differential system.

Because of its easy calculus and interesting attributes, the fractional Brownian motion (fBm) has attracted many scholars. One can go through the works in [4–6] for further details. In certain cases, where the mechanism is not Gaussian, the Rosenblatt process is chosen over fBm. Although the theory of the Rosenblatt process was established during 60's and 70's, significant development has been made in the last decade because of its self-similarity, long-range dependency, and stationary increments.

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Email addresses: upadhyayanjali123@gmail.com (Anjali Upadhyay), mathdma@gmail.com (Surendra Kumar)

In the literature, there are numerous articles that studied various theoretical facets of the Rosenblatt process. Ouahra et al. [35] discussed the qualitative properties of an stochastic delayed neutral functional differential system including impulses, Poisson jump, and the Rosenblatt process. Leonenko and Ahn [7] gave a fruitful result for the rate of convergence of the Rosenblatt process. The distribution property of the Rosenblatt process was investigated by Maejma and Tudor [8]. Sakthivel et al.[9] analyzed an abstract NASO stochastic evolution model with unbounded delay governed by the Rosenblatt process and proved its existence using the Krasnoselskii–Schaefer fixed point theorem, also studied the related autonomous system with bounded delay.

Lakhel and Tlidi [36] employed the Banach fixed point theorem to discuss the existence, uniqueness, and established stability criteria for a neutral stochastic functional differential system with impulses involving variable delays governed by the Rosenblatt process.

On the other hand, Clarke's subdifferential emerges from the applied discipline, namely thermo-viscoelasticity, filtration in porous materials, riveting applications in optimization, and non-smooth analysis [10, 11]. Recently, Vijayakumar [12] considered NASO stochastic inclusions of Clarke's subdifferential form and established the approximate controllability for the proposed systems.

Hernandez and O'Regan [13] introduced the theory of NIIs. Thereafter, many researchers gave various results on differential equations with NIIs. Pierri et al. [14], Yu and Wang [15], Fekan and Wang [16], and many more [17, 18] studied various qualitative properties of differential systems with NIIs. To the best of our incite, no result guarantees the existence of a solution for a NASO stochastic differential inclusions with Clarke's subdifferential including the Rosenblatt process and NIIs.

Let $\mathfrak{Z}$ and $(\mathscr{X}, \|\cdot\|_{\mathscr{X}}, \langle\cdot\rangle_{\mathscr{X}})$ be Hilbert spaces that are real and separable. The notation $L(\mathscr{X}, \mathfrak{Z})$ reflects the space of all bounded linear operators from $\mathscr{X}$ into $\mathfrak{Z}$. Strongly motivated by the above facts and discussions, we examine the subsequent stochastic differential inclusion with unbounded delay and NIIs

$$\begin{cases} d\chi'(\tau) \in [\mathscr{A}(\tau)\chi(\tau) + \partial\Sigma(\tau, \chi(\tau))]d\tau + q(\tau, \chi_\tau)dZ_H(\tau), & \tau \in \bigcup_{k=0}^{\mathcal{M}}(t_k, r_{k+1}]; \\ \chi(\tau) = f_k(\tau, \chi_\tau), & \tau \in \bigcup_{k=1}^{\mathcal{M}}(r_k, t_k]; \\ \chi'(\tau) = g_k(\tau, \chi_\tau), & \tau \in \bigcup_{k=1}^{\mathcal{M}}(r_k, t_k]; \\ \chi(\tau) = \eta(\tau), & \tau \in]-\infty, 0]; \\ \chi'(0) = \xi, \end{cases} \tag{1.1}$$

where, $\chi(\cdot)$ is $\mathfrak{Z}$-valued stochastic process, $\mathscr{A}(\tau) : D(\mathscr{A}(\tau)) \subseteq \mathfrak{Z} \to \mathfrak{Z}$ is closed and linear whose domain is dense in $\mathfrak{Z}$. The $\tau-$segment of χ, χ_τ :$]-\infty, 0] \to \mathfrak{Z}$ is given by $\chi_\tau(\theta) = \chi(\tau+\theta)$; $\theta \in]-\infty, 0]$, and belongs to an abstract phase space $\mathscr{W}$ described in Sect. 2. Let $\mathfrak{J} = [0, \beta]$, $\mathfrak{J}_0 =]-\infty, 0]$. The notation $\partial\Sigma$ represents the Clarke generalized subdifferential (see [10]) of a locally Lipschitz function $\Sigma(\tau, \cdot) : \mathfrak{Z} \to \mathbb{R}$; $q : \mathfrak{J} \times \mathscr{W} \to L_2^0$, $f_k, g_k : (r_k, t_k] \times \mathscr{W} \to \mathfrak{Z}$, $k = 1, \ldots, \mathcal{M}$ are suitable functions. The initial data η is Γ_0-measurable $\mathscr{W}$-valued stochastic process and ξ is Γ_0-measurable $\mathfrak{Z}$-valued stochastic process. Also, η and ξ have finite second moment, and are independent of the Rosenblatt process Z_H.

The points $0 = r_0 < t_0 < r_1 < t_1 < \ldots < t_{\mathcal{M}} < r_{\mathcal{M}+1} = \beta$ are impulsive positions. The impulses begin abruptly at r_k and continue to have an impact on $(r_k, t_k]$. The function $\chi(\cdot)$ takes distinct values in the two subintervals $(r_k, t_k]$, $(t_k, r_{k+1}]$ and is continuous at t_k.

The following is the summary of the rest of the manuscript: Sect. 2 is devoted to basic results, concepts, and Lemmas. The existence result for the proposed system (1.1) is covered in Sect. 3 by using set-valued (multi-valued) fixed point theorem [19]. We have reserved Sect. 4 for an example to show the applicability of the acquired result.

2. Preliminaries

Consider the probability space $(\Omega, \Gamma, \{\Gamma_\tau\}_{\tau\geq 0}, \mathbb{P})$ that is complete with the right continuous increasing sub σ-algebras $\{\Gamma_\tau\}_{\tau\in\mathfrak{J}}$ with $\Gamma_\tau \in \Gamma$ generated by all $\mathbb{P}$-null sets and $\{Z_H(t), t \in [0, \tau]\}$; where $Z_H(\tau)$ represents the Rosenblatt process on $\mathscr{X}$, and $H \in (\frac{1}{2}, 1)$. Let $L_2(\Omega, \mathfrak{Z})$ be the Banach space of strongly measurable, $\mathfrak{Z}$-valued random variables with the norm $\|\chi(\cdot)\|_{L_2} = (\mathbb{E}\|\chi(\cdot)\|^2_{\mathfrak{Z}})^{\frac{1}{2}}$, where $\mathbb{E}\|\chi\|^2 = \int_\Omega \|\chi\|^2 d\mathbb{P} < \infty$. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis for $\mathscr{X}$. The operator $Q \in L(\mathscr{X})$ is defined by $Qe_i = \nu_i e_i$, $i \in \mathbb{N}$ with trace $Tr(Q) = \sum_{i=1}^{\infty} \nu_i < \infty; \nu_i \geq 0$. Denote a sequence of mutually independent Rosenblatt processes by $\{z_i(\tau)\}_i^{\infty}$ on $(\Omega, \Gamma, \mathbb{P})$, which are two-sided and one-dimensional. A $\mathscr{X}$-valued stochastic process $Z_Q(\ell)$ is defined as

$$Z_Q(\ell) = \sum_{i=1}^{\infty} z_i(\ell) Q^{\frac{1}{2}} e_i, \ \ell \geq 0.$$

Moreover, the above series is convergent in $\mathscr{X}$ if $Q \geq 0$ and $Q = Q^*$(adjoint of Q).

Let $\mathscr{X}_0 = Q^{\frac{1}{2}} \mathscr{X}$ be the Hilbert space with the inner product $\langle z_1, z_2 \rangle_{\mathscr{X}_0} = \langle Q^{\frac{1}{2}} z_1, Q^{\frac{1}{2}} z_2 \rangle_{\mathscr{X}}$. Further, let $L_2(\mathscr{X}_0, \mathfrak{Z}) := L_2^0$ be the space of Hilbert–Schmidt operators from $\mathscr{X}_0$ into $\mathfrak{Z}$. Clearly, L_2^0 equipped with the inner product $\langle \Phi_1, \Phi_2 \rangle = \sum_{i=1}^{\infty} \langle \Phi_1 e_i, \Phi_2 e_i \rangle$ is a Hilbert space. Moreover, $\|\Phi\|^2_{L_2^0} = \|\Phi Q^{\frac{1}{2}}\|^2 = Tr(\Phi Q \Phi^*)$.

Let $\mathfrak{I}$ be the interval with the given horizon β. Then the one dimensional Rosenblatt process is represented by [20]

$$Z_H^{\alpha}(\tau) = c(H) \int_0^{\tau} \int_0^{\tau} \Big[\int_{b_1 \vee b_2}^{\tau} \frac{\partial K^{\hat{H}}}{\partial u}(u, b_1) \frac{\partial K^{\hat{H}}}{\partial u}(u, b_2) ds \Big] dB(b_1) dB(b_2)$$

where $B = \{B(\tau) : \tau \in \mathfrak{I}\}$ is the Wiener process, $\hat{H} = \frac{H+1}{2}$, $c(H) = \frac{1}{H+1} \sqrt{\frac{H}{2(2H-1)}}$ and the kernel $K^H(\cdot, \cdot)$ is given by

$$K^H(\ell, s) = \begin{cases} c_H \, s^{\frac{1}{2}-H} \int_s^{\ell} (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, & \ell > s; \\ 0, & \ell \leq s \end{cases}$$

where $c_H = \sqrt{\frac{H(2H-1)}{\mathfrak{B}(2-2H, H-\frac{1}{2})}}$; $\mathfrak{B}(\cdot, \cdot)$ represents the Beta function. The space $PC(\mathfrak{Z})$ formed by all $\mathfrak{Z}$-valued stochastic processes $\{\chi(\tau) : \tau \in \mathfrak{I}\}$ that are Γ_τ-adapted, measurable with χ is continuous at $\tau \neq r_k$, $\chi_{r_k} = \chi_{r_k^-}$, and $\chi_{r_k^+}$ exist for $k = 1, ..., \mathcal{M}$, is a Banach space with $\|\chi\|_{PC} = \Big(\sup_{\beta \geq s \geq 0} \mathbb{E}\|\chi(s)\|^2 \Big)^{\frac{1}{2}}$.

The phase space $(\mathscr{W}, \|\cdot\|_{\mathscr{W}})$ formed by all Γ_0-measurable mappings from $\mathfrak{I}_0$ into $\mathfrak{Z}$ is a semi-normed linear space and the accompanying axioms hold (cf.[21, 22])

(i) If $\chi :]-\infty, \beta] \to \mathfrak{Z}$, $\beta > 0$, is such that $\chi|_{[0,\beta]} \in PC([0, \beta], \mathfrak{Z})$ with $\chi_0 \in \mathscr{W}$, then for all $\tau \in [0, \beta]$ following hold:

 (a) $\chi_\tau \in \mathscr{W}$;
 (b) $\|\chi(\tau)\| \leq J \, \|\chi_\tau\|_{\mathscr{W}}$, where $J > 0$ is a constant;
 (c) $\|\chi_\tau\|_{\mathscr{W}} \leq K(\tau) \sup\{ \|\chi(s)\| : \tau \geq s \geq 0 \} + L(\tau) \|\chi_0\|_{\mathscr{W}}$, where $K, L : \mathbb{R}^+ \cup \{0\} \to [1, \infty)$, K and L are continuous and locally bounded respectively, and are independent of $\chi(\cdot)$.

(i) $\mathscr{W}$ is complete space.

The result given below is extracted from the above axioms:

Lemma 2.1. *[17] Let the process $\chi :] - \infty, \beta] \to \mathfrak{Z}$ be measurable and Γ_τ-adapted with $\chi|_{\mathfrak{I}} \in PC(\mathfrak{Z})$, $\chi_0 = \eta(\tau) \in \mathscr{L}^2_{\Gamma_0}(\Omega, \mathscr{W})$, then*

$$\|\chi_\tau\|_{\mathscr{W}} \leq K_\beta \sup_{\tau \in \mathfrak{I}} \mathbb{E}\|\chi(\tau)\| + L_\beta \|\eta\|_{\mathscr{W}},$$

where $K_\beta = \max_{\tau\in\mathfrak{J}} K(\tau)$ *and* $L_\beta = \sup_{\tau\in\mathfrak{J}} L(\tau)$.

For further development, we describe the following theory of evolution operator.

Definition 2.2. *[30] A mapping* $\mathcal{G} : \mathfrak{J}\times\mathfrak{J} \to L(\mathfrak{Z})$ *is characterized as an evolution operator for* $\chi''(\tau) = \mathscr{A}(\tau)\chi(\tau)$, $\beta \geq s, \tau \geq 0$ *if meet the subsequent conditions:*

(B_1) *The map* $(\tau, s) \mapsto \mathcal{G}(\tau, s)\chi$ *is of class* C^1 *for every* $\chi \in \mathfrak{Z}$, *and*

- *(i)* $\mathcal{G}(\tau, \tau) = 0$ *for every* $\tau \in \mathfrak{J}$,
- *(ii) For every* $\chi \in \mathfrak{Z}$, *for all* $\tau, s \in \mathfrak{J}$,

$$\frac{\partial}{\partial\tau}\mathcal{G}(\tau, s)|_{\tau=s}\,\chi = \chi \text{ and } \frac{\partial}{\partial s}\mathcal{G}(\tau, s)|_{\tau=s}\,\chi = -\chi.$$

(B_2) *For* $\chi \in D(\mathscr{A}(\tau))$, $\mathcal{G}(\tau, s)\chi \in D(\mathscr{A}(\tau))$ *for all* $s, \tau \in \mathfrak{J}$, *the map* $(\tau, s) \mapsto \mathcal{G}(\tau, s)\chi$ *is of class* C^2 *and*

- *(i)* $\frac{\partial^2}{\partial\tau^2}\mathcal{G}(\tau, s)\chi = \mathscr{A}(\tau)\mathcal{G}(\tau, s)\chi$,
- *(ii)* $\frac{\partial^2}{\partial s^2}\mathcal{G}(\tau, s)\chi = \mathcal{G}(\tau, s)\mathscr{A}(s)\chi$,
- *(iii)* $\frac{\partial^2}{\partial s\partial\tau}\mathcal{G}(\tau, s)|_{\tau=s}\,\chi = 0$

(B_3) *For all* $s, \tau \in \mathfrak{J}$, *if* $\chi \in D(\mathscr{A}(\tau))$ *then* $\frac{\partial}{\partial s}\mathcal{G}(\tau, s)\chi \in D(\mathscr{A}(\tau))$, $\frac{\partial^3}{\partial\tau^2\partial s}\mathcal{G}(\tau, s)\chi$, $\frac{\partial^3}{\partial s^2\partial\tau}\mathcal{G}(\tau, s)\chi$ *exist, also*

- *(i)* $\frac{\partial^3}{\partial\tau^2\partial s}\mathcal{G}(\tau, s)\chi = \mathscr{A}(\tau)\frac{\partial}{\partial s}\mathcal{G}(\tau, s)\chi$,
- *(ii)* $\frac{\partial^3}{\partial s^2\partial\tau}\mathcal{G}(\tau, s)\chi = \frac{\partial}{\partial\tau}\mathcal{G}(\tau, s)\mathscr{A}(s)\chi$,

and the map $(\tau, s) \mapsto \mathscr{A}(\tau)\frac{\partial}{\partial s}\mathcal{G}(\tau, s)\chi$ *is continuous.*

Throughout the article, we suppose an evolution operator $\mathcal{G}(\tau, s)$ exists related to the operator $\mathscr{A}(\tau)$. Besides, we present $\mathcal{E}(\tau, s) = -\frac{\partial}{\partial s}\mathcal{G}(\tau, s)$.

We are now presenting some useful definitions for the set-valued map (see [24, 25]). Let $P(\mathfrak{Z})$ denote the family of all non-empty subsets of $\mathfrak{Z}$. For convenience, set:

$$P_{cl}(\mathfrak{Z}) = \{\chi \in P(\mathfrak{Z}) : \chi \text{ is closed}\},\ P_{bd}(\mathfrak{Z}) = \{\chi \in P(\mathfrak{Z}) : \chi \text{ is bounded}\},$$
$$P_{cv}(\mathfrak{Z}) = \{\chi \in P(\mathfrak{Z}) : \chi \text{ is convex}\},\ P_{cp}(\mathfrak{Z}) = \{\chi \in P(\mathfrak{Z}) : \chi \text{ is compact}\},$$

Consider $\mathfrak{Z}_d : P(\mathfrak{Z}) \times P(\mathfrak{Z}) \to \mathbb{R}^+ \cup \{\infty\}$ given by

$$\mathfrak{Z}_d(\mathbb{G}, \mathbb{H}) = \max\Big\{\sup_{u\in\mathbb{G}} d(u, \mathbb{H}),\ \sup_{v\in\mathbb{H}} d(\mathbb{G}, v)\Big\},$$

where $d(u, \mathbb{H}) = \inf_{v\in\mathbb{H}} d(u, v)$, $d(\mathbb{G}, v) = \inf_{u\in\mathbb{G}} d(u, v)$. Then $(P_{bd,cl}(\mathfrak{Z}), \mathfrak{Z}_d)$ is a metric space.

Definition 2.3. *Let* $\Theta : \mathfrak{Z} \to P(\mathfrak{Z})$ *be a set-valued mapping, then*

- *(i)* Θ *is closed (convex) valued if* $\Theta(\chi)$ *is closed (convex) for every* $\chi \in \mathfrak{Z}$.
- *(ii)* Θ *is bounded on bounded sets if* $\Theta(\mathscr{D}) = \cup_{\chi\in\mathscr{D}}\Theta(\chi)$ *is bounded in* $\mathfrak{Z}$ *for all* $\mathscr{D} \in P_{bd}(\mathfrak{Z})$.
- *(iii) If for each* $\chi \in \mathfrak{Z}$, $\Theta(\chi) \neq \emptyset$ *is closed subset of* $\mathfrak{Z}$, *and if for each open set* J *in* $\mathfrak{Z}$ *containing* $\Theta(\chi)$, *there is an open neighbourhood* O *of* χ *such that* $\Theta(O) \subseteq J$, *then* Θ *is characterized as upper semi-continuous (u.s.c.) on* $\mathfrak{Z}$,
- *(iv) If* $\Theta(J)$ *is relatively compact for every* $J \in P_{bd}(\mathfrak{Z})$, *then* Θ *is completely continuous .*

(v) If there is a $\chi \in \mathfrak{Z}$ such that $\chi \in \Theta(\chi)$, then Θ has a fixed element.

Definition 2.4. *A set-valued operator $\Theta : \mathfrak{Z} \to P_{bd,cl}(\mathfrak{Z})$ is known to be contraction if there is $\gamma \in (0,1)$ to ensure that*

$$\mathfrak{Z}_d(\chi_1, \chi_2) \leq \gamma\, d(\chi_1, \chi_2),\ \forall\ \chi_1, \chi_2 \in \mathfrak{Z}.$$

Definition 2.5. *The Clarke generalized directional derivative of a locally Lipschitz functional $\Sigma : \mathfrak{Z} \to \mathbb{R}$ at $z \in \mathfrak{Z}$ in the direction w is defined as*

$$\Sigma^0(z; w) = \lim_{x\to z} \sup_{\varepsilon\to 0^+} \frac{\Sigma(x+\varepsilon w) - \Sigma(x)}{\varepsilon}.$$

The Clarke generalized subdifferential of Σ is a subset of $\mathfrak{Z}^*$, and at a point $z \in \mathfrak{Z}$ is defined as

$$\partial\Sigma(z) = \{z^* \in \mathfrak{Z}^* : \Sigma^0(z; w) \geq \langle z^*, w\rangle, \text{for all } w \in \mathfrak{Z}\}.$$

Lemma 2.6. *[19] Let $\bar{\Theta}_1 : \mathfrak{Z} \to P_{cl,cv,bd}(\mathfrak{Z})$, $\bar{\Theta}_2 : \mathfrak{Z} \to P_{cl,cv}(\mathfrak{Z})$ be set-valued maps satisfying*

(a) $\bar{\Theta}_1$ is a contraction,

(b) $\bar{\Theta}_2$ is u.s.c. and completely continuous.

Then either (i) the inclusion $\lambda\chi \in \bar{\Theta}_1\chi + \bar{\Theta}_2\chi$ has a solution for $\lambda = 1$, or
(ii) the set $\{\chi \in \mathfrak{Z} : \lambda\chi \in \bar{\Theta}_1\chi + \bar{\Theta}_2\chi,\ \lambda > 1\}$ is unbounded.

The following result plays a key role in dealing with the stochastic term.

Lemma 2.7. *[26] Let $\phi : \mathfrak{J} \to L_2^0$ be such that $\sup_{\tau\in\mathfrak{J}} \|\phi\|^2_{L_2^0} < \infty$. Suppose that there is $M > 0$ satisfying $\|\mathcal{G}(\tau, s)\|^2 \leq M$ for all $\tau \geq s$. Then*

$$\mathbb{E}\|\int_0^\tau \mathcal{G}(\tau, s)\phi(s)dZ_H(s)\|^2_{\mathfrak{Z}} \leq c(H)M\tau^{2H}\Big(\sup_{\tau\in\mathfrak{J}} \|\phi\|^2_{L_2^0}\Big).$$

Now we introduce a solution of proposed system (1.1) as follows

Definition 2.8. *An stochastic process $\chi :]-\infty, \beta] \to \mathfrak{Z}$ is called a mild solution for (1.1) if*

1. *the measurable process χ_τ is adapted to Γ_τ, $\tau \geq 0$,*

2. *$\chi = \eta(\tau)$ on $]-\infty, 0]$, satisfying $\|\eta\|^2_{\mathscr{W}} < \infty$, $\chi_\tau \in \mathscr{W}$, $\tau \in \mathfrak{J}$ with $\chi'(0) = \xi \in \mathfrak{Z}$, $\chi|_{\mathfrak{J}} \in PC(\mathfrak{Z})$ and following integral equation hold:*

$$\chi(\tau) = \begin{cases} \mathcal{E}(\tau, 0)\eta(0) + \mathcal{G}(\tau, 0)\xi + \int_0^\tau \mathcal{G}(\tau, s)\rho(s)ds \\ + \int_0^\tau \mathcal{G}(\tau, s)q(s, \chi_s)dZ_H(s), & \tau \in [0, r_1]; \\ f_k(\tau, \chi_\tau), & \tau \in \bigcup_{k=1}^{\mathcal{M}} (r_k, t_k]; \\ \mathcal{E}(\tau, t_k)f_k(t_k, \chi_{t_k}) + \mathcal{G}(\tau, t_k)g_k(t_k, \chi_{t_k}) + \int_{t_k}^\tau \mathcal{G}(\tau, s)\rho(s)ds \\ + \int_{t_k}^\tau \mathcal{G}(\tau, s)q(s, \chi_s)dZ_H(s), & \tau \in \bigcup_{k=1}^{\mathcal{M}} (t_k, r_{k+1}]. \end{cases} \tag{2.1}$$

3. Existence of solution

We start this section by imposing the following conditions on the system parameters:

Suppose assumptions ($A1 - A7$) in Kozak [31] on the operators $\mathscr{A}(\tau), \tau \in \mathfrak{I}$ hold, which ensure the existence of the evolution operator $\mathcal{G}(\tau, s)$ satisfying the conditions $(B_1) - (B_3)$.

($\mathbf{S}_1$) : The operator $\mathcal{G}(\tau, s)$ is compact for all $\tau \geq s$, also there exists M > 0 such that

$$\sup_{(\tau,s)\in\mathfrak{I}\times\mathfrak{I}} \|\mathcal{E}(\tau, s)\| \vee \sup_{(\tau,s)\in\mathfrak{I}\times\mathfrak{I}} \|\mathcal{G}(\tau, s)\| \leq \mathrm{M}, \text{ for all } \tau \geq s.$$

($\mathbf{S}_2$) : The functions $f_k, g_k : (r_k, t_k] \times \mathscr{W} \to \mathfrak{Z}$ are continuous, and also there are $c_k > 0,\ \gamma_k > 0,\ k = 1, \dots \mathcal{M}$ in order that for all $\eta, \eta_1, \eta_2 \in \mathscr{W}$,

$$\mathbb{E}\|f_k(\tau, \eta_1) - f_k(\tau, \eta_2)\|^2_{\mathfrak{Z}} \leq \gamma_k \|\eta_1 - \eta_2\|^2_{\mathscr{W}},\ \mathbb{E}\|f_k(\tau, \eta)\|^2 \leq \gamma_k(1 + \|\eta\|^2_{\mathscr{W}}),$$
$$\mathbb{E}\|g_k(\tau, \eta_1) - g_k(\tau, \eta_2)\|^2_{\mathfrak{Z}} \leq c_k \|\eta_1 - \eta_2\|^2_{\mathscr{W}},\ \mathbb{E}\|g_k(\tau, \eta)\|^2 \leq c_k(1 + \|\eta\|^2_{\mathscr{W}}).$$

($\mathbf{S}_3$) : Let $\Sigma : \mathfrak{I} \times \mathscr{W} \to \mathbb{R}$ be the map such that:

(i) For all $\chi \in \mathfrak{Z}$, $\Sigma(\cdot, \chi)$ is measurable .

(ii) For a.e. $\tau \in \mathfrak{I}$, $\Sigma(\tau, \cdot)$ is locally Lipschitz .

(iii) There is $b_1(\cdot) \in L^1(\mathfrak{I}, \mathbb{R}^+)$ and $0 \leq b_2$ in order that

$$\begin{aligned}\|\partial\Sigma(s, \chi)\|^2 &= \sup\{\|\rho(s)\|^2 |\ \rho(s) \in \partial\Sigma(s, \chi)\} \\ &\leq b_1(s) + b_2\|\chi\|^2 \text{ for all } \chi \in \mathfrak{Z},\ \text{ a.e. } s \in \mathfrak{I}.\end{aligned}$$

($\mathbf{S}_4$) : (i) The function $q(\tau, \cdot) : \mathscr{W} \to L_2^0$ is continuous for all $\tau \in \mathfrak{I}$, and $q(\cdot, \eta) : \mathfrak{I} \to L_2^0$ is strongly measurable for each $\eta \in \mathscr{W}$. Also, there is $M_q > 0$ to ensure that

$$\mathbb{E}\|q(\tau, \eta_1) - q(\tau, \eta_2)\|^2_{L_2^0} \leq M_q \|\eta_1 - \eta_2\|^2_{\mathscr{W}},\ \ \eta_1, \eta_2 \in \mathscr{W}.$$

(ii) There is a continuous function $m_q : [0, \infty) \to (0, \infty)$ that is non decreasing, and $m(\cdot) \in L^1(\mathfrak{I}, \mathbb{R}^+)$ with the aim that

$$\mathbb{E}\|q(\tau, \eta)\|^2_{L_2^0} \leq m(\tau) m_q(\|\eta\|^2_{\mathscr{W}}),\ \ (\tau, \eta) \in \mathfrak{I} \times \mathscr{W}.$$

Consider the set-valued map $S : \mathscr{L}^2(\mathfrak{I}, \mathfrak{Z}) \to 2^{\mathscr{L}^2(\mathfrak{I},\mathfrak{Z})}$given by

$$S_{\Sigma,\chi} = \{\rho \in \mathscr{L}^2(\mathfrak{I}, \mathfrak{Z})|\ \rho(\tau) \in \partial\Sigma(\tau, \chi(\tau))\ a.e.\ \tau \in \mathfrak{I},\ \chi \in \mathscr{L}^2(\mathfrak{I}, \mathfrak{Z})\}.$$

Lemma 3.1. *[27] The set $S_{\Sigma,\chi}$ is non empty, and has convex, weakly compact values for each $\rho \in \mathscr{L}^2(\mathfrak{I}, \mathfrak{Z})$ provided the assumption (S_3) holds.*

Lemma 3.2. *[28] Let the interval $[0, \beta]$ be compact, and the set-valued map Σ satisfies (S_3). Let F be a linear continuous operator from $\mathscr{L}^2([0, \beta], \mathfrak{Z})$ to $C([0, \beta], \mathfrak{Z})$. Then,*

$$F \circ S_\Sigma : C([0, \beta], \mathfrak{Z}) \to P_{cp,cv}(\mathfrak{Z}),\ \chi \to (F \circ S_\Sigma)(\chi) := F(S_{\Sigma,\chi})$$

has closed graph in $C([0, \beta], \mathfrak{Z}) \times C([0, \beta], \mathfrak{Z})$.

Consider the space $\mathscr{W}_\beta = \{\chi :]-\infty,\beta] \to \mathfrak{Z} \ / \ \chi_0 = \eta \in \mathscr{W},\ \xi \in \mathfrak{Z},\ \chi|_{\mathfrak{J}} \in PC(\mathfrak{Z}),\ \sup_{\tau\in\mathfrak{J}} \mathbb{E}\|\chi(\tau)\|^2 < \infty\}$ with the semi-norm $\|\chi\|_\beta = \|\chi_0\|_{\mathscr{W}} + \left(\sup_{\tau\in\mathfrak{J}} \mathbb{E}\|\chi(\tau)\|^2\right)^{1/2}$.

In view of Lemma 2.1, we have

$$\begin{aligned}\|y_\tau + \overline{\eta}_\tau\|^2_{\mathscr{W}} \ &\leq \ 2(\|y_\tau\|^2_{\mathscr{W}} + \|\overline{\eta}_\tau\|^2_{\mathscr{W}}) \leq 4\{K^2_\beta \sup_{\tau\in\mathfrak{J}} \mathbb{E}\|y(s)\|^2 + L^2_\beta\|y_0\|^2_{\mathscr{W}} + K^2_\beta \sup_{\tau\in\mathfrak{J}} \mathbb{E}\|\overline{\eta}(s)\|^2 + L^2_\beta\|\overline{\eta}_0\|^2_{\mathscr{W}}\}\\ &\leq 4\{K^2_\beta \sup_{\tau\in\mathfrak{J}} \mathbb{E}\|y(s)\|^2 + L^2_\beta\|\eta\|^2_{\mathscr{W}}\},\ \tau \in \mathfrak{J},\end{aligned} \tag{3.1}$$

Consider the set-valued map $\Theta : \mathscr{W}_\beta \to P(\mathscr{W}_\beta)$ characterized by $\Theta\chi$, the set of all $\sigma \in \mathscr{W}_\beta$ satisfying

$$\sigma(\tau) = \begin{cases} \eta(\tau), & \tau \in \mathfrak{J}_0;\\ \mathcal{E}(\tau,0)\eta(0) + \mathcal{G}(\tau,0)\xi + \int_0^\tau \mathcal{G}(\tau,s)\rho(s)ds + \int_0^\tau \mathcal{G}(\tau,s)q(s,\chi_s)dZ_H(s), & \tau \in [0,r_1];\\ f_k(\tau,\chi_\tau), & \tau \in \bigcup_{k=1}^{\mathcal{M}}(r_k,t_k];\\ \mathcal{E}(\tau,t_k)f_k(t_k,\chi_{t_k}) + \mathcal{G}(\tau,t_k)g_k(t_k,\chi_{t_k}) + \int_{t_k}^\tau \mathcal{G}(\tau,s)\rho(s)ds & \\ + \int_{t_k}^\tau \mathcal{G}(\tau,s)q(s,\chi_s)dZ_H(s), & \tau \in \bigcup_{k=1}^{\mathcal{M}}(t_k,r_{k+1}], \end{cases} \tag{3.2}$$

where $\rho \in S_{\Sigma,\chi}$. We shall show that Θ has a fixed point in $\mathscr{W}_\beta$ that is a required solution for the system (1.1).

Define $\overline{\eta}(\cdot) :]-\infty,\beta] \to \mathfrak{Z}$ by

$$\overline{\eta}(\tau) = \begin{cases} \eta(\tau), & \tau \in \mathfrak{J}_0;\\ 0, & \tau \in \mathfrak{J}. \end{cases}$$

Obviously, $\overline{\eta} \in \mathscr{W}_\beta$ and $\overline{\eta}_0 = \eta$. Set $\chi(\tau) = \overline{\eta}(\tau) + y(\tau)$, $-\infty < \tau \leq \beta$. Clearly $\chi(\cdot)$ satisfies (2.1) if and only if $y_0 = 0$ and

$$y(\tau) = \begin{cases} \mathcal{E}(\tau,0)\eta(0) + \mathcal{G}(\tau,0)\xi + \int_0^\tau \mathcal{G}(\tau,s)\rho(s)ds + \int_0^\tau \mathcal{G}(\tau,s)q(s,y_s+\overline{\eta}_s)dZ_H(s), & \tau \in [0,r_1];\\ f_k(\tau,y_\tau+\overline{\eta}_\tau), & \tau \in \bigcup_{k=1}^{\mathcal{M}}(r_k,t_k];\\ \mathcal{E}(\tau,t_k)f_k(t_k,y_{t_k}+\overline{\eta}_{t_k}) + \mathcal{G}(\tau,t_k)g_k(t_k,y_{t_k}+\overline{\eta}_{t_k}) + \int_{t_k}^\tau \mathcal{G}(\tau,s)\rho(s)ds & \\ + \int_{t_k}^\tau \mathcal{G}(\tau,s)q(s,y_s+\overline{\eta}_s)dZ_H(s), & \tau \in \bigcup_{k=1}^{\mathcal{M}}(t_k,r_{k+1}], \end{cases} \tag{3.3}$$

Consider the set $\mathscr{W}^0_\beta = \{\chi \in \mathscr{W}_\beta :\ y_0 = 0 \in \mathscr{W}\}$ with the semi-norm given by

$$\|y\|_\beta = \|y_0\|_{\mathscr{W}} + \left(\sup_{s\in\mathfrak{J}} \mathbb{E}\|y(s)\|^2\right)^{1/2} = \left(\sup_{s\in\mathfrak{J}} \mathbb{E}\|y(s)\|^2\right)^{1/2}.$$

Then $(\mathscr{W}^0_\beta, \|\cdot\|_\beta)$ forms a Banach space.

Now suppose that the set-valued map $\bar{\Theta} : \mathscr{W}^0_\beta \to P(\mathscr{W}^0_\beta)$ defined by $\bar{\Theta}y$, the set of all $\bar{\sigma} \in \mathscr{W}^0_\beta$ satisfying $\bar{\sigma}(\tau) = 0$, $\tau \in \mathfrak{J}_0$ and

$$\bar{\sigma}(\tau) = \begin{cases} \mathcal{E}(\tau,0)\eta(0) + \mathcal{G}(\tau,0)\xi + \int_0^\tau \mathcal{G}(\tau,s)\rho(s)ds + \int_0^\tau \mathcal{G}(\tau,s)q(s,y_s+\overline{\eta}_s)dZ_H(s), & \tau \in [0,r_1];\\ f_k(\tau,y_\tau+\overline{\eta}_\tau), & \tau \in \bigcup_{k=1}^{\mathcal{M}}(r_k,t_k];\\ \mathcal{E}(\tau,t_k)f_k(t_k,y_{t_k}+\overline{\eta}_{t_k}) + \mathcal{G}(\tau,t_k)g_k(t_k,y_{t_k}+\overline{\eta}_{t_k}) + \int_{t_k}^\tau \mathcal{G}(\tau,s)\rho(s)ds & \\ + \int_{t_k}^\tau \mathcal{G}(\tau,s)q(s,y_s+\overline{\eta}_s)dZ_H(s), & \tau \in \bigcup_{k=1}^{\mathcal{M}}(t_k,r_{k+1}], \end{cases} \tag{3.4}$$

where $\rho \in S_{\Sigma,y} = \{\rho \in L^2(\mathfrak{J}, L(\mathfrak{Z}, \mathscr{X}) \mid \rho(\tau) \in \partial\Sigma(\tau, y(\tau) + \overline{\eta}_\tau) \ a.e. \tau \in \mathfrak{J}\}$. If $\bar{\Theta}$ has a fixed point in $\mathscr{W}_\beta^0$ then Θ has a fixed point in $\mathscr{W}_\beta^0$. We now assert that $\bar{\Theta}$ fulfils all assumptions of Lemma 2.6. For $\varkappa > 0$, let $D_\varkappa(0, \mathscr{W}_\beta^0) = \{y \in \mathscr{W}_\beta^0 : \ \mathbb{E}\|y\|_\beta^2 \le \varkappa\}$. Clearly, $D_\varkappa \subset \mathscr{W}_\beta^0$ is convex, closed, and bounded. Now in view of inequality (3.1) and Lemma 2.1, it follows that

$$\|y_\tau + \overline{\eta}_\tau\|_{\mathscr{W}}^2 \le 4[K_\beta^2 \varkappa + L_\beta^2 \|\eta\|_{\mathscr{W}}^2] = \varkappa^*, \ \tau \in \mathfrak{J}.$$

Next, split $\bar{\Theta} = \bar{\Theta}_1 + \bar{\Theta}_2$, where

$$(\bar{\Theta}_1 y)(\tau) = \begin{cases} \mathcal{E}(\tau,0)\eta(0) + \mathcal{G}(\tau,0)\xi + \int_0^\tau \mathcal{G}(\tau,s) q(s, y_s + \overline{\eta}_s) dZ_H(s), & \tau \in [0, r_1]; \\ f_k(\tau, y_\tau + \overline{\eta}_\tau), & \tau \in \bigcup_{k=1}^{\mathcal{M}} (r_k, t_k]; \\ \mathcal{E}(\tau, t_k) f_k(t_k, y_{t_k} + \overline{\eta}_{t_k}) + \mathcal{G}(\tau, t_k) g_k(t_k, y_{t_k} + \overline{\eta}_{t_k}) & \\ + \int_{t_k}^\tau \mathcal{G}(\tau,s) q(s, y_s + \overline{\eta}_s) dZ_H(s), & \tau \in \bigcup_{k=1}^{\mathcal{M}} (t_k, r_{k+1}], \end{cases} \tag{3.5}$$

and

$$(\bar{\Theta}_2 y)(\tau) = \begin{cases} \int_0^\tau \mathcal{G}(\tau,s)\rho(s) ds, & \tau \in [0, r_1]; \\ 0, & \tau \in \bigcup_{k=1}^{\mathcal{M}} (r_k, t_k]; \\ \int_{t_k}^\tau \mathcal{G}(\tau,s)\rho(s) ds, & \tau \in \bigcup_{k=1}^{\mathcal{M}} (t_k, r_{k+1}]. \end{cases} \tag{3.6}$$

Lemma 3.3. *If (S_1), (S_2) and (S_4) hold, then $\bar{\Theta}_1$ takes bounded sets into bounded sets in $\mathscr{W}_\beta^0$, and is a contraction on $\mathscr{W}_\beta^0$.*

Proof. **Claim 1: $\bar{\Theta}_1$ maps bounded sets to bounded sets in $\mathscr{W}_\beta^0$.**
Let $y \in D_\varkappa(0, \mathscr{W}_\beta^0)$, then by (S_1), (S_2) and (S_4), for $\tau \in [0, r_1]$, we get

$$\begin{aligned} \mathbb{E}\|y(\tau)\|_{\mathfrak{Z}}^2 &\le 3M\Big[\mathbb{E}\|\eta\|_{\mathfrak{Z}}^2 + \mathbb{E}\|\xi\|_{\mathfrak{Z}}^2 + r_1^{2H} c(H) Tr(Q) \int_0^{r_1} m(s) m_q(\|y_s + \overline{\eta}_s\|_{\mathscr{W}}^2) ds\Big] \\ &\le 3M\Big[\mathbb{E}\|\eta\|_{\mathfrak{Z}}^2 + \mathbb{E}\|\xi\|_{\mathfrak{Z}}^2 + r_1^{2H} c(H) Tr(Q) m_q(\varkappa^*) \|m(\tau)\|_{L^1} := s_0. \end{aligned}$$

For any $\tau \in (r_k, t_k]$, $k = 1, 2, \dots, \mathcal{M}$,

$$\mathbb{E}\|y(\tau)\|_{\mathfrak{Z}}^2 \le \mathbb{E}\|f_k(\tau, y_\tau + \overline{\eta}_\tau)\|_{\mathfrak{Z}}^2 \le \gamma_k(\|y_\tau + \overline{\eta}_\tau\|_{\mathscr{W}}^2 + 1) \le \gamma_k(\varkappa^* + 1) := \zeta_k.$$

Similarly, for $\tau \in (t_k, r_{k+1}]$, $k = 1, 2, \dots, \mathcal{M}$, compute

$$\begin{aligned} \mathbb{E}\|y(\tau)\|_{\mathfrak{Z}}^2 &\le 3[\mathbb{E}\|\mathcal{E}(\tau, t_k) f_k(t_k, y_{t_k} + \overline{\eta}_{t_k})\|_{\mathfrak{Z}}^2 + \mathbb{E}\|\mathcal{G}(\tau, t_k) g_k(t_k, y_{t_k} + \overline{\eta}_{t_k})\|_{\mathfrak{Z}}^2 \\ &\quad + \mathbb{E}\|\int_{t_k}^\tau \mathcal{G}(\tau,s) q(s, y_s + \overline{\eta}_s) dZ_H(s)\|_{\mathfrak{Z}}^2 \\ &\le 3M\Big\{(\gamma_k + c_k)(\|y_{t_k} + \overline{\eta}_{t_k}\|_{\mathscr{W}}^2 + 1) + (r_{k+1} - t_k)^{2H} c(H) M \ Tr(Q) \int_{t_k}^\tau m(s) m_q(\|y_s + \overline{\eta}_s\|_{\mathscr{W}}^2) ds\Big\} \\ &\le 3M\Big\{(\gamma_k + c_k)(\varkappa^* + 1) + (r_{k+1} - t_k)^{2H} c(H) M \ Tr(Q) m_q(\varkappa^*) \|m(\tau)\|_{L^1} := s_k. \end{aligned}$$

Set $\mathcal{N} = \max_{0 \le k \le \mathcal{M}} \{s_k\} + \max_{1 \le k \le \mathcal{M}} \{\zeta_k\}$, we get $\|\bar{\Theta}_1\|_{\mathfrak{Z}}^2 \le \mathcal{N}$.
Claim 2: $\bar{\Theta}_1$ is a contraction on $\mathscr{W}_\beta^0$.

Let $\chi^*,\ \chi^{**} \in \mathscr{W}_\beta^0$. Then for $\tau \in [0, r_1]$, we have

$$\mathbb{E}\|(\bar{\Theta}_1\chi^*)(\tau) - (\bar{\Theta}_1\chi^{**})(\tau)\|_3^2 = \mathbb{E}\|\int_0^\tau \mathcal{G}(\tau,s)[q(s,\chi_s^* + \overline{\eta}_s) - q(s,\chi_s^{**} + \overline{\eta}_s)]dZ_H(s)\|_3^2.$$

Using Lemma 2.1, 2.7, and $(S_4)(i)$, we obtain

$$\begin{aligned}\mathbb{E}\|(\bar{\Theta}_1\chi^*)(\tau) - (\bar{\Theta}_1\chi^{**})(\tau)\|_3^2 &\leq c(H)M\tau^{2H}\mathbb{E}\|q(s,\chi_s^* + \overline{\eta}_s) - q(s,\chi_s^{**} + \overline{\eta}_s)\|_{L_2^0}^2 \\ &\leq c(H)M\beta^{2H}Tr(Q)M_q\|\chi_s^* - \chi_s^{**}\|_{\mathscr{W}}^2 \\ &\leq 2K_\beta^2 c(H)M\beta^{2H}Tr(Q)M_q \sup_{s\in\mathfrak{J}}\mathbb{E}\|\chi^*(s) - \chi^{**}(s)\|_\Upsilon^2 \\ &= 2K_\beta^2 c(H)M\beta^{2H}Tr(Q)M_q\|\chi^* - \chi^{**}\|_{PC}^2\end{aligned}$$

Further, for $\tau \in \bigcup_{i=1}^{\mathcal{M}}(r_k, t_k]$, using Lemma 2.1 and (S_2)(i), we have

$$\begin{aligned}\mathbb{E}\|(\bar{\Theta}_1\chi^*)(\tau) - (\bar{\Theta}_1\chi^{**})(\tau)\|_3^2 &\leq \mathbb{E}\|f_k(\tau,\chi_\tau^* + \overline{\eta}_\tau) - f_k(\tau,\chi_\tau^{**} + \overline{\eta}_\tau)\|_\Upsilon^2 \\ &\leq \gamma_k\|\chi_\tau^* - \chi_\tau^{**}\|_{\mathscr{W}}^2 \\ &\leq 4\gamma_k K_\beta^2 \sup_{s\in\mathfrak{J}}\mathbb{E}\|\chi^*(s) - \chi^{**}(s)\|_\Upsilon^2 \\ &\leq 4\gamma_k K_\beta^2\|\chi^* - \chi^{**}\|_{PC}^2.\end{aligned}$$

Lastly, for $\tau \in \bigcup_{i=1}^{\mathcal{M}}(t_k, r_{k+1}]$,

$$\begin{aligned}\mathbb{E}\|(\bar{\Theta}_1\chi^*)(\tau) - (\bar{\Theta}_1\chi^{**})(\tau)\|_3^2 &\leq 3\|\mathcal{E}(\tau,t_k)\|_3^2\mathbb{E}\|f_k(t_k,\chi_{t_k}^* + \overline{\eta}_{t_k}) - f_k(t_k,\chi_{t_k}^{**} + \overline{\eta}_{t_k})\|_\Upsilon^2 \\ &\quad + 3\|\mathcal{G}(\tau,t_k)\|_3^2\|g_k(t_k,\chi_{t_k}^* + \overline{\eta}_{t_k}) - g_k(t_k,\chi_{t_k}^* + \overline{\eta}_{t_k})\|_3^2 \\ &\quad + 3\mathbb{E}\|\int_{t_k}^\tau \mathcal{G}(\tau,s)[q(s,\chi_s^* + \overline{\eta}_s) - q(s,\chi_s^{**} + \overline{\eta}_s)]dZ_H(s)\|_3^2 \\ &\leq 3M\gamma_k\|\chi_{t_k}^* - \chi_{t_k}^{**}\|_{\mathscr{W}}^2 + 3Mc_k\|\chi_{t_k}^* - \chi_{t_k}^{**}\|_{\mathscr{W}}^2 + 3c(H)Tr(Q)MM_q\beta^{2H}\|\chi_s^* - \chi_s^{**}\|_{\mathscr{W}}^2 \\ &\leq 12MK_\beta^2[\gamma_k + c_k + \beta^{2H}c(H)Tr(Q)M_q]\|\chi^* - \chi^{**}\|_{PC}^2.\end{aligned}$$

Thus for $\tau \in \mathfrak{J}$,

$$\mathbb{E}\|(\bar{\Theta}_1\chi^*)(\tau) - (\bar{\Theta}_1\chi^{**})(\tau)\|_3^2 \leq M_0\|\chi^* - \chi^{**}\|_{PC}^2. \tag{3.7}$$

where $M_0 = \max_{1\leq k\leq\mathcal{M}} 4K_\beta^2[(1+3M)\gamma_k + 3M(c_k + \beta^{2H}c(H)Tr(Q)M_q)] < 1.$

Hence $\bar{\Theta}_1$ is a contraction on $\mathscr{W}_\beta^0$. □

Lemma 3.4. *If (S_1) and (S_3) hold, then $\bar{\Theta}_2$ has convex, compact values, and also is completely continuous.*

Proof. **Claim 1: $\bar{\Theta}_2$ is convex for each $\chi \in \mathscr{W}_\beta^0$.**
If $\hat{\sigma}_1,\ \hat{\sigma}_2 \in \bar{\Theta}_2\chi$, then there are $\rho_1,\ \rho_2 \in S_{\Sigma,y}$ satisfying for any $\tau \in [t_k, r_{k+1}],\ k = 0,1,\ldots,\mathcal{M}$

$$\hat{\sigma}_l(\tau) = \int_{t_k}^\tau \mathcal{G}(\tau,s)\rho_l(s)ds,\ l = 1,2.$$

Let $0 \leq \lambda \leq 1$, then $[\lambda\hat{\sigma}_1 + (1-\lambda)\hat{\sigma}_2](\tau) = \int_{t_k}^{\tau} \mathcal{G}(\tau,s)[\lambda\rho_1(s) + (1-\lambda)\rho_2(s)]ds$.
In view of Lemma 3.1, $S_{\Sigma,y}$ is convex, we have $\lambda\hat{\sigma}_1 + (1-\lambda)\hat{\sigma}_2 \in \bar{\Theta}_2\chi$.
Claim 2: $\bar{\Theta}_2$ takes bounded sets into bounded sets in $\mathscr{W}_\beta^0$.
It is sufficient to establish that there exists $\mathcal{L} > 0$ with the end goal that for each $\hat{\sigma} \in \bar{\Theta}_2 y$, $y \in D_\varkappa(0, \mathscr{W}_\beta^0)$, one has $\|\hat{\sigma}\|_{PC}^2 \leq \mathcal{L}$. If $\hat{\sigma} \in \bar{\Theta}_2 y$, then there exists $\rho \in S_{\Sigma,y}$ for $\tau \in [t_k, r_{k+1}]$, $k = 0, 1, \ldots, \mathcal{M}$ such that

$$\hat{\sigma}(\tau) = \int_{t_k}^{\tau} \mathcal{G}(\tau,s)\rho(s)ds.$$

Now, for $y \in D_\varkappa(0, \mathscr{W}_\beta^0)$,

$$\begin{aligned}
\mathbb{E}\|\hat{\sigma}(\tau)\|_{\Upsilon}^2 = \mathbb{E}\|\int_{t_k}^{\tau} \mathcal{G}(\tau,s)\rho(s)ds\|_{\Upsilon}^2 &\leq \beta M \int_{t_k}^{\tau} \mathbb{E}\|\rho(s)\|_{\Upsilon}^2 ds \\
&\leq \beta M \int_{t_k}^{\tau} [b_1(s) + b_2\mathbb{E}\|y(s) + \bar{\eta}(s)\|_{\mathscr{W}}^2]ds \\
&\leq \beta M[\|b_1\|_{L^1(\mathfrak{I},\mathbb{R}^+)} + 2\beta b_2(\varkappa^* + \|\eta\|_{\mathscr{W}}^2)] = \mathcal{L}.
\end{aligned}$$

Thus for each $y \in D_\varkappa(0, \mathscr{W}_\beta^0)$, we have $\|\hat{\sigma}\|_{PC}^2 \leq \mathcal{L}$.
Claim 3: $\bar{\Theta}_2$ maps bounded sets into equicontinuous sets of $\mathscr{W}_\beta^0$.
For every $y \in D_\varkappa(0, \mathscr{W}_\beta^0)$, $\hat{\sigma} \in \bar{\Theta}_2 y$, there exists $\rho \in S_{\Sigma,y}$ such that for $\tau \in [t_k, r_{k+1}]$, $k = 0, 1, \ldots, \mathcal{M}$

$$\hat{\sigma}(\tau) = \int_{t_k}^{\tau} \mathcal{G}(\tau,s)\rho(s)ds.$$

For $\tau,\ \tau + \varsigma \in [t_k, r_{k+1}]$, $k = 0, 1, \ldots, \mathcal{M}$, $0 < |\varsigma| < \delta$, $\delta > 0$,

$$\begin{aligned}
\mathbb{E}\|\hat{\sigma}(\tau+\varsigma) - \hat{\sigma}(\tau)\|_{\Upsilon}^2 &\leq 2\mathbb{E}\|\int_{t_k}^{\tau} [\mathcal{G}(\tau+\varsigma,s) - \mathcal{G}(\tau,s)]\rho(s)ds\|_{\Upsilon}^2 + 2\mathbb{E}\|\int_{\tau}^{\tau+\varsigma} \mathcal{G}(\tau+\varsigma,s)\rho(s)ds\|_{\Upsilon}^2 \\
&\leq 2(\tau - t_k)\int_{t_k}^{\tau} \|\mathcal{G}(\tau+\varsigma,s) - \mathcal{G}(\tau,s)\|_{\Upsilon}^2 \mathbb{E}\|\rho(s)\|_{\Upsilon}^2 ds + 2\tau \int_{\tau}^{\tau+\varsigma} \|\mathcal{G}(\tau+\varsigma,s)\|_{\Upsilon}^2 \mathbb{E}\|\rho(s)\|_{\Upsilon}^2 ds \\
&\leq 2(r_{k+1} - t_k)\{\|b_1\|_{L^1(\mathfrak{I},\mathbb{R}^+)} + 2(r_{k+1} - t_k)b_2(\varkappa^* + \|\eta\|_{\mathscr{W}}^2)\}\times \\
&\quad \sup_{s\in[t_k,r_{k+1}]} \|\mathcal{G}(\tau+\varsigma,s) - \mathcal{G}(\tau,s)\|_{\Upsilon}^2 + 2\tau M(\|b_1\|_{L^1(\mathfrak{I},\mathbb{R}^+)} + b_2\varkappa^*\tau).
\end{aligned}$$

The compactness of the operator $\mathcal{G}(\tau,s)$ yields the continuity in the uniform operator topology. Thus $\mathbb{E}\|\hat{\sigma}(\tau+\varsigma) - \hat{\sigma}(\tau)\|_{\Upsilon}^2 \to 0$ uniformly independently of $y \in D_\varkappa(0, \mathscr{W}_\beta^0)$ as $\varsigma \to 0$. Hence our claim holds.

Claim 4: $\bar{\Theta}_2$ is a compact set-valued map.
We now assert that $\bar{\Theta}_2$ maps $D_\varkappa(0, \mathscr{W}_\beta^0)$ into a precompact set in $\mathfrak{Z}$. That is, the set $\triangle(\tau) = \{\hat{\sigma}(\tau),\ \hat{\sigma} \in \bar{\Theta}_2 D_\varkappa(0, \mathscr{W}_\beta^0)\}$ is relatively compact in $\mathfrak{Z}$. For $\tau = 0$, $\bar{\Theta}_2 y = 0$ is compact.

If $\tau \in [t_k, r_{k+1}]$, $k = 0, 1, \cdots, n$ then for each $y \in D_\varkappa(0, \mathscr{W}_\beta^0)$ and $\hat{\sigma}(\tau) \in \bar{\Theta}_2 y$, there exists $\rho \in S_{\Sigma,y}$ in order that

$$\hat{\sigma}(\tau) = \int_{t_k}^{\tau} \mathcal{G}(\tau,s)\rho(s)ds.$$

Let $0 < \epsilon < \tau$. Define $\hat{\sigma}_\epsilon(\tau) = \int_{t_k}^{\tau-\epsilon} \mathcal{G}(\tau,s)\rho(s)ds = \mathcal{G}(\tau,\tau-\epsilon)\int_{t_k}^{\tau-\epsilon} \mathcal{G}(\tau-\epsilon,s)\rho(s)ds$.

By (S_1), $\mathcal{G}(\tau,s);\ 0 < s \leq \tau$ is compact. From the boundedness of $\int_{t_k}^{\tau-\epsilon} \mathcal{G}(\tau-\epsilon,s)\rho(s)ds$, we acquire that $\triangle_\epsilon(\tau) = \{\hat{\sigma}(\tau),\ \hat{\sigma} \in \bar{\Theta}_2 D_\varkappa(0, \mathscr{W}_\beta^0)\}$ is relatively compact in $\mathfrak{Z}$.

Furthermore, for $\hat{\sigma}_\epsilon \in \bar{\Theta}_2 D_\varkappa(0, \mathscr{W}_\beta^0)\}$, we obtain

$$\begin{aligned}
\mathbb{E}\|\hat{\sigma}(\tau) - \hat{\sigma}_\epsilon(\tau)\|^2 &\leq \epsilon \int_{\tau-\epsilon}^{\tau} \mathbb{E}\|\mathcal{G}(\tau, s)\rho(s)\|_\Upsilon^2 ds \\
&\leq \epsilon M \int_{\tau-\epsilon}^{\tau} [b_1(s) + b_2(\varkappa^* + \|\eta\|_{\mathscr{W}}^2)] ds \\
&\leq \epsilon M [\epsilon \|b_1\|_{L^1(\mathfrak{J}, \mathbb{R}^+)} + \epsilon b_2(\varkappa^* + \|\eta\|_{\mathscr{W}}^2)] \\
&\to 0, \text{ for sufficiently small positive } \epsilon.
\end{aligned}$$

Thus, we have precompact sets that are arbitrarily close to $\triangle(\tau)$. Hence $\triangle(\tau)$, $\tau > 0$ is totally bounded. Considering **Claim 3** and the Arzela–Ascoli theorem, we infer that $\bar{\Theta}_2$ is a compact operator (completely continuous). □

Lemma 3.5. *If (S_1) and (S_3) hold, then $\bar{\Theta}_2$ has a closed graph.*

Proof. Let $w^{(n)} \to w^*$, $\hat{\sigma}_{(n)} \in \bar{\Theta}_2 w^{(n)}$, $w^{(n)} \in D_\varkappa(0, \mathscr{W}_\beta^0)$ and $\hat{\sigma}_{(n)} \to \hat{\sigma}_*$. Then using the Axiom (i), it follows that

$$\begin{aligned}
\|w_\tau^{(n)} - w_\tau^*\|_{\mathscr{W}}^2 &\leq 2K_\beta^2 \sup\{\|w^{(n)}(s) - w^*(s)\|_{\mathfrak{Z}}^2,\ 0 \leq s \leq \tau\} + 2L_\beta^2 \|w_0^{(n)} - w_0^*\|_{\mathscr{W}}^2 \\
&\leq 2K_\beta^2 \sup_{s \in \mathfrak{J}} \{\|w^{(n)}(s) - w^*(s)\|_{\mathfrak{Z}}^2 \to\ 0 \text{ as } n \to \infty.
\end{aligned}$$

This implies $w_s^{(n)} \to w_s^*$ uniformly as $n \to \infty$ for $s \in]-\infty, \beta]$.

We claim that $\hat{\sigma}_* \in \bar{\Theta}_2 w^*$. For $\hat{\sigma}_{(n)} \in \bar{\Theta}_2 w^{(n)}$, there exists $\rho^{(n)} \in S_{\Sigma, w^{(n)}}$ such that, for $\tau \in [t_k, r_{k+1}]$, $k = 0, 1, \ldots, \mathcal{M}$

$$\hat{\sigma}_{(n)}(\tau) = \int_{t_k}^{\tau} \mathcal{G}(\tau, s)\rho^{(n)}(s) ds.$$

We wish to show that there is $\rho^* \in S_{\Sigma, w^*}$ that insures

$$\hat{\sigma}_*(\tau) = \int_{t_k}^{\tau} \mathcal{G}(\tau, s)\rho^*(s) ds,\ \tau \in [t_k, r_{k+1}],\ k = 0, 1, \ldots, \mathcal{M}.$$

For any $\tau \in [t_k, r_{k+1}]$, $k = 0, 1, 2, \ldots \mathcal{M}$,

$$\begin{aligned}
\|\hat{\sigma}_{(n)}(\tau) - \hat{\sigma}_*(\tau)\|_{PC}^2 &= \| \int_{t_k}^{\tau} \mathcal{G}(\tau, s)[\rho^{(n)}(s) - \rho^*(s)] ds\|_{PC}^2 \\
&\leq (r_{k+1} - t_k) M \int_{t_k}^{\tau} \|\rho^{(n)}(s) - \rho^*(s)\|^2 \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Consider the operator $\Delta : L^2([t_k, r_{k+1}], \mathfrak{Z}) \to C([t_k, r_{k+1}], \mathfrak{Z})$, $k = 0, 1, \ldots, \mathcal{M}$,

$$\Delta(\rho)(\tau) = \int_{t_k}^{\tau} \mathcal{G}(\tau, s)\rho(s) ds.$$

Then $\|\Delta(\rho)\|^2 \leq (r_{k+1} - t_k) M \|\rho(s)\|^2$. This shows that Δ is bounded, which implies Δ is continuous. Lemma 3.2 asserts that the operator $\Delta \circ S_\Sigma$ has a closed graph. Moreover, by the definition of Δ, for $\tau \in [t_k, r_{k+1}]$, $k = 0, 1, \ldots, \mathcal{M}$, we get,

$$\hat{\sigma}_{(n)}(\tau) \in \bar{\Theta}_2(S_{\Sigma, w^{(n)}}).$$

Since $w^{(n)} \to w^*$, for some $\rho^* \in S_{\Sigma,w^*}$, it follows that for any $\tau \in [t_k, r_{k+1}],\ k = 0, 1, \dots \mathcal{M}$, we have

$$\hat{\sigma}_*(\tau) = \int_{t_k}^{\tau} \mathcal{G}(\tau, s)\rho^*(s)ds, \text{ this indicates that } \hat{\sigma}_* \in \bar{\Theta} w^*.$$

This implies operator $\bar{\Theta}_2$ is closed graph. By utilizing Proposition 3.3.12(2) of [33], we get that $\bar{\Theta}_2$ is u.s.c.. Therefore $\bar{\Theta}_2$ satisfies the condition (b) of Lemma 2.6. □

Theorem 3.6. *Let* (S_1)-(S_4) *are fulfilled. Then system* (1.1) *admits at least one solution on* $]-\infty, \beta]$ *provided that*

$$M_0 = \max_{1 \le k \le \mathcal{M}} 4K_\beta^2[\gamma_k + 3M(c_k + \beta^{2H}c(H)Tr(Q)M_q)] < 1, \text{ and}$$
$$\int_0^\beta \max\{c_1^* m(t) + c_2^* b_2\}dt < \int_{\Upsilon(0)}^\infty \frac{ds}{m_q(s)}. \tag{3.8}$$

Proof. We claim that the set $\mho = \{y \in \mathscr{W}_\beta^0 : \lambda y \in \bar{\Theta} y = \bar{\Theta}_1 y + \bar{\Theta}_2 y\}$ is bounded for some $\lambda > 1$ on $[0, \beta]$. Let $y \in \mathscr{W}_\beta^0$ satisfies $\lambda y \in \bar{\Theta} y = \bar{\Theta}_1 y + \bar{\Theta}_2 y$ for some $\lambda > 1$, we obtain

$$y(\tau) = \begin{cases} \frac{1}{\lambda}\Big[\mathcal{E}(\tau,0)\eta(0) + \mathcal{G}(\tau,0)\xi + \int_0^\tau \mathcal{G}(\tau,s)\rho(s)ds + \int_0^\tau \mathcal{G}(\tau,s)q(s, y_s + \overline{\eta}_s)dZ_H(s)\Big], & \tau \in [0, r_1]; \\ \frac{1}{\lambda} f_k(\tau, y_\tau + \overline{\eta}_\tau), & \tau \in \bigcup_{k=1}^{\mathcal{M}} (r_k, t_k]; \\ \frac{1}{\lambda}\Big[\mathcal{E}(\tau,t_k) f_k(t_k, y_{t_k} + \overline{\eta}_{t_k}) + \mathcal{G}(\tau,t_k) g_k(t_k, y_{t_k} + \overline{\eta}_{t_k}) + \int_{t_k}^\tau \mathcal{G}(\tau,s)\rho(s)ds & \\ + \int_{t_k}^\tau \mathcal{G}(\tau,s)q(s, y_s + \overline{\eta}_s)dZ_H(s)\Big], & \tau \in \bigcup_{k=1}^{\mathcal{M}} (t_k, r_{k+1}], \end{cases} \tag{3.9}$$

Thus for $\tau \in [0, r_1]$, we get

$$\begin{aligned} \mathbb{E}\|y(\tau)\|_3^2 \ &\le 4M\Big[\mathbb{E}\|\eta\|_3^2 + \mathbb{E}\|\xi\|_3^2 + r_1 \int_0^{r_1} b_1(s)ds + r_1 b_2 \int_0^{r_1} \mathbb{E}\|y(s)\|_3^2 ds \\ &\quad + r_1^{2H} c(H)Tr(Q) \int_0^{r_1} m(s) m_q(\|y_s + \overline{\eta}_s\|_{\mathscr{W}}^2)ds\Big]. \end{aligned}$$

For any $\tau \in (r_k, t_k],\ k = 1, 2, \dots, \mathcal{M}$, we have

$$\begin{aligned} \mathbb{E}\|y(\tau)\|_3^2 &\le \mathbb{E}\|f_k(\tau, y_\tau + \overline{\eta}_\tau)\|_3^2 \\ &\le \gamma_k(\|y_\tau + \overline{\eta}_\tau\|_{\mathscr{W}}^2 + 1). \end{aligned}$$

Similarly, for $\tau \in (t_k, r_{k+1}],\ k = 1, 2, \dots, \mathcal{M}$, we compute

$$\begin{aligned} \mathbb{E}\|y(\tau)\|_3^2 &\le 4[\mathbb{E}\|\mathcal{E}(\tau,t_k) f_k(t_k, y_{t_k} + \overline{\eta}_{t_k})\|_3^2 + \mathbb{E}\|\mathcal{G}(\tau,t_k) g_k(t_k, y_{t_k} + \overline{\eta}_{t_k})\|_3^2 \\ &\quad + \mathbb{E}\|\int_{t_k}^\tau \mathcal{G}(\tau,s)q(s, y_s + \overline{\eta}_s)dZ_H(s)\|_3^2 + \mathbb{E}\|\int_{t_k}^\tau \mathcal{G}(\tau,s)\rho(s)ds\|_3^2 \\ &\le 4M\Big\{(\gamma_k + c_k)(\|y_{t_k} + \overline{\eta}_{t_k}\|_{\mathscr{W}}^2 + 1) + (r_{k+1} - t_k)^{2H} c(H)M\ Tr(Q) \int_{t_k}^\tau m(s) m_q(\|y_s + \overline{\eta}_s\|_{\mathscr{W}}^2)ds \\ &\quad + (r_{k+1} - t_k)M \int_{t_k}^\tau [b_1(s) + b_2 \mathbb{E}\|y(s) + \overline{\eta}(s)\|_3^2]ds\Big\} \end{aligned}$$

Using Lemma 2.1, we get
$\sup\{\|y_s + \overline{\eta}_s\|_{\mathscr{W}}^2,\ \tau \ge s \ge 0\} \ \le\ 4L_\beta^2 \mathbb{E}\|\eta\|_{\mathscr{W}}^2 + 4K_\beta^2 \sup\{\mathbb{E}\|y(s)\|^2,\ \tau \ge s \ge 0\}$.

Let $\varphi(\tau) = 4L_\beta^2\mathbb{E}\|\eta\|^2_{\mathscr{W}} + 4K_\beta^2 \sup\{\mathbb{E}\|y(s)\|^2,\ \tau \geq s \geq 0\},\ \tau \geq 0$. Thus, for $\tau \in \mathfrak{J}$, we obtain

$$\mathbb{E}\|y(\tau)\|_3^2 \leq \hat{M} + \gamma_k\varphi(\tau) + 4M[\gamma_k\varphi(\tau) + c_k\varphi(\tau)] + 4\beta^{2H}c(H)M\,Tr(Q)\int_0^\tau m(s)m_q(\varphi(s))ds + \beta M\int_0^\tau [b_1(s) + b_2\varphi(s)]ds,$$

where $\widehat{M} = \max\limits_{1\leq k\leq M}[4M(\|\eta\|^2_{\mathscr{W}} + \mathbb{E}\|\xi\|_3^2 + \gamma_k + c_k) + \gamma_k]$.

Also, a simple calculation yields that

$$\varphi(\tau) \leq 4[L_\beta^2\|\eta\|^2 + K_\beta^2\hat{M}] + 4K_\beta^2\Big\{[\gamma_k(1+4M) + 4Mc_k]\varphi(\tau) + 4\beta^{2H}c(H)M\,Tr(Q)\int_0^\tau m(s)m_q(\varphi(s))ds + \beta M\int_0^\tau [b_1(s) + b_2\varphi(s)]ds\Big\}.$$

Using the fact that $\widehat{M_0} = 4K_\beta^2 \max\limits_{1\leq k\leq M}\{\gamma_k(1+4M) + 4Mc_k\} < 1$, we obtain

$$\varphi(\tau) \leq \frac{1}{1-\widehat{M_0}}\{4L_\beta^2\|\eta\|^2_{\mathscr{W}} + 4K_\beta^2\widehat{M}\} + c_1^*\int_0^\tau m(s)m_q(\varphi(s))ds + c_2^*\,b_2\int_0^\tau \varphi(s)ds,$$

where $c_1^* = \frac{4K_\beta^2\beta^{2H}c(H)M\,Tr(Q)}{1-\widehat{M}_0}$, $c_2^* = \frac{\beta M}{1-\widehat{M}_0}$.

Let $c^* = \frac{1}{1-\widehat{M}_0}\{4L_\beta^2\|\eta\|^2_{\mathscr{W}} + 4K_\beta^2\widehat{M} + \beta M\|b_1\|_{L_1(\mathfrak{J},\mathbb{R}^+)}\}$. Then the above inequality can be rewritten as

$$\varphi(\tau) \leq \Upsilon(\tau) = c^* + c_1^*\int_0^\tau m(s)m_q(\varphi(s))ds + c_2^*b_2\int_0^\tau \varphi(s)ds,$$

Also $\Upsilon(0) = c^*$ and

$$\varphi'(\tau) \leq\ c_1^*m(t)m_q(\varphi(\tau)) + c_2^*b_2\varphi(\tau) \leq\ \max\{c_1^*m(\tau),\ c_2^*b_2\}[\Upsilon(\tau) + m_q(\Upsilon(\tau))],\ \tau \in \mathfrak{J}.$$

Thus we get

$$\int_0^\tau \frac{\Upsilon'(s)}{\Upsilon(s) + m_q(\Upsilon(s))}ds \ \leq\ \int_0^\tau \frac{\Upsilon'(s)}{m_q(\Upsilon(s))}ds \ \leq\ \int_0^\beta \max\{c_1^*m(s) + c_2^*b_2\}ds.$$

Moreover,

$$\int_{\Upsilon(0)}^{\Upsilon(\tau)} \frac{ds}{m_q(s)} \leq \int_0^\beta \max\{c_1^*m(s) + c_2^*b_2\}ds < \int_{\Upsilon(0)}^\infty \frac{ds}{m_q(s)}.$$

The above inequality shows that $\Upsilon(\tau)$ is bounded. Therefore, we have $\widetilde{N}$ such that

$$\Upsilon(\tau) \leq \widetilde{N},\ \tau \in \mathfrak{J}.$$

Consequently, $\|y_\tau + \eta_\tau\|^2_{\mathscr{W}} \leq \varphi(\tau) \leq \Upsilon(\tau) \leq \widetilde{N}$, $\tau \in \mathfrak{J}$, where $\widetilde{N}$ depends on $m_q(\cdot)$ and $m(\cdot)$. This proves that $\mho$ is bounded on $[0,\beta]$. Hence, Lemmas 3.3–3.5 and the first assertion of Lemma 2.6 yield that $\bar{\Theta} = \bar{\Theta}_1 + \bar{\Theta}_2$ has a fixed element y^* in $\mathscr{W}_\beta^0$. Set $\chi^*(\tau) = y^*(\tau) + \bar{\eta}(\tau)$, $\tau \in]-\infty,\beta]$. Then χ^* is a fixed point of the operator Θ. Consequently, χ^* is a mild solution of the system (1.1). □

4. An Example

This section is illustrated for the applicability of the above result to a concrete stochastic partial differential inclusions with unbounded delay and Clarke's subdifferential given by

$$\begin{cases} \frac{\partial^2}{\partial\tau^2}\chi(\tau,w) \in \frac{\partial^2}{\partial w^2}\chi(\tau,w) + v(\tau)\frac{\partial}{\partial\tau}\chi(\tau,w) + \partial\Sigma(\tau,w,\chi(\tau,w)) + \int_{-\infty}^{\tau} u(t-\tau)\tilde{u}(\tau,\chi(t,w))dZ_H(t), \\ \quad (\tau,w) \in \bigcup_{k=1}^{\mathcal{M}}(t_k, r_{k+1}] \times [0,\pi]; \\ \chi(\tau,w) = \int_{-\infty}^{\tau}\mu_k(t-\tau)\chi(t,w)dt, \quad (\tau,w) \in \bigcup_{k=1}^{\mathcal{M}}(r_k,t_k] \times [0,\pi]; \\ \chi(\tau,0) = \chi(\tau,\pi) = 0, \ \tau \in (0,\beta]; \\ \chi(\tau,w) = \eta(\tau,w) \in \mathscr{W}, \ (\tau,w) \in]-\infty,0] \times [0,\pi]; \\ \frac{\partial}{\partial\tau}\chi(0,w) = \chi_1(w); \\ \frac{\partial}{\partial\tau}\chi(\tau,w) = \int_{-\infty}^{\tau}\tilde{\mu}_k(t-\tau)\chi(t,w)dt, \quad (\tau,w) \in \bigcup_{k=1}^{\mathcal{M}}(r_k,t_k] \times [0,\pi]; \end{cases} \tag{4.1}$$

where η, χ_1 are continuous. To compose the above system in the abstract form, set $\mathfrak{Z} = \mathscr{Z} = L^2([0,\pi];\mathbb{R})$. Let $\mathscr{H}^2([0,\pi],\mathbb{R})$ be the Sobolev space of all mappings $\chi : [0,\pi] \to \mathbb{R}$ such that $\chi'' \in L^2([0,\pi],\mathbb{R})$. Define $\mathscr{A} : D(\mathscr{A}) \to \mathfrak{Z}$ by $\mathscr{A}\chi(\tau) = \chi''(\tau)$, where $D(\mathscr{A}) = \{\chi \in \mathfrak{Z} : \chi,\ \chi'$ are absolutely continuous, $\chi'' \in \mathfrak{Z},\ \chi(0) = \chi(\pi) = 0\}$. Then, the cosine family $C(\tau)$ and the associated sine function $S(\tau)$ on $\mathfrak{Z}$ are generated by $\mathscr{A}$ and are strongly continuous; also for any $\tau \in \mathbb{R}$, $\|C(\tau)\| \leq 1$ [32]. Define $\widehat{C} : \mathscr{H}^1([0,\pi],\mathbb{R}) \to \mathfrak{Z}$ by $\widehat{C}(\tau)\chi(w) = v(\tau)\chi'(w)$, where $v : [0,1] \to \mathbb{R}$ is Hölder continuous. Define the linear operator $\mathscr{A}(\tau) = \widehat{C}(\tau) + \mathscr{A}$ that is closed also. The operator $\{\mathscr{A}(\tau) : \tau \in \mathfrak{J}\}$ generates the evolution operator $\{\mathcal{G}(\tau,s)\}_{(\tau,s)\in D}$, $D = \{(\tau,s) \in \mathfrak{J}\times\mathfrak{J} : s \leq \tau\}$, see [23]. Moreover, $\mathcal{G}(\cdot,\cdot)$ is well defined and assumption (S_1) hold with $M = 1$.

The map $\Sigma : [0,\pi]\times\mathfrak{J}\times\mathbb{R} \to \mathbb{R}$ is a locally Lipschitz w.r.t. the last variable, which is non-smooth and non-convex. The set-valued function $\partial\Sigma(\tau,w,\phi) : \mathbb{R} \to 2^{\mathbb{R}}$ is non-monotone. To support (S_3) one can take $\Sigma(\phi) = \min\{\varpi_1(\phi), \varpi_2(\phi)\}$, where $\varpi_1, \varpi_2 : \mathbb{R} \to \mathbb{R}$ are convex quadratic functions [34]. Notation $Z_H(\tau)$ stands for the Rosenblatt process that is defined on the complete stochastic space $(\Omega,\Gamma,\mathbb{P})$ and $\frac{1}{2} < H < 1$.

Let the function $\tilde{l} :]-\infty,0] \to \mathbb{R}^+ \cup \{0\}$ be measurable satisfying (g-5)-(g-7) described in [21].

Set $PC_0 \times L^2(\widetilde{l},\mathfrak{Z}) = \Big\{\Pi : \mathfrak{J}_0 \to \mathfrak{Z},\ \Pi(\cdot)$ is Lebesgue measurable on $]-\infty,0)\Big\}$ and

$$\|\Pi\|_{\mathscr{W}} = \|\Pi(0)\| + \Big(\int_{-\infty}^{0}\tilde{l}(s)\|\Pi(s)\|^2 ds\Big)^{\frac{1}{2}}.$$

The space $(\mathscr{W}, \|\cdot\|_{\mathscr{W}}) = (PC_0 \times L^2(\tilde{l},\mathfrak{Z}), \|\cdot\|_{\mathscr{W}})$ satisfies Axioms (i) and (ii), (see [21]).

Suppose that the following conditions hold:

(i) Let $u : \mathbb{R} \to \mathbb{R}$, $\tilde{u} : \mathbb{R}^2 \to \mathbb{R}$ be continuous and $L_u = \Big(\int_{-\infty}^{0}\frac{(u(s))^2}{\tilde{l}(s)}ds\Big)^{1/2} < \infty$, also for $(\tau,x) \in \mathbb{R}^2$, $|\tilde{u}(\tau,x)| \leq \tilde{b}(\tau)|x|$, $\tilde{b} : \mathbb{R} \to \mathbb{R}$ is continuous.

(ii) The functions $\mu_k, \tilde{\mu}_k : \mathbb{R}^2 \to \mathbb{R}$ are continuous and there are mappings $a_k, \tilde{a}_k : \mathbb{R} \to \mathbb{R}$ which are continuous satisfying $|\mu_k(s,x)| \leq a_k(s)$ with $\mathfrak{A}_k = \Big(\int_{-\infty}^{0}\frac{(a_k(s))^2}{\tilde{l}(s)}ds\Big)^{1/2} < \infty$, also $|\tilde{\mu}_k(s,x)| \leq \tilde{a}_k(s)$ with $\tilde{\mathfrak{A}}_k = \Big(\int_{-\infty}^{0}\frac{(\tilde{a}_k(s))^2}{\tilde{l}(s)}ds\Big)^{1/2} < \infty$.

Take $\eta \in \mathscr{W}$ with $\eta(\vartheta)(w) = \eta(\vartheta,w)$, $(\vartheta,w) \in]-\infty,0]\times\mathscr{W}$.

Let $\chi(t)(w) = \chi(t, w)$, define $q : \mathfrak{J} \times \mathscr{W} \to L_2^0,\ f_k, g_k(r_k, t_{k+1}] \times \mathscr{W} \to \mathfrak{Z}$ as

$$q(\tau, \Xi)(w) = \int_{-\infty}^{0} u(t)\tilde{u}(\tau, \Xi(t)(w)dt,$$

$$f_k(\tau, \Xi)(w) = \int_{-\infty}^{0} \mu_k(t)\Xi(t)(w)dt,$$

$$g_k(\tau, \Xi)(w) = \int_{-\infty}^{0} \tilde{\mu}_k(t)\Xi(t)(w)dt.$$

Under the above assumptions the problem (4) can be formulated as (1.1).

From the hypothesis (i), for all $(\tau, \Xi) \in [0, \beta) \times \mathscr{W}$, we have

$$\begin{aligned}
\mathbb{E}\|q(\tau, \Xi)\|^2 &= \mathbb{E}\Big[\Big(\int_0^{\pi}\Big(\int_{-\infty}^{0} u(t)\tilde{u}(\tau, \Xi(t)(w))dt\Big)^2 dw\Big)^{\frac{1}{2}}\Big]^2 \\
&\le \mathbb{E}\Big[\Big(\int_0^{\pi}\Big(\int_{-\infty}^{0} u(t)\tilde{b}(\tau)|\Xi(t)(w)|dt\Big)^2 dw\Big)^{\frac{1}{2}}\Big]^2 \\
&\le \mathbb{E}\Big[\tilde{b}(\tau)\Big(\int_{-\infty}^{0} \frac{(u(t))^2}{\tilde{l}(t)}dt\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{0} \tilde{l}(t)\|\Xi(t)\|^2 dt\Big)^{\frac{1}{2}}\Big]^2 \\
&\le [\tilde{b}(\tau)L_u]^2\|\Xi\|_{\mathscr{W}}^2.
\end{aligned}$$

Also for all $(\tau, \Xi),\ (\tau, \Xi_1) \in (r_k, t_k) \times \mathscr{W}$, we get

$$\begin{aligned}
\mathbb{E}\|g_k(\tau, \Xi) - g_k(\tau, \Xi_1)\|^2 &= \mathbb{E}\Big[\Big(\int_0^{\pi}\Big(\int_{-\infty}^{0} \mu_i(t, w)[\Xi(t)(w) - \Xi_1(t)(w)]dt\Big)^2 dw\Big)^{\frac{1}{2}}\Big]^2 \\
&\le \mathbb{E}\Big[\Big(\int_{-\infty}^{0} \frac{(a_k(t))^2}{\tilde{l}(t)}dt\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{0} \tilde{l}(t)\|\Xi(t) - \Xi_1(t)\|^2 dt\Big)^{\frac{1}{2}}\Big]^2 \\
&\le \Big[\mathfrak{A}_k\Big(\|\Xi(0)\| + \Big(\int_{-\infty}^{0} \tilde{l}(t)\|\Xi(t) - \Xi_1(t)\|^2 dt\Big)^{\frac{1}{2}}\Big)\Big]^2 \\
&\le \mathfrak{A}_k^2\|\Xi - \Xi_1\|_{\mathscr{W}}^2.
\end{aligned}$$

Similarly,

$$\mathbb{E}\|f_k(\tau, \Xi) - f_k(\tau, \Xi_1)\|^2 \le \gamma_k\|\Xi - \Xi_1\|_{\mathscr{W}}^2,\ \gamma_k > 0,\ \text{for all } (\tau, \Xi),\ (\tau, \Xi_1) \in (r_k, t_k) \times \mathscr{W}$$
$$\mathbb{E}\|q(\tau, \Xi) - q(\tau, \Xi_1)\|^2 \le M_q\|\Xi - \Xi_1\|_{\mathscr{W}}^2,\ M_q > 0,\ \text{for all } (\tau, \Xi),\ (\tau, \Xi_1) \in [0, \beta) \times \mathscr{W}$$

Thus all the hypotheses in Theorem 3.6 are followed. Hence, the model (4) admits a solution on $\mathfrak{J}$.

5. Conclusion

In this article, we study a new class of non-autonomous second-order stochastic inclusions of Clarke's subdifferential type involving NIIs, unbounded delay, and the Rosenblatt process. The existence result is deduced by utilizing the fixed point strategy for a set-valued map. The obtained results are illustrated through a concrete example. In the future, it is interesting to study the controllability results (such as approximate controllability, optimal control, and time-optimal control among others) of the associated systems. In our future research work, we will consider the optimal control problem associated with the system (1.1) involving state-dependent delay.

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