Filomat 36:4 (2022), 1203–1214
https://doi.org/10.2298/FIL2204203A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Quasi-Yamabe Solitons and Almost Quasi-Yamabe Solitons on Lightlike Hypersurfaces

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Abstract. In the present paper, we study the quasi-Yamabe solitons and almost quasi-Yamabe solitons on the lightlike hypersurfaces of the semi-Riemannian manifolds endowed with a torse-forming vector field. We show some conditions for the lightlike hypersurfaces to be quasi-Yamabe solitons and almost quasi-Yamabe solitons with the tangential component of the torse-forming vector field on the semi-Riemannian manifolds as the soliton field. In particular, we also specify the conditions for lightlike hypersurfaces of $(n+2)$-dimension semi-Riemannian manifolds of constant curvature to be quasi-Yamabe solitons and almost quasi-Yamabe solitons. Besides, we provide some geometric properties of the lightlike hypersurfaces satisfying quasi-Yamabe solitons, quasi-Yamabe gradient solitons, almost quasi-Yamabe solitons and almost quasi-Yamabe gradient solitons. Furthermore, we investigate properties of screen homothetic lightlike hypersurfaces admitting quasi-Yamabe solitons and almost quasi-Yamabe solitons.

1. Introduction

The notion of Yamabe flow was first introduced by Hamilton [10] in 1988 where the metrics on a manifold evolve according to

$$\frac{\partial}{\partial t}g(t) = -S(t)g(t) \tag{1}$$

where S is the scalar curvature of metric g. In dimension $n = 2$, the Yamabe flow is equivalent to the Ricci flow.

Yamabe soliton is a self-similar solution of the Yamabe flow. A semi-Riemannian manifold $(\bar{M}, \bar{g})$ is a Yamabe soliton if there exists a vector field V on $\bar{M}$ such that

$$\frac{1}{2}\mathcal{L}_V g = (S - \lambda)g + \gamma\pi \otimes \pi \tag{2}$$

where S is the scalar curvature of M, $\mathcal{L}_V$ is the Lie derivative in the direction of V and λ is constant. The Yamabe soliton $(\bar{M}, \bar{g}, V, \lambda)$ is called shrinking, steady or expanding according to $\lambda > 0, \lambda = 0$, or $\lambda < 0$, respectively. If V is a Killing vector field on $\bar{M}$, then the Yamabe soliton $(\bar{M}, \bar{g}, V, \lambda)$ is trivial.

2020 *Mathematics Subject Classification*. Primary 53C50; Secondary 53C25

Keywords. lightlike hypersurface, torse-forming vector field, screen homothetic, quasi-Yamabe soliton, almost quasi-Yamabe soliton

Received: 09 April 2021; Accepted: 22 September 2021

Communicated by Mića S. Stanković

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A Yamabe soliton is said to be a gradient Yamabe soliton if the vector field V is the gradient of some smooth function f on $\bar{M}$. In this case, the equation (2) becomes

$$Hess_f = (S - \lambda)g \tag{3}$$

where $Hess_f$ denotes the Hessian of f.

In 2013, Barbosa and Ribeiro [11] defined almost Yamabe solitons. If λ in equations (2) and (3) is a smooth function on M, then M is called almost Yamabe solitons and gradient almost Yamabe solitons, repectively.

Notion of torse-forming vector field was firstly introduced by Yano[13]. A nowhere vanishing vector field V is said to be torse-forming on $\bar{M}$ if

$$\bar{\nabla}_X V = \mu X + \pi(X)V \tag{4}$$

where $\mu \in C^\infty(M)$ and π is an 1-form.

If the 1-form π in (4) vanishes identically, then V is concircular. Concircular vector fields also known as geodesic vector fields since integral curves of such vector field as geodesic. If $\mu = 1$ and $\pi = 0$ then V is said to be concurrent. The vector field V is recurrent if it satisfies (4) with $\mu = 0$. Also if $\mu = \pi = 0$, the vector field V in (4) is parallel vector field. As a consequence, if the vector field V satisfies (4) with $\pi(V) = 0$, then V is called torqued vector field where μ and π are known as the torqued function and the torqued form of V, respectively [14].

Yamabe solitons and gradient Yamabe solitons have been investigated under some conditions by some mathematicians which can bee seen in [15, 17**? ?** –20]. In particular, almost Yamabe soliton and gradient almost Yamabe solitons and gradient almost Yamabe soliton have been studied by Seko and Maeta [12].

In the present paper, we study the Yamabe solitons and almost Yamabe solitons on lightlike hypersurfaces of semi-Riemannian manifolds endowed with a torse-forming vector field. In this work, we assume that the lightlike hypersurface $(M, g, S(TM))$ of the semi-Riemannian manifold $(\bar{\mathbb{M}}_1^{n+2}, \bar{g})$ equipped with the canonical screen distribution $S(TM)$ such that the local second fundamental h^* on the lightlike hypersurface is symmetric [16], and we take the 1-form π in equation (4) associated with the torse-forming vector field V on $\mathbb{M}_1^{n+2}$.

We organize our present work as follow: In section 2, we provide the geometric properties of the lightlike hypersurface of the semi-Riemannian manifold. We give the Gauss-Weingarten formulas, and the Ricci tensor of the lightlike hypersurface. In section 3, we show some conditions of lightlike hypersurfaces of semi-Riemannian manifolds endowed with a torse-forming vector field to be Yamabe solitons and almost Yamabe solitons. We also show some properties of the lightlike hypersurfaces satisfying gradient Yamabe solitons and gradient almost Yamabe solitons. In section 4, we particularly specify the conditions of lightlike hypersurfaces of $(n + 2)$-dimensional semi-Riemannian manifold of constant curvature to be Yamabe solitons and almost Yamabe solitons and show their properties. In the last section, we focus on the geometric properties of the screen homothetic lightlike hypersurfaces admitting Yamabe solitons and almost Yamabe solitons.

2. Preliminaries

Let $(M^n, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ and $\bar{\nabla}$ be the metric (Levi-Civita) connection on $\bar{M}$ with respect to $\bar{g}$. The tangent bundle $T\bar{M}$ splits into three bundle spaces on M as follow

$$T\bar{M} = RadTM \oplus S(TM) \oplus ltr(TM) = TM \oplus ltr(TM). \tag{5}$$

where $RadTM, S(TM), ltr(TM)$ are called radiant, screen distribution and lightlike transversal vector bundle spaces of M, respectively.

The Gauss-Weingarten formula are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{6}$$

$$\bar{\nabla}_X V = -A_V X + D_X V \tag{7}$$

for any $X, Y \in \Gamma(TM)$ and $V \in ltr(TM)$.
Locally, suppose $\{\xi, N\}$ is a pair of section on $\mathcal{U} \subset M$. Define a symmetric bilinear form B and a 1-form τ on M by

$$B(X, Y) = \bar{g}(h(X, Y), \xi), \quad \tau(X) = \bar{g}(D_x N, \xi) \tag{8}$$

such that $h(X, Y) = B(X, Y)N$ and $D_X N = \tau(X)N$ for any $X, Y \in \Gamma(TM)$ on $\mathcal{U}$. Hence, the equations (6) and (7) become

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N \tag{9}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N. \tag{10}$$

Here, B is the second local fundamental form of M since it is the only component of h on $\mathcal{U}$ with respect to N.
Let P denote the projection of TM on $S(TM)$. Then we have the following decompositions

$$\begin{aligned} \nabla_X PY &= \nabla^*_X PY + C(X, PY)\xi \\ \nabla_\xi X &= -A^*_\xi X - \tau(X)\xi, \end{aligned} \tag{11}$$

where $\nabla^*_X PY$ and $A^*_\xi X$ belong to $S(TM)$, C is a 1-form on $\mathcal{U}$. In general, the induced metric connection of M is not compatible with the induced metric g. So that we have

$$(\nabla_Z g)(X, Y) = B(Z, Y)\eta(X) + B(X, Z)\eta(Y), \tag{12}$$

where

$$\eta(X) = \bar{g}(X, N). \tag{13}$$

We also have the equations

$$g(A_N X, PY) = C(X, PY), \quad \bar{g}(A_N X, N) = 0 \tag{14}$$

$$g(A^*_\xi X, PY) = B(X, PY), \quad \bar{g}(A^*_\xi X, N) = 0. \tag{15}$$

The mean curvature of lightlike hypersurface (M, g) is given by [9]

$$\alpha = \sum_{a=1}^{n} C(e_i, e_j), \tag{16}$$

where $\{e_1, ..., e_n\}$ is an orthonormal basis of $S(TM)$.

Let R and $\bar{R}$ be the curvature tensor of the lightlike hypersurface $(M, g, S(TM))$ and the ambient semi-Riemannian manifold $(\bar{M}, \bar{g})$, respectively. Then, we have

$$\bar{R}(X, Y)Z = R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z). \tag{17}$$

Suppose $\{e_1, ..., e_n, \xi, N\}$ is a quasi orthonormal basis on $T\bar{M}$, where $\{e_1, ..., e_n\}$ is an orthonormal basis of $\Gamma(S(TM))$. The induced Ricci tensor of M is given by

$$Ric(X, Y) = \sum_{j=1}^{n} g(R(e_j, X)Y, e_j) + \bar{g}(R(\xi, X)Y, N) \tag{18}$$

for any $X, Y \in TM$.

Let $\bar{R}ic$ and Ric denote the Ricci tensor of $\bar{M}$ and induced Ricci type tensor of M. Then we have

$$Ric^{(0,2)}(X,Y) = \bar{R}ic(X,Y) + B(X,Y)trA_N - g(A_NX, A^*_\xi Y) - \bar{g}(\bar{R}(\xi,Y)X,N) \tag{19}$$

for all $X,Y \in \Gamma(TM)$. It is clear that the induced Ricci type tensor $Ric^{(0,2)}(X,Y)$ given in equation (19) is not symmetric so that it has no geometric or physical meaning. The induced Ricci type tensor $Ric(X,Y)$ is called induced Ricci tensor if it is symmetric. Atindogbe [8] has formulate the induced Ricci tensor of the lightlight hypersurface of $(m+2)$-dimensional semi-Riemanian manifold by

$$Ric(X,Y) = \bar{R}ic(X,Y) + B(X,Y)trA_N - \frac{1}{2}\{\eta(\bar{R}(\xi,Y)X) + \eta(\bar{R}(\xi,X)Y) + g(A_NX,A^*_\xi Y) + g(A_NY,A^*_\xi X)\} \tag{20}$$

Let M be a lightlike hypersurface of $(n+2)$-dimensional semi-Riemannian manifold $\bar{M}$ of constant curvature c. Then, we have $\bar{R}ic(X,Y) = (n+1)c\bar{g}(X,Y)$ and $\eta(\bar{R}(\xi,Y)X) = cg(X,Y)$. As a consequence, from equation (20), we find

$$Ric(X,Y) = ncg(X,Y) + B(X,Y)\alpha - \frac{1}{2}\{g(A_NX,A^*_\xi Y) + g(A_NY,A^*_\xi X)\} \tag{21}$$

where $\alpha = \Sigma_{i=1}\epsilon_i g(A_N w_i, w_i)$, for $\{w_i\}, i \in \{1,2,...,m-1\}$ and $X,Y \in \Gamma(TM)$.

Furthermore, M is called totally umbilical if there exists a smooth function F such that

$$B(X,Y) = Fg(X,Y), \quad X,Y \in \Gamma(TM). \tag{22}$$

for all vector field $X,Y \in \Gamma(TM)$ [9, 16].

3. Lightlike hypersurfaces as Yamabe solitons and almost Yamabe solitons

In this section, we show some conditions for a lightlike hypersurfaces of semi-Riemannian manifolds endowed with a torse-forming vector field to be a quasi-Yamabe soliton and an almost quasi-Yamabe soliton. We also show some geometric properties of the lightlike hypersurface satisfying quasi-Yamabe solitons, quasi-Yamabe gradient solitons, almost quasi-Yamabe solitons and almost quasi-Yamabe gradient solitons.

Theorem 3.1. *Let $(M,g,S(TM))$ be a lightlike hypersurface of semi-Riemannian manifold $(\bar{M},\bar{g})$. (M,g,V^T,γ,λ) is a quasi-Yamabe soliton or almost quasi-Yamabe soitons if and only if the scalar curvature of M satisfies*

$$(S-\lambda-\mu)g(X,Y) = \frac{1}{2}(B(X,V^T)\eta(Y) + B(V^T,Y)\eta(X)) + (1-\gamma)\pi(X)\pi(Y) - \rho g(A_NX,Y) \tag{23}$$

for $X,Y \in TM$.

Proof. Let $\varphi: M \to \bar{M}$ denotes the isometric immersion. Then, for vector field $V \in \Gamma(TM)$ by equation (5), we have

$$V = V^T + \rho N \tag{24}$$

where $V^T \in \Gamma(TM)$, $N \in ltr(TM)$ and $\rho = \bar{g}(V,\xi)$.

Since V is a torse-forming vector field on $\Gamma(T\bar{M})$, then by Gauss-Weingarten formulas, we have

$$\begin{aligned}\mu X + \pi(X)V &= \bar{\nabla}_X V^T + \bar{\nabla}_X(\rho N)\\ &= \nabla_X V^T + B(X,V^T)N + X(\rho)N - \rho A_N X + \rho\tau(X)N.\end{aligned}$$

Comparing the tangential and the transfersal components of the equation above yields

$$\nabla_X V^T = \mu X + \pi(X)V^T - \rho A_N X \tag{25}$$

$$\rho\pi(X) = \rho\tau(X) + B(X, V^T) - X(\rho). \tag{26}$$

Note that $\pi(X) = \bar{g}(X, V) = g(X, V^T)$. Therefore, from equations (12) and (25) we obtain

$$\begin{aligned}(\mathcal{L}_{V^T} g)(X, Y) =& (\nabla_{V^T} g)(X, Y) + g(\nabla_X V^T, Y) + g(X, \nabla_Y V^T) \\ =& B(X, V^T)\eta(Y) + B(V^T, Y)\eta(X) + 2\mu g(X, Y) + 2\pi(X)\pi(Y) - 2\rho g(A_N X, Y).\end{aligned} \tag{27}$$

The lightlike hypersurface $(M, g, V^T, \gamma, \lambda)$ is quasi-Yamabe solitons or almost quasi-Yamabe solitons if and only if it satisfies equation (2). Therefore, by using equation (27), we can easily show that $(M, g, V^T, \gamma, \lambda)$ is Yamabe solitons or almost Yamabe solitons if and only if equation (23) holds. □

If a lightlike hypersurface is totally geodesic, then the second fundamental h vanishes which implies $A^*_\xi = 0$ [16]. Therefore, we have have the following corollary:

Corollary 3.2. *Let $(M, g, S(TM))$ be a totally geodesic lightlike hypersurface of semi-Riemannian manifold $(\bar{M}, \bar{g})$. $(M, g, V^T, \gamma, \lambda)$ is a quasi-Yamabe soliton or almost quasi-Yamabe soitons if and only if the scalar curvature of (M, g) satisfies*

$$(S - \lambda - \mu)g(X, Y) = (1 - \gamma)\pi(X)\pi(Y) - \rho g(A_N X, Y) \tag{28}$$

for $X, Y \in TM$.

Theorem 3.3. *Every totally geodesic lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is trivial quasi-Yamabe soliton and trivial almost quasi-Yamabe soliton.*

Proof. Let $(M, g, S(TM))$ be a totally geodesic lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then, by equation (8), the symmetric bilinear form B vanishes everywhere. As a consequence, by equation (12) the induced connection ∇ is a metric connection. Therefore, we have

$$0 = (\nabla_X g)(\xi, Z) = X(g(\xi, g)) - g(\nabla_X \xi, Z) - g(\xi, \nabla_X Z) = -g(\nabla_X \xi, Z).$$

Therefore, we find

$$(\mathcal{L}_\xi g)(X, Y) = (\nabla_\xi g)(X, Y) + g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0$$

which implies ξ to be Killing vector field and (M, g, ξ, λ) to be trivial Yamabe solitons and almost Yamabe solitons. □

Theorem 3.4. *Let a lightlike hypersurface $(M, g, V^T, \gamma, \lambda)$ admit quasi-Yamabe solitons or almost quasi-Yamabe solitons. Then, the mean curvature of M is given by*

$$\alpha = \frac{1}{\rho}((1 - \gamma)\|V^T\|^2 - n(S - \lambda - \mu)). \tag{29}$$

Proof. Let $\{e_1, \ldots e_n\}$ be the orthonormal basis of $S(TM)$. Taking the trace of equation (23) and using equation (16), we have

$$\begin{aligned}(S - \lambda - \mu)\sum_{i=1}^{n} g(e_i, e_i) =& \sum_{i=1}^{n}\{\frac{1}{2}(B(e_i, V^T)\eta(e_i) + B(V^T, e_i)\eta(e_i)) + (1 - \gamma)\pi(e_i)\pi(e_i) - \rho g(A_N e_i, e_i)\} \\ n(S - \lambda - \mu) =& (1 - \gamma)\|V^T\|^2 - \rho\alpha\end{aligned}$$

which is nothing but equation (29). □

Theorem 3.5. *Let a lightlike hypersurface (M, g, V^T, λ) be quasi-Yamabe solitons or almost quasi-Yamabe solitons. Then,*

(i) M is shrinking if $\alpha > \frac{1}{\rho}(n(\mu - S) + (1-\gamma)\|V^T\|^2)$,

(ii) M is steady if $\alpha = \frac{1}{\rho}(n(\mu - S) + (1-\gamma)\|V^T\|^2)$,

(iii) M is expanding if $\alpha > \frac{1}{\rho}(n(\mu - S) + (1-\gamma)\|V^T\|^2)$

Proof. From equation (29), we have

$$\lambda = S - \mu - \frac{1}{n}((1-\gamma)\|V^T\|^2 - \rho\alpha). \tag{30}$$

Since $(M, g, V^T, \gamma, \lambda)$ is shrinking, steady or expanding if $\lambda > 0, \lambda = 0$ or $\lambda < 0$, respectively, the by equation (30), the proof is completed. □

Theorem 3.6. *Let a lightlike hypersurface (M, g, V^T, λ) be quasi-Yamabe solitons or almost quasi-Yamabe solitons. (M, g, V^T, λ) is trivial if and only if the mean curvature of M is given by*

$$\alpha = \frac{1}{\rho}(n\mu + \|V^T\|^2). \tag{31}$$

Proof. (M, g, V^T, λ) is trivial Yamabe solitons or almost Yamabe solitons if and only if V^T is a Killing vector field. Therefore, from equation (27), we have

$$2\rho g(A_N X, Y) = B(X, V^T)\eta(Y) + B(V^T, Y)\eta(X) + 2\mu g(X, Y) + 2\pi(X)\pi(Y)$$

Taking the trace of this equation corresponding to the orthonormal basis of $S(TM)$ and using equation (19), we obtain

$$2\rho\alpha = 2n\mu + 2\|V^T\|^2$$

which implies equation (31). □

Theorem 3.7. *Let a lightlike hypersurface (M, g, V^T, λ) be a quasi-Yamabe solitons or almost quasi-Yamabe solitons. If M is a quasi-Yamabe gradient solitons or almost quasi-Yamabe gradient soliton, then the mean curvature of M is given by*

$$\alpha = \frac{1}{\rho}((1-\gamma)\|\nabla f\|^2 - n(S - \lambda - \mu)) \tag{32}$$

where $V^T = \nabla f$.

Proof. Since the quasi-Yamabe soliton (M, g, V^T, λ) is a quasi-Yamabe gradient soliton or gradient almost Yamabe soliton, then there exist a smooth function f on M such that V^T is the gradient of f that is, $V^T = \nabla f$. Let $\{e_1, \ldots, e_n, \xi\}$ be an quasi-orthonormal basis of TM, then

$$\begin{aligned}\nabla_i\nabla_j f =& g(\nabla_i \nabla f, e_j)\\ =& g(\mu e_i + \nabla_i f \nabla f - \rho A_n e_i, e_j)\\ =& \mu\delta_{ij} + \nabla_i f \nabla_j f - \rho g(A_N e_i, e_j)\end{aligned}$$

where δ_{ij} is a cronecker delta. Contracting this equation, we have

$$\nabla_i\nabla_i f = \mu + \|\nabla_i f\|^2 - \rho g(A_N e_i, e_i) \tag{33}$$

and

$$\nabla_\xi \nabla_\xi f = \|\nabla_\xi f\|^2. \tag{34}$$

As a consequence,

$$\Delta f = n\mu + \|\nabla f\|^2 - \rho\alpha \tag{35}$$

where α is the mean curvature of M. On the other hand, taking the trace of equation (4) with respect to the quasi orthonormal basis of $S(TM)$, we have

$$\Delta f = n(S - \lambda) + \gamma\|\nabla f\|^2. \tag{36}$$

From equations (35) and (36), we find

$$(1-\gamma)\|\nabla f\|^2 - n(S - \lambda - \mu) = \rho\alpha \tag{37}$$

which implies equation (32). □

4. Lightlike hypersurface of semi-Riemannian manifolds of constant curvature as quasi-Yamabe solitons and almost quasi-Yamabe solitons

In this section, we spesify some conditions for lightlike hypersurfaces of the $(n+2)$-dimensional semi-Riemannian manifolds of constant curvature endowed with a torse-forming vector field and particularly lightlike hypersurfaces of semi-Euclidean space endowed with a torse-forming vector field to be quasi-Yamabe solitons and almost quasi-Yamabe solitons. We also some properties of these lightlike hypersurfaces satisfying quasi-Yamabe solitons and almost quasi-Yamabe solitons.

Theorem 4.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(n+2)$-dimensional semi-Riemannian manifolds of constant curvature c endowed with a torse-forming vector field. (M, g, V^T, λ) is quasi-Yamabe solitons or almost quasi-Yamabe solitons if and only if it satisfies*

$$(n^2c - tr(A^*_\xi)\alpha - tr(A^*_\xi A_N) - \lambda - \mu)X = \frac{1}{2}(B(X, V^T)V^T + \eta(X)A^*_\xi V^T) + (1-\gamma)\pi(X)V^T - \rho A_N X \tag{38}$$

for all $X \in \Gamma(TM)$.

Proof. Let $\{e_1, \ldots, e_n, \xi\}$ be the quasi-orthonormal basis of TM and $\{e_1, \ldots, e_n\}$ be the orthonormal basis of screen distribution $S(TM)$. Taking the trace of equation (21), we have

$$S = n^2c - tr(A^*_\xi)\alpha - trace(A^*_\xi A_N). \tag{39}$$

(M, g, V^T, λ) is Yamabe solitons or almost Yamabe solitons if and only if it satisfies equation (2). Therefore, frome equations (23) and (39), we find

$$\begin{aligned}(n^2c - tr(A^*_\xi)\alpha - tr(A^*_\xi A_N) - \lambda)g(X, Y) =& \frac{1}{2}(B(X, V^T)\eta(Y) + B(V^T, Y)\eta(X)) + (1-\gamma)\pi(X)\pi(Y) \\ &+ \mu g(X, Y) - \rho g(A_N X, Y)\end{aligned} \tag{40}$$

which implies equation (38) □

In case $(\bar{M}, \bar{g})$ is an $(n+2)$-dimensional semi-Euclidean space, then $c = 0$. Hence, we have the following corollary:

Corollary 4.2. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(n+2)$-dimensional semi-Euclidean space endowed with a torse-forming vector field. (M, g, V^T, λ) is quasi-Yamabe solitons or almost quasi-Yamabe solitons if and only if it satisfies*

$$(tr(A^*_\xi)\alpha - tr(A^*_\xi A_N) - \lambda - \mu)X = \frac{1}{2}(B(X, V^T)V^T + \eta(X)A^*_\xi V^T) + (1-\gamma)\pi(X)V^T - \rho A_N X$$

for all $X \in \Gamma(TM)$.

Theorem 4.3. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(n+2)$-dimensional semi-Riemannian manifolds of constant curvature c endowed with a torse-forming vector field. If (M, g, V^T, λ) is quasi-Yamabe solitons or almost quasi-Yamabe solitons, then the mean curvature of M is given by*

$$\alpha = \frac{n(n^2c - tr(A^*_\xi A_N) - \lambda - \mu) + (\gamma - 1)\|V^T\|^2}{n \cdot tr(A^*_\xi) - \rho} \tag{41}$$

Proof. Let $\{e_1, \dots e_n\}$ be the orthonormal basis of $S(TM)$. Taking the trace of equation (40) and using equation (16), we have

$$\begin{aligned} n(n^2c - \alpha \cdot tr(A^*_\xi) - tr(A^*_\xi A_N) - \lambda - \mu) &= (1-\gamma)\|V^T\|^2 - \rho\alpha \\ n(n^2c - tr(A^*_\xi A_N) - \lambda - \mu) + (\gamma - 1)\|V^T\|^2 &= (n \cdot tr(A^*_\xi) - \rho)\alpha \end{aligned}$$

which is nothing but equation (41). □

Corollary 4.4. *Let the lightlike hypersurface (M, g, V^T, λ) of $(n+2)$-dimensional semi-Riemannian manifolds of constant curvature c endowed with a torse-forming vector field be quasi-Yamabe solitons or almost quasi-Yamabe solitons. Then,*

(i) M is shrinking if $\alpha < \frac{n(n^2c - tr(A^*_\xi A_N) - \mu) + (\gamma-1)\|V^T\|^2}{n \cdot tr(A^*_\xi) - \rho}$,

(ii) M is steady if $\alpha = \frac{n(n^2c - tr(A^*_\xi A_N) - \mu) + (\gamma-1)\|V^T\|^2}{n \cdot tr(A^*_\xi) - \rho}$,

(iii) M is expanding if $\alpha > \frac{n(n^2c - tr(A^*_\xi A_N) - \mu) + (\gamma-1)\|V^T\|^2}{n \cdot tr(A^*_\xi) - \rho}$.

Proof. Since the lightlike hypersurface (M, g, V^T, λ) of $(n+2)$-dimensional semi-Riemannian manifolds of constant curvature c endowed with a torse-forming vector field is quasi-Yamabe solitons or almost quasi-Yamabe solitons, then from equation (41), by direct calculation we have

$$\lambda = n^2c - tr(A^*_\xi A_N) - \mu - \frac{(n \cdot tr(A^*_\xi) - \rho)\alpha + (\gamma - 1)\|V^T\|^2}{n}.$$

As (M, g, V^T, λ) is shrinking, steady or expanding if $\lambda > 0, \lambda = 0$ or $\lambda < 0$, respectively, then our assertions holds. □

Theorem 4.5. *Let $(M, g, S(TM))$ be a totally geodesic lightlike hypersurface of $(n+2)$-dimensional semi-Riemannian manifolds of constant curvature c endowed with a torse-forming vector field. (M, g, V^T, λ) is quasi-Yamabe solitons or almost quasi-Yamabe solitons if and only if it satisfies*

$$(n^2c - \lambda - \mu)X = (1-\gamma)\pi(X)V^T - \rho A_N X \tag{42}$$

for all $X \in \Gamma(TM)$.

Proof. If $(M, g, S(TM))$ is a totally geodesic lightlike hypersurface, then the second fundamental h vanishes. As a result, the symmetric bilinear form and the shape opeator A^*_ξ also vanish. Therefore, from equation (21), we have

$$Ric(X, Y) = ncg(X, Y)$$

which implies

$$S = n^2 c. \tag{43}$$

In addition, from equation (27), we have

$$(\mathcal{L}_{V^T} g)(X, Y) = 2\mu g(X, Y) + 2\pi(X)\pi(Y) - 2\rho g(A_N X, Y) \tag{44}$$

Applying equations (43) and (44) to (2), we find

$$(n^2 c - \lambda - \mu)g(X, Y) + \gamma\pi(X)\pi(Y) = \pi(X)\pi(Y) - \rho g(A_N X, Y) \tag{45}$$

which implies equation (42). □

Corollary 4.6. *Let (M, g) be a totally geodesic lightlike hypersurface of $(n + 2)$-dimensional semi-Riemannian manifolds of constant curvature c endowed with a torse-forming vector field. If (M, g, V^T, λ) is quasi-Yamabe solitons or almost quasi-Yamabe solitons, then the mean curvature of M is given by*

$$\alpha = \frac{(1 - \gamma)||V^T||^2 - n(n^2 c - \lambda - \mu)}{\rho}$$

Corollary 4.7. *Let a totally geodesic lightlike hypersurface (M, g, V^T, λ) of $(n + 2)$-dimensional semi-Riemannian manifold of constant curvature c endowed with a torse-forming vector field be quasi-Yamabe solitons or almost quasi-Yamabe solitons. Then,*

(i) M is shrinking if $\alpha > \frac{(1-\gamma)||V^T||^2 - n(n^2 c - \mu)}{\rho}$,

(ii) M is steady if $\alpha = \frac{(1-\gamma)||V^T||^2 - n(n^2 c - \mu)}{\rho}$,

(iii) M is expanding if $\alpha < \frac{(1-\gamma)||V^T||^2 - n(n^2 c - \mu)}{\rho}$.

Theorem 4.8. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(n + 2)$-dimensional semi-Riemannian manifolds of constant curvature c endowed with a torse-forming vector field. If (M, g, f, λ) is quasi-Yamabe gradient solitons or almost quasi-Yamabe gradient solitons with $V^T = \nabla f$, then it satisfies*

$$\alpha = \frac{n(n^2 c - tr(A^*_\xi A_N) - \lambda - \mu) + (\gamma - 1)||V^T||^2}{n \cdot tr(A^*_\xi) - \rho} \tag{46}$$

where α is the mean curvature of M.

Proof. Suppose that the lightlike hypersurface (M, g, f, λ) is quasi-Yamabe gradient solitons or almost quasi-Yamabe gradient solitons with $V^T = \nabla f$. Applying equation (39) to equation (36), we have

$$-n(n^2 c - tr(A^*_\xi)\alpha - tr(A^*_\xi A_N) - \lambda - \mu) + (1 - \gamma)||\nabla f||^2 = \rho\alpha \tag{47}$$

which is equivalent to

$$n(n^2 c - tr(A^*_\xi A_N) - \lambda - \mu) + (\gamma - 1)||V^T||^2 = (n \cdot tr(A^*_\xi) - \rho)\alpha. \tag{48}$$

Hence, the equation (46) holds. □

Corollary 4.9. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(n+2)$-dimensional semi-Euclidean space endowed with a torse-forming vector field. If (M, g, f, λ) is quasi-Yamabe gradientsolitons or almost quasi-Yamabe gradient solitons with $V^T = \nabla f$, then the man curvature of M is given by*

$$\alpha = \frac{n(tr(A^*_\xi A_N) + \lambda + \mu) + (1-\gamma)\|V^T\|^2}{\rho - n \cdot tr(A^*_\xi)} \tag{49}$$

for all $X \in \Gamma(TM)$.

5. Yamabe solitons and almost Yamabe solitons on screen homomthetic lightlike hypersurfaces

In this section, we show some geometric properties of the screen homothetic lightlike hypersurface $(M, g, S(TM))$ satisfying quasi-Yamabe solitons and almost quasi-Yamabe solitons of semi-Riemannian manifolds endowed with a torse-forming vector field.

Definition 5.1. *A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be screen conformal if*

$$A_N = \varphi A^*_\xi, \tag{50}$$

where A_N and A^*_ξ are the shape operators of M and $S(TM)$, respectively and φ is a non-vanishing smooth function on M.

It is easy to see that (50) is equivalent to

$$C(X, PY) = \varphi B(X, Y), \tag{51}$$

for all $X, Y \in \Gamma(TM)$. In case φ is a non-constant on M, the lightlike hypersurface M is called screen homothetic. Furthermore, if $\varphi = 0$, i.e., $C = A_N = 0$, then M is called totally geodesic.

Theorem 5.2. *Let a screen homothetic lightlike hypersurface (M, g, V^T, λ) of an $(n+2)$-dimensional semi-Riemannian manifolds $(\bar{M}, \bar{g})$ endowed with a torse-forming vector field be quasi-Yamabe solitons or almost quasi-Yamabe solitons. Then the principal curvature of M is given by*

$$\kappa_i = \frac{1}{\rho\varphi}((1-\gamma)\pi(e_i)^2 + \lambda + \mu - S) \tag{52}$$

for $i = 1, \ldots, n$.

Proof. Since the screen homothetic lightlike hypersurface (M, g, V^T, λ) of an $(n+2)$-dimensional semi-Riemannian manifoldd $(\bar{M}, \bar{g})$ endowed with a torse-forming vector field admit quasi-Yamabe solitons or almost quasi-Yamabe solitons, then, by using equation (50) and (23), we have

$$(S - \lambda - \mu)g(X, Y) = \frac{1}{2}(B(X, V^T)\eta(Y) + B(V^T, Y)\eta(X)) + (1-\gamma)\pi(X)\pi(Y) - \rho\varphi g(A_\xi X, Y) \tag{53}$$

for all $X, Y \in \Gamma(TM)$.
Let $\{\xi, N\}$ be the canonical null pair on M such that 1-form τ vanishes. Since ξ is the eigenvector field of A^*_ξ corresponding to the eigenvalue 0 and A^*_ξ is $\Gamma(S(TM))$-valued real symmetric, A^*_ξ has n orthonormal eigenvector field in $S(TM)$ and is diagonalizable. Suppose $\{e_1, ..., e_n, \xi\}$ is a frame field of eigen vectors of A^*_ξ and $\{e_1, ..., e_n\}$ is the orthonormal frame field of $S(TM)$. Then we have,

$$A^*_\xi e_i = k_i e_i, \quad 1 \leq i \leq n, \tag{54}$$

where k_i is the screen principal curvature.
Therefore, applying equation (54) to equation (53), we have

$$(S - \lambda - \mu)\delta_{ij} = (1-\gamma)\pi(e_i)\pi(e_j) - \rho\varphi\kappa_i\delta_{ij}$$

which implies equation (52). □

Theorem 5.2 shows that the screen homothetic lightlike hypersurface M admitting quasi-Yamabe solitons or almost quasi-Yamabe solitons has only one principal curvature. As a result, we have the following corollary:

Corollary 5.3. *Every conformal homothetic lightlike hypersurface* (M, g, V^T, λ) *admitting Yamabe solitons or almost quasi-Yamabe solitons is totally umbilical.*

Theorem 5.4. *Let a screen homothetic lightlike hypersurface* $(M, g, S(TM))$ *of semi-Riemannian manifold* $(\bar{M}, \bar{g})$ *endowed with a torse-forming vector field be totally umbilical. The lightlike hypersurface* (M, g, V^T, λ) *is a quasi-Yamabe soliton or almost quasi-Yamabe soliton if and only if*

$$(S - \rho\varphi F - \lambda - \mu)X = \frac{F}{2}(\pi(X)N + \eta(X)V^T) + (1 - \gamma)\pi(X)V^T \tag{55}$$

for all $X \in \Gamma(TM)$.

Proof. Since the lightlike hypersurface (M, g) is screen conformal and totally umbilical, then by applying equations (15), (22) and (50) to (23), we have

$$(S - \lambda - \mu)g(X, Y) = \frac{1}{2}(F\pi(X)\eta(Y) + F\pi(Y)\eta(X)) + (1 - \gamma)\pi(X)\pi(Y) - \rho\varphi F g(X, Y).$$

This equation is equivalent to

$$(S + \rho\varphi F - \lambda - \mu)g(X, Y) = \frac{1}{2}(F\pi(X)\eta(Y) + F\pi(Y)\eta(X)) + (1 - \gamma)\pi(X)\pi(Y)$$

which implies equation (55) □

By the same way as proof of Theorem 5.2, we have the following theorem:

Theorem 5.5. *Let a conformal homothetic lightlike hypersurface* (M, g, f, λ) *with* $V^T = \nabla f$ *of an* $(n+2)$*-dimensional semi-Riemannian manifolds* $(\bar{M}, \bar{g})$ *endowed with a torse-forming vector field be quasi-Yamabe gradient solitons or almost quasi-Yamabe gradient solitons. Then the principal curvature of* M *is given by*

$$\kappa_i = \frac{1}{\rho\varphi}(\pi(e_i)^2 + \lambda + \mu - S) \tag{56}$$

for $i = 1, \ldots, n$.

Corollary 5.6. *A conformal homothetic lightlike hypersurface* (M, g, V^T, λ) *admitting quasi-Yamabe gradient solitons or almost quasi-Yamabe gradient solitons is totally umbilical.*

Theorem 5.7. *Let the screen conformal lightlike hypersurface* (M, g, V^T, λ) *be quasi-Yamabe solitons or almost Yamabe solitons. If* (M, g, f, λ) *admits trivial quasi-Yamabe solitons or trivial almost quasi-Yamabe gradient solitons, then* M *is totally umbilical hypersurface.*

Proof. Since (M, g, f, λ) is quasi-Yamabe gradient solitons or trivial almost quasi-Yamabe gradient solitons, then we have $\nabla f = V^T$, where ∇ denotes the gradient. If (M, g, f, λ) is trivial, then $f = \textit{constant}$ which implies $V^T = 0$. Therefore, from equation (25), we get

$$A_N X = \frac{\mu}{\rho} X.$$

As M is screen conformal, then by equations (50) and (51), we find

$$A^*_\xi X = \frac{\mu}{\rho\varphi} X$$

which leads us to equation (20). □

Acknowledgements.

The author states that this work does not have any conflict of interest. No funds, grants, or other support was received for conducting this study.

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