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Double Points and Universal Covers

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Abstract. In this note we study the double points set of a particular covering map of an open manifold, and we present a new procedure for building universal covering spaces of such manifolds. This is done by means of an arborescent construction, starting from a presentation of the manifold as a non-compact simplicial complex with pairwise identified faces. The proof uses the so-called "zipping theory" of Poénaru which helps the understanding of the topology of the quotient manifold resulted from the combinatorial presentation.

1. Introduction

One of the oldest ways to represent in a simple way a PL closed n-manifold M^n is that of considering a polyhedral ball modulo the orbits of a fixed point-free involution on the $(n-1)$-simplices of ∂M^n. More precisely, one starts with a PL n-ball Δ whose boundary is triangulated with an even number of $(n-1)$-simplices. Then, in this set of $(n-1)$-simplices, $\{h_1, h_2, \dots, h_{2p}\}$, one considers an appropriate fixed-point free involution r. Finally, one glues each h_i to $r(h_i)$ via a well-chosen simplicial isomorphism. The quotient space Δ/ρ_r obtained by this process, will be exactly M^n. Now, if $\Delta_L = \cup_{l\in L} l\Delta$ is the tree of fundamental domains of the free monoid L generated by the identifications of the $(n-1)$-simplices of $\partial\Delta$, the universal covering space of M^n can also be obtained as a quotient of Δ_L by an opportune equivalence relation "forced" by the singularities of the natural map from Δ_L to M^n, conceived by V. Poénaru in [3], and successfully exploited in [5–7] (and, more recently, in [1, 2]).

In this note we will present similar representations for any open n-manifold V^n and its universal covering space. In broad lines it works in this way. Since the polyhedral n-ball Δ of above may also be viewed as the n-dimensional regular neighbourhood of a point, we will replace Δ by T, the n-dimensional regular neighbourhood of a properly embedded non compact tree $T^1 \subset V^n$, with empty "boundary" (namely an infinite tree whose all endpoints are at the infinity). We will also consider a triangulation τ of ∂T (which actually is a $(n-1)$-sphere with open disks removed, corresponding to the ends of T) and a suitable fixed point-free involution j on the finite set of $(n-1)$-simplices $\{t_1, t_2, \dots, t_{2p}\}$ of τ in such a way that if one identifies the simplexes t_i with $j(t_i)$ (by means of a pertinent linear isomorphism), one gets a quotient space T/ρ_j which is exactly V^n. We will also prove that the universal covering space of V^n may be constructed as a quotient of a tree-like object T^∞, obtained by unrolling T along its faces, which will be the analogous of Δ_L for closed manifolds.

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Approximately, the definition of T^∞ can be schematised as follows: $T^\infty = T \cup_{j_i} j_i T \cup_{j_i j_k} j_i j_k T \cup \cdots$, where any $j_i j_k \ldots j_r j_s$ identifies two corresponding faces in $j_i j_k \ldots j_r T$ and in $j_i j_k \ldots j_r j_s T$. Of course, by construction, this tree-like object has lots of singularities and we will make use of Poénaru's (Φ/Ψ)-theory from [3], which is a practical strategy for getting rid of them, but still preserving some useful topological information. The equivalence relation Φ is the standard equivalence relation associated to a map f, where $(x, y) \in \Phi$ means that $f(x) = f(y)$, while $\Psi(f)$ is an equivalence relation that is the smallest possible such relation killing all the singularities of the map f.

In our first result (Theorem 3.2) we will show that that $\widetilde{V^n} \cong T^\infty/\Psi(f^\infty)$, where f^∞ is the natural map from T^∞ to V^n sending any copy of $T \subset T^\infty$ to $T \subset V^n$. In the second result (Theorem 3.5) we will actually prove that $\Psi(f^\infty) = \Phi(f^\infty)$, which implies, in particular, that once one has killed all the singularities of f^∞, there are no more double points left (or, in other words, the cheapest way to kill all the singularities is to kill all the double points).

2. Preliminaries

2.1. *The equivalence relation Ψ forced by the singularities*

In Section 2 of [3], Poénaru considered and investigated the double points structure of the very general situation of a non-degenerate simplicial map $f : X \to M^3$, where M^3 is a triangulated 3-dimensional manifold without boundary, and X is a not necessarily locally finite simplicial complex of dimension ≤ 3 (here *non-degenerate* means that for any simplex σ of X, $\dim f(\sigma) = \dim \sigma$).

In that paper he introduced and studied two equivalence relations $\Psi(f) \subset \Phi(f) \subset X \times X$, where $\Phi(f)$ is the "ordinary" equivalence relation $(x, y) \in \Phi(f) \iff f(x) = f(y)$, whereas $\Psi(f)$ is the "smallest" equivalence relation, compatible with f, which kills all the possible singularities of f (where an equivalence relation $\mathcal{R}$ is *compatible* if the quotient $X/\mathcal{R}$ remains a simplicial complex, together with its induced map to M^3). Recall also that a point $z \in X$ is a *singularity* ($z \in \mathrm{Sing}(f)$) if there exist two different simplexes $\sigma_1, \sigma_2 \subset X$ with $z \in \sigma_1 \cap \sigma_2$ and such that $f(\sigma_1) = f(\sigma_2)$. Clearly, the quotient space $X/\Phi(f)$ is nothing but $f(X)$. On the other hand, the equivalence relation Ψ gives rise to a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & M^3 \\ & \searrow \qquad \nearrow_{f_1} & \\ & X/\Psi(f) & \end{array} \qquad (1)$$

where f_1 is an *immersion* (i.e. without singularities, namely $\mathrm{Sing}(f_1) = \emptyset$) and no smaller equivalence relation, compatible with f, fulfils this condition.

This statement and various other properties of $\Psi(f)$ are proved and explained in details in [3] (see also [1] for a simple applications of this theory). Here we want just to mention that while the standard quotient map from X to $X/\Phi(f)$ forgets, in general, any topological information, Lemma 2.4 of [3] shows that the natural canonical map from $\pi_1(X)$ to $\pi_1(X/\Psi(f))$ is actually surjective, which specifically means that $\pi_1(X/\Psi(f)) = 0$ whenever X is simply connected. This is a very important feature that will allows us to obtain universal covering spaces.

Finally, note also that, although the paper [3] deals with 3-dimensional manifolds, all the results remain valid in any dimension $n \geq 3$ (indeed dimension 3 was important just for the applications of the theory for the Poincaré Conjecture and/or the simple connectivity at infinity of universal covers of closed 3-dimensional manifolds).

2.2. *An application of the* (Φ/Ψ)*-theory: closed manifolds*

This little theory can be nicely applied and exploited in the context of closed manifolds in order to obtain a different construction of their universal covering spaces (see [5, 6]). This was actually the main tool Poénaru has exploited in order to transform the geometric information given by the fundamental group of the manifold (such as being almost-convex, hyperbolic or combable), into topological conditions of its universal cover.

Let M^n be a closed n-manifold, and consider a triangulation τ of M^n together with its dual cellular decomposition $\tau^\star$. A general procedure for representing the n-manifold M^n as an appropriate quotient of the combinatorial object Δ (the n-dimensional PL ball) is to start by considering a maximal tree Λ of the 1-skeleton of $\tau^\star$, and attach, along its edges, all the $(n-1)$-simplices of τ. The space obtained Δ is a collapsible n-dimensional simplicial complex endowed with a simplicial map $g : \Delta \to M^n$, and such that $\Delta/\rho = M^n$.

In this way, when we consider $\Delta_L = \cup_{l\in L} l\Delta$, that is the tree of fundamental domains of the free monoid L generated by the identifications of the $(n-1)$-simplices of $\partial\Delta$, every $l\Delta$ can be considered as a triangulated simplicial complex just as Δ, and hence the whole Δ_L can be viewed as a simplicial complex with a simplicial non-degenerate map $g_\infty : \Delta_L \to M^n$ which sends each $l\Delta$ onto $g(\Delta)$. This *arborescent* space is obviously not locally-finite, but it turns out that $\Delta_L/\Psi(g_\infty)$ is actually homeomorphic to $\widetilde{M}^n$ (see [6]).

Remark 2.1. The space Δ_L has a configuration which is based on the Cayley graph of $\pi_1 M^n$, and, if, in the definition of $\Delta_L = \cup_{l\in L} l\Delta$, one considers only reduced words, then the set of singularities of the map $g_\infty : \Delta_L \to M^n$ (namely the points where Δ_L is not a manifold) is the $(n-2)$-skeleton of Δ_L, otherwise it would be the whole $(n-1)$-skeleton.

2.3. *Other applications*

The results we will present now should serve as a reminder of how the Φ/Ψ-manipulation was used by Poénaru in his work in differential topology and geometric group theory.

A smooth open n-manifold M^n is said to be *Dehn-exhaustible* if for every compact subset $k \subset M^n$ we can find a compact bounded n-manifold K^n with $\pi_1 K^n = 0$, entering in the following commutative diagram

$$\begin{array}{ccc} k & \xrightarrow{\quad i \quad} & M^n \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle f} & \\ & K^n & \end{array}$$

which is such that: i is the canonical inclusion and j is an inclusion too, f is a smooth immersion, and the following so-called *Dehn-condition* is fulfilled: $j(k) \cap M_2(f) = \emptyset$, where $M_2(f)$ is the set of points $x \in K^n$ such that $\text{card}\{f^{-1}f(x)\} > 1$.

(Note that the last condition is similar to the one in the renowned Dehn's lemma). In [4], Poénaru proved his own version of Dehn's lemma: *"Any open simply-connected 3-manifold V^3 which is Dehn-exhaustible, is simply connected at infinity"*.

He then used the Φ/Ψ-theory for universal coverings of closed manifolds, together with the Dehn-type Lemma of above in order to prove the following well-known classical result [5]: *"Let M^3 be a closed 3-manifold with $\pi_1 M^3 = G$. Assume G is almost convex (or Gromov-hyperbolic). Then $\widetilde{M}^3$ is simply connected at infinity"*.

One of the main tool he used in his work was the notion of "inverse-representation", which heavily uses the $\Phi = \Psi$ condition. The first "representation-result" from [6] (the so-called "Collapsible Pseudo-Spine Representation Theorem") states that: *"Given a homotopy* 3*-sphere Σ^3, one can construct a* REPRESENTATION *$f : K^2 \to \Sigma^3$, where K^2 is a finite* 2*-complex and f a non-degenerate simplicial map with controlled singularities , such that the complement of $f(K^2)$ is a finite collection of open* 3*-cells, K^2 is collapsible and* $\Psi(f) = \Phi(f)$*"*.

Afterwards, trying to adapt this kind of result to open 3-manifolds, Poénaru and Tanasi [7] gave an extension of these ideas to the case of simply-connected open 3-manifolds V^3, introducing the notion of almost-arborescent representation.

All these results have been obtained with the help of the equivalence relation Ψ in order to be able to push away all the singularities of the representation map one needs to work with.

3. Main results

3.1. Open manifolds and universal covers

In this section we will adapt the construction of Section 2.2 for open n-manifolds. The starting point is the following statement:

Proposition 3.1. *Given an open, connected, triangulated n-manifold V^n, there are a triangulated, connected, non compact n-manifold T, with non empty boundary, and a simplicial map $F : T \to V^n$, with the following properties:*

- *The restriction of F to any simplex σ is an isomorphism between σ and $F(\sigma)$.*
- *There is a proper embedding of an infinite, locally finite tree, $i : T^1 \to \mathrm{int(T)} = \mathrm{T} - \partial\mathrm{T}$, such that T is a n-dimensional regular neighbourhood of $i(T^1)$, and the endpoints of T^1 lie at the infinity of T.*
- *The map F is surjective, and $F|_{\mathrm{int(T)}}$ is an embedding.*
- *$F(\mathrm{intT}) \cap \mathrm{F}(\partial\mathrm{T}) = \emptyset$.*
- *If we denote by S the set of all $(n-1)$-dimensional simplexes of ∂T, and if $\sigma \in S$, then there is exactly one other element $j\sigma \in S$, different from σ, such that $F(\sigma) = F(j\sigma)$.*

Proof. In order to obtain the result it suffices to pickup a maximal tree T^1 of tetrahedra of the triangulation of V^n and then to consider its regular neighbourhood T. □

From the last point of Proposition 3.1, we have on S a fixed point-free involution $j : S \to S$ and for any $\sigma \in S$ a linear isomorphism $\lambda_{(\sigma,j\sigma)} : \sigma \to j\sigma$ with $\lambda_{(j\sigma,\sigma)} = \lambda^{-1}_{(\sigma,j\sigma)}$ and with obvious compatibility conditions around the edges of ∂T.

Thus, the data $\{S, j, \lambda\}$ induces an equivalence relation ρ on T and actually one has the equality $\rho = \Phi(F)$. Hence $V^n = T/\rho$ and we will call T the *fundamental domain*.

We will consider now the free monoid G which is generated by $S = \{h_1, h_2, \dots, h_{2p}\}$ and by 1, and the n-dimensional non locally finite, tree-like, simplicial complex T^∞, obtained as follows. We start with the disjoined union $\sum_{x\in G} xT$ and then, for each $x \in G$ and $h \in S$, we identify the h-face of xT to the jh-face of $(xh)T$. The quotient space is our T^∞ and the definition of T^∞ may be schematised by the following symbolical formula

$$T^\infty = T \cup_{hi} h_i T \cup_{h_i h_j} h_i h_j T \cup \cdots . \tag{2}$$

There is a tautological map $f^\infty : T^\infty \to V^n$ sending each $xT \subset T^\infty$ identically onto $F(T) \subset V^n$, where $F : T \to V^n$ is the map of Proposition 3.1.

Theorem 3.2. *The natural arrow*

$$f_1^\infty : T^\infty/\Psi(f^\infty) \to V^n \tag{3}$$

is the universal covering map for V^n (i.e. $T^\infty/\Psi(f^\infty) \cong \widetilde{V}^n$).

Proof. The arborescent space T^∞ is obviously simply connected and hence so is $T^\infty/\Psi(f^\infty)$. Also, it is easy to prove that f_1^∞ is a local homeomorphism, because it is an immersion by (1), and because T^∞, as well as $T^\infty/\Psi(f^\infty)$, have no free-faces. Furthermore, $T^\infty/\Psi(f^\infty)$ is complete, in the sense that each infinite word $h_{i_1}h_{i_2}h_{i_3}\cdots$ can be represented in T^∞ and hence in $T^\infty/\Psi(f^\infty)$ by a continuous chain of fundamental domains, starting in $1 \cdot T$ and going to infinity. The conclusion follows from these three facts. □

We have so obtained a new reinterpretation of the universal covering space of an open manifold, but we do not have a manageable method for obtaining our equivalence relation Ψ. This will be done in the next section.

3.2. An effective construction of Ψ

For an arbitrary open n-manifold V^n we have then the following commutative diagram

$$
\begin{array}{ccccc}
 & & T^\infty/\Phi(f^\infty) & & \\
 & \nearrow & & \searrow^{id} & \\
T^\infty & & \xrightarrow{\quad f^\infty \quad} & & V^n \\
 & \searrow & & \nearrow_{f_1^\infty} & \\
 & & T^\infty/\Psi(f^\infty) & &
\end{array}
\tag{4}
$$

Now, if we assume that V^n is simply connected (and from now on it will be assumed all along the paper), the combination of the diagram above and of Theorem 3.2 tells us that we have the equality $\Phi(f^\infty) = \Psi(f^\infty)$, because $V^n = T^\infty/\Psi(f^\infty)$.

Remember now that a point $X \in T^\infty$ is a singularity for f^∞ (and one writes $X \in \mathrm{Sing}(f^\infty)$) if and only if there exist two fundamental domains $x'T, x''T \subset T^\infty$, with x', x'' distinct elements of G, such that:

- $X \in x'T \cap x''T$,
- there exist two small neighbourhoods of X, $U' \subset x'T$ and $U'' \subset x''T$ such that $f^\infty(U') = f^\infty(U'')$. In such a case we will say that $\mathrm{germ}(f^\infty|_{x'T})_X = \mathrm{germ}\,(f^\infty|_{x''T})_X$.

Remark 3.3. Each xT is just another copy of T, which is being sent identically onto $F(T) \subset V^3$ by the map f^∞. Also, the map $f^\infty|_{\mathrm{int}(xT)} : xT \to V^n$ is not a homeomorphism, whereas $f^\infty|_{\mathrm{int}(xT)} : \mathrm{int}(xT) \to \mathrm{int}(T) \subset V^n$ is one.

If we label once for all the vertices of $F(T) \subset V^n$ by $v_1, v_2, \cdots, v_\alpha, \cdots$ this automatically labels the vertices of each xT. If our singularity $X \in x'T \cap x''T$ is a vertex, it has the same label in $x'T$ and in $x''T$.

We introduce now the set $\textsc{sing}(T^\infty)$ of SINGULARITIES of the space T^∞ counted with multiplicities. By definition, an element of $\textsc{sing}(T^\infty)$ is a triple $(X; x'T, x''T)$ where:

- X is a vertex, an edge or a face of T^∞,
- $x'T$ and $x''T$ are two distinct fundamental domains such that $X \in x'T \cap x''T$, and $\mathrm{germ}(f^\infty|_{x'T})_X = \mathrm{germ}\,(f^\infty|_{x''T})_X$.

Remark 3.4.

- We do not distinguish between the SINGULARITIES $(X; x'T, x''T)$ and $(X; x''T, x'T)$.
- Every point $p \in X$, where $(X; x'T, x''T)$ is a SINGULARITY, belongs clearly to $\mathrm{Sing}(f^\infty)$. But one should not mix up such singularities (belonging to $\mathrm{Sing}(f^\infty)$) with SINGULARITIES (belonging to $\textsc{sing}(T^\infty)$).
- Like before, if $(X; x'T, x''T)$ is a SINGULARITY, then X is the same vertex, or edge, or face, whatever considered in $x'T$ or in $x''T$.
- Even if we have something like $\sigma_1 =$ (Vertex (or Edge); $x'T, x''T$) which is part of a larger SINGULARITY $\sigma_2 =$ (Edge (or Face); $x'T, x''T$), we will anyway count σ_1 and σ_2 as distinct SINGULARITIES.

The next theorem provides an explicit description of the equality $\Phi(f^\infty) = \Psi(f^\infty)$ of above.

Theorem 3.5. *There is a well ordered (possibly transfinite) sequence of successive quotient spaces of T^∞, obtained by folding maps; each of these operations identifies two fundamental domains with non-void intersection, and which are part of a* SINGULARITY *(at the source of the corresponding folding map). In this way we get a sequence of spaces*

$$
T^\infty \xrightarrow{\rho_1} T^\infty(\sigma_1) \xrightarrow{\rho_2} T^\infty(\sigma_1)(\sigma_2) \xrightarrow{\rho_3} \cdots \tag{5}
$$

with the following two properties:

- *Corresponding to the well-ordered sequence of ordinals, one has:*
  - *A first element $\sigma_1 \in \textsc{Sing}(T^\infty)$ with $\sigma_1(u_1; x'_1T, x''_1T)$ and its associated projection map p_1 which identifies x'_1T to x''_1T.*
  - *A second element $\sigma_2 \in \textsc{Sing}(T^\infty(\sigma_1))$ with $\sigma_2(u_2; x'_2T, x''_2T)$ and its associated projection map p_2 which identifies x'_2T to x''_2T.*
  - $\cdots\cdots$
- *Each of the objects appearing in the sequence above is a union of fundamental domains. This process eventually ends when there are no more* SINGULARITIES *left. When that happens, we are left with a quotient space of T^∞ which contains a unique fundamental domain. This quotient space is exactly*

$$T/\Psi(f^\infty) = T/\Phi(f^\infty) = V^n.$$

Proof. Firstly, we set up an inductive process of folding maps which kills successively all the SINGULARITIES $(u_i; x'T, x''T)$. Since T^∞ has no free faces (by which we mean that the various fundamental domains of T which are incident to a given face τ have f^∞-images which occupy both sides of $f^\infty\tau$), this process also kills all the SINGULARITIES of the type (Vertex; Face, Face) and (Edge; Face, Face). It is not hard to see that it actually kills all the SINGULARITIES of the type (Vertex; Edge, Edge), too.

In other words, by the time we have killed all the SINGULARITIES $(u_i; x'T, x''T)$, there are no more SINGULARITIES $(X; x'T, x''T)$ left. So we can finally apply the equality $V^n = T^\infty/\Psi(f^\infty)$ of above in order to get our conclusion. □

Proposition 3.6. *One can choose the folding maps $\rho_1, \rho_2, \cdots$ of Theorem 3.5 in such a way that the sequence (5) does not continue beyond the first infinite ordinal ω.*

Proof. Denote by $Z_n \subset T^\infty$ the part of T^∞ obtained by taking only the xT's where x is a word of length $\leq n$ in G. We have then the sequence

$$Z_1 \subset Z_2 \subset \cdots \subset Z_n \subset Z_{n+1} \subset \cdots T^\infty$$

which in fact exhausts T^∞. For each Z_n, we consider the equivalence relations $\Psi_n = \Psi(f^\infty|_{Z_n})$ and also $\Psi'_n \subset \Psi_n$ which is the equivalence relation generated by all the folding maps which kill the SINGULARITIES of Z_n. Since Z_n has finitely many T's, there exists a number $n_1 \gg n$ with the property that $Z_n/\Psi'_{n_1} \to Z_n/\Psi'_{n_1+1} \to \cdots$ are all bijective. We denote now $(n_1)_1$ by n_2, $(n_2)_1$ by n_3, and so on. Then, the commutative diagram

$$\begin{array}{ccc} Z_n/\Psi'_{n_1} & \hookrightarrow & Z_{n_1}/\Psi'_{n_1} \\ \downarrow \simeq & & \downarrow \\ Z_n/\Psi'_{n_2} & \hookrightarrow & Z_{n_1}/\Psi'_{n_2} \end{array}$$

tells us that $Z_n/\Psi'_{n_1} \to Z_{n_1}/\Psi'_{n_2}$ is actually injective.

So we have a sequence of embeddings

$$Z_n/\Psi'_{n_1} \subset Z_{n_1}/\Psi'_{n_2} \subset Z_{n_2}/\Psi'_{n_3} \subset \cdots \tag{6}$$

and the union of the objects in (6) is a quotient space of T^∞, which we denote by T^∞/R. The various maps $Z_n/\Psi'_{n_1} \to V^n$, $Z_{n_1}/\Psi'_{n_2} \to V^n, \cdots$ are all compatible and induce, then, a well-defined map $\phi : T^\infty/R \to V^n$, which is itself compatible with f^∞. A local analysis shows that if for high enough $N \gg n$ we kill all the SINGULARITIES of Z_N, this automatically also kills all the singularities of $f^\infty|_{Z_n}$. It follows that the map ϕ is an immersion, and since a priori $R \subset \Psi(f^\infty)$, we actually have the equality $R = \Psi(f^\infty)$. Hence $\Psi'_{n_1}|_{Z_n} = \Psi(f^\infty)|_{Z_n}$, $\Psi'_{n_2}|_{Z_{n_1}} = \Psi(f^\infty)|_{Z_{n_1}}$.

We are able now to produce a sequence as in (5) but modelled on the first transfinite ordinal ω. This goes as follows: we choose to successively kill only SINGULARITIES of Z_n until we realise $\Psi'_{n_1}|_{Z_n}$, then only SINGULARITIES of Z_{n_1} until we realise $\Psi'_{n_2}|_{Z_{n_1}}$, and so on. This ends our lemma. □

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