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New Areas for Applications of Contractive Mappings

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Abstract. In this paper, as a new application of fixed point theorems, we utilize some new contractions to study the existence and uniqueness intervals of some different classes of nonlinear boundary eigenvalue problems in integer and fractional order.

1. Introduction

Since many practical problems in applied sciences can be transformed to a problem of finding fixed points of nonlinear mappings, fixed point theory play a key role in nonlinear analysis. One of the most basic fixed point theorem is the Banach theorem. During the last few decades, several famous extensions of this theorem have been proved [1–7]. It is known that many natural phenomena lead to a boundary eigenvalue problem. For example heat equation, wave equation, Bessel's equation, advection dispersion equation, Heun's equation, Schrodinger equation can be mentioned in this regard. Therefore this makes it clear that the boundary eigenvalue problems are of broad interest and there is a developed theory of them in the literature of differential equations. On the other hand, regularly, these problems should establish the eigenvalue interval for the existence of unique positive solution for the eigenvalue problem. There are, albeit limited methods to study the eigenvalue problems, such as asymptotic methods [10]. Some authors [3, 8, 9, 11, 13–16, 24, 25] have used u_0-positive operators to study the theory of boundary eigenvalue problems. A nonlinear generalization of the Laplace operator, known as p-Laplacian operator, is widely used in analyzing mathematical models of physical and scientific phenomena. In recent years the research of boundary value problems with (generalized) p-Laplacian operators has become attractive [16, 18, 21–23, 30]. Also, p-Laplacian operators have used to investigate the existence of positive solutions of eigenvalue problems [22, 23]. One of the limited methods to study of eigenvalue problems is using of fixed point theorems. But it should be noted that most fixed point theorems can not be used for this purpose. Almost all results are related to Guo-Krasnosel'skii fixed Point theorem. On the other hand there are very few articles on the uniqueness of the solution of eigenvalue problems. In 2012 Samet et al [26] introduced the class of $\alpha - \psi$-contraction mappings and studied the existence of fixed points of such mappings. In [27] we have an extension to the results of this paper. For more results in this direction we refer the reader to [11, 16, 19–21, 27, 29–38]. In this paper we use $\alpha - \psi$-contraction mappings theory to the study of two

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cases of nonlinear boundary eigenvalue problems. First, we consider the following boundary eigenvalue problem

$$w^{''}(\varsigma) + \lambda a(\varsigma) f(w(\varsigma)) = 0 \ , \ \varsigma \in (0,1),$$
$$w^{'}(0) = 0 \ , w(1) = \kappa \int_0^{\eta} w(\varrho) d\varrho, \tag{1}$$

where $0 < \eta < 1$ and $0 < \kappa < \frac{1}{\eta}$ and obtain intervals of λ on which a solution of (1) exists and then looking for uniqueness intervals of λ. In the next step, we generalize using of $\alpha - \psi$-contraction mappings to the study of existence and uniqueness eigenvalue intervals of the following boundary eigenvalue problem of Volterra type on infinite interval and arbitrary fractional order

$$\begin{cases} D^{\kappa} w(\varsigma) + \lambda f(\varsigma, w(\varsigma), Sw(\varsigma)) = 0, \quad 3 < \kappa \le 4, \\ w(0) = w^{'}(0) = w^{''}(0) = 0, \quad D^{\kappa-1} w(\infty) = \zeta I^{\iota} w(\mu), \quad \iota > 0, \end{cases} \tag{2}$$

where D^{κ} denotes the fractional derivative of order κ in Riemann-Liouville definition, I^{ι} is the fractional integral of order ι with the same definition and $(Sw)(\varsigma) = \int_0^{\varsigma} k(\varsigma, \varrho) w(\varrho) d\varrho$ such that $k(\varsigma, \varrho) \in C[D, \mathrm{R}], D = \{(\varsigma, \varrho) \in \mathrm{R}^2 | 0 \le \varrho \le \varsigma\}$.

In throughout of the paper we denote $\chi = [0, 1]$.

2. Preliminaries

Definition 2.1. *([18–20]) The Riemann-Liouville fractional derivative of order $\kappa > 0$ of a continuous function $f : (0, +\infty) \to \mathrm{R}$ is given by*

$$\left(D^{\kappa}_{0,+} f\right)(x) = \frac{1}{\Gamma(m-\kappa)} \left(\frac{d}{dx}\right)^m \int_0^x (x-t)^{m-\kappa-1} f(t) dt,$$

where $m = [\kappa] + 1$, $[\kappa]$ denote the integer part of number κ, provided that the right side is pointwise define on $(0, +\infty)$.

Definition 2.2. *([18–20]) The Riemann-Liouville fractional integral of a function $f : (0, +\infty) \to \mathrm{R}$ is given by*

$$\left(I^{\kappa}_{0,+} f\right)(x) = \frac{1}{\Gamma(\kappa)} \int_0^x (x-t)^{\kappa-1} f(t) dt, \quad \kappa > 0.$$

Definition 2.3. *([26]) Suppose Ψ be the family of all functions $\psi : [0, \infty) \to [0, \infty)$ with the following properties:*

P_1) ψ is nondecreasing;

P_2) let ψ^k be the k-th iterate of ψ, then $\sum_{k=0}^{\infty} \psi^k(\varsigma) < \infty$, for all $\varsigma > 0$.

A function $\psi \in \Psi$ is called a (c)-comparison function.

Definition 2.4. *([26]) Let $\varphi : \mathrm{M} \to \mathrm{M}$ be a given mapping where we supposed (M, d) is a metric space. A function φ is called an $\alpha - \psi$-contraction, if there exist a (c)-comparison function $\psi \in \Psi$ and a function $\alpha : \mathrm{M} \times \mathrm{M} \to \mathrm{R}$ such that*

$$\alpha(\nu, \omega) d(\varphi \nu, \varphi \omega) \le \psi(d(\nu, \omega)), \qquad \forall \ \nu, \omega \in \mathrm{M}. \tag{3}$$

Definition 2.5. *([26]) Let $M \neq \emptyset$, $\varphi : \mathrm{M} \to \mathrm{M}$ and $\alpha : \mathrm{M} \times \mathrm{M} \to \mathrm{R}$, we say that φ is α-admissible if*

$$\alpha(\nu, \omega) \ge 1 \Rightarrow \alpha(\varphi \nu, \varphi \omega) \ge 1, \quad \nu, \omega \in \mathrm{M}.$$

Theorem 2.6. *([27]) Let there exists $\alpha : \mathrm{M} \times \mathrm{M} \to \mathrm{R}$ such that the followings hold* *i) for α and ψ the inequality (3) holds;*

ii) φ is α-admissible;

iii) $\exists\, \nu_0 \in \mathrm{M}$ with $\alpha(\nu_0, \varphi\nu_0) \geq 1$;

iv) φ is continuous or

v) for any sequence $\{\nu_n\}_{n\in\mathbb{N}}$, where $\nu_n \to \nu$, if $\alpha(\nu_n, \nu_{n+1}) \geq 1$, then $\alpha(\nu_n, \nu) \geq 1$.

Then φ has a fixed point. In addition, if for arbitrary pair $(u, v) \in \mathrm{M} \times \mathrm{M}$, there exists $w \in \mathrm{M}$ such that $\alpha(u, w) \geq 1$, $\alpha(v, w) \geq 1$, then the fixed point of φ is a unique fixed point.

We denote the set of fixed points of a mapping $\varphi : \mathrm{M} \to \mathrm{M}$ by $Fix(\varphi)$, that is

$$Fix(\varphi) = \{\nu : \nu = \varphi\nu\}.$$

We will use the following lemma, frequently.

Lemma 2.7. *([27]) For any $\psi \in \Psi$,*

i) $\psi(\varsigma) < \varsigma$, for $\varsigma > 0$;

ii) $\psi(0) = 0$;

iii) at $\varsigma = 0$, ψ is continuous.

We define the following set Σ_ψ, for any $\psi \in \Psi$, as

$$\Sigma_\psi = \{\sigma \in (0, \infty) : \sigma\psi \in \Psi\}.$$

We continue with the following proposition.

Proposition 2.8. *([27]) Let (M, d) be a metric space, also $\varphi : \mathrm{M} \to \mathrm{M}$ an $\alpha - \psi$ contraction mapping where $\alpha : \mathrm{M} \times \mathrm{M} \to \mathrm{R}$ and $\psi \in \Psi$. Suppose that there exist $\sigma \in \sum_\psi$ such that for some positive integer p, there exists a sequence $\{\xi_i\}_{i=0}^{p} \subset \mathrm{M}$ with the following properties*

$$\begin{aligned} &\xi_0 = \nu_0, \quad \xi_p = \varphi\nu_0, \quad \alpha(\varphi^n\xi_i, \varphi^n\xi_{i+1}) \geq \sigma^{-1}, \quad n \in \mathbb{N}, \\ &i = 1, 2, \cdots, p-1. \end{aligned} \tag{4}$$

Then $\{\varphi^n\nu_0\}$ is a Cauchy sequence in (M, d).

Theorem 2.9. *([27]) Let $\varphi : \mathrm{M} \to \mathrm{M}$ be a given mapping and (M, d) be a complete metric space. Suppose φ is an $\alpha - \psi$–contraction for $\alpha : \mathrm{M} \times \mathrm{M} \to \mathrm{R}$ and $\psi \in \Psi$. Moreover (4) is satisfied. Then $\{\varphi^n\nu_0\}$ converges to some $\nu^* \in \mathrm{M}$. In addition, if there exists a subsequence $\{\varphi^{\theta(n)}\nu_0\}$ of $\{\varphi^n\nu_0\}$ such that*

$$\lim_{n\to\infty} \alpha(\varphi^{\theta(n)}\nu_0, \nu^*) = l \in (0, \infty).$$

Then ν^* is a fixed point of φ.

Theorem 2.10. *([27]) Let φ is an $\alpha - \psi$ contraction mapping, moreover*

i) $Fix(\varsigma) \neq \emptyset$;

ii) let $(\nu, \omega) \in Fix(\varsigma) \times Fix(\varsigma)$ be an arbitrary pair with $\nu \neq \omega$ and $\alpha(\nu, \omega) < 1$, then there exist $\eta \in \Sigma_\psi$ and $\{\zeta_i(\nu, \omega)\}_{i=0}^{q} \subset \mathrm{M}$ (q is positive integer) such that

$$\begin{aligned} &\zeta_0(\nu, \omega) = \nu, \quad \zeta_q(\nu, \omega) = \omega, \quad \alpha(\varphi^n\zeta_i(\nu, \omega), \varphi^n\zeta_{i+1}(\nu, \omega)) \geq \eta^{-1},\ n \in \mathbb{N} \\ &\text{and } \ i = 1, 2, \ldots, q-1. \end{aligned} \tag{5}$$

Then φ has a unique fixed point.

Lemma 2.11. *([17]) Let $\kappa\eta \neq 1$. Then for $w \in C(\chi, \mathrm{R})$, the problem*

$$\begin{aligned} &w''(\varsigma) + w(\varsigma) = 0 \,, \varsigma \in (0,1), \\ &w'(0) = 0 \,, w(1) = \kappa \int_0^{\eta} w(\varrho) d\varrho, \end{aligned} \tag{6}$$

has a unique solution

$$w(\varsigma) = \int_0^1 \mathcal{G}(\varsigma, \varrho) v(\varrho) d\varrho,$$

where $\mathcal{G}(\varsigma, \varrho) : \chi \times \chi \to \mathrm{R}$ is the Green's function defined by

$$\mathcal{G}(\varsigma, \varrho) = \frac{1}{2(1-\kappa\eta)} \begin{cases} 2(1-\varrho) - \kappa(\eta-\varrho)^2 - 2(1-\kappa\eta)(\varsigma-\varrho), & \varrho \leq min\{\eta, \varsigma\}, \\ 2(1-\varrho) - \kappa(\eta-\varrho)^2, & \varsigma \leq \varrho \leq \eta, \\ 2(1-\varrho) - 2(1-\kappa\eta)(\varsigma-\varrho), & \eta \leq \varrho \leq \varsigma, \\ 2(1-\varrho), & max\{\eta, \varsigma\} \leq \varrho. \end{cases} \tag{7}$$

We define, $g(\varrho) = \dfrac{1}{1-\kappa\eta}(1-\varrho),\ \varrho \in \chi$, then we have the following two lemmas.

Lemma 2.12. *([17]) Let $0 < \eta < 1$ and $0 < \kappa < \dfrac{1}{\eta}$, then the Green's function in (7) satisfies:*

$$0 \leq \mathcal{G}(\varsigma, \varrho) \leq g(\varrho),$$

for $\varrho, \varsigma \in \chi$.

Lemma 2.13. *([17]) Let $0 < \eta < 1$ and $0 < \kappa < \dfrac{1}{\eta}$. Then for $(\varsigma, \varrho) \in [0, \eta] \times \chi$, $\mathcal{G}(\varsigma, \varrho) \geq \theta g(\varrho)$, where*

$$0 < \theta = 1 - \eta < 1. \tag{8}$$

3. Main result

Let $E = C(\chi, \mathrm{R}^{+N})$, then E is a Banach space. We define a cone K by

$$K = \{w \in E | w = (w_1, w_2 \ldots, w_N) \geq 0, \max_i \{\min_{\varsigma \in [0,\eta]} w_i(\varsigma)\} \geq \theta \|w\|_\infty\},$$

where θ is defined in (8).
Now, we use $\alpha - \psi$-contraction mappings to the study of two cases of nonlinear boundary eigenvalue problems.

3.1. Existence and uniqueness results

In this section we use $\alpha - \psi$-contraction mappings to study the boundary eigenvalue problem of the form(1). From Lemma 2.11, it is sufficient to consider the following integral equation

$$v_i(\varsigma) = \lambda \int_0^1 \mathcal{G}(\varsigma, \varrho) a(\varrho) f_i(v(\varrho)) d\varrho, \qquad \varsigma \in \chi, i = 1, 2, \ldots, N, \tag{9}$$

where $v = (v_1, v_2 \ldots, v_N) \in E$, $a : \chi \to \mathrm{R}^+$, $f_i : \mathrm{R}^N \to \mathrm{R}^+$ and $\mathcal{G}$ is the same defined in (7). We endow E with a metric defined by

$$d(v, \omega) = max\{\|v(\varsigma) - \omega(\varsigma)\| : \varsigma \in \chi\}, \quad (v, \omega) \in E \times E.$$

Obviously, (E, d) is a complete metric space. Let us define

$$\|\nu\|_\infty = max\{\|\nu(\varsigma)\| : \varsigma \in \chi\}, \ \ \nu \in E.$$

We will endow R^N with the partial order

$$(w_1, w_2, ..., w_N) \leq_{R^N} (z_1, z_2, ..., z_N) \Leftrightarrow w_i < z_i, \ \ i = 1, 2, 3, \ldots, N.$$

Consider the following assumptions,

i) for $f : R^N \to R^{+N}$ defined by $f(a) = (f_1(a), f_2(a), \ldots, f_n(a))$ and $\varrho \in \chi$,

$$\|f(\nu(\varrho)) - f(\omega(\varrho))\| \leq L\|\nu - \omega\|,$$

where $L > 0$ is a constant;

ii) $\nu = (\nu_1, \nu_1, \ldots, \nu_N) \leq_{R^N} (\omega_1, \omega_2, \ldots, \omega_N) = \omega$ implies $f_i(\nu) \leq f_i(\omega)$;

iii) $\exists$ a constant $P > 0$ such that $a(\varrho) \leq P$;

iv) $\exists\ \nu_0 \in E$ such that

$$\nu_0(\varsigma) \leq_{R^N} \lambda \int_0^1 \mathcal{G}(\varsigma, \varrho) a(\varrho) f(\nu_0(\varrho)) d\varrho. \tag{10}$$

Theorem 3.1. *Suppose that conditions (i)-(iv) are satisfied. Let* $0 < \lambda < \frac{2(1-\kappa\eta)}{LP}$, *then the integral equation (9) has a unique continuous solution* $\nu^* \in E$.

Proof. *First we define a mapping* φ *by*

$$\varphi\nu(\varsigma) = \lambda \int_0^1 \mathcal{G}(\varsigma, \varrho) a(\varrho) f(\nu(\varrho)) d\varrho,$$

and then for the proof we consider the following steps.
Step 1. *To show that* $\varphi(\nu) = (\varphi_1(\nu), \varphi_2(\nu), \ldots, \varphi_N(\nu)) : K \to K$ *we write,*

$$\min_{\varsigma\in\chi} \varphi_i \nu(\varsigma) = \min_{\varsigma\in\chi} \lambda \int_0^1 \mathcal{G}(\varsigma, \varrho) a(\varrho) f_i(\nu(\varrho)) d\varrho$$
$$\geq \min \lambda \int_0^1 \theta g(\varrho) a(\varrho) f_i(\nu(\varrho)) d\varrho, \ i = 1, 2, \ldots, N. \tag{11}$$

On the other hand

$$\max_i\{\min_{\varsigma\in\chi} \varphi_i \nu(\varsigma)\} \leq \max_i\{\varphi_i \nu(\varsigma)\} = \|\varphi\nu\|_\infty \leq \max\{\lambda \int_0^1 g(\varrho) a(\varrho) f_i(\nu(\varrho)) d\varrho\}.$$

Thus, we obtain

$$\max_i\{\min_{\varsigma\in\chi} \varphi_i \nu(\varsigma)\} \geq \theta\|\varphi\nu\|,$$

which implies $\varphi : K \to K$.
Step 2. *To prove that* φ *is an* $\alpha - \psi$*-contraction, define the function* $\alpha : E \times E \to R$ *by the following*

$$\alpha(\nu, \omega) = \begin{cases} 1 & if \ \ \nu(\varsigma) \leq_{R^N} \omega(\varsigma), \quad \varsigma \in \chi \\ 0 & otherwise. \end{cases} \tag{12}$$

Consider the function $\psi : [0,\infty) \to [0,\infty)$ *by*

$$\psi(\varsigma) = \frac{\lambda LP}{2(1-\kappa\eta)}\varsigma, \quad \varsigma > 0.$$

It is obvious that $\psi \in \Psi$. *We show that*

$$\alpha(\nu,\omega)d(\varphi\nu,\varphi\omega) \leq \psi(d(\nu,\omega)), \quad \nu,\omega \in E.$$

Note that the above inequality holds immediately if the condition $\nu(\varsigma) \leq_{\mathbb{R}^N} \omega(\varsigma)$ *is not satisfied. Therefore we suppose that* $\nu(\varsigma) \leq_{\mathbb{R}^N} \omega(\varsigma)$, *for all* $\varsigma \in \chi$. *In this case, for all* $\varsigma \in \chi$, *we have* $\alpha(\nu,\omega)d(\varphi\nu,\varphi\omega) \leq \psi(d(\nu,\omega))$, *then*

$$\begin{aligned}
\|\varphi\nu(\varsigma) - \varphi\omega(\varsigma)\| &= \|\lambda \int_0^1 \mathcal{G}(\varsigma,\varrho)a(\varrho)f(\nu(\varrho))d\varrho - \lambda \int_0^1 \mathcal{G}(\varsigma,\varrho)a(\varrho)f(\omega(\varrho))d\varrho\| \\
&\leq \lambda \int_0^1 \mathcal{G}(\varsigma,\varrho)a(\varrho)\|f(\nu(\varrho)) - f(\omega(\varrho))\|d\varrho \\
&\leq \lambda \int_0^1 \mathcal{G}(\varsigma,\varrho)a(\varrho)L\|\nu(\varrho) - \omega(\varrho)\|d\varrho \\
&\leq \lambda L \int_0^1 \mathcal{G}(\varsigma,\varrho)a(\varrho)d(\nu,\omega)d\varrho \\
&\leq \lambda LP \int_0^1 \mathcal{G}(\varsigma,\varrho)d(\nu,\omega)d\varrho \\
&\leq \lambda LP \int_0^1 \frac{1-\varrho}{1-\kappa\eta}d\varrho\, d(\nu,\omega) \\
&= \lambda LP \frac{1}{2}.\frac{1}{1-\kappa\eta}d(\nu,\omega) = \psi(d(\nu,\omega)). \qquad (13)
\end{aligned}$$

Then φ *is an* $\alpha-\psi$*-contraction.*
Step 3. *From (iv), we have* $\alpha(\nu_0,\varphi\nu_0) = 1$, *since* f *is increasing and* $\mathcal{G}(\varsigma,\varrho), a(\varrho) > 0$, *so by induction, we obtain easily* $\alpha(\varphi^n\nu_0,\varphi^{n+1}\nu_0) = 1,\ n \in \mathbb{N}$.
Step(4). *Convergence of the Picard sequence* $\{\varphi^n\nu_0\}$. *Evidently there exists a subsequence* $\{\varphi^{\theta(n)}\nu_0\}$ *of* $\{\varphi^n\nu_0\}$ *such that*

$$\lim_{n\to\infty} \alpha(\varphi^{\theta(n)}\nu_0,\nu^*) = l \in (0,\infty).$$

Using Theorems 2.8, 2.9 and from step (3), we deduce that ν^* *is a fixed point of* φ, *in other words* $\nu^* \in E$ *is a solution to the integral equation (9), which implies it is a solution of the desired problem.*
Step 5. *In order to prove uniqueness of the solution, we choose an arbitrary pair* $(z,q) \in E \times E$, *given by*

$$z(\varsigma) = (z_1(\varsigma), z_2(\varsigma)\cdots, z_N(\varsigma)), \quad q(\varsigma) = (q_1(\varsigma), q_2(\varsigma)\cdots, q_N(\varsigma)),\ \varsigma \in \chi.$$

Now for any $i = 1,2,...,N$, *we define*

$$m_i(\varsigma) = max\{z_i(\varsigma), q_i(\varsigma)\}, \quad \varsigma \in \chi.$$

Obviously, one has $\alpha(z,m) = \alpha(q,m) = 1$. *Therefore, uniqueness follows immediately from Theorem 2.10.* □

Example 3.2. *Consider the following eigenvalue problem,*

$$w''(\varsigma) + \lambda a(\varsigma)\frac{w^2+1}{w^2+2} = 0, \quad 0 < \varsigma < 1, \qquad (14)$$

$$w'(0) = 0\ , w(1) = 2\int_0^{\frac{1}{3}} w(\varrho)d\varrho,$$

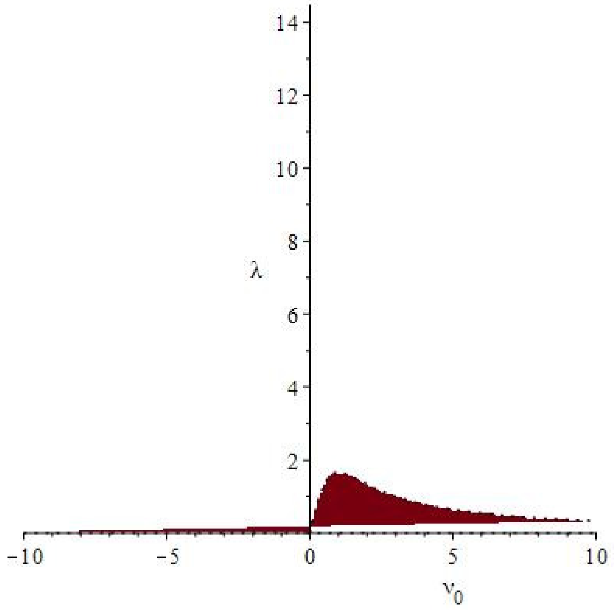

Figure 1: Domain of λ for existence and uniqueness of solution of problem 14.

where $\kappa = 2, \eta = \dfrac{1}{3}, a(\varsigma) = \dfrac{1}{5}, f(w) = \dfrac{w^2+1}{w^2+2}$. *Since* $|\dfrac{\partial f}{\partial w}| < \frac{23}{100}$, *then* f *is a Lipschitz function with Lipschitz constant* $L = \frac{23}{100}$. *According to the notations of the Theorem 3.1, one has* $P = \frac{1}{5}$ *and therefore for* $0 < \lambda < 14.493$, *the problem (14) has a positive solution. On the other hand for uniqueness of solution, by (10) we should choose* $\nu_0(\varsigma)$ *such that:*

$$\nu_0(\varsigma) \leq \frac{\lambda}{5}\int_0^1 \mathcal{G}(\varsigma,\varrho)\frac{\nu_0^2(\varrho)+1}{\nu_0^2(\varrho)+2}d\varrho. \tag{15}$$

Without loss of generality, we suppose that, ν_0 *is constant and therefore from (10), for uniqueness of the solution we should solve the following system of inequalities*

$$\nu_0 \leq \frac{3\lambda}{10}\,\frac{\nu_0^2+1}{\nu_0^2+2}, \qquad 0 < \lambda < 14.493. \tag{16}$$

From Fig.1. it is obvious that approximately for $0 < \lambda < 1.88$, *we have the uniqueness of solution.*

3.2. Existence and uniqueness of solution of fractional Volterra type eigenvalue problems

In this section, we consider the following fractional Volterra type problem on infinite interval

$$\begin{cases} D^{\kappa}w(\varsigma) + \lambda f(\varsigma, w(\varsigma), Sw(\varsigma)) = 0, & 3 < \kappa \leq 4, \\ w(0) = w'(0) = w''(0) = 0, \quad D^{\kappa-1}w(\infty) = \zeta I^{\iota}w(\mu), & \iota > 0, \end{cases} \tag{17}$$

where $\varsigma \in J = [0,\infty), f \in C[J \times \mathrm{R} \times \mathrm{R}, \mathrm{R}^+]$, $\zeta \in \mathrm{R}$, $\mu \in J$, D^{κ} denotes the fractional derivative of order κ in Riemann-Liouville definition, I^{ι} is the fractional integral of order ι, with the same definition and $(Sw)(\varsigma) = \int_0^{\varsigma} k(\varsigma,\varrho)w(\varrho)d\varrho$ such that $k(\varsigma,\varrho) \in C[D, \mathrm{R}^+], D = \{(\varsigma,\varrho) \in \mathrm{R}^2 | 0 \leq \varrho \leq \varsigma\}$. Define the space

$$E = \{w \in C(J,\mathrm{R}) : sup_{\varsigma\in J}\frac{|w(\varsigma)|}{1+\varsigma^{\kappa-1}} < +\infty\},$$

equipped with the norm

$$\|w\|_E = sup_{\varsigma\in J}\frac{|w(\varsigma)|}{1+\varsigma^{\kappa-1}}.$$

It is obvious that E is a Banach space.
First, Let us consider the following lemma.

Lemma 3.3. *([28]) Let $h \in C(J,J)$ such that $\int_0^{+\infty} h(\varrho)d\varrho < \infty$. If $\Gamma(\kappa+\iota) \neq \zeta\mu^{\kappa+\iota-1}$, then the following boundary value problem of fractional order with integral condition*

$$\begin{cases} D^{\kappa}w(\varsigma) + h(\varsigma) = 0, \\ w(0) = w'(0) = w''(0) = 0, \ \ D^{\kappa-1}w(\infty) = \zeta I^{\iota}w(\mu), \ \ \iota > 0, \end{cases} \tag{18}$$

has a unique solution

$$w(\varsigma) = \int_0^{+\infty} \mathcal{G}(\varsigma,\varrho)h(\varrho)d\varrho,$$

where

$$\mathcal{G}(\varsigma,\varrho) = \frac{1}{\Delta}\begin{cases} \left[\Gamma(\kappa+\iota) - \zeta(\mu-\varrho)^{\kappa+\iota-1}\right]\varsigma^{\kappa-1} & \\ -\left[\Gamma(\kappa+\iota) - \zeta\mu^{\kappa+\iota-1}\right](\varsigma-\varrho)^{\kappa-1} & \varrho \le \varsigma, \varrho \le \mu \\ \left[\Gamma(\kappa+\iota) - \zeta(\mu-\varrho)^{\kappa+\iota-1}\right]\varsigma^{\kappa-1} & 0 \le \varsigma \le \varrho \le \mu \\ \Gamma(\kappa+\iota)\left[\varsigma^{\kappa-1} - (\varsigma-\varrho)^{\kappa-1}\right] + \zeta\mu^{\kappa+\iota-1}(\varsigma-\varrho)^{\kappa-1} & 0 \le \mu \le \varrho \le \varsigma \\ \Gamma(\kappa+\iota)\varsigma^{\kappa-1} & \varrho \ge \varsigma, \varrho \ge \mu \end{cases}.$$

and

$$\Delta = \Gamma(\kappa)\left[\Gamma(\kappa+\iota) - \zeta\mu^{\kappa+\iota-1}\right].$$

Remark 3.4. *([28]) For $(\varrho,\varsigma) \in J \times J$, suppose that $\Gamma(\kappa+\iota) > \zeta\mu^{\kappa+\iota-1}$, $(\zeta \ge 0,)$ holds, then one has*

$$0 \le \frac{\mathcal{G}(\varsigma,\varrho)}{1+\varsigma^{\kappa-1}} \le \frac{\Gamma(\kappa+\iota)}{\Gamma(\kappa)[\Gamma(\kappa+\iota) - \zeta\mu^{\kappa+\iota-1}]} := F. \tag{19}$$

Let E be a Banach space. Consider the following cone;

$$K = \{w \in E | w \ge 0, \inf_{\varsigma\in J} w(\varsigma) \ge \frac{m}{F}\|w\|\|\}, \tag{20}$$

where $m = \inf_{\varsigma\in J} G(\varsigma,\varrho)$.

In this section we study the following integral equation of Volterra type

$$\nu(\varsigma) = \lambda\int_0^{\infty} \mathcal{G}(\varsigma,\varrho)f(\varrho,\nu(\varrho),S\nu(\varrho))d\varrho, \tag{21}$$

with the following assumption

i) $|f(\varrho,\nu(\varrho),S\nu(\varrho)) - f(\varrho,w(\varrho),Sw(\varrho))| \le \frac{L}{(\varrho^2+1)}d(\nu,\omega)$;

ii) for all $\varsigma \in J$, $\nu,\omega \in \mathrm{R}^3$, $\nu \underset{\mathrm{R}^3}{\le} \omega \Rightarrow f(\nu) \le f(\omega)$;

iii) there exist $\nu_0 \in \mathrm{M}$ such that

$$\nu_0(\varsigma)\le\lambda\int_0^{\infty} \mathcal{G}(\varsigma,\varrho)f(\varrho,\nu_0(\varrho),S\nu_0(\varrho))d\varrho. \tag{22}$$

Theorem 3.5. *Suppose that conditions $(i)-(iii)$ are satisfied. Let $0<\lambda<\frac{1}{FL\pi}$, then the integral equation of Volterra type (21) has a unique continuous solution $\nu^*\in C(J,\mathrm{R})$.*

Proof. First, let us introduce the associated mapping φ, by the following

$$\varphi\nu(\varsigma)=\lambda\int_0^\infty \mathcal{G}(\varsigma,\varrho)f(\varrho,\nu(\varrho),S\nu(\varrho))d\varrho,$$

then for the proof we consider the following steps.

Step 1. First we show $\varphi:K\to K$, we have

$$\inf_{\varsigma\in\chi}\varphi\nu(\varsigma)=\inf\lambda\int_0^\infty \mathcal{G}(\varsigma,\varrho)f(\varrho,\nu(\varrho),S\nu(\varrho))d\varrho,$$

on the other hand

$$\begin{aligned}\frac{m}{F}\|\varphi\nu\|&=\sup_{\varsigma\in\chi}\lambda\int_0^\infty \frac{m}{F}\frac{\mathcal{G}(\varsigma,\varrho)}{1+\varsigma^{\kappa-1}}f(\varrho,\nu(\varrho),S\nu(\varrho))d\varrho\\&\le\sup_{\varsigma\in\chi}\lambda\int_0^\infty m\frac{1+\varsigma^{\kappa-1}}{\mathcal{G}(\varsigma,\varrho)}\frac{\mathcal{G}(\varsigma,\varrho)}{1+\varsigma^{\kappa-1}}f(\varrho,\nu(\varrho),S\nu(\varrho))d\varrho\\&=\lambda\int_0^\infty mf(\varrho,\nu(\varrho),S\nu(\varrho))d\varrho\\&\le\inf_{\varsigma\in\chi}\lambda\int_0^\infty \mathcal{G}(\varsigma,\varrho)f(\varrho,\nu(\varrho),S\nu(\varrho))d\varrho\\&=\inf_{\varsigma\in\chi}\varphi\nu(\varsigma).\end{aligned}$$

Which implies $\varphi:K\to K$.

Step 2. To prove that φ is an $\alpha-\psi$ contraction, let us define the function $\alpha:\mathrm{M}\times\mathrm{M}\to\mathrm{R}$ by the following

$$\alpha(\nu,\omega)=\begin{cases}1 & \text{if } \nu(\varsigma)\le\omega(\varsigma),\\ 0 & \textit{otherwise}.\end{cases}$$

Also we consider the function $\psi:J\to J$ define by

$$\psi(\varsigma)=\lambda FL\pi\varsigma,\quad \varsigma>0.$$

Now we show that $\psi\in\Psi$. Suppose, $\nu(\varsigma)\le\omega(\varsigma)$, for all $\varsigma\in J$. In this case, we have

$$\alpha(\nu,\omega)d(\varphi\nu,\varphi\omega)\le\psi(d(\nu,\omega)),\quad \nu,\omega\in E.$$

Indeed we have

$$\begin{aligned}|\varphi\nu(\varsigma)-\varphi\omega(\varsigma)|&=|\lambda\int_0^\infty \mathcal{G}(\varsigma,\varrho)f(\varrho,\nu(\varrho),S\nu(\varrho))d\varrho\\&\quad-\lambda\int_0^\infty \mathcal{G}(\varsigma,\varrho)f(\varrho,w(\varrho),Sw(\varrho))d\varrho|\\&\le\lambda\int_0^\infty \mathcal{G}(\varsigma,\varrho)|f(\varrho,\nu(\varrho),S(\nu(\varrho))-f(\varrho,w(\varrho),Sw(\varrho))|d\varrho\\&\le\lambda\int_0^\infty F(1+\varsigma^{\kappa-1})|f(\varrho,\nu(\varrho),S(\nu(\varrho))-f(\varrho,w(\varrho),Sw(\varrho))|d\varrho\\&\le\lambda F(1+\varsigma^{\kappa-1})\int_0^\infty \frac{L}{1+\varrho^2}d(\nu,\omega)d\varrho\\&=\lambda FL\pi(1+\varsigma^{\kappa-1})d(\nu,\omega).\end{aligned}$$

Then

$$\frac{|\varphi\nu(\varsigma)-\varphi\omega(\varsigma)|}{1+\varsigma^{\kappa-1}} \leq FL\pi d(\nu,\omega)=\psi(d(\nu,\omega)).$$

Therefore $\alpha(\nu,\omega)d(\varphi\nu,\varphi\omega) \leq \psi(d(\nu,\omega))$.
Step 3. From (iii), we have $\alpha(\nu_0,\varphi\nu_0)=1$, since f is increasing and $\mathcal{G}(\varsigma,\varrho), a(\varrho)>0$, so by induction, we obtain easily $\alpha(\varphi^n\nu_0,\varphi^{n+1}\nu_0)=1,\ n\in\mathbb{N}$.
Step 4. Evidently there exists a subsequence $\{\varphi^{\theta(n)}\nu_0\}$ of $\{\varphi^n\nu_0\}$ such that

$$\lim_{n\to\infty}\alpha(\varphi^{\theta(n)}\nu_0,\nu^*)=l\in(0,\infty).$$

Using Theorems 2.8, 2.9 and from step (3), we deduce that ν^* is a fixed point of φ, in other words $\nu^*\in E$ is a solution to the integral equation (18), which implies it is a solution of the desired problem.
Step 5. In order to prove uniqueness of the solution, we choose an arbitrary pair $(z,q)\in E\times E$, we define

$$m(\varsigma)=max\{z(\varsigma),q(\varsigma)\},\quad \varsigma\in J.$$

Obviously, one has $\alpha(z,m)=\alpha(q,m)=1$. Therefore, uniqueness follows immediately from Theorem 2.10. □

4. Conclusion

Literature review shows that the study of boundary eigenvalue problems by fixed point theorems is limited to the use of few contractions on cones and often does not include uniqueness results. In this paper we have used some new $\alpha-\psi$-contraction mappings to obtain existence and uniqueness results for eigenvalue intervals of two classes of nonlinear boundary eigenvalue problems in integer and fractional orders.

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