Filomat 36:4 (2022), 1125–1141
https://doi.org/10.2298/FIL2204125W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The $\mathfrak{m}$-WG Inverse in Minkowski Space

Hui Wu[a], Hongxing Wang[b], Hongwei Jin[b]

[a]*School of Mathematics and Physics, Hechi University, Yizhou 546300, China*
[b]*School of Mathematics and Physics, Guangxi University for Nationalities, Nanning 530006, China*

Abstract. In this paper, we introduce the $\mathfrak{m}$-WG inverse in Minkowski space. Firstly, we show the existence and the uniqueness of the $\mathfrak{m}$-WG inverse. Secondly, we give representations of the $\mathfrak{m}$-WG inverse. Thirdly, we characterize the $\mathfrak{m}$-WG inverse by applying a bordered matrix. In addition, we extend the generalized Cayley-Hamilton theorem to the $\mathfrak{m}$-WG inverse matrix. Finally, we apply the $\mathfrak{m}$-WG inverse to solve linear equations in Minkowski space.

1. Introduction

The set of $n \times n$ complex matrices will be denoted by $\mathbb{C}_{n,n}$. We use symbols A^*, $\mathcal{R}(A)$, and $\mathrm{rk}(A)$ for the conjugate transpose, range space (or column space), and rank of $A \in \mathbb{C}_{n,n}$, respectively. In addition, $k = \mathrm{Ind}(A)$ denotes the index of A, which is defined as the minimal positive integer k such that $\mathrm{rk}\left(A^{k+1}\right) = \mathrm{rk}\left(A^k\right)$. Denote

$$\mathbb{C}_n^{\mathrm{CM}} = \left\{A \mid A \in \mathbb{C}_{n,n},\ \mathrm{rk}(A^2) = \mathrm{rk}(A)\right\}.$$

The classical Minkowski space is a fictitious four-dimensions space-time, which is named by the german mathematician Hermann Minkowski. Formally, it is a four dimensional real vector space equipped with non-degenerate, symmetric bilinear form with the signature $(+, -, -, -)$. Then it is often denoted by $\mathbb{R}^{1,3}$, in which the metric matrix is $\mathcal{G} = \mathrm{Diag}(1, -I_3)$.

In order to solve Xing's [1] study on polarization of light, Renardy needed to apply singular value decomposition of matrix in Minkowski space. In 1996, Renardy [2] introduce singular value decomposition in the Minkowski space $\mathcal{M}$, and proposed the Minkowski adjoint of a matrix $A \in \mathbb{C}_{n,n}$, which is defined as $A^\sim = GA^*G$. The Minkowski metric matrix can be written as

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix}, \tag{1.1}$$

2020 *Mathematics Subject Classification*. 15A09, 15A24, 15A29
Keywords. Minkowski space, $\mathfrak{m}$-WG inverse, bordered matrix, generalized Cayley-Hamilton theorem
Received: 02 April 2021; Revised: 26 October 2021; Accepted: 31 October 2021
Communicated by Dijana Mosić
Research supported by the Basic Ability Improvement Project for Middle-Aged and Young Teachers of Universities in Guangxi [No. 2022KY0610](the first author), and the Special Fund for Science and Technological Bases and Talents of Guangxi [No. GUIKE AD19245148] and the Special Fund for Bagui Scholars of Guangxi [No. 2016A17](the second author), and the National Natural Science Foundation of China [No. 12061015], Guangxi Natural Science Foundation [No. 2018GXNSFDA281023], and the Special Fund for Science and Technological Bases and Talents of Guangxi [No. GUIKE AD21220024] (the third author).
Email addresses: huiwumath168@163.com (Hui Wu), winghongxing0902@163.com (Hongxing Wang), hw-jin@hotmail.com (Hongwei Jin)

it is easily seen that $G = G^*$ and $G^2 = I_n$. Let $A, B \in \mathbb{C}_{n,n}$, it is obvious that $(AB)^\sim = B^\sim A^\sim$ and $(A^\sim)^\sim = A$.

In 2000, Meenakshi [3] studied the generalized inverse in the Minkowski space $\mathcal{M}$, and got its existence conditions.

The *Minkowski inverse* of a matrix $A \in \mathbb{C}_{n,n}$ in $\mathcal{M}$ is defined as the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying the followings [3]:

$$(1)\ AXA = A, (2)\ XAX = X, (3^{\mathfrak{m}})\ \ (AX)^\sim = AX, (4^{\mathfrak{m}})\ \ (XA)^\sim = XA.$$

The Minkowski inverse of A is denoted by $A^{\mathfrak{m}}$. It is worthy to notice that the Minkowski inverse $A^{\mathfrak{m}}$ exists if and only if

$$\mathrm{rk}\,(A^\sim A) = \mathrm{rk}\,(AA^\sim) = \mathrm{rk}(A),\ [3].$$

Furthermore, Kiliçman and Al-Zhour [6, 7] studied generalized the weighed Minkowski inverse in $\mathcal{M}$. In [19], Wang, Li and Liu defined the $\mathfrak{m}$-core inverse in $\mathcal{M}$. Let $A \in \mathbb{C}_n^{\mathrm{CM}}$, the $\mathfrak{m}$-core inverse of A is defined as the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying the followings

$$(1)\ \ AXA = A,\ \ \left(2^l\right)\ AX^2 = X,\ \ \ (3^{\mathfrak{m}})\ (AX)^\sim = AX,$$

and is denoted by $A^{\textcircled{m}}$. By using the SVD and the Hartwig-Spindelböck decomposition, Wang et al [19] concluded that A is $\mathfrak{m}$-core invertible if and only if

$$\mathrm{rk}(A^\sim A) = \mathrm{rk}(A).$$

Furthermore, let $A \in \mathbb{C}_{n,n}$ with $\mathrm{rk}\,(A) = r$ and the Hartwig-Spindelböck's decomposition [8] of A be as

$$A = V\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} V^*, \tag{1.2}$$

where $V \in \mathbb{C}_{n,n}$ is unitary, $\Sigma = Diag(\sigma_1, \sigma_2, ..., \sigma_r)$ is a diagonal matrix whose diagonal elements are singular values of A, $\sigma_1 \geq \cdots \geq \sigma_r > 0$, and $K \in \mathbb{C}_{r,r}$, $L \in \mathbb{C}_{r,n-r}$ satisfy $KK^* + LL^* = I_r$.

Then $\widehat{G_1}$ is invertible if and only if $\mathrm{rk}\,(A^\sim A) = r$, where $\widehat{G_1} \in \mathbb{C}_{r,r}$ and

$$V^*GV = \begin{bmatrix} \widehat{G_1} & \widehat{G_2} \\ \widehat{G_3} & \widehat{G_4} \end{bmatrix}. \tag{1.3}$$

And $A^{\textcircled{m}}$ can be written as the form

$$A^{\textcircled{m}} = V\begin{bmatrix} (\Sigma K)^{-1}\widehat{G}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*G. \tag{1.4}$$

Later, Wang, Wu and Liu [20] promoted related research and introduced the $\mathfrak{m}$-core-EP inverse. Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$. The $\mathfrak{m}$-core-EP inverse of A in $\mathcal{M}$ is defined as the unique solution satisfying the following equations

$$(1)XAX = X, (2^k)XA^{k+1} = A^k, (3^{\mathfrak{m}})\,(AX)^\sim = AX, (4^r)\,\mathcal{R}(X) \subseteq \mathcal{R}\left(A^k\right),$$

and denoted by $A^{\textcircled{E}}$. In addition, a matrix A is $\mathfrak{m}$-core-EP invertible if and only if

$$\mathrm{rk}((A^k)^\sim A^k) = \mathrm{rk}(A^k). \tag{1.5}$$

It is easy to prove that $A^{\textcircled{E}}$ and $A^{\textcircled{m}}$ are equal when the index of A is less than or equal to 1. Since the SVD and the Hartwig-Spindelböck decomposition are not suitable for studying the $\mathfrak{m}$-core-EP inverse, then Wang applied the core-EP decomposition for studying the $\mathfrak{m}$-core-EP inverse. Furtehermore, by applying the core-EP decomposition, Wang et al [20] got several sufficient and necessary conditions for the existence of the $\mathfrak{m}$-core-EP inverse and considered some related issues.

The other a couple of corresponding generalized inverse is core-EP inverse. In [9], K. Manjunatha Prasad and K.S. Mohana gave the core-EP inverse. The core-EP inverse of A is defined as

$$\left(1^k\right) XA^{k+1} = A^k, (2) XAX = X, (3)(AX)^* = AX, (4^r)\,\mathcal{R}(X) \subseteq \mathcal{R}(A^k),$$

and is denoted by $A^{\textcircled{\dagger}}$, where $\mathrm{Ind}(A) = k$.

In 2018, Wang and Chen [12] defined weak group (WG) inverse for square matrices of an arbitrary index. The WG inverse of $A \in \mathbb{C}_{n,n}$ is the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying

$$\left(2^l\right)\ AX^2 = X,\ (3^c)\ AX = A^{\textcircled{\dagger}}A. \tag{1.6}$$

and it denoted by $A^{\textcircled{w}}$. The WG inverse is a new kind of generalized group inverse, which is different from the group inverse, and it is true for square matrices of an arbitrary index.

In recent years, many scholars have drawn their interest in the WG inverse. In [13], Wang and Liu proposed the concept of the WG matrix on the basis of the WG inverse. In [14], Ferreyra, Orquera and Thome generalize the WG inverse to rectangular matrices and gave properties of the weighted WG inverse. In [16], Zhou et al proposed the WG inverse in proper *-rings and gave a new equivalent characterization of the WG inverse. In [15], Xu et al gave concept and properties of generalized WG inverse. In [17], Mosić and Zhang studied the weighted WG inverse in Hilbert space. In [18], Mosić and Stanimirović gave new representations and characterizations for the WG inverse, applied SMS algorithm to compute the WG inverse and applied the WG inverse to solve linear equations.

A commonly used tool is the core-EP decomposition [10]. Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{rk}\left(A^k\right) = r$ and $\mathrm{Ind}(A) = k$. Then $A = A_1 + A_2$ and

$$A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*, \tag{1.7}$$

where $A_1 \in \mathbb{C}_n^{\mathrm{CM}}$, $A_2^k = 0$, and $A_1^* A_2 = A_2 A_1 = 0$. Furthermore, there exists an $n \times n$ unitary matrix U such that

$$A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^* \text{ and } A_2 = U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*, \tag{1.8}$$

where $S \in \mathbb{C}_{r,n-r}$, $T \in \mathbb{C}_{r,r}$ is invertible, $N \in \mathbb{C}_{n-r,n-r}$ is nilpotent, and $N^k = 0$.

When $A \in \mathbb{C}_n^{\mathrm{CM}}$, it is obvious that $N = 0$ and

$$A = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*. \tag{1.9}$$

In [20], we see that $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$ is $\mathfrak{m}$-core-EP invertible if and only if $G_1 \in \mathbb{C}_{r,r}$ is invertible, where

$$U^*GU = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}, \tag{1.10}$$

and U is as in (1.7). Furthermore,

$$A^{\textcircled{\mathfrak{m}}} = U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G. \tag{1.11}$$

The aim of this paper is to consider the WG inverse in Minkowski space, we also investigate the $\mathfrak{m}$-WG inverse for square matrices of an arbitrary index. In addition, we give the representations, properties, and applications of the $\mathfrak{m}$-WG inverse.

2. The $\mathfrak{m}$-WG inverse in Minkowski space

Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$ be of the form (1.7), and $T \in \mathbb{C}_{r,r}$ be invertible, then

$$A^k = U\begin{bmatrix} T^k & \widehat{T} \\ 0 & 0 \end{bmatrix} U^*, \tag{2.1}$$

where $\widehat{T} = \sum_{i=1}^{k} T^{i-1} S N^{k-i}$.

Definition 2.1. *Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$, $\text{rk}\left(A^k\right) = \text{rk}\left(\left(A^k\right)^{\sim} A^k\right)$. The $\mathfrak{m}$-WG inverse of A in $\mathcal{M}$ is defined as solution of*

$$\left(2^l\right)\ AX^2 = X,\ (3^{\mathfrak{c}})\ AX = A^{\circledS}A, \tag{2.2}$$

and is denoted by A^{W}.

Theorem 2.2. *Let A be as in Definition 2.1. The $\mathfrak{m}$-WG inverse of matrix A is unique.*

Proof. Suppose that X and Y satisfy (2.2), then we obtain

$$X = AX^2 = A^{\circledS}AX = A^{\circledS}A^{\circledS}A = A^{\circledS}AY = AY^2 = Y,$$

therefore, the $\mathfrak{m}$-WG inverse of matrix A is unique. □

Theorem 2.3. *The $\mathfrak{m}$-WG inverse of matrix A can be expressed as*

$$A^{W} = U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix} U^*, \tag{2.3}$$

where A is as in Definition 2.1, G, G_1 and G_2 are as in (1.10).

Proof. Let

$$X = U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix} U^*. \tag{2.4}$$

By applying (1.7) and (2.4), we have

$$\begin{aligned} AX^2 &= U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix} U^*U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix} U^* \\ &= U\begin{bmatrix} I_r & T^{-1}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix} U^* \\ &= U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix} U^* = X. \end{aligned}$$

By applying (1.7), (1.11) and (2.4), we have

$$\begin{aligned} AX &= U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix} U^* \\ &= U\begin{bmatrix} I_r & T^{-1}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix} U^*, \end{aligned}$$

$$
\begin{aligned}
A^{\textcircled{m}}A &= U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*GU \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* \\
&= U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* \\
&= U\begin{bmatrix} I_r & T^{-1}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix} U^*.
\end{aligned}
$$

Therefore, we obtain $AX = A^{\textcircled{m}}A$. From the above, we know that X satisfies the above equations. □

Remark 2.4. *Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$. Then A is $\mathfrak{m}$-WG invertible if and only if*

$$
\mathrm{rk}((A^k)^{\sim}A^k) = \mathrm{rk}(A^k). \tag{2.5}
$$

Proof. By applying (2.2) and (2.3), it is easy to that if the $\mathfrak{m}$-core-EP inverse exists, the $\mathfrak{m}$-WG inverse exists. By applying (1.7), we obtain (2.5). □

As is known to all that matrix equation and matrix decomposition are important methods to describe generalized inverses. Next, we apply the matrix equation and the matrix decomposition to give the equivalent characterization of the $\mathfrak{m}$-WG inverse.

Theorem 2.5. *Let A be as in Definition 2.1. Then the following statements are equivalent:*

(i). $AX^2 = X$, $AX = A^{\textcircled{m}}A$;

(ii). $(A^k)^{\sim}A^2X = (A^k)^{\sim}A$, $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$;

(iii). $XA^{k+1} = A^k$, $AX^2 = X$, $(A^k)^{\sim}A^2X = (A^k)^{\sim}A$.

Proof. $(i) \Leftrightarrow (ii)$: Let $A \in \mathbb{C}_{n,n}$ be of the form (1.7). Suppose that X is satisfying statement (ii), and denoted by

$$
X = U\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} U^*.
$$

Since $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, we obtain $X = A^kY$. Let

$$
Y = U\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} U^*,
$$

that is,

$$
U\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} U^* = U\begin{bmatrix} T^k & \widehat{T} \\ 0 & 0 \end{bmatrix} U^*U\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} U^* = U\begin{bmatrix} T^kY_{11} + \widehat{T}Y_{21} & T^kY_{12} + \widehat{T}Y_{22} \\ 0 & 0 \end{bmatrix} U^*.
$$

Therefore, we obtain $X_{21} = 0$ and $X_{22} = 0$. By applying $(A^k)^{\sim}A^2X = (A^k)^{\sim}A$, we have

$$
\begin{aligned}
GU\begin{bmatrix} (T^k)^* & 0 \\ \widehat{T}^* & 0 \end{bmatrix} U^*GU\begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*U\begin{bmatrix} X_{11} & X_{12} \\ 0 & 0 \end{bmatrix} U^* \\
= GU\begin{bmatrix} (T^k)^* & 0 \\ \widehat{T}^* & 0 \end{bmatrix} U^*GU\begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*,
\end{aligned}
$$

$$
\begin{bmatrix} (T^k)^* & 0 \\ \widehat{T}^* & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}^2\begin{bmatrix} X_{11} & X_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (T^k)^* & 0 \\ \widehat{T}^* & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix},
$$

$$
\begin{bmatrix} (T^k)^*G_1T^2X_{11} & (T^k)^*G_1T^2X_{12} \\ \widehat{T}^*G_1T^2X_{11} & \widehat{T}^*G_1T^2X_{12} \end{bmatrix} = \begin{bmatrix} (T^k)^*G_1T & (T^k)^*G_1S + (T^k)^*G_2N \\ \widehat{T}^*G_1T & \widehat{T}^*G_1S + \widehat{T}^*G_2N \end{bmatrix}.
$$

Therefore, $X_{11} = T^{-1}$ and $X_{12} = T^{-2}(S + G_1^{-1}G_2N)$. Then, we obtain

$$X = U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix}U^*. \tag{2.6}$$

By applying (2.3) and (2.6), it is obvious that statement (*i*) and statement (*ii*) are the same solution. Since the solution of the statement (*i*) is unique, then we obtain X is the unique solution satisfying the statement (*ii*). Therefore, statement (*i*) and statement (*ii*) are equivalent.

(*i*) ⇔ (*iii*): The proof is similar to the above. □

From the above, we know that statements (*i*), (*ii*), and (*iii*) are equivalent. Therefore, statements (*ii*) and (*iii*) can also define the $\mathfrak{m}$-WG inverse of A.

3. Characterizations and representations of the $\mathfrak{m}$-WG inverse

In this section, we mainly use matrix decomposition and matrix equation to give characterization of the $\mathfrak{m}$-WG inverse.

Lemma 3.1. *Let $A \in \mathbb{C}_n^{\mathrm{CM}}$ be as given in (1.9), then*

$$A^{\#} = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^*. \tag{3.1}$$

Corollary 3.2. *The following statements are true.*

(i). *If $A \in \mathbb{C}_n^{\mathrm{CM}}$, then $A^W = A^{\#}$.*

(ii). *If $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$, $\mathrm{rk}\left(A^k\right) = \mathrm{rk}\left(\left(A^k\right)^{\sim} A^k\right)$, then $A^W \in A\{2\}$.*

Proof. (i)When $A \in \mathbb{C}_n^{\mathrm{CM}}$, we have $N = 0$. By applying (2.3) and (3.1), we have $A^W = A^{\#}$.

(ii) Let $A \in \mathbb{C}_{n,n}$ be of the form (1.7), and applying (2.3), we have

$$\begin{aligned} A^W A A^W &= U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} I_r & T^{-1}S + T^{-2}(S + G_1^{-1}G_2N)N \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix}U^* = A^W. \end{aligned}$$

Then, we obtain $A^W \in A\{2\}$. □

Next, we take an example to illustrate that the $\mathfrak{m}$-WG inverse is different from the core-EP inverse, the $\mathfrak{m}$-core-EP inverse and the WG inverse.

Example 3.3. *Let* $A = \begin{bmatrix} 0 & \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{2}{3} & 0 \end{bmatrix}$ *with* $\mathrm{Ind}(A) = 2$. *There exists a unitary matrix*

$$U = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix},$$

such that

$$A = U\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} U^*.$$

By calculating, we obtain $A^{\textcircled{\dagger}}$ *and* $A^{\textcircled{E}}$ *are*

$$A^{\textcircled{\dagger}} = \begin{bmatrix} \frac{4}{9} & \frac{2}{9} & -\frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ -\frac{4}{9} & -\frac{2}{9} & \frac{4}{9} \end{bmatrix} \text{ and } A^{\textcircled{E}} = \begin{bmatrix} -4 & 2 & -4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}.$$

Besides, $A^{\textcircled{w}}$ *and* A^W *are*

$$A^{\textcircled{w}} = \begin{bmatrix} \frac{2}{9} & \frac{10}{9} & -\frac{2}{9} \\ \frac{1}{9} & \frac{5}{9} & -\frac{1}{9} \\ -\frac{2}{9} & -\frac{10}{9} & \frac{2}{9} \end{bmatrix} \text{ and } A^W = \begin{bmatrix} 2 & -\frac{2}{3} & \frac{2}{3} \\ 1 & -\frac{1}{3} & \frac{1}{3} \\ -2 & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$

In [12], we see that $A^{\textcircled{w}} = A_1^{\#}$, where A_1 is of the form (1.8). In the following, by applying the $\mathfrak{m}$-core-EP decomposition of A, we can obtain similar results.

Lemma 3.4 ([20]). *Let A be as in Definition 2.1. Then the $\mathfrak{m}$-core-EP Decomposition of A can be expressed as $A = \widehat{A_1} + \widehat{A_2}$, where*

(i). $\widehat{A_1} \in \mathbb{C}_n^{\text{CM}}$ *with* $\operatorname{rk}\left(\widehat{A_1}\right) = \operatorname{rk}\left(\widehat{A_1}^{\sim}\widehat{A_1}\right)$;

(ii). $\widehat{A_2}^{k} = 0$;

(iii). $\widehat{A_1}^{\sim}\widehat{A_2} = \widehat{A_2}\widehat{A_1} = 0$.

Furthermore, $\widehat{A_1}$ *and* $\widehat{A_2}$ *have the forms*

$$\widehat{A_1} = U\begin{bmatrix} T & S + G_1^{-1}G_2N \\ 0 & 0 \end{bmatrix} U^* \text{ and } \widehat{A_2} = U\begin{bmatrix} 0 & -G_1^{-1}G_2N \\ 0 & N \end{bmatrix} U^*. \tag{3.2}$$

Theorem 3.5. *Let $A \in \mathbb{C}_{n,n}$ with $\operatorname{Ind}(A) = k$, $\operatorname{rk}\left(A^k\right) = \operatorname{rk}\left(\left(A^k\right)^{\sim} A^k\right)$. Then*

$$A^W = \widehat{A_1}^{\#}, \tag{3.3}$$

where $A \in \mathbb{C}_{n,n}$ be of the form (1.8).

Proof. Let $\widehat{A_1}$ be as in (3.2), by applying Theorem 2.3 and Lemma 3.1, we derive (3.3). □

Theorem 3.6. *Let A be as in Definition 2.1. Then*

$$A^W = (AA^{\textcircled{E}}A)^{\#} = (A^{\textcircled{E}})^2 A = \left(A^2\right)^{\textcircled{E}} A. \tag{3.4}$$

Proof. Let $A \in \mathbb{C}_{n,n}$ be of the form (1.7), and $A^{\textcircled{E}}$ be as in (1.11). Then

$$AA^{\textcircled{E}}A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* = U\begin{bmatrix} T & S + G_1^{-1}G_2N \\ 0 & 0 \end{bmatrix} U^*,$$

It follows from Lemma 3.1 that,

$$(AA^{\textcircled{E}}A)^{\#} = \left(U\begin{bmatrix} T & S + G_1^{-1}G_2N \\ 0 & 0 \end{bmatrix} U^*\right)^{\#} = U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix} U^*.$$

And

$$(A^{\textcircled{E}})^2 = U\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}\begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G = U\begin{bmatrix} T^{-2}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G,$$

$$\left(A^2\right)^{\textcircled{E}} = \left(U\begin{bmatrix} T^2 & TS+SN \\ 0 & N^2 \end{bmatrix}U^*\right)^{\textcircled{E}} = U\begin{bmatrix} T^{-2}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G,$$

therefore,

$$\begin{aligned} (A^{\textcircled{E}})^2A = (A^2)^{\textcircled{E}}A &= U\begin{bmatrix} T^{-2}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^{-1} & T^{-2}(S+G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix}U^*. \end{aligned}$$

By applying (2.3), we obtain (3.4). □

Lemma 3.7 ([20]). *Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$, $\mathrm{rk}(A^k) = \mathrm{rk}((A^k)^{\sim}A^k) = \mathrm{rk}(A_1A_1^{\sim}) = r$. Then*

$$A^{\textcircled{E}} = A_1^{\sharp}A_1A_1^{m}. \tag{3.5}$$

Corollary 3.8. *Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$, $\mathrm{rk}\left(A^k\right) = \mathrm{rk}\left(\left(A^k\right)^{\sim}A^k\right) = \mathrm{rk}\left(A_1A_1^{\sim}\right) = r$. Then*

$$A^{W} = A_1^{\sharp}A_1^{m}A.$$

Proof. By applying (3.4) and (3.5), we can obtain $A^W = A_1^{\sharp}A_1A_1^{m}A_1^{\sharp}A_1A_1^{m}A = A_1^{\sharp}A_1A_1^{m}A_1A_1^{\sharp}A_1^{m}A = A_1^{\sharp}A_1A_1^{\sharp}A_1^{m}A = A_1^{\sharp}A_1^{m}A$. □

Lemma 3.9 ([20]). *Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$, $\mathrm{rk}\left(A^k\right) = \mathrm{rk}\left(\left(A^k\right)^{\sim}A^k\right)$. Then $(A^k)^{\textcircled{w}}$ can be written as the form*

$$(A^k)^{\textcircled{w}} = U\begin{bmatrix} (T^k)^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*G, \tag{3.6}$$

where A be as in (1.7).

Theorem 3.10. *Let A be as in Definition 2.1. Then*

$$A^W = A^k\left(A^{k+2}\right)^{\textcircled{w}}A = \left(A^2P_{A^k}\right)^{\textcircled{w}}A. \tag{3.7}$$

Proof. Let $A \in \mathbb{C}_{n,n}$ be of the form (1.7). By applying (2.1) and (3.6), we have

$$\begin{aligned} A^k(A^{k+2})^{\textcircled{w}}A &= U\begin{bmatrix} T^k & \widehat{T} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^{-(k+2)}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^{-1} & T^{-2}(S+G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix}U^* = A^W, \\ P_{A^k} &= A^k\left(A^k\right)^{\dagger} = U\begin{bmatrix} I_{\mathrm{rk}(A^k)} & 0 \\ 0 & 0 \end{bmatrix}U^*, \\ (A^2P_{A^k})^{\textcircled{w}}A &= \left(U\begin{bmatrix} T^2 & 0 \\ 0 & 0 \end{bmatrix}U^*\right)^{\textcircled{w}}U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^{-2}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^{-1} & T^{-2}(S+G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix}U^* = A^W, \end{aligned}$$

where G, G_1 and G_2 are as in (1.10). Hence, we obtain (3.7). □

Corollary 3.11. *Let A be as in Definition 2.1. Then,* $\mathcal{R}(A^W) = \mathcal{R}(A^k)$.

Proof. Let $A \in \mathbb{C}_{n,n}$ be of the form (1.7), by applying (3.7), we obtian $\mathcal{R}(A^W) \subseteq \mathcal{R}(A^k)$.
Next, we just need to verify $\mathcal{R}(A^k) \subseteq \mathcal{R}(A^W)$. Let any $x = A^k y \in \mathcal{R}(A^k)$, since

$$\begin{aligned} A^k &= U\begin{bmatrix} T^k & \widehat{T} \\ 0 & 0 \end{bmatrix} U^* \\ &= U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix} U^*U\begin{bmatrix} T^{k+1} & T\widehat{T} \\ 0 & 0 \end{bmatrix} U^* \\ &= A^W U\begin{bmatrix} T^{k+1} & T\widehat{T} \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} x = A^k y &= U\begin{bmatrix} T^k & \widehat{T} \\ 0 & 0 \end{bmatrix} U^* y \\ &= U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix} U^*U\begin{bmatrix} T^{k+1} & T\widehat{T} \\ 0 & 0 \end{bmatrix} U^* y \\ &= A^W U\begin{bmatrix} T^{k+1} & T\widehat{T} \\ 0 & 0 \end{bmatrix} U^* y. \end{aligned}$$

Let

$$z = U\begin{bmatrix} T^{k+1} & T\widehat{T} \\ 0 & 0 \end{bmatrix} U^* y.$$

We have $x = A^W z$ and then $x \in \mathcal{R}(A^W)$, from which we have $\mathcal{R}(A^k) \subseteq \mathcal{R}(A^W)$. Then, $\mathcal{R}(A^k) = \mathcal{R}(A^W)$. □

4. The $\mathfrak{m}$-WG inverse included in certain bordered matrix

As is known to all that if A is an invertible matrix, then $X = A^{-1}$ is the unique matrix statisfy following rank equality

$$\operatorname{rk}\left(\begin{bmatrix} A & I \\ I & X \end{bmatrix}\right) = \operatorname{rk}(A).$$

In this section, we investigate the $\mathfrak{m}$-WG inverse A^W of A and give an analogous result of the $\mathfrak{m}$-WG inverse A^W of A. Firstly, we give the following lemma.

Lemma 4.1 ([21]). *Let A an $n \times n$ matrix and let M be a $2n \times 2n$ matrix partitioned as* $M = \begin{bmatrix} A & AT \\ SA & B \end{bmatrix}$. *Then*

$$\operatorname{rk}(M) = \operatorname{rk}(A) + \operatorname{rk}(B - SAT).$$

Theorem 4.2. *Let* $A \in \mathbb{C}_{n,n}$ *with* $\operatorname{Ind}(A) = k$, $\operatorname{rk}\left(A^k\right) = \operatorname{rk}\left(\left(A^k\right)^{\sim} A^k\right) = r$. *Then there exist a unique matrix X such that*

$$(A^k)^{\sim} A^2 X = 0,\ XA^k = 0,\ X^2 = X,\ \operatorname{rk}(X) = n - r, \tag{4.1}$$

a unique matrix Y such that

$$(A^k)^{\sim} AY = 0,\ YA^k = 0,\ Y^2 = Y,\ \operatorname{rk}(Y) = n - r, \tag{4.2}$$

and a unique matrix Z such that

$$\mathrm{rk}\left(\begin{bmatrix} A & I-Y \\ I-X & Z \end{bmatrix}\right) = \mathrm{rk}(A). \tag{4.3}$$

The matrix Z is the $\mathfrak{m}$*-WG inverse A^{W} of A. Furthermore, we have*

$$X = I - A^{W}A, \quad Y = I - AA^{W}.$$

Proof. Let us assume that A has the form (1.7), and A^{W} is as in (2.3). It is easy to verify that the block matrix

$$X = U\begin{bmatrix} 0 & -T^{-1}S - T^{-2}(S + G_1^{-1}G_2N)N \\ 0 & I_{n-r} \end{bmatrix}U^{*} = I - A^{W}A \tag{4.4}$$

satisfies the condition (4.1). Next, we prove the uniqueness of X. Firstly, we assume that both X and X_1 satisfy (4.1). Let $X_1 = UX_0U^{*}$, and X_0 can be partitioned as the following form

$$X_0 = \begin{bmatrix} E & F \\ K & H \end{bmatrix}, \tag{4.5}$$

where E is an $r \times r$ matrix. On the basis of $X_1A^{k} = 0$, by applying (4.5) and (2.1), we obtain

$$\begin{bmatrix} E & F \\ K & H \end{bmatrix}\begin{bmatrix} T^{k} & \widehat{T} \\ 0 & 0 \end{bmatrix} = 0.$$

As a result, $E = 0$ and $K = 0$. Furthermore, after observing $X_1^2 = X_1$ and $\mathrm{rk}(X_1) = n - r$, it is easily obtain that $H^2 = H$, $F = FH$ and $\mathrm{rk}(H) = n - r$. Therefore, H is nonsingular and $H = I_{n-r}$.

On the other hand, by applying (4.1), we can obtain

$$\begin{aligned} (A^{k})^{\sim}A^{2}X_1 &= GU\begin{bmatrix} (T^{k})^{*} & 0 \\ \widehat{T}^{*} & 0 \end{bmatrix}U^{*}GU\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^{*}U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^{*}U\begin{bmatrix} 0 & F \\ 0 & I_{n-r} \end{bmatrix}U^{*} \\ &= GU\begin{bmatrix} 0 & (T^{k})^{*}G_1T^{2}F + (T^{k})^{*}G_1(TS + SN) + (T^{k})^{*}G_2N^{2} \\ 0 & \widehat{T}^{*}G_1T^{2}F + \widehat{T}^{*}G_1(TS + SN) + \widehat{T}^{*}G_2N^{2} \end{bmatrix}U^{*} = 0. \end{aligned}$$

Since T and G_1 are nonsingular, it follows that $(T^{k})^{*}G_1T^{2}F + (T^{k})^{*}G_1(TS + SN) + (T^{k})^{*}G_2N^{2} = 0$ and further $F = -T^{-2}(TS + SN + G_1^{-1}G_2N^{2})$.

Thus, $X_1 = X$. By using a similar way, we can also prove property (4.2), which Y is given by

$$Y = U\begin{bmatrix} 0 & -T^{-1}(S + G_1^{-1}G_2N) \\ 0 & I_{n-r} \end{bmatrix}U^{*} = I - AA^{W}. \tag{4.6}$$

The matrices $X = I - A^{W}A$ and $Y = I - AA^{W}$ satisfy

$$\begin{bmatrix} A & I-Y \\ I-X & Z \end{bmatrix} = \begin{bmatrix} A & AA^{W} \\ A^{W}A & Z \end{bmatrix}.$$

By applying Lemma 4.1 and (4.3), then

$$Z = A^{W}AA^{W} = A^{W}.$$

The above proof is completed. □

In the following, by applying $X = I - A^{W}A$ and $Y = I - AA^{W}$, we give another characterization of the $\mathfrak{m}$-WG inverse.

Theorem 4.3. *Let $A \in \mathbb{C}_{n,n}$ with* $\mathrm{Ind}(A) = k$, $\mathrm{rk}\left(A^k\right) = \mathrm{rk}\left(\left(A^k\right)^{\sim} A^k\right) = r$. *Then*

$$A^W = (A - X)^{-1}(I - Y) = (A + X)^{-1}(I - Y), \tag{4.7}$$

where $X = I - A^W A$ and $Y = I - AA^W$.

Proof. Let A be of the form (1.7), by applying (4.4) and (4.6), we obtain

$$\begin{aligned} A - X &= U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* - U\begin{bmatrix} 0 & -T^{-1}S - T^{-2}(S + G_1^{-1}G_2N)N \\ 0 & I_{n-r} \end{bmatrix}U^* \\ &= U\begin{bmatrix} T & S + T^{-1}S + T^{-2}(S + G_1^{-1}G_2N)N \\ 0 & N - I_{n-r} \end{bmatrix}U^*. \end{aligned}$$

Since T and $N - I_{n-r}$ are nonsingular, then

$$(A - X)^{-1} = U\begin{bmatrix} T^{-1} & -T^{-1}[S + T^{-1}S + T^{-2}(S + G_1^{-1}G_2N)N](N - I_{n-r})^{-1} \\ 0 & (N - I_{n-r})^{-1} \end{bmatrix}U^*,$$

and

$$\begin{aligned} (A - X)^{-1}(I - Y) &= U\begin{bmatrix} T^{-1} & -T^{-1}[S+T^{-1}S+T^{-2}(S+G_1^{-1}G_2N)N](N-I_{n-r})^{-1} \\ 0 & (N-I_{n-r})^{-1} \end{bmatrix}U^*U\begin{bmatrix} I_r & T^{-1}(S+G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix}U^* = A^W. \end{aligned}$$

By applying the similar method, we can also obtain the property $A^W = (A + X)^{-1}(I - Y)$. Thus,

$$A^W = (A - X)^{-1}(I - Y) = (A + X)^{-1}(I - Y),$$

which confirms the representations (4.7). □

In the following, we give an example to verify the results of Theorem 4.2.

Example 4.4. *Let*

$$A = \begin{bmatrix} 0 & 4 & -1 \\ -1 & 3 & -1 \\ -2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

satisfying $rk(A) = 2$ and $rk(A^2) = rk(A^3) = 1$. Therefore, we know that $k = \mathrm{Ind}(A) = 2$. The $\mathfrak{m}$-WG inverse of A is given by

$$\begin{aligned} A^W &= U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix}U^* \\ &= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{9} & \frac{2}{9} \\ \frac{1}{3} & -\frac{1}{9} & \frac{1}{9} \\ -\frac{2}{3} & \frac{2}{9} & -\frac{2}{9} \end{bmatrix}. \end{aligned}$$

The block matrix

$$B = \begin{bmatrix} A & I - Y \\ I - X & Z \end{bmatrix} = \begin{bmatrix} A & AA^W \\ A^WA & A^W \end{bmatrix} = \begin{bmatrix} 0 & 4 & -1 & 2 & -\frac{2}{3} & \frac{2}{3} \\ -1 & 3 & -1 & 1 & -\frac{1}{3} & \frac{1}{3} \\ -2 & -2 & 0 & -2 & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{9} & \frac{14}{9} & -\frac{4}{9} & \frac{2}{3} & -\frac{2}{9} & \frac{2}{9} \\ -\frac{1}{9} & \frac{7}{9} & -\frac{2}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{9} \\ \frac{2}{9} & -\frac{14}{9} & \frac{4}{9} & -\frac{2}{3} & \frac{2}{9} & -\frac{2}{9} \end{bmatrix}$$

satisfies $rk(B) = rk(A) = 2$*. Furthermore, the matrix*

$$X = I - A^W A = \begin{bmatrix} \frac{11}{9} & -\frac{14}{9} & \frac{4}{9} \\ \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ -\frac{2}{9} & \frac{14}{9} & \frac{5}{9} \end{bmatrix}$$

satisfies(4.1). In addition, one can verify that

$$Y = I - AA^W = \begin{bmatrix} -1 & \frac{2}{3} & -\frac{2}{3} \\ -1 & \frac{4}{3} & -\frac{1}{3} \\ 2 & -\frac{2}{3} & \frac{5}{3} \end{bmatrix}$$

satisfies(4.2).

5. Generalized Cayley-Hamilton theorem for the $\mathfrak{m}$-WG inverse matrix

In this section, generalized Cayley-Hamilton theorem will be extended to the $\mathfrak{m}$-WG inverse matrix. By assumption, matrix A is singular, i.e. $det(A) = 0$.

Lemma 5.1 (Cayley-Hamilton theorem, [23]). *Let* $A \in \mathbb{C}_{n,n}$*, the characteristic polynomial of* A *be*

$$p_A(s) = det(sI_n - A) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0.$$

Then

$$p_A(A) = A^n + a_{n-1}A^{n-1} + ... + a_1 A + a_0 I_n,$$

if A *is singular, then* $a_0 = 0$*.*

Theorem 5.2. *Let* $A \in \mathbb{C}_{n,n}$ *with* $\mathrm{Ind}(A) = k$, $\mathrm{rk}\left(A^k\right) = \mathrm{rk}\left(\left(A^k\right)^{\sim} A^k\right)$*, the characteristic polynomial of* A *be*

$$p_A(s) = det(sI_n - A) = s^n + a_{n-1}s^{n-1} + ... + a_1 s.$$

Then

$$A^W + a_{n-1}(A^W)^2 + ... + a_1(A^W)^n = 0. \tag{5.1}$$

Proof. Since $A \in \mathbb{C}_{n,n}$ is singular, by applying Cayley-Hamilton theorem, we have

$$A^n + a_{n-1}A^{n-1} + ... + a_1 A = 0. \tag{5.2}$$

Postmultiplying both sides of (5.2) with $(A^W)^{n+1}$, one has

$$A^n(A^W)^{n+1} + a_{n-1}A^{n-1}(A^W)^{n+1} + ... + a_1 A(A^W)^{n+1} = 0. \tag{5.3}$$

By applying (1.7) and (2.3), we have $A^W = A(A^W)^2$. Therefore, $A(A^W)^{n+1} = A(A^W)^2(A^W)^{n-1} = A^W(A^W)^{n-1} = (A^W)^n$. By applying similar ways, we obtain $A^2(A^W)^{n+1} = (A^W)^{n-1}, ..., A^{n-1}(A^W)^{n+1} = (A^W)^2, A^n(A^W)^{n+1} = A^W$. Substituting above equality into (5.3), we obtain (5.1). □

Next, we give an example to verify Theorem 5.2.

Example 5.3. *Let* $A = \begin{bmatrix} 0 & \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{2}{3} & 0 \end{bmatrix}$ *with* $\mathrm{Ind}(A) = 2$. *By calculating, we obtain*

$$A^W = \begin{bmatrix} 2 & -\frac{2}{3} & \frac{2}{3} \\ 1 & -\frac{1}{3} & \frac{1}{3} \\ -2 & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$

Then

$$det(sI_3 - A) = \begin{vmatrix} s & -\frac{4}{3} & \frac{1}{3} \\ \frac{1}{3} & s-1 & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & s \end{vmatrix} = s^3 - s^2.$$

By applying Cayley-Hamilton theorem, we have

$$A^3 - A^2 = \begin{bmatrix} 0 & \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{2}{3} & 0 \end{bmatrix}^3 - \begin{bmatrix} 0 & \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{2}{3} & 0 \end{bmatrix}^2 = 0.$$

By applying Theorem 5.2, we have

$$(A^W)^2 - (A^W)^3 = \begin{bmatrix} 2 & -\frac{2}{3} & \frac{2}{3} \\ 1 & -\frac{1}{3} & \frac{1}{3} \\ -2 & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}^2 - \begin{bmatrix} 2 & -\frac{2}{3} & \frac{2}{3} \\ 1 & -\frac{1}{3} & \frac{1}{3} \\ -2 & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}^3 = 0.$$

In the following, we extend generalized Cayley-Hamilton theorem to the $\mathfrak{m}$-WG inverse. Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$, $\mathrm{rk}\left(A^k\right) = \mathrm{rk}\left(\left(A^k\right)^{\sim} A^k\right)$, by applying Lemma 5.1 and Theorem 5.2, we have

$$\begin{aligned} det(sI_n - A^W) &= det(sI_n - U \begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix} U^*) \\ &= s^{n-rk(A^k)} det(sI_{rk(A^k)} - T^{-1}). \end{aligned} \tag{5.4}$$

Let the characteristic polynomial of T^{-1} be

$$\begin{aligned} p_{T^{-1}}(s) &= det(sI_{rk(A^k)} - T^{-1}) \\ &= s^{rk(A^k)} + b_{n-1}s^{rk(A^k)-1} + ... + b_{n-rk(A^k)+1}s + b_{n-rk(A^k)}. \end{aligned} \tag{5.5}$$

By applying (5.4) and (5.5), we obtain the following Theorem 5.4.

Theorem 5.4. *Let* $A \in \mathbb{C}_{n,n}$ *with* $\mathrm{Ind}(A) = k$, $\mathrm{rk}\left(A^k\right) = \mathrm{rk}\left(\left(A^k\right)^{\sim} A^k\right)$, *the characteristic polynomial of* A^W *be*

$$p_{A^W}(s) = det(sI_n - A^W) = s^n + b_{n-1}s^{n-1} + ... + b_{n-rk(A^k)}s^{n-rk(A^k)}.$$

Then

$$p_{A^W(A^W)} = (A^W)^n + b_{n-1}(A^W)^{n-1} + ... + b_{n-rk(A^k)}(A^W)^{n-rk(A^k)} = 0,$$

where $b_{n-1}, ..., b_{n-rk(A^k)}$ *be as in (5.5).*

In the following, we give an example to verify Theorem 5.4.

Example 5.5. *Let* $A = \begin{bmatrix} \frac{2\sqrt{48}-3}{3\sqrt{48}} & \frac{\sqrt{48}+3}{3\sqrt{48}} & \frac{\sqrt{48}+3}{3\sqrt{48}} & \frac{4\sqrt{2}-1}{\sqrt{48}} \\ \frac{\sqrt{48}-6\sqrt{2}+3}{3\sqrt{48}} & \frac{2\sqrt{48}+6\sqrt{2}-3}{3\sqrt{48}} & \frac{-\sqrt{48}+6\sqrt{2}-3}{3\sqrt{48}} & \frac{2\sqrt{2}+1}{\sqrt{48}} \\ \frac{\sqrt{48}+6\sqrt{2}+3}{3\sqrt{48}} & \frac{-\sqrt{48}-6\sqrt{2}-3}{3\sqrt{48}} & \frac{2\sqrt{48}-6\sqrt{2}-3}{3\sqrt{48}} & \frac{2\sqrt{2}+1}{\sqrt{48}} \\ \frac{3}{\sqrt{48}} & -\frac{3}{\sqrt{48}} & -\frac{3}{\sqrt{48}} & \frac{3}{\sqrt{48}} \end{bmatrix}$ *with* $\mathrm{Ind}(A) = 2$*. There exists a unitary matrix*

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & -\frac{1}{2} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{12}} & -\frac{1}{2} \\ 0 & 0 & \frac{3}{\sqrt{12}} & \frac{1}{2} \end{bmatrix},$$

such that

$$A = U\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}U^*.$$

By calculating, we obtain

$$U^*GU = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{3} & -\frac{1}{\sqrt{18}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{18}} & -\frac{5}{6} & -\frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix}.$$

where $G_1 = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{3} \end{bmatrix}$, $G_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{18}} & \frac{1}{\sqrt{6}} \end{bmatrix}$, $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
Thus, we obtain

$$\begin{aligned} A^W &= U\begin{bmatrix} T^{-1} & T^{-2}(S + G_1^{-1}G_2N) \\ 0 & 0 \end{bmatrix}U^* \\ &= \begin{bmatrix} \frac{2-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{2-\sqrt{2}}{6} \\ \frac{\sqrt{24}-3\sqrt{2}-6}{3\sqrt{24}} & \frac{2\sqrt{24}+3\sqrt{2}+6}{3\sqrt{24}} & -\frac{\sqrt{24}-3\sqrt{2}-6}{3\sqrt{24}} & \frac{2-\sqrt{2}}{\sqrt{24}} \\ \frac{\sqrt{24}-3\sqrt{2}+6}{3\sqrt{24}} & -\frac{\sqrt{24}-3\sqrt{2}+6}{3\sqrt{24}} & \frac{\sqrt{24}+3\sqrt{2}-6}{3\sqrt{24}} & \frac{2-\sqrt{2}}{\sqrt{24}} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Then

$$p_{T^{-1}}(s) = s^2 - 2s + 1, \quad p_{A^W}(s) = s^2(s^2 - 2s + 1) = s^4 - 2s^3 + s^2.$$

And

$$\begin{aligned} &\begin{bmatrix} \frac{2-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{2-\sqrt{2}}{6} \\ \frac{\sqrt{24}-3\sqrt{2}-6}{3\sqrt{24}} & \frac{2\sqrt{24}+3\sqrt{2}+6}{3\sqrt{24}} & -\frac{\sqrt{24}-3\sqrt{2}-6}{3\sqrt{24}} & \frac{2-\sqrt{2}}{\sqrt{24}} \\ \frac{\sqrt{24}-3\sqrt{2}+6}{3\sqrt{24}} & -\frac{\sqrt{24}-3\sqrt{2}+6}{3\sqrt{24}} & \frac{\sqrt{24}+3\sqrt{2}-6}{3\sqrt{24}} & \frac{2-\sqrt{2}}{\sqrt{24}} \\ 0 & 0 & 0 & 0 \end{bmatrix}^4 - 2\begin{bmatrix} \frac{2-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{2-\sqrt{2}}{6} \\ \frac{\sqrt{24}-3\sqrt{2}-6}{3\sqrt{24}} & \frac{2\sqrt{24}+3\sqrt{2}+6}{3\sqrt{24}} & -\frac{\sqrt{24}-3\sqrt{2}-6}{3\sqrt{24}} & \frac{2-\sqrt{2}}{\sqrt{24}} \\ \frac{\sqrt{24}-3\sqrt{2}+6}{3\sqrt{24}} & -\frac{\sqrt{24}-3\sqrt{2}+6}{3\sqrt{24}} & \frac{\sqrt{24}+3\sqrt{2}-6}{3\sqrt{24}} & \frac{2-\sqrt{2}}{\sqrt{24}} \\ 0 & 0 & 0 & 0 \end{bmatrix}^3 \\ &+ \begin{bmatrix} \frac{2-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{2-\sqrt{2}}{6} \\ \frac{\sqrt{24}-3\sqrt{2}-6}{3\sqrt{24}} & \frac{2\sqrt{24}+3\sqrt{2}+6}{3\sqrt{24}} & -\frac{\sqrt{24}-3\sqrt{2}-6}{3\sqrt{24}} & \frac{2-\sqrt{2}}{\sqrt{24}} \\ \frac{\sqrt{24}-3\sqrt{2}+6}{3\sqrt{24}} & -\frac{\sqrt{24}-3\sqrt{2}+6}{3\sqrt{24}} & \frac{\sqrt{24}+3\sqrt{2}-6}{3\sqrt{24}} & \frac{2-\sqrt{2}}{\sqrt{24}} \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = 0. \end{aligned}$$

Therefore, $(A^W)^4 - 2(A^W)^3 + (A^W)^2 = 0.$

6. Applications of the $\mathfrak{m}$-WG inverse

In [18], Mosić and Stanimirović applied the WG inverse to solve linear equations. The following matrix equation

$$(A^{k+2})^*A^2x = (A^{k+2})^*Ab,\ b \in \mathbb{C}_{n,1},$$

is consistent and its general soluytion is

$$x = A^{\circledR}b + (I - A^{\circledR}A)y,$$

where $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$, for arbitrary $y \in \mathbb{C}_{n,1}$.

In the following, by using the $\mathfrak{m}$-WG inverse, we give the general solutions of the following matrix equation in Minkowski space

$$(A^k)^{\sim}A^2x = (A^k)^{\sim}Ab,\ b \in \mathbb{C}_{n,1},$$

where $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$, $\mathrm{rk}\left(A^k\right) = \mathrm{rk}\left(\left(A^k\right)^{\sim} A^k\right)$.

Theorem 6.1. *Let A be as in Definition 2.1. Then the equation*

$$(A^k)^{\sim}A^2x = (A^k)^{\sim}Ab,\ b \in \mathbb{C}_{n,1}, \tag{6.1}$$

is consistent and its general solutions is

$$x = A^{W}b + \left(I - A^{W}A\right)y, \tag{6.2}$$

for arbitrary $y \in \mathbb{C}_{n,1}$.

Proof. Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$ be of the form (1.7), A^W and U^*GU be of the form (2.3) and (1.10), respectively. Since $\mathrm{rk}\left(A^k\right) = \mathrm{rk}\left(\left(A^k\right)^{\sim} A^k\right)$, G_1 and T are invertible. Denote

$$U^*x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ U^*b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ and } A^Wb = U\begin{bmatrix} T^{-1}b_1 + T^{-2}(S + G_1^{-1}G_2N)b_2 \\ 0 \end{bmatrix}, \tag{6.3}$$

where x_1, b_1 and $T^{-1}b_1 + T^{-2}(S + G_1^{-1}G_2N)b_2 \in \mathbb{C}_{r,1}$. By applying (1.7) and (1.10), we obtain

$$\begin{aligned}
&(A^k)^{\sim}A^2x - (A^k)^{\sim}Ab \\
&= GU\begin{bmatrix} (T^k)^* & 0 \\ \widehat{T^*} & 0 \end{bmatrix} U^*GU \begin{bmatrix} T^2 & TS + SN \\ 0 & N^2 \end{bmatrix} U^*x - GU\begin{bmatrix} (T^k)^* & 0 \\ \widehat{T^*} & 0 \end{bmatrix} U^*GU \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*b \\
&= GU\begin{bmatrix} (T^k)^* & 0 \\ \widehat{T^*} & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}\left(\begin{bmatrix} T^2 & TS + SN \\ 0 & N^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) \\
&= GU\begin{bmatrix} (T^k)^*G_1T^2x_1 + ((T^k)^*G_1TS + (T^k)^*G_1SN + (T^k)^*G_2N^2)x_2 \\ -(T^k)^*G_1Tb_1 - ((T^k)^*G_1S + (T^k)^*G_2N)b_2 \\ \widehat{T^*}G_1T^2x_1 + (\widehat{T^*}G_1TS + \widehat{T^*}G_1SN + \widehat{T^*}G_2N^2)x_2 \\ -\widehat{T^*}G_1Tb_1 - (\widehat{T^*}G_1S + \widehat{T^*}G_2N)b_2 \end{bmatrix}.
\end{aligned} \tag{6.4}$$

On account of G_1 and T are nonsingular, then we have

$$x_1 = T^{-1}b_1 + T^{-2}(S + G_1^{-1}G_2N)b_2 - T^{-2}(TS + SN + G_1^{-1}G_2N^2)x_2$$

such that

$$(T^k)^* G_1 T^2 x_1 + ((T^k)^* G_1 TS + (T^k)^* G_1 SN + (T^k)^* G_2 N^2) x_2 \\ - (T^k)^* G_1 T b_1 - ((T^k)^* G_1 S + (T^k)^* G_2 N) b_2 = 0$$

and

$$\widehat{T^*} G_1 T^2 x_1 + (\widehat{T^*} G_1 TS + \widehat{T^*} G_1 SN + \widehat{T^*} G_2 N^2) x_2 - \widehat{T^*} G_1 T b_1 - (\widehat{T^*} G_1 S + \widehat{T^*} G_2 N) b_2 = 0,$$

that is, there exists x such that $(A^k)^{\sim} A^2 x = (A^k)^{\sim} Ab$. Hence, we obtain the equation (6.1) is consistent.

By applying (6.3) and (6.4), then we have

$$x = U \begin{bmatrix} T^{-1} b_1 + T^{-2}(S + G_1^{-1} G_2 N) b_2 - T^{-2}(TS + SN + G_1^{-1} G_2 N^2) x_2 \\ x_2 \end{bmatrix}, \tag{6.5}$$

for arbitrary $x_2 \in \mathbb{C}_{n-r,1}$. By applying (1.7) and (2.3), we can easily get

$$I - A^W A = U \begin{bmatrix} 0 & -T^{-2}(TS + SN + G_1^{-1} G_2 N^2) \\ 0 & I_{n-r} \end{bmatrix} U^*. \tag{6.6}$$

Therefore, applying (6.3), (6.5), (6.6) and a simple computation shows

$$\begin{aligned} x &= U \begin{bmatrix} T^{-1} b_1 + T^{-2}(S + G_1^{-1} G_2 N) b_2 \\ 0 \end{bmatrix} + U \begin{bmatrix} -T^{-2}(TS + SN + G_1^{-1} G_2 N^2) x_2 \\ x_2 \end{bmatrix} \\ &= A^W b + (I - A^W A) y, \end{aligned}$$

where $x_2 \in \mathbb{C}_{n-r,1}$ and $y \in \mathbb{C}_{n,1}$ are arbitrary. Therefore, we get the general solutions (6.2). □

Acknowledgements

The authors would like to thank the referees and Professor Dijana Mosić for their helpful suggestions to the improvement of this paper.

References

[1] Z. Xing. On the Deterministic and Non-deterministic Mueller Matrix, Optica Acta International Journal of Optics 39 (1992) 461-484.
[2] M. Renardy. Singular value decomposition in Minkowski space, Linear Algebra and its Applications 236 (1996) 53-58.
[3] A. R. Meenakshi. Generalized inverses of matrices in Minkowski space, Proc. Nat. Seminar Alg. Appln. Annamalai University, Annamalainagar 57 (2000) 1-14.
[4] A. Ben-Israel, T. N. E. Greville. Generalized Inverses: Theory and Applications, Springer-Verlag, New York, second ed., 2003.
[5] G. Wang, Y. Wei, S. Qiao. Generalized Inverses: Theory and Computations, Beijing: Science Press, 2018.
[6] A. Kılıçman, Z. Al-Zhour. The representation and approximation for the weighted Minkowski inverse in Minkowski space, Mathematical and Computer Modelling 47 (2008) 363-371.
[7] Z. Al-Zhour. Extension and generalization properties of the weighted Minkowski inverse in a Minkowski space for an arbitrary matrix, Computers and Mathematics with Applications 70 (2015) 954-961.
[8] R. E. Hartwig, K. Spindelböck. Matrices for which A^* and $A^\dagger$ commute, Linear and Multilinear Algebra 14 (1983) 241-256.
[9] K. M. Prasad, K. S. Mohana. Core-EP inverse, Linear and Multilinear Algebra 62 (2014) 792-802.
[10] H. Wang. Core-EP decomposition and its applications, Linear Algebra and its Applications 508 (2016) 289-300.
[11] H. Ma, P. S. Stanimirovi. Characterizations, approximation and perturbations of the core-EP inverse, Applied Mathematics and Computation 397 (2019) 404-417.
[12] H. Wang, J. Chen. Weak group inverse, Open Mathematics 16 (2018) 1218-1232.
[13] H. Wang, X. Liu. The weak group matrix, Aequationes Mathematicae 93 (2019) 1261-1273.
[14] D. E. Ferreyra, V. Orquera, N. Thome. A weak group inverse for rectangular matrices, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matematicas 113 (2019) 3727-3740.
[15] S. Xu, H. Wang, J. Chen, X. Chen, T. Zhao. Generalized WG inverse, Journal of Algebra and Its Applications 20 (2021) 2150072.
[16] M. Zhou, J. Chen, Y. Zhou. Weak group inverses in proper *-rings, Journal of Algebra and Its Applications 19 (2020) 2050238.
[17] D. Mosić, D. Zhang. Weighted weak group inverse for Hilbert space operators, Frontiers of Mathematics in China 15 (2020) 709-726.

[18] D. Mosić, P. S. Stanimirović. Representations for the weak group inverse, Applied Mathematics and Computation 397 (2021) 125957.
[19] H. Wang, N. Li, X. Liu. The $\mathfrak{m}$-core inverse and its applications, Linear and Multilinear Algebra 69 (2021) 2491-2509.
[20] H. Wang, H. Wu, X. Liu. The $\mathfrak{m}$-core-EP inverse in Minkowski space, Bulletin of the Iranian Mathematical Society. https://doi.org/10.1007/s41980-021-00619-2 (to appear)
[21] G. Marsaglia, G. Styan. Equalities and inequalities for ranks of matrices, Linear and Multilinear Algebra 2 (1974) 269-292.
[22] T. Kaczorek. Cayley-Hamilton theorem for Drazin inverse matrix and standard inverse matrices, Nephron Clinical Practice 64 (2016) 793-797.
[23] R. A. Horn, C. R. Johnson. Matrix analysis, Cambridge University Press, Cambridge, second ed., 2013.