Existence of Mild Solutions to Semilinear Fractional Evolution Equation Using Krasnoselskii Fixed Point Theorem

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Abstract. This paper is devoted to study the existence and stability of mild solutions for semilinear fractional evolution equations with a nonlocal final condition. The analysis is based on analytic semigroup theory, Krasnoselskii fixed point theorem, and a special probability density function. An application to a time fractional diffusion equation with nonlocal final condition is also given.

1. Introduction

Let H be a Hilbert space, and $A : D(A) \subset H \to H$ be the infinitesimal generator of a compact analytic semigroup $\{S(t)\}_{t \geq 0}$ of uniformly bounded linear operators on H. This work is devoted to study the semilinear fractional evolution equation

$$\partial_t^r u(t) = Au(t) + F(t, u(t)), \quad 0 \leq t < T, \tag{1}$$

which is equipped with the nonlocal final condition

$$u(T) - G(u) = u_T. \tag{2}$$

Here, notation ∂_t^r stands for the right Caputo's derivative of fractional derivative order $0 < r < 1$, defined by (see [1]-[8])

$$\partial_t^r u(t) = \frac{-1}{\Gamma(1-r)} \int_t^T (s-t)^{-r} u'(s) ds, \quad 0 < r < 1,$$

where Γ is the Gamma function. The functions F, G and u_T are specified later.

Let us shortly recall the history of the problem. It is well-known that fractional evolution equations have gained much attention. These equations can be dealt by developing mathematical tools of semigroups of bounded linear operators on Banach spaces, referred to G.M. Mophou and G.M. N'Guérékata [14], D.

Baleanu and R.P. Agarwal [17], R.P. Agarwal, M. Benchohra, S. Hamani [18], P. Chen, Y. Li and H. Fan [19], Wahash-Panchal-Abdo [40], Ardjounia-Djoudi [41], etc. In [9], K. Balachandran and J.Y. Park studied the existence and uniqueness of a solution to the abstract fractional semilinear evolution equation

$$\partial_t^r u(t) = A(t)u(t) + f(t, u(t)), \quad t \in (0, T), \tag{3}$$
$$u(0) + g(u) = u_0, \tag{4}$$

where $0 < r < 1$, T is a postive real number, X is a Hilbert space, $A(t)$ is a bounded linear operator for each $t \in [0, T]$, and $f : [0, T] \times X \to X$ is continuous with respect to the second variable. In [10], Y. Zhou and F. Jiao proved the existence of a solution to the following problem

$$\partial_t^q u(t) = Au(t) + f(t, u(t)), \quad 0 < t \le T, \tag{5}$$
$$u(0) + g(u) = u_0 \tag{6}$$

where $0 < q < 1$, T is a postive real number, X is a Banach space, $f : [0, T] \times X \to X$ is a given functions satisfying some assumptions and A is the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t \ge 0}$ of linear bounded operators on X, i.e.

$$Au = \lim_{t \to 0^+} \frac{T(t)u - u}{t} \quad \text{in } X.$$

In [20], M.M. Borai considered a one-sided stable probability density of Wright type functions to obtain the fundamental solution of a fractional evolution equations class. A. Debbouche, D. Baleanu and R.P. Agarwal [21], F. Li, J. Liang and H.K. Xu [38], R. Wang, J. Liu and D. Chen [23] developed this tool to solve fractional evolution equations equipped with non-local initial conditions. N.I. Mahmudov and S. Zorlu [24] discussed compact analytic semigroups instead of strongly semigroups on the approximate controllability. P. Chen, Y. Li, Q. Chen and B. Feng [25] also investigated the initial value problem of fractional evolution equations with noncompact semigroup. Let us refer the reader to some interesting papers on fractional models using fixed point theory can be found by E. Karapinar and his colleagues [42–46].

However, there is only few results that study final value problems for time fractional equations with nonlocal final conditions, which appear in many sciences and play an important role in fractional differential equations. In [26], Mohammed M. Matar considered the existence of solutions to the following fractional integro-differential equation

$$\frac{\partial^\beta}{\partial t^\beta} x(t) = f\left(t, x(t), I_t^\beta x(t)\right), \quad 0 < t < T, \tag{7}$$
$$x(T) = x_T, \tag{8}$$

where $0 < \beta < 1$, I_t^β is the fractional integral of order β, see [1]-[5], X is a Banach space, $x_T \in X$ and $f : [0, T] \times (C([0, T], X))^2 \to C([0, T], X)$ is a given function satisfying some specific assumptions. In [27], Y. Hu and S. Peng established the existence and uniqueness of the following problem backward stochastic differential equation driven by fractional Brownian motions

$$dy_t = -f(t, \eta_t, y_t, z_t)dt - z_t dB_t^H, \tag{9}$$
$$y_T = \xi, \tag{10}$$

where $\{B_t^H, t \ge 0\}$ is the fractional Brownian motion with zero mean, and

$$\eta_t = \eta_0 + b_t + \int_0^t \sigma_s dB_s^H$$

with η_0, b_t, σ_s are deterministic constants or functions. In [28], N. El Karoui, S. Peng, M. C. Quenez show an application of a backward problem in finance. Some more results can be found in [29], [30], [31], [32],

and references therein. The study of fractional evolution equations is urgent due to the applications in the random model of living matter in biology; predict occurrences of the earthquakes; electroencephalograph problems of restoring epilepsy points in the human brain; determine remote electromagnetic waves in the universe, etc.

This paper aims to study the existence and stability of mild solutions for semilinear evolution equations with nonlocal final conditions. It is divided into four parts. In Section 2, we present some definitions of fractional calculus, functional analysis, and propose the fraction version of the Gronwall inequality. The main results are presented in section 3. An application of our methods to a nonlinear problem for time fractional diffusion equations will be addressed in the last section.

2. Preliminaries

In this section, we introduce the concept of mild solutions, based on the integral formulation for solutions. Then, we discuss bounded properties of the solution operators. Finally, we recall some useful theorems of fixed point theory. By taking inspiration from the idea of the paper [15], we will transform the model (3)-(4) with right-sided Caputo derivative ∂_t^r to the another model with the left-sided Caputo derivative $^*\partial_t^r$. More specifically, by noting that A is a sefl-adjoint operator on the Hilbert space H and using Proposition 3.1 in [15], we get the following abstract differential equation

$$\partial_t^r v(t) = Av(t) + F(t, v(t)), \quad v(T) = b \in H, \tag{11}$$

on the interval $[0, T]$ is given by

$$v(t) = \mathcal{P}_r(T - t)\, b + \int_t^T (s - t)^{r-1} Q_r(s - t) F(s, v(s)) ds,$$

where the operator $\mathcal{P}_r(t)$, $Q_r(t)$, $t \geq 0$, are defined by

$$\mathcal{P}_r(t) := \int_0^\infty \frac{\xi^{-1-\frac{1}{r}} \mathcal{D}_r(\xi^{-\frac{1}{r}})}{r} \mathcal{S}(t^r \xi) d\xi, \tag{12}$$

$$Q_r(t) := \int_0^\infty r\xi^{-\frac{1}{r}} \mathcal{D}_r(\xi^{-\frac{1}{r}}) \mathcal{S}(t^r \xi) d\xi, \tag{13}$$

and

$$\mathcal{D}_r(\xi) := \frac{1}{r} \sum_{n=1}^\infty (-1)^{n-1} \xi^{-nr-1} \frac{\Gamma(nr + 1)}{n!} \sin(n\pi r), \tag{14}$$

for all $\xi > 0$. The function $\mathcal{D}_r(\xi)$ is also the Laplace transform $\mathcal{L}(e^{-\lambda^r})$, see [35]. We notice that the (non-negative) function $\mathcal{D}_r$ satisfies $\int_0^\infty \mathcal{D}_r(\xi) d\xi = 1$ and furthermore

$$\int_0^\infty \xi^{-s} \mathcal{D}_r(\xi) d\xi = \frac{\Gamma(1 + \frac{s}{r})}{\Gamma(1 + s)}, \ \forall s > -1. \tag{15}$$

By the above arguments, we then have a formula for the solutions of the Problem (3)-(4) as follows

$$u(t) = \mathcal{P}_r(T - t)\Big(u_T + G(u)\Big) + \int_t^T (s - t)^{r-1} Q_r(s - t) F(s, u(s)) ds. \tag{16}$$

To make all the details more clear, we present the concept of mild solutions in the following definition. Let us denote by $\mathcal{C}_H := C([0, T]; H)$ the space of all continuous functions from $[0, T]$ into H corresponding to the supremum norm $\|x\|_{\mathcal{C}_H} = \sup_{t \in [0,T]} \|x(t)\|$ for all $x \in \mathcal{C}_H$, where $\|\cdot\|$ denotes the norm on H.

Definition 2.1. *If a function $u \in \mathcal{C}_H$ satisfies Equation (16), then it is called a mild solution of Problem (1)-(2).*

In this paper, we denote $\mathcal{L}_H$ the Banach space of all linear bounded operators on H. In the following lemma, we presents appropriate properties of the solution operators $\mathcal{P}_r, Q_r$ in $\mathcal{L}_H$, which are certainly used to establish the existence of mild solutions. A brief demonstration of this lemma is included in Appendix.

Lemma 2.2. *The following statements are true:*

   i) $\mathcal{P}_r(t)$ and $Q_r(t)$ are bounded linear operators for any $t \geq 0$ with respect to

$$\|\mathcal{P}_r(t)\|_{\mathcal{L}_H} \leq M \qquad and \qquad \|Q_r(t)\|_{\mathcal{L}_H} \leq \frac{rM}{\Gamma(1+r)}. \tag{17}$$

   for any $t \geq 0$ provided that $M = \sup_{t \geq 0} \|\mathcal{S}(t)\|_{\mathcal{L}_H}$.

  ii) $\mathcal{P}_r(t)$ and $Q_r(t)$ are strongly continuous for any $t \geq 0$.

  iii) $\mathcal{P}_r(t)$ and $Q_r(t)$ are compact operators for any $t > 0$.

2.1. Some fixed point theorems

In this part, we recall some useful theorems in establishing the existence of mild solutions to Problem (1)-(2). First, a fixed point theorem of Krasnoselskii will be recalled in the following lemma, which can be found in [39, Chapter 1]. This helps to obtain the existence of the equation $\mathcal{T}_1 x + \mathcal{T}_2 x = x$ in Banach spaces.

Lemma 2.3 (A fixed point theorem of Krasnoselskii). *Let U be a closed convex and nonempty subset of Banach space B. Let $\mathcal{T}_1, \mathcal{T}_2$ be two operators map U into $\mathbb{R}$ such that*

  a) $\mathcal{T}_1 x + \mathcal{T}_2 y \in U$ whenever $x, y \in U$;

  b) $\mathcal{T}_1$ is a contraction mapping; and

  c) $\mathcal{T}_2$ is completely continuous.

Then, the operator $\mathcal{T} := \mathcal{T}_1 + \mathcal{T}_2$ has a fixed point.

In the next lemma, the relatively compact criterion in the space $\mathcal{C}_H$ will be presented, which plays an important role in checking completely continuous mapping.

Lemma 2.4 (Ascoli-Arzela theorem). *A subset U of $\mathcal{C}_H$ is relatively compact in $\mathcal{C}_H$ if and only if the following statements are true*

  a) U is uniformly bounded and equicontinuous;

  b) For each $t \in [0, T]$, the set $U(t) := \{x(t)|x \in U\}$ is relatively compact in X.

2.2. Grönwall inequality

The Grönwall inequality plays a key role in proving the existence and estimating solutions for differential equations. Let us recall the following version, which was given in [16, Lemma 3.1].

Lemma 2.5 (Grönwall inequality). *Suppose that u, a, b, k are nonnegative and integrable functions of the variable $t \in [0, T]$. If*

$$u(t) \leq a(t) + b(t) \int_t^T k(s)u(s)ds, \ 0 \leq t \leq T,$$

then there holds that

$$u(t) \leq a(t) + b(t) \int_t^T a(s)k(s) \exp\left(\int_t^\tau b(\tau)k(\tau)d\tau\right)ds, \ 0 \leq t \leq T.$$

In particular, if $a(t) \equiv a$, $b(t) \equiv b$, and $k(t) \equiv 1$, then

$$u(t) \leq ae^{b(T-t)}, \ 0 \leq t \leq T.$$

3. Existence and stability of mild solutions

In this section, we present main results of our work, which will be obtained by considering the following assumptions on F, G. The nonlinearity F is assumed to satisfy two first ones. The last one introduces an assumption for the nonlinearity G.

(A1) For each $t \in [0, T]$, the function $F(t, \cdot) : H \to H$ is continuous. For each $\psi \in H$, the function $F(\cdot, \psi) : [0, T] \to H$ is strongly measurable.

(A2) There exist a constant $r_1 \in [0, r)$ and $p \in L^{1/r_1}([0, T], \mathbb{R}^+)$ such that

$$\|F(t, \psi)\| \le p(t),$$

for all $\psi \in H$ and almost all $t \in [0, T]$.

(A3) $G : \mathcal{C}_H \to H$, and there exists a positive constant K such that $\|G(u) - G(v)\| \le K \|u - v\|_{\mathcal{C}_H}$, for all $u, v \in \mathcal{C}_H$.

Let us give some useful explanations on the above assumptions. For each $t \in [0, T]$ and $v \in \mathcal{C}_H$, we note that $F(t, v(t))$ is a function of t and $v(t)$, where $v(t)$ takes value in H. On the other hand, $G(v)$ is a function of v (not of $v(t)$ or $v(T)$) and takes value in H. In Section 4, an application of Problem (1)-(2) is discussed, where $H = L^2(\Omega)$, and G is given by

$$G(v) := \sum_{k=1}^m \mu_k v(T_k, \cdot), \quad \forall v \in \mathcal{C}_{L^2(\Omega)},$$

where $\mu_k > 0$, $T_k \in (0, T)$ are given numbers, and Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$.

3.1. Existence of a mild solution

We firstly obtain the existence of a mild solution in the following theorem, where bound properties of $\mathcal{P}_r$, $\mathcal{Q}_r$ in Lemma 2.2 and fixed point theory are used.

Theorem 3.1. *Assume that the assumptions (A1)-(A3) are hold and $MK < 1$. If $u_T \in H$, then Problem (1)-(2) has a mild solution in $\mathcal{C}_H$.*

Proof. This theorem will be proved by applying the fixed point theorem of Krasnoselskii. We firstly begin with some basic settings. Let us set $U = \{x \in \mathcal{C}_H : \|x\|_{\mathcal{C}_H} \le R\}$, where the radius R is given by

$$R = \frac{M}{1 - MK}\left(\|u_T\| + \|G(0)\| + C_0 T^{\frac{r-r_1}{1-r_1}} \|p\|_{L^{\frac{1}{r_1}}([0,T];\mathbb{R})} \right), \tag{18}$$

and $C_0 := \frac{r(1-r_1)^{1-r_1}}{(r-r_1)^{1-r_1}\Gamma(1+r)}$. This is a closed convex and nonempty ball of $\mathcal{C}_H$ clearly. On this ball, we define the mapping $\mathcal{T}$ by $\mathcal{T} := \mathcal{T}_1 + \mathcal{T}_2$, where

$$\mathcal{T}_1 x(t) := \mathcal{P}_r(T - t)\big(u_T + G(x)\big), \tag{19}$$

$$\mathcal{T}_2 x(t) := \int_t^T (s - t)^{r-1} \mathcal{Q}_r(s - t) F(s, x(s)) ds, \tag{20}$$

for all $x \in U$ and $0 \le t \le T$. This proof will be split into the following steps.

Step 1. Proving $\mathcal{T}$ is well-defined. For this purpose, we will estimate $\mathcal{T}_1 x(t)$ as follows. For all $x \in U$ and $t \in [0, T]$, using the assumptions (A3), and Part i of Lemma 2.2 gives the estimates

$$\|\mathcal{T}_1 x(t)\| \le \|\mathcal{P}_r(T - t)\|_{\mathcal{L}_H} \|u_T\| + \|\mathcal{P}_r(T - t)\|_{\mathcal{L}_H} \|G(x)\|$$

$$\le M\big(\|u_T\| + K \|x\|_{\mathcal{C}_H} + \|G(0)\| \big)$$

$$\le MKR + M \|u_T\|_H + M \|G(0)\|.$$

Next, we will estimate the term $\mathcal{T}_2 x(t)$. Let us take arbitrarily $x, y \in U$. Then, by observing the assumption (A1) and Part i of Lemma 2.2, we see that

$$\int_t^T \left\| (s-t)^{r-1} \mathcal{Q}_r(s-t) F(s, y(s)) \right\| ds \leq \int_t^T (s-t)^{r-1} \left\| \mathcal{Q}_r(s-t) \right\|_{\mathcal{L}_H} \left\| F(s, y(s)) \right\| ds,$$

which deduces the estimate

$$\int_t^T \left\| (s-t)^{r-1} \mathcal{Q}_r(s-t) F(s, y(s)) \right\| ds \leq \frac{rM}{\Gamma(1+r)} \int_t^T (s-t)^{r-1} p(s) ds. \tag{21}$$

On the other hand, it is easy to see that $s \mapsto (s-t)^{(r-1)/(1-r_1)}$ is Lebesgue integrable. This implies that the function $s \mapsto (s-t)^{r-1}$ belongs to $L^{1/(1-r_1)}([t, T]; \mathbb{R})$. So, from the assumption (A2), we can apply the Hölder inequality as

$$\int_t^T (s-t)^{r-1} p(s) ds \leq \left(\int_t^T (s-t)^{\frac{r-1}{1-r_1}} ds \right)^{1-r_1} \left(\int_t^T p^{\frac{1}{r_1}}(s) ds \right)^{r_1}, \tag{22}$$

where p belongs to $L^{1/r_1}([0, T]; \mathbb{R})$. By combining the estimates (21), (22) together, we now derive

$$\int_t^T \left\| (s-t)^{r-1} \mathcal{Q}_r(s-t) F(s, y(s)) \right\| ds \leq M C_0 (T-t)^{r-r_1} \left\| p \right\|_{L^{1/r_1}([0,T];\mathbb{R})}. \tag{23}$$

We then conclude that the function $s \mapsto \left\| (s-t)^{r-1} \mathcal{Q}_r(s-t) F(s, y(s)) \right\|$ is Lebesgue integrable on the interval $[t, T]$, for each $t \in [0, T]$. Therefore, the abstract function $s \mapsto (s-t)^{r-1} \mathcal{Q}_r(s-t) F(s, y(s))$ is Bochner integrable on $[t, T]$. Consequently, $\mathcal{T}_1(t)$ and $\mathcal{T}_2(t)$ are defined for all $t \in [0, T]$. Furthermore, for all $x, y \in U$ and $0 \leq t \leq T$, we imply from estimates for $\mathcal{T}_1, \mathcal{T}_2$ in the above arguments that

$$\begin{aligned}
\left\| \mathcal{T}_1 x(t) + \mathcal{T}_2 y(t) \right\| &\leq MKR + M \left\| u_T \right\|_H + M \left\| G(0) \right\| \\
&\quad + M C_0 (T-t)^{r-r_1} \left\| p \right\|_{L^{1/r_1}([0,T];\mathbb{R})} \leq R,
\end{aligned} \tag{24}$$

and so the mapping $\mathcal{T}$ is well-defined on U.

Step 2. Proving $\mathcal{T}_1$ is a contraction mapping on U. By applying the second part of Lemma 2.2 and the assumption (A3), we can see that

$$\left\| \mathcal{T}_1 x(t) - \mathcal{T}_1 y(t) \right\| \leq \left\| \mathcal{P}_r(T-t) \right\|_{\mathcal{L}_H} \left\| G(x) - G(y) \right\| \leq MK \left\| x - y \right\|_{\mathcal{C}_H}$$

for all x, y in U. Since $MK < 1$, $\mathcal{T}_1$ is a contraction mapping.

Step 3. Proving $\mathcal{T}_2$ is completely continuous. For the sake of convenience, we divide this step into the following sub-steps.

• *Firstly, we will prove $\mathcal{T}_2$ is continuous on U.* Let $x_n \in U, n = 1, 2, \ldots$ such that the sequence $\{x_n\}$ converges to x in U as $n \to \infty$, i.e., $\|x_n - x\|_{\mathcal{C}_H} \to 0$. Then $\|x_n(s) - x(s)\| \to 0$ for all $s \in [0, T]$. By applying the assumption (A1), we deduce

$$\| F(s, x_n(s)) - F(s, x(s)) \| \to 0, \quad \forall s \in [0, T]. \tag{25}$$

Making uses of Lemma 2.2 and the assumption (A2) together, and then applying the Hölder inequality, one can check the chain

$$\begin{aligned}
\| \mathcal{T}_2 x_n(t) - \mathcal{T}_2 x(t) \| &\leq \int_t^T (s-t)^{r-1} \| \mathcal{Q}_r(s-t) \|_{\mathcal{L}_H} \| F(s, x_n(s)) - F(s, x(s)) \| ds \\
&\leq \frac{rM}{\Gamma(1+r)} \int_t^T (s-t)^{r-1} \| F(s, x_n(s)) - F(s, x(s)) \| ds \\
&\leq \frac{rM}{\Gamma(1+r)} \theta_T \left(\int_0^T \| F(s, x_n(s)) - F(s, x(s)) \|^{\frac{1}{r_1}} ds \right)^{r_1},
\end{aligned} \tag{26}$$

where we note

$$\left(\int_t^T (s-t)^{\frac{r-1}{1-r_1}} ds\right)^{1-r_1} \le \frac{(1-r_1)^{1-r_1}}{(r-r_1)^{1-r_1}} T^{r-r_1} =: \theta_T. \tag{27}$$

Here, the assumption (A2) and triangle inequality allow that the norm of $F(s, x_n(s)) - F(s, x(s))$ is bounded by $2p(s)$, where $p \in L^{1/r_1}([0,T]; \mathbb{R})$. This implies that $F(s, x_n(s)) - F(s, x(s))$ contains in $L^{1/r_1}([0,T]; H)$. Therefore, taking supremum two sides of (26) on $[0,T]$ gives

$$\|\mathcal{T}_2 x_n - \mathcal{T}_2 x\|_{\mathcal{C}_H} \le MC_0 T^{r-r_1} \left(\int_0^T \|F(s, x_n(s)) - F(s, x(s))\|^{\frac{1}{r_1}} ds\right)^{r_1}, \tag{28}$$

where C_0 was given by (18). Let us combine the estimates (25), (26), and apply the Lebesgue's dominated convergence theorem to obtain that $\mathcal{T}_2 x_n - \mathcal{T}_2 x$ converges to 0 in $\mathcal{C}_H$. Summarily, the mapping $\mathcal{T}_2$ is continuous on U.

• *Secondly, we will prove $\mathcal{T}_2(U) := \{\mathcal{T}_2 x \mid x \in U\}$ is uniformly bounded and equicontinuous.* For this purpose, it should be noticed that the the uniform boundedness of $\mathcal{T}_2(U)$ is obvious since $\mathcal{T}_2 x \in U$ for all $x \in U$. Hence, we are going to prove the equicontinuity of $\mathcal{T}_2(U)$. For all $0 \le t_1 < t_2 \le T$, we have

$$
\mathcal{T}_2 x(t_1) - \mathcal{T}_2 x(t_2) = \underbrace{\int_{t_1}^{t_2} (s-t_1)^{r-1} Q_r(s-t_1) F(s, x(s)) ds}_{:=\varphi_1}
$$

$$
+ \underbrace{\int_{t_2}^T \left((s-t_1)^{r-1} Q_r(s-t_1) - (s-t_2)^{r-1} Q_r(s-t_1)\right) F(s, x(s)) ds}_{:=\varphi_2}
$$

$$
+ \underbrace{\int_{t_2}^T \left((s-t_2)^{r-1} Q_r(s-t_1) - (s-t_2)^{r-1} Q_r(s-t_2)\right) F(s, x(s)) ds}_{:=\varphi_3}.
$$

By making uses of the assumption (A2) and the Hölder inequality similarly as (23), the first term φ_1 can be estimated as follows

$$\|\varphi_1\| \le MC_0 (t_2 - t_1)^{r-r_1} \|p\|_{L^{1/r_1}([0,T];\mathbb{R})}, \tag{29}$$

where the constant C_0 was also given by (18). We now estimate the second term φ_2. It is useful to recall the inequality $(a-b)^\lambda \le a^\lambda - b^\lambda$ for all $a > b$ and $\lambda > 1$. Applying this inequality with respect to $\lambda = \frac{1}{1-r_1} > 1$ gives that

$$\|\varphi_2\| \le \frac{Mr}{\Gamma(1+r)} \left\{\int_{t_2}^T \left|(s-t_1)^{r-1} - (s-t_2)^{r-1}\right|^{\frac{1}{1-r_1}} ds\right\}^{1-r_1} \|p\|_{L^{1/r_1}([0,T];\mathbb{R})}$$

$$\le \frac{Mr}{\Gamma(1+r)} \left\{\int_{t_2}^T \left((s-t_2)^{\frac{r-1}{1-r_1}} - (s-t_1)^{\frac{r-1}{1-r_1}}\right) ds\right\}^{1-r_1} \|p\|_{L^{1/r_1}([0,T];\mathbb{R})},$$

where the assumption (A2) and the Hölder inequality have been used consecutively. Taking some simple computations, we then obtain

$$\|\varphi_2\| \le MC_0 \left((T-t_2)^{\frac{r-r_1}{1-r_1}} - (T-t_1)^{\frac{r-r_1}{1-r_1}} + (t_2-t_1)^{\frac{r-r_1}{1-r_1}}\right)^{1-r_1} \|p\|_{L^{1/r_1}([0,T];\mathbb{R})}.$$

Let us proceed to estimate the last term φ_3. In the case $t_2 = T$, the integrand of φ_3 is Bochner integrable (see Step 1), which yields that $\|\varphi_3\| = 0$. We consider the case $t_2 < T$. By taking a positive real number $\epsilon_0 \in (t_2, T)$, we can write

$$\varphi_3 = \int_{t_2}^{t_2+\epsilon} (s - t_2)^{r-1} \big(Q_r(s - t_1) - Q_r(s - t_2)\big) F(s, x(s)) ds$$

$$+ \int_{t_2+\epsilon}^{T} (s - t_2)^{r-1} \big(Q_r(s - t_1) - Q_r(s - t_2)\big) F(s, x(s)) ds := \varphi_{31} + \varphi_{32}, \tag{30}$$

for any $0 < \epsilon < \epsilon_0$. Since the norm of $Q_r(s)$ in H is bounded by $Mr/\Gamma(r+1)$, we can estimate $Q_r(s-t_1)-Q_r(s-t_2)$ in H by $2Mr/\Gamma(r+1)$. Therefore, the term φ_{31} can be estimated as

$$\|\varphi_{31}\| \leq \frac{2Mr}{\Gamma(r+1)} \int_{t_2}^{t_2+\epsilon} (s - t_2)^{r-1} \|F(s, x(s))\| ds \leq 2MC_0 \|p\|_{L^{1/r_1}([0,T];\mathbb{R})} \epsilon^{r-r_1}.$$

Moreover, by the strongly continuous property of Q_r in Part ii of Lemma 2.2, the supremum $\|Q_r(s - t_1) - Q_r(s - t_2)\|_{\mathcal{L}_H}$ on $[t_2 + \epsilon, T]$ exists finitely. Therefore, by the Hölder inequality and also some fundamental computations, the following chain is obvious

$$\|\varphi_{32}\| \leq \sup_{s \in [t_2+\epsilon,T]} \|Q_r(s - t_1) - Q_r(s - t_2)\|_{\mathcal{L}_H} \int_{t_2+\epsilon}^{T} (s - t_2)^{r-1} \|F(s, w(s))\| ds$$

$$\leq \sup_{s \in [t_2+\epsilon,T]} \|Q_r(s - t_1) - Q_r(s - t_2)\|_{\mathcal{L}_H} C_1 \Big((T - t_2)^{\frac{r-r_1}{1-r_1}} - \epsilon^{\frac{r-r_1}{1-r_1}}\Big)^{1-r_1},$$

where $C_1 = \frac{(1-r_1)^{1-r_1}}{(r-r_1)^{1-r_1}} \|p\|_{L^{1/r_1}([0,T];\mathbb{R})}$.

Taking the above estimates for φ_1, φ_2, and φ_{31}, φ_{32} together, we consequently obtain

$$\|\mathcal{T}_2 x(t_1) - \mathcal{T}_2 x(t_2)\|$$

$$\leq MC_0 \|p\|_{L^{1/r_1}([0,T];\mathbb{R})} (t_2 - t_1)^{r-r_1} + 2MC_0 \|p\|_{L^{1/r_1}([0,T];\mathbb{R})} \epsilon^{r-r_1}$$

$$+ MC_0 \|p\|_{L^{1/r_1}([0,T];\mathbb{R})} \Big((T - t_2)^{\frac{r-r_1}{1-r_1}} - (T - t_1)^{\frac{r-r_1}{1-r_1}} + (t_2 - t_1)^{\frac{r-r_1}{1-r_1}}\Big)^{1-r_1}$$

$$+ C_1 \sup_{s \in [t_2+\epsilon,T]} \|Q_r(s - t_1) - Q_r(s - t_2)\|_{\mathcal{L}_H} \Big((T - t_2)^{\frac{r-r_1}{1-r_1}} - \epsilon^{\frac{r-r_1}{1-r_1}}\Big)^{1-r_1}.$$

The strongly continuous property of Q_r in Part ii of Lemma 2.2 also yields that limit of the latter supremum is zero as $t_2 - t_1$ approaches zero. Let us take the limit both sides of the above estimates as $t_2 - t_1 \to 0$, which accordingly deduces that

$$\lim_{t_2-t_1 \to 0} \|\mathcal{T}_2 x(t_1) - \mathcal{T}_2 x(t_2)\| \leq 2MC_0 \|p\|_{L^{1/r_1}([0,T];\mathbb{R})} \epsilon^{r-r_1}, \tag{31}$$

for all $\epsilon > 0$. As a consequence, taking the limit both sides of (31) as $\epsilon \to 0^+$ implies that $\lim_{t_2-t_1 \to 0} \|\mathcal{T}_2 x(t_1) - \mathcal{T}_2 x(t_2)\| = 0$. Notice that the above convergence does not depend on x. Therefore, the set $\mathcal{T}_2(U)$ is equicontinuous.

• *Thirdly, we will prove that the set $\{\mathcal{T}_2 x(t) \,|\, x \in U\}$ is relatively compact for any $t \geq 0$*. The case $t = 0$ is trivial. So, it is only necessary to consider the case $t > 0$. For any $\epsilon > 0$, we now define

$$_\epsilon Q_r(t) := \int_{\epsilon}^{\infty} r\xi^{-\frac{1}{r}} \mathcal{D}_r(\xi^{-\frac{1}{r}}) S(t^r \xi) d\xi,$$

$$_\epsilon \mathcal{T}_2 x(t) := \int_{t+\epsilon}^{T} (s - t)^{r-1} \, _\epsilon Q_r(s - t) F(s, x(s)) ds.$$

We recall that $\{\mathcal{S}(t) : t \geq 0\}$ is a semigroup. Hence, we can write $\mathcal{S}(t^r\xi)$ as an action of $\mathcal{S}(\epsilon^{1+r})$ on $\mathcal{S}((s-t)^r\xi - \epsilon^{1+r})$, which reads

$$_\epsilon Q_r(s-t) = \mathcal{S}(\epsilon^{1+r}) \int_\epsilon^\infty r\xi^{-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})\mathcal{S}((s-t)^r\xi - \epsilon^{1+r})d\xi,$$

for all $s > t + \epsilon$. Form the above equality, we can express $_\epsilon\mathcal{T}_2x(t)$ in term of $\mathcal{S}$ and $\mathcal{D}_r$. Indeed, there holds

$$_\epsilon\mathcal{T}_2x(t) = \mathcal{S}(\epsilon^{1+r}) \int_{t+\epsilon}^T \int_\epsilon^\infty (s-t)^{r-1}r\xi^{-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})\mathcal{S}((s-t)^r\xi - \epsilon^{1+r})F(s,x(s))d\xi ds.$$

We will show that the following set

$$\left\{ \int_{t+\epsilon}^T \int_\epsilon^\infty (s-t)^{r-1}r\xi^{-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})\mathcal{S}\left((s-t)^r\xi - \epsilon^{1+r}\right)F(s,x(s))d\xi ds \;\middle|\; \|x\| \leq R \right\}$$

is bounded. By the making use of the Hölder inequality analogously as (22), we have

$$\left\| \int_{t+\epsilon}^T \int_\epsilon^\infty (s-t)^{r-1}r\xi^{-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})\mathcal{S}\left((s-t)^r\xi - \epsilon^{1+r}\right)F(s,x(s))d\xi ds \right\|$$

$$\leq M \int_{t+\epsilon}^T \int_\epsilon^\infty (s-t)^{r-1}r\xi^{-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})\|F(s,x(s))\|d\xi ds$$

$$\leq M \int_t^T (s-t)^{r-1}\|F(s,x(s))\|ds \times \int_0^\infty r\xi^{-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})d\xi$$

$$\leq M \frac{r(1-r_1)^{1-r_1}}{(r-r_1)^{1-r_1}\Gamma(1+r)}(T-t)^{r-r_1} \|p\|_{L^{1/r_1}([0,T];\mathbb{R})} \frac{r^2}{\Gamma(1+r)}. \tag{32}$$

Moreover, we also recall that, $\mathcal{S}(t)$ is a compact operator for each $t > 0$. So, $\mathcal{S}(\epsilon^{1+r})$ is compact. We imply from the formula of $_\epsilon\mathcal{T}_2x(t)$ that the set $\{_\epsilon\mathcal{T}_2x(t) \,|\, x \in U\}$ is relatively compact. Summarily, the proof will be finished by showing that the norm $\|\mathcal{T}_2x(t) - _\epsilon\mathcal{T}_2x(t)\|$ tends to zero as ϵ approaches zero. One has

$$\mathcal{T}_2x(t) - _\epsilon\mathcal{T}_2x(t) = \int_t^{t+\epsilon} (s-t)^{r-1} Q_r(s-t)F(s,x(s))ds$$

$$+ \int_{t+\epsilon}^T (s-t)^{r-1}\left(Q_r(s-t) - _\epsilon Q_r(s-t)\right)F(s,x(s))ds.$$

By the same way as estimating φ_{31}, we can estimate the first term above as follows

$$\left\| \int_t^{t+\epsilon} (s-t)^{r-1}Q_r(s-t)F(s,x(s))ds \right\| \leq MC_0\epsilon^{r-r_1} \|p\|_{L^{1/r_1}([0,T];\mathbb{R})},$$

where the assumption (A2) and the Hölder inequality have been employed. On the other hand, according to the definition of Q_r and the fundamental theorem of Calculus, we can write the difference $Q_r(s-t) - _\epsilon Q_r(s-t)$ as $\int_0^\epsilon r\xi^{-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})\mathcal{S}((s-t)^r\xi)d\xi$. Therefore, by using the property (15), and then applying the assumption (A2), the Hölder inequality again, we obtain the following chain of estimates

$$\left\| \int_{t+\epsilon}^T (s-t)^{r-1}\left(Q_r(s-t) - _\epsilon Q_r(s-t)\right)F(s,x(s))ds \right\|$$

$$= \left\| \int_{t+\epsilon}^T \int_0^\epsilon (s-t)^{r-1}r\xi^{-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})\mathcal{S}((s-t)^r\xi)F(s,x(s))d\xi ds \right\|$$

$$\leq M \frac{r(1-r_1)^{1-r_1}}{(r-r_1)^{1-r_1}\Gamma(1+r)}(T-t)^{r-r_1} \|p\|_{L^{1/r_1}([0,T];\mathbb{R})} \int_0^\epsilon r\xi^{-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})d\xi,$$

where the latter right hand side converges to zero as ϵ approaches zero. $\square$

Remark 3.1.1. *A natural question may arise that why cannot Banach mapping theorem be directly applied to establish the existence of mild solutions? In fact, in order to use the Banach theorem, it requires Lipschitz continuity of the nonlinearity F, which did not assume in the assumptions (A1) and (A2).*

3.2. Stability of the mild solution

In this part, we will obtain the stability of the mild solution with respect to u_T. For this purpose, we will consider the assumption (A1b) below instead of (A1), which gives a Lipschitz continuity of the nonlinearity F.

(A1b) The function $F : [0, T] \times H \to H$ such that

$$\|F(t, \psi_1) - F(t, \psi_2)\| \le K_F \|\psi_1 - \psi_2\|, \quad \forall t \in [0, T], \ \psi_1, \psi_2 \in H,$$

where K_F does not depend on t, ψ_1, ψ_2. Moreover, the function $F(\cdot, \psi) : [0, T] \to H$ is strongly measurable for each $\psi \in H$.

The above assumption is clearly stronger than (A1) in the sense: if F satisfies (A1b) then it satisfies (A1). Consequently, Theorem 3.1 also holds if we replace (A1) by (A1b).

Theorem 3.2 (Stability). *Assume that the assumptions (A1b), (A2), (A3) are satisfied, and K is small enough. Let x_{u_T} be the mild solution of Problem (1)-(2) corresponding to $u_T \in H$, which was obtained in Theorem 3.1. Then, there exists $C > 0$ such that*

$$\left\| x_{u_T} - x_{\widetilde{u}_T} \right\|_{\mathcal{C}_H} \le C \left\| u_T - \widetilde{u}_T \right\|, \quad \forall u_T, \widetilde{u}_T \in H.$$

Proof. We note that Theorem 3.1 can be applied if we take K small enough such that $MK < 1$. Besides, the mild solution $x_{u_T} \in \mathcal{C}_H$ satisfies the equation $\mathcal{T} x_{u_T}(t) = x_{u_T}(t)$ or $\mathcal{T}_1 x_{u_T}(t) + \mathcal{T}_2 x_{u_T}(t) = x_{u_T}(t)$ for all $t \in [0, T]$, where the estimate for $\mathcal{T}_1 x_{u_T}, \mathcal{T}_2 x_{u_T}$ can be established analogously as Step 1 in the proof of Theorem 3.1. Firstly, the assumptions (A1b) and (A3) deduce that

$$
\begin{aligned}
\left\| \mathcal{T}_1 x_{u_T}(t) - \mathcal{T}_1 x_{\widetilde{u}_T}(t) \right\| &\le \|\mathcal{P}_r(T - t)\|_{\mathcal{L}_H} \left\| u_T - \widetilde{u}_T \right\| \\
&\quad + \|\mathcal{P}_r(T - t)\|_{\mathcal{L}_H} \left\| G(x_{u_T}) - G(x_{\widetilde{u}_T}) \right\| \\
&\le M \left(\left\| u_T - \widetilde{u}_T \right\| + K \left\| x_{u_T} - x_{\widetilde{u}_T} \right\|_{\mathcal{C}_H} \right).
\end{aligned}
$$

By the same techniques as (26), we have

$$
\begin{aligned}
\left\| \mathcal{T}_2 x_{u_T}(t) - \mathcal{T}_2 x_{\widetilde{u}_T}(t) \right\| &\le \int_t^T (s - t)^{r-1} \|\mathcal{Q}_r(s - t)\|_{\mathcal{L}_H} \left\| F(s, x_{u_T}(s)) - F(s, x_{\widetilde{u}_T}(s)) \right\| ds \\
&\le \frac{rM}{\Gamma(1 + r)} \int_t^T (s - t)^{r-1} \left\| F(s, x_{u_T}(s)) - F(s, x_{\widetilde{u}_T}(s)) \right\| ds \\
&\le \frac{rM}{\Gamma(1 + r)} \theta_T \left(\int_t^T \left\| F(s, x_{u_T}(s)) - F(s, x_{\widetilde{u}_T}(s)) \right\|^{\frac{1}{r_1}} ds \right)^{r_1},
\end{aligned}
$$

where the notation θ_T is given by (27). Here, the number $1/r_1$ is strictly greater than 1. By combining the above arguments, we obtain

$$
\begin{aligned}
\left\| x_{u_T}(t) - x_{\widetilde{u}_T}(t) \right\| &\le M \left(\left\| u_T - \widetilde{u}_T \right\| + K \left\| x_{u_T} - x_{\widetilde{u}_T} \right\|_{\mathcal{C}_H} \right) \\
&\quad + \frac{rM}{\Gamma(1 + r)} \theta_T \left(\int_t^T \left\| F(s, x_{u_T}(s)) - F(s, x_{\widetilde{u}_T}(s)) \right\|^{\frac{1}{r_1}} ds \right)^{r_1} \\
&\le M \left(\left\| u_T - \widetilde{u}_T \right\| + K \left\| x_{u_T} - x_{\widetilde{u}_T} \right\|_{\mathcal{C}_H} \right) \\
&\quad + \frac{rM}{\Gamma(1 + r)} \theta_T K_F \left(\int_t^T \left\| x_{u_T}(s) - x_{\widetilde{u}_T}(s) \right\|^{\frac{1}{r_1}} ds \right)^{r_1}.
\end{aligned}
$$

We now recall the inequality: for all $a, b \geq 0$, $q \geq 1$, there holds $(a + b)^q \leq 2^{q-1}(a^q + b^q)$. Since $1/r_1$ is greater than 1, one can apply this inequality to obtain the following chain

$$\left\| x_{u_T}(t) - x_{\widetilde{u}_T}(t) \right\|^{\frac{1}{r_1}} \leq 2^{\frac{2-2r_1}{r_1}} M^{\frac{1}{r_1}} \left(\left\| u_T - \widetilde{u}_T \right\|^{\frac{1}{r_1}} + K^{\frac{1}{r_1}} \left\| x_{u_T} - x_{\widetilde{u}_T} \right\|_{\mathcal{C}_H}^{\frac{1}{r_1}} \right)$$
$$+ \left(\frac{rM}{\Gamma(1+r)} \theta_T K_F \right)^{\frac{1}{r_1}} \int_t^T \left\| x_{u_T}(s) - x_{\widetilde{u}_T}(s) \right\|^{\frac{1}{r_1}} ds.$$

Then, by applying the Grönwall inequality in Lemma 2.5, one accordingly deduces that

$$\left\| x_{u_T}(t) - x_{\widetilde{u}_T}(t) \right\|^{\frac{1}{r_1}} \leq C_2 \left(\left\| u_T - \widetilde{u}_T \right\|^{\frac{1}{r_1}} + K^{\frac{1}{r_1}} \left\| x_{u_T} - x_{\widetilde{u}_T} \right\|_{\mathcal{C}_H}^{\frac{1}{r_1}} \right) e^{C_3(T-t)},$$

where $C_2 = 2^{\frac{2-2r_1}{r_1}} M^{\frac{1}{r_1}}$, and $C_3 = \left(\frac{rM}{\Gamma(1+r)} \theta_T K_F \right)^{\frac{1}{r_1}}$. Let us taking the supremum two sides of the above estimate with respect to $t \in [0, T]$, and choose K small enough such that $C_2 K^{1/r_1} e^{C_3 T} < 1$. Then, one can find a positive constant C_4 such that

$$\left\| x_{u_T} - x_{\widetilde{u}_T} \right\|_{\mathcal{C}_H} \leq C_4 \left\| u_T - \widetilde{u}_T \right\|,$$

namely, the theorem is proved.   □

4. Application

In this section, we present an application of our main results corresponding to some specific cases of the nonlinear functions F, G. Let Ω be a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$. We consider the time fractional diffusion equation

$$\begin{cases} \partial_t^r u(t,x) - \Delta u(t,x) &= F(t, u(t,x)), & 0 \leq t < 1, \, x \in \Omega, \\ u(t,x) &= 0, & 0 \leq t < 1, \, x \in \partial\Omega, \end{cases} \tag{33}$$

subjected to the nonlocal final condition

$$u(1,x) + \sum_{k=1}^m \mu_k u(T_k, x) = \varphi(x), \quad x \in \Omega, \tag{34}$$

where Δ is the Laplace operator defined on the domain $D(A) = C(\overline{\Omega}) \cap H^2(\Omega)$, the functions F, u_T and numbers $\mu_k > 0$, $T_k \in (0, T)$, $1 \leq k \leq m$, are given.

The nonlocal final condition (34) can be used to describe diffusion phenomena, where a small amount of gas diffuses in a transparent tube. Suppose that we observe the diffusion via the surface of the tube. In the case too little gas can be measured at the final time $T = 1$, we may measure the diffusion at some added points $T_1, T_2, ..., T_m$ in the interval $(0, 1)$. The measurement $u(1, x) + \sum_{k=1}^m \mu_k u(T_k, x)$ may be more accurate than $u(1, x)$. We refer the reader to the papers [36–38] for more discussions on nonlocal conditions.

We focus on establishing the existence of mild solutions to Problem (33)-(34) by applying Theorem 3.1 under some suitable assumptions. This will be presented in Theorem 4.1, where we need the following assumptions on F.

(H1) For each $t \in [0, T]$, the function $F(t, \cdot) : L^2(\Omega) \to L^2(\Omega)$ is continuous. For each $\psi \in L^2(\Omega)$, the function $F(\cdot, \psi) : [0, T] \to L^2(\Omega)$ is strongly measurable.

(H2) The exists a constant $r_1 \in [0, r)$ and $p \in L^{1/r_1}([0, T], \mathbb{R}^+)$ such that

$$\|F(t, \psi)\| \leq p(t),$$

for all $\psi \in L^2(\Omega)$ and almost all $t \in [0, T]$.

(H1b) The function $F : [0, T] \times L^2(\Omega) \to L^2(\Omega)$ such that

$$\|F(t, \psi_1) - F(t, \psi_2)\| \le K_F \|\psi_1 - \psi_2\|, \quad \forall t \in [0, T], \, \psi_1, \psi_2 \in L^2(\Omega),$$

where K_F does not depend on t, ψ_1, ψ_2. Moreover, the function $F(\cdot, \psi) : [0, T] \to L^2(\Omega)$ is strongly measurable for each $\psi \in L^2(\Omega)$.

Theorem 4.1. *Assume that (H1), (H2) are satisfied. If $\sum_{k=1}^{m} \mu_k < 1$ and $\varphi \in L^2(\Omega)$, then Problem (33)-(34) has a mild solution u_φ in $\mathcal{C}_{L^2(\Omega)}$.*

Furthermore, if the assumption (H1) is replaced by (H1b) and $\sum_{k=1}^{m} \mu_k$ is small enough, then the mild solution is stable corresponding to

$$\left\| u_\varphi - u_{\widetilde{\varphi}} \right\|_{\mathcal{C}_{L^2(\Omega)}} \lesssim \left\| \varphi - \widetilde{\varphi} \right\|, \quad \forall \varphi, \widetilde{\varphi} \in L^2(\Omega).$$

Proof. By comparing with Problem (1)-(2), we have $A := \Delta$, and

$$(G(v))(x) := -\sum_{k=1}^{m} \mu_k v(T_k, x), \quad x \in \Omega.$$

The semigroup $\{\mathcal{S}(t)\}_{t \ge 0}$ is formulated by $\mathcal{S}(t) := e^{t\Delta}$, which is compact analytic. Moreover, since $M = \sup_{t \ge 0} \|\mathcal{S}(t)\|_{\mathcal{L}_{L^2(\Omega)}}$, we then have $M \le 1$. For all $v_1, v_2 \in \mathcal{C}_{L^2(\Omega)}$, it is easy to see that

$$\|G(v_1) - G(v_2)\|_{L^2(\Omega)} \le \sum_{k=1}^{m} \mu_k \|v_1(T_k, \cdot) - v_2(T_k, \cdot)\|_{L^2(\Omega)} \le K \|v_1 - v_2\|_{\mathcal{C}_{L^2(\Omega)}},$$

where $K := \sum_{k=1}^{m} \mu_k$. By the assumptions (H1), (H2), the condition $\sum_{k=1}^{m} \mu_k < 1$ is sufficient to apply Theorem 3.1, which shows the existence of a mild solution u_φ to Problem (33)-(34) in the space $\mathcal{C}_{L^2(\Omega)}$. Moreover, the stability can be obtained by applying Theorem 3.2. $\square$

Remark 4.1.1. *If the first eigenvalue of the operator Δ is $-\lambda_1 < 0$, then $M = e^{-\lambda_1}$.*

Remark 4.1.2. *If the amount of gas, measured at the final time $T = 1$, is sufficient to determine the diffusion, no additional measurements are required at $T_1, T_2, ..., T_m$. This also leads to the fact that $G = 0$, and that the problem becomes a final value problem for time fraction diffusion equations, which has been widely applied in real world problems.*

Appendix

Proof. [Proof of Lemma 2.2] In order to prove the first part, we will employ the property (15) of the density function $\mathcal{D}_r$. Indeed, this property ensures that

$$\|\mathcal{P}_r(t)\|_{\mathcal{L}_H} \le M \int_0^\infty \frac{\xi^{-1-\frac{1}{r}} \mathcal{D}_r(\xi^{-\frac{1}{r}})}{r} d\xi = M \int_0^\infty \frac{(\xi^{-r})^{-1-\frac{1}{r}} \mathcal{D}_r(\xi)}{r} r\xi^{-1-r} d\xi = M.$$

Similarly, we also have $\|\mathcal{P}_r(t)\|_{\mathcal{L}_H} \le \dfrac{Mr^2}{\Gamma(1+r)}$ by using (15), namely, Part i is proved. Part ii can be proved by applying (15) also. We now proceed to prove Part iii. Let us fix real numbers $t > 0$ and $R > 0$. In order to show that $P_r(t)$ is compact, we need to show that the set $I(t) := \{\mathcal{P}_r(t)x \,|\, \|x\| \le R\}$ is relatively compact in H. Firstly, for any $\epsilon > 0$, we define

$$_\epsilon\mathcal{P}_r(t) := \int_\epsilon^\infty \frac{\xi^{-1-\frac{1}{r}} \mathcal{D}_r(\xi^{-\frac{1}{r}})}{r} \mathcal{S}(t^r \xi) d\xi.$$

Since $\mathcal{S}$ is a semigroup, we then have

$$_{\epsilon}\mathcal{P}_r(t)x = \mathcal{S}(t^r\epsilon)\int_{\epsilon}^{\infty}\frac{\xi^{-1-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})}{r}\mathcal{S}(t^r\xi - t^r\epsilon)xd\xi. \tag{35}$$

By using Part i, we notice that the following set

$$\left\{\int_{\epsilon}^{\infty}\frac{\xi^{-1-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})}{r}\mathcal{S}(t^r\xi - t^r\epsilon)xd\xi \;\middle|\; \|x\| \leq R\right\}$$

is bounded. Hence, it follows from the equality (35) and the compactness of the operator $\mathcal{S}(\epsilon^r\xi)$ that the set $I_{\epsilon}(t) := \{_{\epsilon}\mathcal{P}_r(t)x \,|\, \|x\| \leq R\}$ is relatively compact in H.

Moreover, it is obvious that

$$\mathcal{P}_r(t)x - {}_{\epsilon}\mathcal{P}_r(t)x = \int_0^{\epsilon}\frac{\xi^{-1-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})}{r}\mathcal{S}(t^r\xi)xd\xi. \tag{36}$$

We recall that $\|\mathcal{S}(t)\|_{\mathcal{L}_H} \leq M$ for all $t \geq 0$. Therefore, the norm $\|\mathcal{P}_r(t)x - {}_{\epsilon}\mathcal{P}_r(t)x\|$ is bounded by $MR\int_0^{\epsilon} r^{-1}\xi^{-1-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})d\xi$. The integrable property of the integrand $\xi^{-1-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})$ implies that the integral $\int_0^{\epsilon} r^{-1}\xi^{-1-\frac{1}{r}}\mathcal{D}_r(\xi^{-\frac{1}{r}})d\xi$ tends 0 as ϵ approaches 0. This leads to

$$\left\|\mathcal{P}_r(t)x - {}_{\epsilon}\mathcal{P}_r(t)x\right\| \longrightarrow 0$$

as $\epsilon \to 0$. Consequently, the set $I(t) := \{\mathcal{P}_r(t)x \,|\, |x| \leq R\}$ is relatively compact in H. The relative compactness of $Q_r(t)$ can be proved similarly. $\square$

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