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Canonical Almost Geodesic Mappings of the First Type onto Generalized Ricci Symmetric Spaces

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Abstract. In the paper we consider canonical almost geodesic mappings of spaces with affine connection onto m-Ricci-symmetric spaces. In particular, we studied in detail canonical almost geodesic mappings of the first type of spaces with affine connections onto 2- and 3-Ricci-symmetric spaces. In either case the main equations for the mappings have been obtained as a closed mixed system of PDEs of Cauchy type. We have found the maximum number of essential parameters which the solution of the system depends on.

1. Introduction

The paper is devoted to further study of the theory of almost geodesic mappings of affinely connected spaces. The theory, as well as other theories of diffeomorphisms, goes back to the paper [17] of T. Levi-Civita, in which the problem on the search for Riemannian spaces with common geodesics was stated and solved in a special coordinate system. We note a remarkable fact that this problem is related to the study of equations of dynamics of mechanical systems.

The theory of geodesic mappings has been developed by T. Thomas, H. Weyl, P.A. Shirokov, A.S. Solodovnikov, N.S. Sinyukov, A.V. Aminova, J. Mikeš, and others. Issues arisen by the exploration were studied by V.F. Kagan, G. Vrançeanu, Ya.L. Shapiro, D.V. Vedenyapin et al. The authors discover special classes of $(n-2)$-spaces.

In [26], A.Z. Petrov introduced the notion of quasi-geodesic mappings. In particular, holomorphically projective mappings of Kählerian spaces are special quasi-geodesic mappings; they were examined by T. Otsuki and Y. Tashiro, M. Prvanović, J. Mikeš and others.

A natural generalization of these classes of mappings is the class of almost geodesic mappings introduced by Sinyukov (see [30–33]). He also specified three types of almost geodesic mappings π_1, π_2, π_3. The problem of completeness of classification had long remained unresolved.

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The theory of almost geodesic mappings was developed by V.S. Sobchuk [34, 35], V.S. Shadnyi [36], N.Y. Yablonskaya [42], V.E. Berezovski, J. Mikeš [1–14, 23, 24, 39], Lj.S. Velimirović, N. Vesić, M.S. Stankovič, M.Z. Petrović [27–29, 37, 38, 40, 41] et al.

It is known [32] that almost geodesic mappings allow to model processes in some settings, including energetic conditions (which are described by some spaces) if there were no external forces, by replacing the processes (and corresponding spaces) by others in other settings and there are external forces. In particular, almost geodesic mappings and transformations are used in theories of gravity [16].

N.S. Sinyukov [33] proved that the main equations for canonical almost geodesic mappings of the first type of spaces with affine connections onto Ricci-symmetric spaces can be written as a closed system of partial differential equations of Cauchy type in covariant derivatives. It follows that the family of spaces onto which an affinely connected space admits canonical almost geodesic mappings of the type π_1, depends on a finite number of essential parameters.

On the other hand, geodesic mappings are a special case of the canonical almost geodesic mappings of the type π_1. The main equations of geodesic mappings of affinely connected spaces can not be reduced to a closed system of partial differential equations of Cauchy type, because the general solution depends on n arbitrary functions.

Hence in the general case the main equations of canonical almost geodesic mappings of the type π_1 of affinely connected spaces also can not be reduced to a closed system of partial differential equations of Cauchy type.

The papers [8, 13, 14] are devoted to the study of canonical almost geodesic mappings of the first type of spaces with affine connections onto Riemannian spaces and onto generalized Ricci-symmetric spaces. The main equations of the above said mappings were reduced to closed systems of partial differential equations of Cauchy type. These results substantively supplemented the results obtained by N.S. Sinyukov [33] for canonical almost geodesic mappings of the first type of spaces with affine connections onto Ricci-symmetric spaces.

In seventy of 20 century V.R. Kaygorodov [15] studied geometry of generalized symmetric and recurrent spaces. Results of geodesic and holomorphically projective mappings onto these spaces obtained J. Mikeš [18, 19, 25], J. Mikeš and V.S. Sobchuk [25], see [20–23, 32].

In this paper, the main equations for almost geodesic mappings of spaces with affine connections onto 2-, 3- and m-Ricci-symmetric spaces are obtained as closed system Cauchy-type differential equations in covariant derivatives. These results are a generalization of the above said results.

2. Basic definitions of almost geodesic mappings of spaces with affine connections.

Let us recall the basic definition, formulas and theorems of the theory presented in [22, 23, 31–33].

Consider a space A_n with an affine connection Γ^h_{ij} without torsion. The space is referred to coordinates $x^1, x^2, \ldots, x^n$.

A curve l: $x^h = x^h(t)$ in the space A_n is a *geodesic* when its tangent vector $\lambda^h(t) = dx^h(t)/dt$ satisfies the equations

$$\lambda^h_1 = \rho(t)\cdot\lambda^h, \quad \text{where} \quad \lambda^h_1 \equiv \lambda^h_{,\alpha}\lambda^\alpha = \frac{d\lambda^h(t)}{dt} + \Gamma^h_{\alpha\beta}(x(t))\lambda^\alpha(t)\lambda^\beta(t),$$

and $\rho(t)$ is a function of t. We denote by comma " , " the covariant derivative with respect to the connection of the space A_n.

A curve in the space A_n ($n > 2$) is an *almost geodesic* when its tangent vector $\lambda^h(t) = dx^h(t)/dt$ satisfies the equations

$$\lambda^h_2 = a(t)\cdot\lambda^h + b(t)\cdot\lambda^h_1,$$

where $\lambda^h_2 \equiv \lambda^h_{1,\alpha}\lambda^\alpha$, and $a(t)$, $b(t)$ are functions of the parameter t.

We say that a diffeomorphism $f\colon A_n \to \overline{A}_n$ is an *almost geodesic mapping* if any geodesic curve of A_n is mapped under f onto an almost geodesic curve in $\overline{A}_n$.

Suppose, that a space A_n with affine connection $\Gamma^h_{ij}(x)$ admits a mapping f onto a space $\overline{A}_n$ with affine connection $\overline{\Gamma}^h_{ij}(x)$, and the spaces are referred to the common coordinate system $x^1, x^2, \dots, x^n$.

The tensor

$$P^h_{ij}(x) = \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x), \tag{1}$$

is called a *deformation tensor* of the connections $\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ with respect to the mapping f. The symbols $\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ are components of affine connections of the spaces A_n and $\overline{A}_n$ respectively.

According to [30–33], a necessary and sufficient condition for the mapping of a space A_n onto a space $\overline{A}_n$ to be almost geodesic is that the deformation tensor $P^h_{ij}(x)$ of the mapping f in the common coordinate system $x^1, x^2, \dots, x^n$ has to satisfy the condition

$$A^h_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma = a \cdot P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + b \cdot \lambda^h,$$

where λ^h is an arbitrary vector, a and b are certain functions of variables $x^1, x^2, \dots, x^n$ and $\lambda^1, \lambda^2, \dots, \lambda^n$. The tensor A^h_{ijk} is defined as

$$A^h_{ijk} \overset{\text{def}}{=} P^h_{ij,k} + P^\alpha_{ij}P^h_{\alpha k}.$$

Almost geodesic mappings of spaces with affine connections were introduced by N.S. Sinyukov [30–33]. He distinguished three kinds of almost geodesic mappings, namely π_1, π_2 and π_3 characterized, by following conditions for the deformation tensor.

A mapping $f : A_n \to \overline{A}_n$ is called

- an *almost geodesic of type* π_1, if it satisfies the following condition

$$A^h_{(ijk)} = \delta^h_{(i}a_{jk)} + b_{(i}P^h_{jk)}, \tag{2}$$

  where a_{ij} is a symmetric tensor, b_i is a covariant vector, and δ^h_i is the Kronecker delta. We denote by the round parentheses an operation called symmetrization without division with respect to the indices.

- an *almost geodesic of type* π_2, if the conditions $P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)}$, $F^h_{(i,j)} + F^h_\alpha F^\alpha_{(i}\varphi_{j)} = \delta^h_{(i}\mu_{j)} + F^h_{(i}\rho_{j)}$ holds. Here ψ_i, φ_i, μ_i, and ρ_i are covectors, F^h_i is a tensor of type $(1,1)$.

- an *almost geodesic of type* π_3, if the conditions $P^h_{ij} = \delta^h_{(i}\psi_{j)} + \theta^h a_{ij}$, $\theta^h_{,i} = \rho\delta^h_i + \theta^h a_i$ holds.
  Here θ^h is a torse forming vector, ψ_i, a_i are covectors, a_{ij} is a symmetric tensor and ρ is a function.

V. Berezovski and J. Mikeš [5, 6] proved that for $n > 5$ other types of almost geodesic mappings except π_1, π_2, and π_3 do not exist. The types of almost geodesic mappings π_1, π_2, π_3 can intersect. It is proved that if an almost geodesic mapping f is simultaneously π_1 and π_2, then f is a mapping of affine connection spaces with preserved linear complex of geodesic lines. If the mapping f is simultaneously π_1 and π_3, then f is a mapping of affine connection spaces with preserved quadratic complex of geodesic lines [6].

If in the equation (2) the condition $b_i \equiv 0$ holds, then the almost geodesic mapping π_1 is called *canonical*. Hence the conditions for the deformation tensor are given by

$$P^h_{(ij,k)} + P^\alpha_{(ij}P^h_{k)\alpha} = \delta^h_{(i}a_{jk)}. \tag{3}$$

It is known [33] that any almost geodesic mapping π_1 can be written as the composition of a canonical almost geodesic mapping of type π_1 and a geodesic mapping. The latter may be referred to as a trivial almost geodesic mapping and hence it can be neglected.

Also it is known that the equation (3) can be written in the form

$$3\left(P^h_{ij,k} + P^h_{k\alpha}P^\alpha_{ij}\right) = R^h_{(ij)k} - \overline{R}^h_{(ij)k} + \delta^h_{(k}a_{ij)}, \tag{4}$$

where R^h_{ijk} and $\overline{R}^h_{ijk}$ are the Riemann tensors of the spaces A_n and $\overline{A}_n$ respectively.

3. Canonical almost geodesic mappings of the first type of spaces with affine connections onto 2-Ricci-symmetric spaces

A space $\overline{A}_n$ with an affine connection is called 2-*Ricci-symmetric* if its Ricci tensor satisfies the condition

$$\overline{R}_{ij;km} = 0. \tag{5}$$

By the symbol " ; " we denote a covariant derivative with respect to the connection of the space $\overline{A}_n$.

Let us consider the canonical almost geodesic mappings of type π_1 of spaces A_n with affine connections onto 2-Ricci-symmetric spaces $\overline{A}_n$, which are determined by the equations (4).

Obviously, the equations (4) can be considered as a Cauchy type system of equations in the space A_n respective the unknown functions P^h_{ij}. The integrability conditions of (4) are

$$\begin{aligned}\overline{R}^h_{(ij)[k,l]} = R^h_{(ij)[k,l]} + \delta^h_{(i}a_{jk),l} - \delta^h_{(i}a_{jl),k} - 3(-P^\alpha_{ij}\overline{R}^h_{\alpha kl} + P^h_{\alpha j}R^\alpha_{ikl} + P^h_{i\alpha}R^\alpha_{jkl})\\ -P^h_{\alpha k}(R^\alpha_{(ij)l} - \overline{R}^\alpha_{(ij)l} + \delta^\alpha_{(i}a_{jl)}) + P^h_{\alpha l}(R^\alpha_{(ij)k} - \overline{R}^\alpha_{(ij)k} + \delta^\alpha_{(i}a_{jk)}),\end{aligned} \tag{6}$$

where by the brackets $[k,l]$ we denote an operation called antisymmetrization (or, alternation) without division with respect to the indices k and l.

Hence from (1) we have

$$\overline{R}^h_{ijk,l} = \overline{R}^h_{ijk;l} - P^h_{l\alpha}\overline{R}^\alpha_{ijk} + P^\alpha_{il}\overline{R}^h_{\alpha jk} + P^\alpha_{jl}\overline{R}^h_{i\alpha k} + P^\alpha_{kl}\overline{R}^h_{ij\alpha}. \tag{7}$$

Taking into account (7), in the left-hand side we replace the covariant derivative of the tensor $\overline{R}^h_{ijk}$ with respect to the connection of A_n by the covariant derivative of the same tensor with respect to the connection of $\overline{A}_n$. We have

$$\begin{aligned}\overline{R}^h_{(ij)[k;l]} = R^h_{(ij)[k,l]} + \delta^h_{(i}a_{jk),l} - \delta^h_{(i}a_{jl),k} - 3(-P^\alpha_{ij}\overline{R}^h_{\alpha kl} + P^h_{\alpha j}R^\alpha_{ikl} + P^h_{i\alpha}R^\alpha_{jkl})\\ -P^h_{\alpha k}(R^\alpha_{(ij)l} + \delta^\alpha_{(i}a_{jl)}) + P^h_{\alpha l}(R^\alpha_{(ij)k} + \delta^\alpha_{(i}a_{jk)}) - P^\alpha_{l(i}\overline{R}^h_{|\alpha|j)k} - P^\alpha_{l(i}\overline{R}^h_{j)\alpha k} + P^\alpha_{k(i}\overline{R}^h_{|\alpha|j)l} + P^\alpha_{k(i}\overline{R}^h_{j)\alpha l}.\end{aligned} \tag{8}$$

Using the Bianchi identity, we write (8) in the form

$$\overline{R}^h_{ilk;j} + \overline{R}^h_{jlk;i} = \delta^h_{(i}a_{jk),l} - \delta^h_{(i}a_{jl),k} + \rho^h_{ijkl}, \tag{9}$$

where

$$\begin{aligned}\rho^h_{ijkl} = R^h_{(ij)[k,l]} - 3(-P^\alpha_{ij}\overline{R}^h_{\alpha kl} + P^h_{\alpha j}R^\alpha_{ikl} + P^h_{i\alpha}R^\alpha_{jkl}) - P^h_{\alpha k}(R^\alpha_{(ij)l} + \delta^\alpha_{(i}a_{jl)}) + P^h_{\alpha l}(R^\alpha_{(ij)k} + \delta^\alpha_{(i}a_{jk)})\\ -P^\alpha_{l(i}\overline{R}^h_{|\alpha|j)k} - P^\alpha_{l(i}\overline{R}^h_{j)\alpha k} + P^\alpha_{k(i}\overline{R}^h_{|\alpha|j)l} + P^\alpha_{k(i}\overline{R}^h_{j)\alpha l}.\end{aligned}$$

From (1) we have also

$$(\overline{R}^h_{ilk;j})_{,m} = \overline{R}^h_{ilk;jm} - P^h_{\alpha m}\overline{R}^\alpha_{ilk;j} + P^\alpha_{im}\overline{R}^h_{\alpha lk;j} + P^\alpha_{lm}\overline{R}^h_{i\alpha k;j} + P^\alpha_{km}\overline{R}^h_{il\alpha;j} + P^\alpha_{jm}\overline{R}^h_{ilk;\alpha}. \tag{10}$$

We differentiate the equations (9) covariantly with respect to the connection of A_n and in the left-hand side in consequence of (10) express the covariant derivative with respect to the connection of A_n in terms of the covariant derivative with respect to the connection of $\overline{A}_n$. Taking account of (7), we obtain

$$\overline{R}^h_{ilk;jm} + \overline{R}^h_{jlk;im} = \delta^h_{(i}a_{jk),lm} - \delta^h_{(i}a_{jl),km} + \rho^h_{ijklm}, \tag{11}$$

where

$$\begin{aligned}\rho^h_{ijklm} = \rho^h_{ijkl,m} + P^h_{m\alpha}\Big(\overline{R}^\alpha_{(i|lk|,j)} - P^\alpha_{\beta(j}\overline{R}^\beta_{i)lk} + 2P^\beta_{ij}\overline{R}^\alpha_{\beta lk} + P^\beta_{l(j}\overline{R}^\alpha_{i)\beta k} + P^\beta_{k(j}\overline{R}^\alpha_{i)l\beta}\Big)\\ -P^\alpha_{ml}\Big(\overline{R}^h_{(i|\alpha k|,j)} - P^h_{\beta(j}\overline{R}^\beta_{i)\alpha k} + 2P^\beta_{ij}\overline{R}^h_{\beta\alpha k} + P^\beta_{\alpha(j}\overline{R}^h_{i)\beta k} + P^\beta_{k(j}\overline{R}^h_{i)\alpha\beta}\Big)\\ -P^\alpha_{mk}\Big(\overline{R}^h_{(i|l\alpha|,j)} - P^h_{\beta(j}\overline{R}^\beta_{i)l\alpha} + 2P^\beta_{ij}\overline{R}^h_{\beta l\alpha} + P^\beta_{l(j}\overline{R}^h_{i)\beta\alpha} + P^\beta_{\alpha(j}\overline{R}^h_{i)l\beta}\Big)\\ -P^\alpha_{mi}\Big(\overline{R}^h_{(\alpha|lk|,j)} - P^h_{\beta(j}\overline{R}^\beta_{\alpha)lk} + 2P^\beta_{\alpha j}\overline{R}^h_{\beta lk} + P^\beta_{l(\alpha}\overline{R}^h_{j)\beta k} + P^\beta_{k(j}\overline{R}^h_{\alpha)l\beta}\Big)\\ -P^\alpha_{mj}\Big(\overline{R}^h_{(i|lk|,\alpha)} - P^h_{\beta(i}\overline{R}^\beta_{\alpha)lk} + 2P^\beta_{\alpha i}\overline{R}^h_{\beta lk} + P^\beta_{l(\alpha}\overline{R}^h_{i)\beta k} + P^\beta_{k(i}\overline{R}^h_{\alpha)l\beta}\Big).\end{aligned} \tag{12}$$

We introduce the tensors a_{ijk} and $\overline{R}^h_{ijkm}$ defined by

$$a_{ij,k} = a_{ijk}, \tag{13}$$

$$\overline{R}^h_{ijk,m} = \overline{R}^h_{ijkm}. \tag{14}$$

We have assumed that in the right-hand side of (12) the covariant derivatives of the tensors P^h_{ij}, a_{ij}, $\overline{R}^h_{ijk}$ with respect to the connection of A_n are expressed in according to the formulas (4), (13) and (14). In other words, the tensor ρ^h_{ijklm} depends on the unknown tensors P^h_{ij}, a_{ij}, a_{ijk}, $\overline{R}^h_{ijk}$, $\overline{R}^h_{ijkm}$, and on some tensors, which are assumed to be known.

Since in this case the space $\overline{A}_n$ is 2-Ricci-symmetric, it follows that the Ricci tensor $\overline{R}_{ij}$ satisfies the condition (5).

Contracting the equations (11) for h and k, from (5) we get

$$(n+1)\, a_{ij,lm} - a_{li,jm} - a_{lj,im} = -\rho^{\alpha}_{ij\alpha lm}. \tag{15}$$

We symmetrize the equation (15) in the indices l and i. Then, interchanging the indices l and j, we get

$$a_{il,jm} + a_{lj,im} = -\frac{1}{n}\,\rho^{\alpha}_{(i|l\alpha|j)m} + \frac{2}{n}\,a_{ij,lm}. \tag{16}$$

The equation (15) because of (13) and (16) can be written in the form

$$\frac{n^2+n-2}{n}\,a_{ijl,m} = -\rho^{\alpha}_{ij\alpha lm} - \frac{1}{n}\,\rho^{\alpha}_{(i|l\alpha|j)m}. \tag{17}$$

Taking into account (13), we write (11) in the form

$$\overline{R}^h_{ilk;jm} + \overline{R}^h_{jlk;im} = \mu^h_{ilkjm}, \tag{18}$$

where

$$\mu^h_{ilkjm} = \delta^h_{(i}a_{jk)l,m} - \delta^h_{(i}a_{jl)k,m} + \rho^h_{ijklm}. \tag{19}$$

We have assumed that in the right-hand side of (19) the covariant derivatives of the tensor a_{ijk} with respect to the connection of A_n are expressed in according to (17). In other words, the tensor μ^h_{ilkjm} depends on the unknown tensors P^h_{ij}, a_{ij}, a_{ijk}, $\overline{R}^h_{ijk}$, $\overline{R}^h_{ijkm}$, and on some tensors, which are assumed to be known.

Let us alternate (18) with respect to the indicies i and m. Taking into account the Ricci identity we obtain

$$\overline{R}^h_{ilk;mj} - \overline{R}^h_{mlk;ij} = t^h_{ilkmj}, \tag{20}$$

$$\begin{aligned} t^h_{ilkmj} = \mu^h_{[i|lkj|m]} &- \overline{R}^{\alpha}_{ilk}\overline{R}^h_{\alpha mj} + \overline{R}^h_{\alpha lk}\overline{R}^{\alpha}_{imj} + \overline{R}^h_{i\alpha k}\overline{R}^{\alpha}_{lmj} + \overline{R}^h_{il\alpha}\overline{R}^{\alpha}_{kmj} \\ &- \overline{R}^{\alpha}_{mlk}\overline{R}^h_{\alpha ji} + \overline{R}^h_{\alpha lk}\overline{R}^{\alpha}_{mji} + \overline{R}^h_{m\alpha k}\overline{R}^{\alpha}_{lji} + \overline{R}^h_{ml\alpha}\overline{R}^{\alpha}_{kji} \\ &- \overline{R}^{\alpha}_{jlk}\overline{R}^h_{\alpha mi} + \overline{R}^h_{\alpha lk}\overline{R}^{\alpha}_{jmi} + \overline{R}^h_{j\alpha k}\overline{R}^{\alpha}_{lmi} + \overline{R}^h_{jl\alpha}\overline{R}^{\alpha}_{kmi}. \end{aligned}$$

Interchanging the indices m and j in the equation (20), we obtain

$$\overline{R}^h_{ilk;jm} - \overline{R}^h_{jlk;im} = t^h_{ilkjm}. \tag{21}$$

If the equations (18) and (21) be added, we have

$$2\overline{R}^h_{ilk;jm} = \mu^h_{ilkjm} + t^h_{ilkjm}. \tag{22}$$

In consequence of (7) and (10) in the left-hand side of (22) we express the covariant derivative with respect to the connection of $\overline{A}_n$ in terms of the covariant derivative with respect to the connection of A_n. Taking account of (14), we obtain

$$\begin{aligned} 2\overline{R}^h_{ilkj,m} = \mu^h_{ilkjm} + t^h_{ilkjm} - 2\Big(&P^h_{j\alpha,m}\overline{R}^\alpha_{ilk} - P^\alpha_{ij,m}\overline{R}^h_{\alpha lk} - P^\alpha_{lj,m}\overline{R}^h_{i\alpha k} - P^\alpha_{kj,m}\overline{R}^h_{il\alpha} \\ &+P^h_{j\alpha}\overline{R}^\alpha_{ilkm} - P^\alpha_{ij}\overline{R}^h_{\alpha lkm} - P^\alpha_{lj}\overline{R}^h_{i\alpha km} - P^\alpha_{kj}\overline{R}^h_{il\alpha m} \\ &+P^h_{\alpha m}\Big(\overline{R}^\alpha_{ilkj} + P^\alpha_{j\beta}\overline{R}^\beta_{ilk} - P^\beta_{ij}\overline{R}^\alpha_{\beta lk} - P^\beta_{lj}\overline{R}^\alpha_{i\beta k} - P^\beta_{kj}\overline{R}^\alpha_{il\beta}\Big) \\ &-P^\alpha_{im}\Big(\overline{R}^h_{\alpha lkj} + P^h_{j\beta}\overline{R}^\beta_{\alpha lk} - P^\beta_{\alpha j}\overline{R}^h_{\beta lk} - P^\beta_{lj}\overline{R}^h_{\alpha\beta k} - P^\beta_{kj}\overline{R}^h_{\alpha l\beta}\Big) \\ &-P^\alpha_{lm}\Big(\overline{R}^h_{i\alpha kj} + P^h_{j\beta}\overline{R}^\beta_{i\alpha k} - P^\beta_{ij}\overline{R}^h_{\beta\alpha k} + P^\beta_{\alpha j}\overline{R}^h_{i\beta k} + P^\beta_{kj}\overline{R}^h_{i\alpha\beta}\Big) \\ &-P^\alpha_{km}\Big(\overline{R}^h_{il\alpha j} + P^h_{\beta j}\overline{R}^\beta_{il\alpha} - P^\beta_{ij}\overline{R}^h_{\beta l\alpha} - P^\beta_{lj}\overline{R}^h_{i\beta\alpha} - P^\beta_{\alpha j}\overline{R}^h_{il\beta}\Big) \\ &-P^\alpha_{jm}\Big(\overline{R}^h_{ilk\alpha} + P^h_{\beta\alpha}\overline{R}^\beta_{ilk} - P^\beta_{i\alpha}\overline{R}^h_{\beta lk} - P^\beta_{l\alpha}\overline{R}^h_{i\beta k} - P^\beta_{k\alpha}\overline{R}^h_{il\beta}\Big)\Big). \end{aligned} \tag{23}$$

We have assumed that in the right-hand side of (23) the covariant derivatives of the tensor P^h_{ij} with respect to the connection of A_n are expressed in according to (4).

Obviously, in the space A_n the equations (4), (13), (14), (17) and (23) form a closed mixed system of PDEs of Cauchy type with respect to the functions $P^h_{ij}(x)$, $a_{ij}(x)$, $a_{ijk}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$. The functions must satisfy the algebraic conditions

$$P^h_{ij}(x) = P^h_{ji}(x), \qquad a_{ij}(x) = a_{ji}(x), \qquad \overline{R}^h_{i(jk)}(x) = \overline{R}^h_{(ijk)}(x) = 0. \tag{24}$$

Hence we have proved

Theorem 3.1. *In order that a space A_n with an affine connection admit an almost geodesic mappings of type π_1 onto a 2-Ricci-symmetric space $\overline{A}_n$, it is necessary and sufficient that the mixed system of differential equations of Cauchy type in covariant derivatives* (4), (13), (14), (17), (23) *and* (24) *have a solution with respect to the unknown functions* $P^h_{ij}(x)$, $a_{ij}(x)$, $a_{ijk}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$.

The general solution of the mixed system of Cauchy type (4), (13), (14), (17), (23) and (24) depends on no more than $\frac{1}{2}n(2n+1)(n+1) + \frac{1}{3}n^2(n+1)(n^2-1)$ essential parameters.

4. Canonical almost geodesic mappings of the first type of spaces with affine connections onto 3-Ricci-symmetric spaces

A space $\overline{A}_n$ with an affine connection is called *3-Ricci-symmetric* if its Ricci tensor satisfies the condition

$$\overline{R}_{ij;kml} = 0. \tag{25}$$

Let us consider the canonical almost geodesic mappings of type π_1 of spaces A_n with affine connections onto 3-Ricci-symmetric spaces $\overline{A}_n$, which are determined by the equations (4). Exploring the conditions of integrability of (4), we obtain the equations (11) and (23).

Because of (1) we have also

$$\Big(\overline{R}^h_{ilk;jm}\Big)_{,q} = \overline{R}^h_{ilk;jmq} - P^h_{\alpha q}\overline{R}^\alpha_{ilk;jm} + P^\alpha_{iq}\overline{R}^h_{\alpha lk;jm} + P^\alpha_{lq}\overline{R}^h_{i\alpha k;jm} + P^\alpha_{kq}\overline{R}^h_{il\alpha;jm} + P^\alpha_{jq}\overline{R}^h_{ilk;\alpha m} + P^\alpha_{mq}\overline{R}^h_{ilk;j\alpha}. \tag{26}$$

We differentiate the equations (11) covariantly with respect to the connection of A_n and in the left-hand side in consequence of (26) express the covariant derivative with respect to the connection of A_n in terms of the covariant derivative with respect to the connection of $\overline{A}_n$. We get

$$\overline{R}^h_{ilk;jmq} + \overline{R}^h_{jlk;imq} = \delta^h_{(i}a_{jk),lmq} - \delta^h_{(i}a_{jl),kmq} + \theta^h_{ilkjmq}, \tag{27}$$

where

$$\theta^h_{ilkjmq} = \rho^h_{ijklm,q} + P^h_{\alpha q}\overline{R}^\alpha_{(i|lk|;j)m} - P^\alpha_{q(i}\overline{R}^h_{(|\alpha lk|;j)m} - P^\alpha_{lq}\overline{R}^h_{(i|\alpha k|;j)m} - P^\alpha_{kq}\overline{R}^h_{(i|l\alpha|;j)m} - P^\alpha_{q(j}\overline{R}^h_{i)lk;\alpha m} - P^\alpha_{mq}\overline{R}^h_{(i|lk|;j)\alpha}. \quad (28)$$

We have assumed that in the right-hand side of (28) the corresponding covariant derivatives are expressed in according to (4), (7), (10), (13), (14) and (23). Hence, the tensor μ^h_{ilkjm} is expressed in terms of the unknown tensors P^h_{ij}, a_{ij}, a_{ijk}, $a_{ijk,m}$ $\overline{R}^h_{ijk}$, $\overline{R}^h_{ijkm}$, and of some tensors, which are assumed to be known.

We introduce the tensor a_{ijkm} defined by

$$a_{ijk,m} = a_{ijkm}. \quad (29)$$

Since the space $\overline{A}_n$ is 3-Ricci-symmetric, it follows that the Ricci tensor $\overline{R}_{ij}$ satisfies the condition (25).

Contracting the equations (27) for h and k, because of (25) we get

$$(n+1)\, a_{ij,lmq} - a_{li,jmq} - a_{lj,imq} = -\theta^\alpha_{il\alpha jmq}. \quad (30)$$

Symmetrizing the equation (30) in the indices l and i and interchanging the indices l and j, we get

$$a_{il,jmq} + a_{lj,imq} = -\frac{1}{n}\,\theta^\alpha_{(ij)\alpha lmq} + \frac{2}{n}\,a_{ij,lmq}. \quad (31)$$

In consequence of (13), (29) and (31) the equation (30) can be written in the form

$$\frac{n^2+n-2}{n}\,a_{ijlm,q} = -\theta^\alpha_{il\alpha jmq} - \frac{1}{n}\,\theta^\alpha_{(ij)\alpha lmq}. \quad (32)$$

Obviously, in the space A_n the equations (4), (13), (14), (23), (29) and (32) form a closed system of PDEs of Cauchy type with respect to the functions $P^h_{ij}(x)$, $a_{ij}(x)$, $a_{ijk}(x)$, $a_{ijkm}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$. The functions must satisfy the algebraic conditions (24).

Hence we have proved

Theorem 4.1. *In order that a space A_n with an affine connection admit an almost geodesic mappings of type π_1 onto a 3-Ricci-symmetric space $\overline{A}_n$, it is necessary and sufficient that the mixed system of differential equations of Cauchy type in covariant derivatives* (4), (13), (14), (23), (29), (32) *and* (24) *have a solution with respect to the unknown functions $P^h_{ij}(x)$, $a_{ij}(x)$, $a_{ijk}(x)$, $a_{ijkm}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$.*

The general solution of the mixed system of Cauchy type (4), (13), (14), (23), (29), (32) and (24) depends on no more than $\frac{1}{2}n(n+1)^2 + \frac{1}{3}n^2(n+1)(n^2-1)$ essential parameters.

5. Canonical almost geodesic mappings of the first type of spaces with affine connections onto *m*-Ricci-symmetric spaces

A space $\overline{A}_n$ with an affine connection is called *m-Ricci-symmetric* if its Ricci tensor satisfies the condition

$$\overline{R}_{ij;\rho_1\rho_2\ldots\rho_m} = 0. \quad (33)$$

Of course 2-Ricci-symmetric spaces and 3-Ricci-symmetric spaces are special cases of m-Ricci-symmetric spaces.

Let us consider the canonical almost geodesic mappings of type π_1 of spaces A_n with affine connections onto m-Ricci-symmetric spaces $\overline{A}_n$ ($m > 3$), which are determined by the equations (4). Exploring the conditions of integrability of (4), we obtain the equations (11) and (23).

Let us differentiate the equations (11) covariantly with respect to the connection of A_n $(m-2)$ times and in the left-hand side in consequence of (1) express the covariant derivative with respect to the connection of A_n in terms of the covariant derivative with respect to the connection of $\overline{A}_n$. We get

$$\overline{R}^h_{ilk;\rho_1\rho_2\ldots\rho_m} + \overline{R}^h_{\rho_1 lk;i\rho_2\ldots\rho_m} = \delta^h_{(i}a_{\rho_1 k),l\rho_2\ldots\rho_m} - \delta^h_{(i}a_{\rho_1 l),k\rho_2\ldots\rho_m} + \Omega^h_{ilk\rho_1\rho_2\ldots\rho_m}. \tag{34}$$

Taking into account (1), (4), (14) and (23), it is readily shown that the tensor $\Omega^h_{ilk\rho_1\rho_2\ldots\rho_m}$ depends on the unknown tensors P^h_{ij}, a_{ij}, a_{ij,ρ_1}, $a_{ij,\rho_1\rho_2}$, …, $a_{ij,\rho_1\rho_2\ldots\rho_{m-1}}$, $\overline{R}^h_{ijk}$, $\overline{R}^h_{ijkm}$, and on some tensors, which are assumed to be known.

Contracting (34) for h and k, because of (33) we get

$$(n+1)\,a_{i\rho_1,l\rho_2\ldots\rho_m} - a_{il,\rho_1\rho_2\ldots\rho_m} - a_{l\rho_1,i\rho_2\ldots\rho_m} = -\,\Omega^\alpha_{il\alpha\rho_1\rho_2\ldots\rho_m}. \tag{35}$$

Symmetrizing the equation (35) in the indices l and i and interchanging the indices l and ρ_1, we get

$$a_{il,\rho_1\rho_2\ldots\rho_m} + a_{l\rho_1,i\rho_2\ldots\rho_m} = -\,\frac{1}{n}\,\Omega^\alpha_{(i\rho_1)\alpha l\rho_2\ldots\rho_m} + \frac{2}{n}\,a_{i\rho_1,l\rho_2\ldots\rho_m}. \tag{36}$$

From the equation (35) in consequence of (36) we have

$$\frac{n^2+n-2}{n}\,a_{i\rho_1,l\rho_2\ldots\rho_m} = -\,\Omega^\alpha_{il\alpha\rho_1\rho_2\ldots\rho_m} - \frac{1}{n}\,\Omega^\alpha_{(i\rho_1)\alpha l\rho_2\ldots\rho_m}. \tag{37}$$

We introduce the tensors $a_{il\rho_1\rho_2\rho_3}, \ldots, a_{il\rho_1\rho_2\ldots\rho_{m-1}}$ defined by

$$a_{il\rho_1\rho_2\,,\,\rho_3} = a_{il\rho_1\rho_2\rho_3}, \quad \ldots\,, \quad a_{il\rho_1\rho_2\rho_3\ldots\rho_{m-2}\,,\,\rho_{m-1}} = a_{il\rho_1\rho_2\rho_3\ldots\rho_{m-2}\rho_{m-1}}. \tag{38}$$

Writing the equation (37) in consequence of (38), we obtain

$$\frac{n^2+n-2}{n}\,a_{i\rho_1 l\rho_2\ldots\rho_{m-1},\rho_m} = -\,\Omega^\alpha_{il\alpha\rho_1\rho_2\ldots\rho_{m-1}\rho_m} - \frac{1}{n}\,\Omega^\alpha_{(i\rho_1)\alpha l\rho_2\ldots\rho_{m-1}\rho_m}. \tag{39}$$

The right-hand side of (39) depends on the unknown tensors P^h_{ij}, a_{ij}, $a_{ij\rho_1}$, $a_{ij\rho_1\rho_2}$, …, $a_{ij\rho_1\rho_2\ldots\rho_{m-1}}$, $\overline{R}^h_{ijk}$, $\overline{R}^h_{ijkm}$.

Obviously, in the space A_n the equations (4), (13), (14), (23), (29), (38), (39) form a closed system of PDEs of Cauchy type with respect to the functions $P^h_{ij}(x)$, $a_{ij}(x)$, $a_{ij\rho_1}(x)$, $a_{ij\rho_1\rho_2}(x)$, …, $a_{ij\rho_1\rho_2\ldots\rho_{m-1}}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$. The functions must satisfy the algebraic conditions (24).

Hence we have proved

Theorem 5.1. *In order that a space A_n with an affine connection admit an almost geodesic mappings of type π_1 onto a m-Ricci-symmetric space $\overline{A}_n$, it is necessary and sufficient that the mixed system of differential equations of Cauchy type in covariant derivatives* (4), (13), (14), (23), (29), (38), (39) *and* (24) *have a solution with respect to the unknown functions $P^h_{ij}(x)$, $a_{ij}(x)$, $a_{ij\rho_1}(x)$, $a_{ij\rho_1\rho_2}(x)$, …, $a_{ij\rho_1\rho_2\ldots\rho_{m-1}}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$.*

Of course in the Theorem 5.1 we have assumed that $m > 3$, since other cases had been considered above.

The general solution of the mixed system of Cauchy type (4), (13), (14), (23), (29), (38), (39) and (24) depends on no more than $\frac{1}{2}n(n+1)(1+2n+n^2+n^3+\ldots+n^{m-1}) + \frac{1}{3}n^2(n+1)(n^2-1)$ essential parameters.

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