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Generalized Inverses and the Solutions of Constructed Equations

Chunmei Peng[a], Hainan Zhou[a], Junchao Wei[a]

[a] *School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu 225002, P.R.China*

Abstract. In this paper, by virtue of the properties of the generalized inverses of the elements in rings with involution, we construct the related equations. By discussing the solutions of these equations, the generalized invertible elements are characterized.

1. Introduction

Let R be an associative ring with 1, and let $a \in R$. a is said to be group invertible if there exists $a^{\#} \in R$ such that

$$aa^{\#}a = a, \qquad a^{\#}aa^{\#} = a^{\#}, \qquad aa^{\#} = a^{\#}a.$$

The element $a^{\#}$ is called a group inverse of a, which is uniquely determined by the above equations [1]. We denote the set of all group invertible elements of R by $R^{\#}$.

An involution in R is an anti-isomorphism $* : R \to R$, $a \mapsto a^*$ of degree 2, that is,

$$(a^*)^* = a, \qquad (a+b)^* = a^* + b^*, \qquad (ab)^* = b^*a^*.$$

If $a^*a = aa^*$, the element a is called normal [5]. In [6, 7], we discussed many properties of normal elements.

An element $a^+ \in R$ is called the Moore-Penrose inverse (or MP-inverse) [1] of a, if

$$aa^+a = a, \quad a^+aa^+ = a^+, \quad (aa^+)^* = aa^+, \quad (a^+a)^* = a^+a.$$

If a^+ exists, then it is unique.

Let $a \in R$. Then a is called partial isometry if $a = aa^*a$. Clearly, $a \in R^+$ is partial isometry if and only if $a^* = a^+$. Denote by R^{PI} the set of all partial isometries of R. If $a \in R^{\#} \cap R^+$ and $a^{\#} = a^+$, then a is called an EP element. We write R^{EP} to denote the set of all EP elements of R. If $a \in R^{EP} \cap R^{PI}$, then a is called a strongly EP element [8]. Also we write R^{SEP} for the set of all strongly EP elements of R.

In this paper, we provide some equivalent conditions for an element to be an EP element, normal element, partial isometry and strongly EP element by using solutions of some equations.

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Email addresses: pengchunmei0411@163.com (Chunmei Peng), 1909576069@qq.com (Hainan Zhou), jcweiyz@126.com (Junchao Wei)

2. Generalized inverses and the solutions of equation in a fixed set

Lemma 2.1. *([3])Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equavilent:*
1) $a \in R^{EP}$;
2) $a^{+} = aa^{+}a^{+}$;
3) $a^{+} = a^{+}a^{+}a$;
4) $a^{+}a = a^{\#}a$;
5) $aa^{+} = aa^{\#}$;
6) $a = a^{+}a^{2}$;
7) $a = a^{2}a^{+}$.

Lemma 2.2. *Let $a \in R^{\#} \cap R^{+}$. Then (1) $a^{+}a^{*}(a^{\#})^{*} = a^{+} = (a^{\#})^{*}a^{*}a^{+}$.*
(2) $(a^{+})^{*}aa^{\#} = (a^{+})^{*} = a^{\#}a(a^{+})^{*}$.

Lemma 2.3. *([3, Theorem 1.2.1]) Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{EP}$ if and only if $aa^{+}a^{\#} = a^{+}a^{\#}a$.*

Lemma 2.3 implies us to construct the following equation:

$$xa^{+}a^{\#} = a^{+}a^{\#}x. \tag{1}$$

Theorem 2.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{EP}$ if and only if the equation (1) has at least one solution in $\chi_a =: \{a, a^{\#}, a^{+}, a^{*}, (a^{\#})^{*}, (a^{+})^{*}\}$.*

Proof. *" $\Longrightarrow$ " Assume that $a \in R^{EP}$, then, by Lemma 2.3, we have $aa^{+}a^{\#} = a^{+}a^{\#}a$, which implies $x = a$ is a solution of the equation (1) in χ_a.*
" $\Longleftarrow$ " 1) If $x = a$ is a solution, then $aa^{+}a^{\#} = a^{+}a^{\#}a$. Hence we have $a \in R^{EP}$ by Lemma 2.3.

2) If $x = a^{\#}$, then $a^{\#}a^{+}a^{\#} = a^{+}a^{\#}a^{\#}$, that is $(a^{\#})^{3} = a^{+}(a^{\#})^{2}$. Multiplying the equality on the right by a^{3}, we have $a^{\#}a = a^{+}a$. By Lemma 2.1, one has $a \in R^{EP}$.

3) If $x = a^{+}$, then $a^{+}a^{+}a^{\#} = a^{+}a^{\#}a^{+}$, it follows that $a^{+}a^{\#}a^{+}(1-a^{+}a) = a^{+}a^{+}a^{\#}(1-a^{+}a) = 0$. Multiplying the equality on the left by $a^{*}a^{3}$, we have $a^{*}(1-a^{+}a) = 0$. Applying the involution on the last equality, we have $(1-a^{+}a)a = 0$, this gives $a = a^{+}a^{2}$. Hence $a \in R^{EP}$ by Lemma 2.1.

4) If $x = a^{*}$, then $a^{*}a^{+}a^{\#} = a^{+}a^{\#}a^{*}$. Multiplying the equality on the right by $1-aa^{+}$, we have $a^{*}a^{+}a^{\#}(1-aa^{+}) = 0$. Multiplying the last equality on the left by $a^{3}(a^{\#})^{*}$, it follows from Lemma 2.2 that $a(1-aa^{+}) = 0$, which implies $a = a^{2}a^{+}$. Hence $a \in R^{EP}$ by Lemma 2.1.

5) If $x = (a^{\#})^{*}$, then $(a^{\#})^{*}a^{+}a^{\#} = a^{+}a^{\#}(a^{\#})^{*}$. Multiplying the equality on the right by $1-aa^{+}$, we have $(a^{\#})^{*}a^{+}a^{\#}(1-aa^{+}) = a^{+}a^{\#}(a^{\#})^{*}(1-aa^{+}) = 0$. Multiplying the last equality on the left by $a^{3}a^{*}$, we have $a(1-aa^{+}) = 0$ by Lemma 2.2. It follows that $a = a^{2}a^{+}$, by Lemma 2.1 we have $a \in R^{EP}$.

6) If $x = (a^{+})^{*}$, then $(a^{+})^{*}a^{+}a^{\#} = a^{+}a^{\#}(a^{+})^{*}$. Multiplying the equality on the left by $1-a^{+}a$, we have $(1-a^{+}a)(a^{+})^{*}a^{+}a^{\#} = (1-a^{+}a)a^{+}a^{\#}(a^{\#})^{*} = 0$. Multiplying the last equality on the right by $a^{2}a^{*}a$, we have $(1-a^{+}a)a = 0$. Hence $a \in R^{EP}$ by Lemma 2.1. □

Lemma 2.5. *[2, Lemma 1.7] Let $a \in R^{+}$. If a is normal, then $a \in R^{EP}$.*

Lemma 2.6. *[8, Lemma 2.11] Let $a \in R^{\#} \cap R^{+}$ and $x \in R$.* (1) *If $xa^{+}a^{+} = 0$, then $xa^{+} = 0$.*
(2) *If $a^{+}a^{+}x = 0$, then $a^{+}x = 0$.*

Lemma 2.7. *Let $a \in R^{\#} \cap R^{+}$ and $x \in R$.* (1) *If $xa^{+}a^{*} = 0$, then $xa^{+} = 0$.*
(2) *If $a^{*}a^{+}x = 0$, then $a^{+}x = 0$.*

Proof. (1) *Since $a^{+}a^{+}a = a^{+}a^{*}(a^{+})^{*}$, $xa^{+}a^{+} = xa^{+}a^{+}(aa^{+}) = xa^{+}a^{*}(a^{+})^{*}a^{+} = 0$. By Lemma 2.6, we have $xa^{+} = 0$.*
(2) *Similar to* (1). □

Theorem 2.8. *Let $a \in R^{\#} \cap R^{+}$. Then a is normal if and only if the following equation* (2) *has at least one solution in χ_a.*

$$xa^{+}a^{*} = a^{\#}a^{*}x. \tag{2}$$

Proof. " $\Longrightarrow$ " Assume that a is normal, then $aa^* = a^*a$ and $a \in R^{EP}$ by Lemma 2.5. Hence $a^{\#}a^*a = a^{\#}aa^* = aa^{\#}a^* = aa^{+}a^*$, it follows that $x = a$ is a solution of the equation (2) in χ_a.

" $\Longleftarrow$ " (1) If $x = a$ is a solution, then $aa^{+}a^* = a^{\#}a^*a$. Multiplying the equality on the right by aa^{+}, we have $a^{\#}a^*a = a^{\#}a^*a^2a^{+}$. Multiplying the last equality on the left by $(a^{+})^*a^{+}a^2$, we have $a = a^2a^{+}$. By Lemma 2.1, one has $a \in R^{EP}$. It follows that $a^* = a^{+}aa^* = aa^{+}a^* = a^{\#}a^*a$, hence $aa^* = aa^{\#}a^*a = a^{\#}aa^*a = a^{+}aa^*a = a^*a$, which implies a is normal.

(2) If $x = a^{\#}$, then $a^{\#}a^{+}a^* = a^{\#}a^*a^{\#}$. Multiplying the equality on the left by $a^{+}a^2$, we have $a^{+}a^* = a^*a^{\#}$. Hence $a \in R^{EP}$ by [3, Theorem 2.1]. It follows that $a^* = a^{+}aa^* = aa^{+}a^* = aa^*a^{\#}$ and $aa^* = aa^*a^{\#}a = a^*a$. Thus a is normal.

(3) If $x = a^{+}$, then $a^{+}a^{+}a^* = a^{\#}a^*a^{+}$. Multiplying the equality on the left by aa^{+}, we have $a^{+}a^{+}a^* = aa^{+}a^{+}a^{+}a^*$. By Lemma 2.7, one gets $a^{+}a^{+} = aa^{+}a^{+}a^{+}$. By Lemma 2.6, one has $a^{+} = aa^{+}a^{+}$. By Lemma 2.1 we obtain $a \in R^{EP}$. Hence $x = a^{+} = a^{\#}$ is a solution, by (2), a is normal.

(4) If $x = a^*$, then $a^*a^{+}a^* = a^{\#}a^*a^*$, this gives $(a^* - a^{\#}a^*a)a^{+}a^* = 0$. By Lemma 2.7 we have $(a^* - a^{\#}a^*a)a^{+} = 0$, that is $a^*a^{+} = a^{\#}a^*$. Similar to the proof of (2) we know that a is normal.

(5) If $x = (a^{+})^*$, then $(a^{+})^*a^{+}a^* = a^{\#}a^*(a^{+})^* = a^{\#}$, it follows that $a^*a^{\#} = a^*(a^{+})^*a^{+}a^* = a^{+}aa^{+}a^* = a^{+}a^*$. By (2), we have a is normal.

(6) If $x = (a^{\#})^*$, then $(a^{\#})^*a^{+}a^* = a^{\#}a^*(a^{\#})^*$. Multiplying the equality on the left by $a^{+}a$, we have $a^{\#}a^*(a^{\#})^* = a^{+}aa^{\#}a^*(a^{\#})^*$. Multiplying the equality on the right by $a^{+}a$, we have $a^{\#} = a^{+}aa^{\#}$. Hence $a \in R^{EP}$, this gives $x = (a^{\#})^* = (a^{+})^*$ is a solution. By (5), a is normal. □

Theorem 2.9. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{EP}$ if and only if the following equation* (3) *has at least one solution in $\chi_a^2 =: \{(u, v)|u, v \in \chi_a\}$.*

$$xya^{+} = a^{\#}yx. \tag{3}$$

Proof. " $\Longrightarrow$ " Assume that $a \in R^{EP}$. Choose $x = y = a$, then $a^2a^{+} = a = a^{\#}a^2$. Hence $(x, y) = (a, a)$ is the solution of the equation (3).

" $\Longleftarrow$ " (1) If $(x, y) = (a, a)$ is a solution, then $a^2a^{+} = a^{\#}a^2 = a$. By Lemma 2.1, $a \in R^{EP}$.

(2) If $(x, y) = (a^{\#}, a)$, then $a^{\#}aa^{+} = a^{\#}aa^{\#} = a^{\#}$. By Lemma 2.1, $a \in R^{EP}$.

(3) If $(x, y) = (a^{+}, a)$, then $a^{+} = a^{+}aa^{+} = a^{\#}aa^{+}$. By Lemma 2.1, $a \in R^{EP}$.

(4) If $(x, y) = (a^*, a)$, then $a^* = a^*aa^{+} = a^{\#}aa^*$. By [3, Theorem 2.1], $a \in R^{EP}$.

(5) If $(x, y) = ((a^{+})^*, a)$, then $(a^{+})^*aa^{+} = a^{\#}a(a^{+})^*$. By Lemma 2.2, one gets $(a^{+})^*aa^{+} = (a^{+})^*$. Applying the involution on the equality, we have $a^{+} = aa^{+}a^{+}$. By Lemma 2.1, $a \in R^{EP}$.

(6) If $(x, y) = ((a^{\#})^*, a)$, then $(a^{\#})^* = (a^{\#})^*aa^{+} = a^{\#}a(a^{\#})^*$. Multiplying the equality on the right by $(a^*)^2$, we have $a^* = a^{\#}aa^*$. By (4), $a \in R^{EP}$.

(7) If $(x, y) = (a, a^{\#})$, then $aa^{\#}a^{+} = a^{\#}a^{\#}a = a^{\#}$. By Lemma 2.1, $a \in R^{EP}$.

(8) If $(x, y) = (a^{\#}, a^{\#})$, then $a^{\#}a^{\#}a^{+} = a^{\#}a^{\#}a^{\#}$. Multiplying the equality on the left by a^3, we have $aa^{+} = aa^{\#}$. By Lemma 2.1, $a \in R^{EP}$.

(9) If $(x, y) = (a^{+}, a^{\#})$, then $a^{+}a^{\#}a^{+} = a^{\#}a^{\#}a^{+}$. Multiplying the equality on the right by a^3, we have $a^{+}a = a^{\#}a$. By Lemma 2.1, one yields $a \in R^{EP}$.

(10) If $(x, y) = (a^*, a^{\#})$, then $a^*a^{\#}a^{+} = a^{\#}a^{\#}a^*$. Multiplying the equality on the left by $a^{+}a$, we have $a^{\#}a^{\#}a^* = a^{+}a^{\#}a^*$. Multiplying the last equality on the right by $(a^{+})^*a^2$, we have $a^{\#}a = a^{+}a$. By Lemma 2.1, $a \in R^{EP}$.

(11) If $(x, y) = ((a^{+})^*, a^{\#})$, then $(a^{+})^*a^{\#}a^{+} = a^{\#}a^{\#}(a^{+})^*$. Multiplying the equality on the right by aa^{+}, we have $a^{\#}a^{\#}(a^{+})^* = a^{\#}a^{\#}(a^{+})^*aa^{+}$. Multiplying the last equality on the left by a^2, one gets $(a^{+})^* = (a^{+})^*aa^{+}$ by Lemma 2.2. Hence we have $a^{+} = aa^{+}a^{+}$. By Lemma 2.1, $a \in R^{EP}$.

(12) If $(x, y) = ((a^{\#})^*, a^{\#})$, then $(a^{\#})^*a^{\#}a^{+} = a^{\#}a^{\#}(a^{\#})^*$. Multiplying the equality on the left by $a^{+}a$, we have $a^{\#}a^{\#}(a^{\#})^* = a^{+}a^{\#}(a^{\#})^*$. Multiplying the last equality on the right by $a^*a^{+}a$, we have $a^{\#}a^{\#} = a^{+}a^{\#}$. Hence $a \in R^{EP}$ by [3, Theorem 2.1].

(13) *If* $(x, y) = (a, a^+)$, *then* $aa^+a^+ = a^\#a^+a = a^\#$. *Hence* $a \in R^{EP}$ *by [3, Theorem 2.1].*

(14) *If* $(x, y) = (a^\#, a^+)$, *then* $a^\#a^+a^+ = a^\#a^+a^\#$. *Multiplying the equality on the right by* aa^+, *we have* $a^\#a^+a^\# = a^\#a^+a^\#aa^+$. *Multiplying the equality on the left by* a^2, *we have* $a^\#aa^+ = a^\#$. *Hence* $a \in R^{EP}$.

(15) *If* $(x, y) = (a^+, a^+)$, *then* $a^+a^+a^+ = a^\#a^+a^+$. *By Lemma 2.6, we have* $a^+a^+ = a^\#a^+$. *Multiplying the equality on the right by a, we have* $a^+a^+a = a^\#$. *Then multiplying the equality on the left by* a^+a, *we have* $a^+aa^\# = a^\#$. *Hence* $a \in R^{EP}$ *by [3, Theorem 2.1].*

(16) *If* $(x, y) = (a^*, a^+)$, *then* $a^*a^+a^+ = a^\#a^+a^*$. *Multiplying the equality on the left by* a^+a, *we have* $a^+aa^\#a^+a^* = a^\#a^+a^*$. *Multiplying the last equality on the right by* $(a^\#)^*a$, *one gets* $a^+aa^\# = a^\#$ *by Lemma 2.2. Hence* $a \in R^{EP}$.

(17) *If* $(x, y) = ((a^+)^*, a^+)$, *then* $(a^+)^*a^+a^+ = a^\#a^+(a^+)^*$. *Multiplying the equality on the right by* a^+a, *we have* $(a^+)^*a^+a^+ = (a^+)^*a^+a^+a^+a$. *Multiplying the last equality on the left by* a^*, *one obtains* $a^+a^+a^+a = a^+a^+$. *By Lemma 2.6,* $a^+a^+a = a^+$. *Hence* $a \in R^{EP}$ *by Lemma 2.1.*

(18) *If* $(x, y) = ((a^\#)^*, a^+)$, *then* $(a^\#)^*a^+a^+ = a^\#a^+(a^\#)^*$. *Multiplying the equality on the left by* a^+a, *we have* $a^\#a^+(a^\#)^* = a^+aa^\#a^+(a^\#)^*$. *Multiplying the last equality on the right by* a^*a, *we have* $a^+aa^\# = a^\#$. *Hence* $a \in R^{EP}$.

(19) *If* $(x, y) = (a, a^*)$, *then* $aa^*a^+ = a^\#a^*a$. *Multiplying the equality on the right by* a^+a, *we have* $aa^*a^+ = aa^*a^+a^+a$. *Multiplying the last equality on the left by* $a^+(a^+)^*a^+$, *we have* $a^+a^+ = a^+a^+a^+a$. *By Lemma 2.6, one has* $a^+ = a^+a^+a$. *Hence* $a \in R^{EP}$ *by Lemma 2.1.*

(20) *If* $(x, y) = (a^\#, a^*)$, *then* $a^\#a^*a^+ = a^\#a^*a^\#$. *Multiplying the equality on the left by* $a^+(a^+)^*a^+a^2$, *we have* $a^+a^+ = a^+a^\#$. *It follows from [3, Theorem 2.1] that* $a \in R^{EP}$.

(21) *If* $(x, y) = (a^+, a^*)$, *then* $a^+a^*a^+ = a^\#a^*a^+$. *By Lemma 2.6, one has* $a^+a^* = a^\#a^*$. *Hence* $a \in R^{EP}$ *by [3, Theorem 2.1].*

(22) *If* $(x, y) = (a^*, a^*)$, *then* $a^*a^*a^+ = a^\#a^*a^*$. *Multiplying the equality on the left by* a^+a, *we have* $a^\#a^*a^* = a^+aa^\#a^*a^*$. *Multiplying the equality on the right by* $(a^\#)^*(a^\#)^*a^+a$, *we have* $a^\#a^+a = a^+aa^\#a^+a$, *that is* $a^\# = a^+aa^\#$. *Hence* $a \in R^{EP}$ *by Lemma 2.1.*

(23) *If* $(x, y) = ((a^+)^*, a^*)$, *then* $(a^+)^*a^*a^+ = a^\#a^*(a^+)^*$, *that is* $aa^+a^+ = a^\#a^+a = a^\#$. *Hence* $a \in R^{EP}$.

(24) *If* $(x, y) = ((a^\#)^*, a^*)$, *then* $(a^\#)^*a^*a^+ = a^\#a^*(a^\#)^*$, *that is* $a^+ = a^\#a^*(a^\#)^*$. *Multiplying the equality on the left by* $1 - aa^+$, *we have* $(1 - aa^+)a^+ = (1 - aa^+)a^\#a^*(a^\#)^* = 0$, *this gives* $a^+ = aa^+a^+$. *Hence* $a \in R^{EP}$ *by Lemma 2.1.*

(25) *If* $(x, y) = (a, (a^+)^*)$, *then* $a(a^+)^*a^+ = a^\#(a^+)^*a$. *Multiplying the equality on the right by* $1 - aa^+$, *we have* $a^\#(a^+)^*a(1 - aa^+) = 0$. *Multiplying the last equality on the left by* $a^\#aa^*a$, *we have* $a(1 - aa^+) = 0$, *that is* $a = a^2a^+$. *Hence* $a \in R^{EP}$.

(26) *If* $(x, y) = (a^\#, (a^+)^*)$, *then* $a^\#(a^+)^*a^+ = a^\#(a^+)^*a^\#$. *Multiplying the equality on the left by* a^*a, *we have* $a^+ = a^+aa^\#$. *It follows from [3, Theorem 2.1] that* $a \in R^{EP}$.

(27) *If* $(x, y) = (a^+, (a^+)^*)$, *then* $a^+(a^+)^*a^+ = a^\#(a^+)^*a^+$. *Multiplying the equality on the right by* aa^*, *we have* $a^+ = a^\#aa^+$. *Hence* $a \in R^{EP}$ *by [3, Theorem 2.1].*

(28) *If* $(x, y) = (a^*, (a^+)^*)$, *then* $a^*(a^+)^*a^+ = a^\#(a^+)^*a^*$, *that is* $a^+ = a^\#aa^+$. *Hence* $a \in R^{EP}$.

(29) *If* $(x, y) = ((a^+)^*, (a^+)^*)$, *then* $(a^+)^*(a^+)^*a^+ = a^\#(a^+)^*(a^+)^*$. *Multiplying the equality on the right by* a^+a, *we have* $(a^+)^*(a^+)^*a^+ = (a^+)^*(a^+)^*a^+a^+a$. *Applying the involution on the equality, we have* $(a^+)^*a^+a^+ = a^+a(a^+)^*a^+a^+$. *By Lemma 2.6 we have* $(a^+)^*a^+ = a^+a(a^+)^*a^+$. *Again applying the involution on the last equality, we have* $(a^+)^*a^+ = (a^+)^*a^+a^+a$. *Then multiplying the equality on the left by* a^*, *we have* $a^+ = a^+a^+a$. *By Lemma 2.1,* $a \in R^{EP}$.

(30) *If* $(x, y) = ((a^\#)^*, (a^+)^*)$, *then* $(a^\#)^*(a^+)^*a^+ = a^\#(a^+)^*(a^\#)^*$. *Multiplying the equality on the left by* a^+a, *we have* $a^\#(a^+)^*(a^\#)^* = a^+aa^\#(a^+)^*(a^\#)^* = a^+(a^+)^*(a^\#)^*$. *Multiplying the last equality on the right by* $(a^*)^2$, *we have* $a^\#aa^+ = a^+$. *Hence* $a \in R^{EP}$.

(31) *If* $(x, y) = (a, (a^\#)^*)$, *then* $a(a^\#)^*a^+ = a^\#(a^\#)^*a$. *Multiplying the equality on the right by* aa^+, *we have* $a^\#(a^\#)^*a = a^\#(a^\#)^*a^2a^+$. *Then multiplying the last equality on the left by* a^+a^2, *we have* $(a^\#)^*a = (a^\#)^*a^2a^+$. *Again multiplying the last equality on the left by* aa^+a^*, *we have* $a = a^2a^+$. *By Lemma 2.1,* $a \in R^{EP}$.

(32) *If* $(x, y) = (a^\#, (a^\#)^*)$, *then* $a^\#(a^\#)^*a^+ = a^\#(a^\#)^*a^\#$. *Multiplying the equality on the right by* aa^+, *we have* $a^\#(a^\#)^*a^\# = a^\#(a^\#)^*a^\#aa^+$. *Multiplying the last equality on the left by* $aa^+a^*a^+a^2$, *we have* $a^\# = aa^+a^\# = aa^+a^\#aa^+ = a^\#aa^+$. *It follows that* $a \in R^{EP}$.

(33) *If* $(x, y) = (a^+, (a^\#)^*)$, *then* $a^+(a^\#)^*a^+ = a^\#(a^\#)^*a^+$. *Multiplying the equality on the right by* aa^*, *we have* $a^+ = a^\#(a^\#)^*a^*$. *Then multiplying the last equality on the left by* $1 - aa^+$, *we have* $(1 - aa^+)a^+ = (1 - aa^+)a^\#(a^\#)^*a^* = 0$. *Hence* $a \in R^{EP}$.

(34) *If* $(x, y) = (a^*, (a^\#)^*)$, *then* $a^*(a^\#)^*a^+ = a^\#(a^\#)^*a^*$. *By Lemma 2.2, we have* $a^+ = a^\#(a^\#)^*a^*$. *By* (33), *we have* $a \in R^{EP}$.

(35) *If* $(x, y) = ((a^+)^*, (a^\#)^*)$, *then* $(a^+)^*(a^\#)^*a^+ = a^\#(a^\#)^*(a^+)^*$. *Multiplying the equality on the right by* a^+a, *we have* $(a^+)^*(a^\#)^*a^+ = (a^+)^*(a^\#)^*a^+a^+a$. *Then multiplying the last equality on the left by* a^*a^*, *we have* $a^+ = a^+a^+a$ *by Lemma 2.2. Hence* $a \in R^{EP}$ *by Lemma 2.1.*

(36) *If* $(x, y) = ((a^\#)^*, (a^\#)^*)$, *then* $(a^\#)^*(a^\#)^*a^+ = a^\#(a^\#)^*(a^\#)^*$. *Multiplying the last equality on the left by* a^+a, *we have* $a^\#(a^\#)^*(a^\#)^* = a^+aa^\#(a^\#)^*(a^\#)^*$. *Then multiplying the equality on the right by* $(a^*)^3(a^+)^*$, *we have* $a^\# = a^+aa^\#$. *Hence* $a \in R^{EP}$. □

3. Generalized inverses and the general solutions of equation

The equation (1) can be generalized as follows

$$xa^+a^\# = a^+a^\#y. \tag{4}$$

Lemma 3.1. *Let* $a \in R^\# \cap R^+$. *Then* $a^+a^\# \in R^{EP}$, *and* $(a^+a^\#)^+ = a^+a^3$.

Proof. *It is a routine verification.* □

Theorem 3.2. *Let* $a \in R^+ \cap R^\#$. *Then the general solution of the equation* (4) *is given by the following formula.*

$$\begin{cases} x = -a^+apa^+a^3 + u - ua^+a \\ quy = -a^+a^3pa^+a + z - a^+az \end{cases}, \text{where } p, u, z \in R. \tag{5}$$

Proof. *First, we prove that the formula* (5) *is the solution of the equation* (4).

In fact, $(-a^+apa^+a^3 + u - ua^+a)a^+a^\# = -a^+apa^+a + ua^+a^\# - ua^+aa^+a^\# = -a^+apa^+a$;

$a^+a^\#(-a^+a^3pa^+a + z - a^+az) = -a^+apa^+a + a^+a^\#z - a^+a^\#a^+az = -a^+apa^+a$;

Hence the formula (5) *is the solution of the equation* (4).

Next, we prove that all solutions of the euqation (4) *can be written in the form of* (5).

Suppose that $x = x_0, y = y_0$ *is a solution of the equation* (4), *then* $x_0a^+a^\# = a^+a^\#y_0$. *By Lemma 3.1, we have*

$-a^+a(-a^+a^\#y_0)a^+a^3 + x_0 - x_0a^+a = a^+a^\#y_0a^+a^3 + x_0 - x_0a^+a = x_0a^+a^\#a^+a^3 + x_0 - x_0a^+a = x_0a^+a + x_0 - x_0a^+a = x_0$;

$-a^+a^3(-x_0a^+a^\#)a^+a + y_0 - a^+ay_0 = a^+a^3x_0a^+a^\# + y_0 - a^+ay_0 = a^+a^3(a^+a^\#y_0) + y_0 - a^+ay_0 = a^+ay_0 + y_0 - a^+ay_0 = y_0$.

Hence the general solution of the equation (4) *is given by the formula* (5). □

The formula (5) can be changed as follows.

$$\begin{cases} x = -aa^+pa^+a^3 + u - ua^+a \\ y = -a^+a^3pa^+a + z - a^+az \end{cases}, \text{where } p, u, z \in R. \tag{6}$$

Corollary 3.3. *Let* $a \in R^\# \cap R^+$. *Then* $a \in R^{EP}$ *if and only if the general solution of the equation* (4) *is given by formula* (6).

Proof. ″ $\Longrightarrow$ ″ *Assume that* $a \in R^{EP}$, *then* $aa^+ = a^+a$. *Hence the formula* (5) *is same as* (6). *By Theorem 3.2, we know that the general solution of the equation* (4) *is given by formula* (6).

″ $\Longleftarrow$ ″ *Assume that the general solution of the equation* (4) *is given by formula* (6). *Then*

$$(-aa^+pa^+a^3 + u - ua^+a)a^+a^\# = a^+a^\#(-a^+a^3pa^+a + z - a^+az).$$

Hence $aa^+pa^+a = a^+apa^+a$ *for any* $p \in R$. *Especially choose* $p = a$, *we have* $a = a^+a^2$. *Hence* $a \in R^{EP}$ *by Lemma 2.1.* □

Similarly, we have the following corollary.

Corollary 3.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{EP}$ if and only if the general solution of the equation* (4) *is given by*

$$\begin{cases} x = -a^{+}apa^{+}a^{3} + u - ua^{+}a \\ y = -a^{+}a^{3}paa^{+} + z - a^{+}az \end{cases}, \text{where } p, u, z \in R. \tag{7}$$

Theorem 3.5. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if the general solution of the equation* (4) *is given by*

$$\begin{cases} x = -a^{*}apa^{+}a^{3} + u - ua^{+}a \\ y = -a^{+}a^{3}pa^{+}a + z - a^{+}az \end{cases}, \text{where } p, u, z \in R. \tag{8}$$

Proof. $'' \Longrightarrow ''$ Assume that $a \in R^{PI}$, then $a^{+} = a^{*}$, this infers the formula (5) is same as (8). Hence, by Theorem 3.2, one knows that the general solution of equation (4) is given by the formula (8).

$'' \Longleftarrow ''$ Assume that the formula (8) is the general solution of the equation (4). Then

$$(-a^{*}apa^{+}a^{3} + u - ua^{+}a)a^{+}a^{\#} = a^{+}a^{\#}(-a^{+}a^{3}pa^{+}a + z - a^{+}az).$$

It follows that $a^{*}apa^{+}a = a^{+}apa^{+}a$ for all $p \in R$. Especially, choose $p = 1$, we have $a^{*}a = a^{+}a$. Hence $a \in R^{PI}$ by [5, Theorem 2.2]. □

4. Generalized and inverse problem of the general solution

We don't know the general solution of which equation is given by the formula (6) and (7). However, we can construct the following equation.

$$xa^{+}a^{\#} = aa^{+}a^{+}a^{\#}y. \tag{9}$$

The following lemma follows from Lemma 2.2.

Lemma 4.1. *Let $a \in R^{\#} \cap R^{+}$. Then $(aa^{\#})^{*}aa^{+}a^{+} = a^{+} = a^{+}a^{+}a(aa^{\#})^{*}$.*

Theorem 4.2. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution of the equation* (9) *is given by*

$$\begin{cases} x = -aa^{+}pa^{+}a^{3} + u - ua^{+}a \\ y = -a^{+}a^{3}pa^{+}a + z - a^{+}az \end{cases}, \text{where } u, z \in R \text{ and } p \in R \text{ with } a^{+}p = a^{+}a^{+}ap. \tag{10}$$

Proof. First, the formula (10) *is the general solution of the equation* (9)*. In fact,*

$$(-aa^{+}pa^{+}a^{3} + u - ua^{+}a)a^{+}a^{\#} = -aa^{+}pa^{+}a = -aa^{+}a^{+}apa^{+}a = aa^{+}a^{+}a^{\#}(-a^{+}a^{3}pa^{+}a + z - a^{+}az).$$

Next, let $x = x_0$ and $y = y_0$ be a solution of the equation (9)*. Then $x_0a^{+}a^{\#} = aa^{+}a^{+}a^{\#}y_0$. Choose $p = -a^{+}a^{\#}y_0$, $u = x_0$ and $z = y_0$. Then by Lemma 4.1, we have*

$$-aa^{+}pa^{+}a^{3} + u - ua^{+}a = -aa^{+}(-a^{+}a^{\#}y_0)a^{+}a^{3} + x_0 - x_0a^{+}a = x_0a^{+}a + x_0 - x_0a^{+}a = x_0,$$

$$-a^{+}a^{3}pa^{+}a + z - a^{+}az = -a^{+}a^{3}(-a^{+}a^{\#}y_0)a^{+}a + y_0 - a^{+}ay_0 = a^{+}a^{3}(aa^{\#})^{*}(aa^{+}a^{+}a^{\#}y_0)a^{+}a + y_0$$
$$-a^{+}ay_0 = a^{+}a^{3}(aa^{\#})^{*}(x_0a^{+}a^{\#})a^{+}a + y_0 - a^{+}ay_0 = a^{+}a^{3}(aa^{\#})^{*}(x_0a^{+}a^{\#}) + y_0 - a^{+}ay_0$$
$$= a^{+}a^{3}(a^{+}a^{\#}y_0) + y_0 - a^{+}ay_0 = a^{+}ay_0 + y_0 - a^{+}ay_0 = y_0,$$

$$a^{+}a^{+}ap = a^{+}a^{+}a(-a^{+}a^{\#}y_0) = -a^{+}a^{+}a^{\#}y_0 = a^{+}p.$$

Hence the general solution of the equation (9) *is given by the formula* (10). □

Question 4.3. *Let $a \in R^{\#} \cap R^{+}$. We consider if p which appears in the formula* (10) *takes all the elements of R, then $a \in R^{EP}$?*

$$xa^{+}a^{\#}(aa^{\#})^{*} = a^{+}a^{\#}y. \tag{11}$$

Similar to Theorem 4.1, we have the following theorem.

Theorem 4.4. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution of the equation* (11) *is given by*

$$\begin{cases} x = -a^{+}apa^{+}a^{3} + u - ua^{+}a \\ y = -a^{+}a^{3}paa^{+} + z - a^{+}az \end{cases}, where \ \ u, z \in R \ \ and \ \ p \in R \ \ with \ \ pa^{+} = paa^{+}a^{+}. \tag{12}$$

5. Generalized inverses and invertible elements

Lemma 4.1 implies the following lemma.

Lemma 5.1. *Let $a \in R^{\#} \cap R^{+}$. Then $(aa^{\#})^{*}a \in R^{EP}$ and $((aa^{\#})^{*}a)^{+} = a^{+}a^{+}a$.*

It is well known that if $a \in R^{\#}$, then $a + 1 - aa^{\#} \in R^{-1}$ and $(a + 1 - aa^{\#})^{-1} = a^{\#} + 1 - aa^{\#}$. Hence Lemma 5.1 leads to the following lemma.

Lemma 5.2. *Let $a \in R^{\#} \cap R^{+}$. Then $(aa^{\#})^{*}a + 1 - a^{+}a \in R^{-1}$ and $((aa^{\#})^{*}a + 1 - a^{+}a)^{-1} = a^{+}a^{+}a + 1 - a^{+}a$.*

The following theorem follows from Lemma 2.1 and Lemma 5.2.

Theorem 5.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{EP}$ if and only if $((aa^{\#})^{*}a + 1 - a^{+}a)^{-1} = a^{+} + 1 - a^{+}a$.*

Noting that for any $a \in R^{\#} \cap R^{+}$, $a \in R^{PI}$ if and only if $a^{*}a^{+}a = a^{+}a^{+}a$. Hence we have the following theorem by Lemma 5.2.

Theorem 5.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if $((aa^{\#})^{*}a + 1 - a^{+}a)^{-1} = a^{*}a^{+}a + 1 - a^{+}a$.*

Since $a \in R^{SEP}$ if and only if $a^{*} = a^{*}a^{+}a$, Theorem 5.4 infers the following corollary.

Corollary 5.5. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $((aa^{\#})^{*}a + 1 - a^{+}a)^{-1} = a^{*} + 1 - a^{+}a$.*

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