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The Second Regularized Trace of Even Order Differential Operators with Operator Coefficient

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Abstract.
In this paper, we investigate the spectrum of the self adjoint operator L defined by

$$L := (-1)^r \frac{d^{2r}}{dx^{2r}} + A + Q(x),$$

where A is a self adjoint operator, $Q(x)$ is a nuclear operator in a separable Hilbert space, and we derive asymptotic formulas for the sum of eigenvalues of the operator L.

1. Introduction

The theory of regularized traces of differential operators began with the study of Gelfand and Levitan [12]. They calculated the trace formula for the sum of substraction of eigenvalues of two self adjoint operators. After this primary work, many mathematicians concentrated on this theory in a large scale.
Dikiy [13], Halberg and Kramer [4], Levitan [2] and some others studied the regularized traces of scalar differential operators. The list of works on the subject was given in the works Levitan and Sargsyan [3] and Fulton and Pruess [16], but a few of these works were about the regularized trace of differential operators with operator coefficient. Chalilova [15] calculated regularized trace of Sturm Lioville operator with bounded operator coefficient. Adıgüzelov [5] computed regularized trace of the difference of two Sturm-Liouville operators with bounded operator coefficient given in the semi-axis. Maksudov, Bayramoglu and Adıgüzelov [10] found a formula for the regularized trace of Sturm-Liouville operators with unbounded operator coefficient under the Dirichlet boundary conditions. Bayramoglu and Adıgüzelov [14] obtained the regularized trace of second order singular differential operator with bounded operator coefficient. Furthermore Adıgüzelov and Bakşi [6], Adıgüzelov and Sezer [7], [8] and Sen, Bayramov and Orucoglu [9] investigated the regularized trace formulas of differential operator with operator coefficient.
Although most of the previous researches on the subject dealt with regularized trace of second order differential operators, we focused on higher order differential operators. It is clear that our study advances the formulation of regularized trace that the prior manuscripts has proved. This paper aims to explore the second regularized trace of higher order differential operators with operator coefficient.

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Let us begin by recalling some definitions and properties:
Let H be an infinite dimensional separable Hilbert space. We denote the inner products in H by $(.,.)$ and the norm in H by $\|.\|$. Let $H_1 = L_2(0,\pi;\, H)$ denote the set of all functions f from $[0,\pi]$ into H which are strongly measurable and satisfy the condition $\int_0^\pi \left\|f(x)\right\|^2 dx < \infty$.The space H_1 is a linear space. If the inner product of arbitrary two elements f and g of the space H_1 is defined as $(f,g)_{H_1} = \int_0^\pi \left(f(x), g(x)\right) dx$, then H_1 becomes an infinite dimensional separable Hilbert space [1]. The norm in the space H_1 is denoted by $\|.\|_1$.
$\sigma_\infty(H)$ denotes the set of all compact operators from H to H. If $T \in \sigma_\infty(H)$, then T^*T is a nonnegative self adjoint operator and $(T^*T)^{\frac{1}{2}} \in \sigma_\infty(H)$ [11]. Let the nonzero eigenvalues of the operator $(T^*T)^{\frac{1}{2}}$ be $\{s_j\}_{j=1}^k$ $(0 \le k \le \infty)$ such that $s_1 \ge s_2 \ge ... \ge s_k$ according to its multiplicity number. Since $(T^*T)^{\frac{1}{2}}$ is non negative, s_k's are positive numbers. The numbers s_k are called s-numbers of the operator T. If $k < \infty$, then $s_j = 0$ $(j = k+1, k+2, ...)$ will be accepted. s-numbers of the operator T are also denoted by $s_j(T)$ $(j = 1, 2, ...)$. Here $s_1(T) = \|T\|$.
If T is a normal operator, then $s_j(T) = |\lambda_j(T)|$ $(j = 1, 2, ..., k)$ [11]. Here, $\{\lambda_1(T), \lambda_2(T), ..., \lambda_k(T)\}$ is an ordering of all nonzero eigenvalues of the operator T according to $|\lambda_1(T)| \ge |\lambda_2(T)| \ge ... \ge |\lambda_k(T)|$. σ_p or $\sigma_p(H)$ denotes the set of all compact operators, the s-numbers of which satisfy the condition $\sum\limits_{j=1}^{\infty} s_j^p(T) < \infty$ $(p \ge 1)$. The set σ_p $(p \ge 1)$ is a separable Banach space with respect to the norm $\|T\|_{\sigma_p(H)} = \left[\sum\limits_{j=1}^{\infty} s_j^p(T)\right]^{\frac{1}{p}}$, for $T \in \sigma_p(H)$ [11]. $\sigma_1(H)$ is the set of all operators $T \in \sigma_\infty(H)$, the s-numbers of which satisfy the condition $\sum\limits_{j=1}^{\infty} s_j(T) < \infty$. If an operator belongs to $\sigma_1(H)$, then it is called a nuclear operator. If the operators $T \in \sigma_p(H)$ and $B \in B(H)$, then $TB, BT \in \sigma_p(H)$ and $\|TB\|_{\sigma_p(H)} \le \|B\|\|T\|_{\sigma_p(H)}$, $\|BT\|_{\sigma_p(H)} \le \|B\|\|T\|_{\sigma_p(H)}$.
If T is a nuclear operator and $\{e_j\}_{j=1}^\infty \subset H$ is any orthonormal basis, then the series $\sum\limits_{j=1}^{\infty}\left(Te_j, e_j\right)$ is convergent and the sum of the series does not depend on the choice of the basis $\{e_j\}_{j=1}^\infty$. The sum of this series is said to be matrix trace of the operator T denoted by trT. Moreover

$$trT = \sum_{j=1}^{\nu(T)} \lambda_j(T)$$

[11]. Here, each eigenvalue counted according to its algebraic multiplicity number. $\nu(T)$ denotes the sum of algebraic multiplicity of non-zero eigenvalues of the operator T [11]. The sum of the series $\sum\limits_{j=1}^{\nu(T)} \lambda_j(T)$ is called spectral trace of the operator T.
Now, let us return to our problem. Consider the differential expression

$$\ell_0(y) = (-1)^r y^{(2r)}(x) + Ay(x)$$

in the space $H_1 = L_2(0,\pi;\, H)$. Here, the densely defined operator $A : D(A) \to H$ satisfies the conditions $A = A^* \ge I$ (I is unit operator in H) and $A^{-1} \in \sigma_\infty(H)$. Let $\{\gamma_n\}_1^\infty$ be an ordering of all eigenvalues of A according to $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_n \le \cdots$ and φ_n the corresponding orthonormal eigenfunctions. Here, each eigenvalue counted according to its multiplicity number.
Let D_0 be a subset of the space H_1 . A function $y(x) \in D_0$, if $y(x)$ satisfies the following conditions:
(y1) $y(x)$ has continuous derivative of the $(2r)$th-order with respect to the norm in the space H for every $x \in [0,\pi]$,
(y2) $Ay(x)$ is continuous with respect to the norm of the space H on $[0,\pi]$,
(y3) $y'(0) = y'''(0) = \cdots = y^{(2r-1)}(0) = y(\pi) = y''(\pi) = \cdots = y^{(2r-2)}(\pi) = 0$ $(r = 1, 2, \ldots, m)$.
Here, $\overline{D_0} = H_1$. Define the linear operator $L_0' : D_0 \to H_1$ as $L_0' y := \ell_0(y)$.

The construction above gives that $L_0^{'}$ is symmetric. The eigenvalues of $L_0^{'}$ are $\left(k+\frac{1}{2}\right)^{2r}+\gamma_j$ $(k=0,1,2,\ldots;j=1,2,\ldots)$ and $\sqrt{\frac{2}{\pi}}\varphi_j\cos\left(k+\frac{1}{2}\right)x$ the corresponding orthonormal eigenvectors.

We can see that the orthonormal eigenvector system of the symmetric operator $L_0^{'}$ is an orthonormal basis in the space H_1. We denote the closure of $L_0^{'}$ by $L_0 : D(L_0) \to H_1$. Since the orthonormal eigenvector system of the operator $L_0^{'}$ is an orthonormal basis in the space H_1, L_0 is a self adjoint operator.

Let $Q(x)$ defined on $[0,\pi]$ be an operator function satisfying the following conditions:

(Q1) $Q(x)$ has weak derivative of $(2r+2)$th order and

$Q^{(2i+1)}(0)=Q^{(2i+1)}(\pi)=0 \qquad (i=0,1,2,\ldots,r)$

(Q2) $Q^{(i)}(x):H\to H \qquad (i=0,1,2,\ldots,2r+2)$ are self-adjoint operators for every $x\in[0,\pi]$, $AQ''(x), Q^{(2r+2)}(x)\in\sigma_1(H)$ and the functions $\|AQ''(x)\|_{\sigma_1(H)}$, $\|Q^{(2r+2)}(x)\|_{\sigma_1(H)}$ are bounded and measurable in the interval $[0,\pi]$.

Define the operator $L : D(L_0)\to H_1$ as follows

$$L = L_0 + Q.$$

The operators L_0 and L are self adjoint operators and have purely discrete spectrum [6]. We denote the resolvent sets of L_0, L by $\rho(L_0)$, $\rho(L)$ and the resolvent operators of L_0, L by $R_\lambda^0=(L_0-\lambda I)^{-1}$, $R_\lambda=(L-\lambda I)^{-1}$, respectively. Also, we denote the eigenvalues of the operators L_0 and L by $\{\mu_n\}_1^\infty$ and $\{\lambda_n\}_1^\infty$ satisfying the inequalities $\mu_1\le\mu_2\le\cdots\le\mu_n\le\cdots$ and $\lambda_1\le\lambda_2\le\cdots\le\lambda_n\le\cdots$.

If $\gamma_j\sim aj^\alpha \quad (a>0, 0<\alpha<\infty)$ as $j\to\infty$, then

$$\mu_n,\lambda_n\sim d_1 n^{\frac{2r\alpha}{2r+\alpha}}, \tag{1.1}$$

as $n\to\infty$ [7]. Here, d_1 is a positive constant. By using the asymptotic formula (1.1), there exists a subsequence n_p of positive integers such that

$$\mu_q-\mu_{n_p}\ge d_2\left(q^{\frac{2r\alpha}{2r+\alpha}}-n_p^{\frac{2r\alpha}{2r+\alpha}}\right),\quad (q=n_p+1,n_p+2,\ldots)$$

where d_2 is a positive constant.

In the work [8], the formula in the form

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\left[\lambda_q-\mu_q-\frac{1}{\pi}\int_0^\pi\left(Q(x)\varphi_{j_q},\varphi_{j_q}\right)dx\right]=\frac{1}{4}\left(trQ(0)+trQ(\pi)\right)-\frac{1}{2\pi}\int_0^\pi trQ(x)dx$$

is obtained for the first regularized trace of the operator L. In this present work, we find a formula in the form

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\left(\lambda_q^2-\mu_q^2-2\sum_{s=2}^{m}(-1)^s s^{-1}Res_{\lambda=\mu_q}tr\left[\lambda(QR_\lambda^0)^s\right]-\frac{2\mu_q}{\pi}\int_0^\pi\left(Q(x)\varphi_{j_q},\varphi_{j_q}\right)dx\right)$$

$$=(-1)^r2^{-1-2r}\left[trQ^{(2r)}(0)-trQ^{(2r)}(\pi)\right]+\frac{1}{2}\left[trAQ(0)-trAQ(\pi)\right]. \tag{1.2}$$

The left hand side of equality (1.2) is called *the second regularized trace of the differential operator L.*

2. Main Results

The main purpose of this section is to obtain the second trace formula for the operator L. Now, we find the relations between resolvents and eigenvalues of the operators L_0 and L.

If $\alpha > \frac{2r}{2r-1}$ and $\lambda \neq \lambda_q, \mu_q$ $(q = 1, 2, ...)$, then by (1.1), R_λ^0 and R_λ are trace class operators. Hence

$$tr(R_\lambda - R_\lambda^0) = trR_\lambda - trR_\lambda^0 = \sum_{q=1}^{\infty}\Big(\frac{1}{\lambda_q - \lambda} - \frac{1}{\mu_q - \lambda}\Big).$$

If this equality is multiply with $\frac{\lambda^2}{2\pi i}$ and integrated on the circle $|\lambda| = b_p = \frac{1}{2}(\mu_{n_p} + \mu_{n_p+1})$, then we have following equality

$$\frac{1}{2\pi i}\int_{|\lambda|=b_p} \lambda^2 tr\Big(R_\lambda - R_\lambda^0\Big)d\lambda = \frac{1}{2\pi i}\int_{|\lambda|=b_p} \sum_{q=1}^{\infty}\Big(\frac{\lambda^2}{\lambda_q - \lambda}\Big)d\lambda - \frac{1}{2\pi i}\int_{|\lambda|=b_p} \sum_{q=1}^{\infty}\Big(\frac{\lambda^2}{\mu_q - \lambda}\Big)d\lambda. \tag{2.1}$$

We can see that for the large values of p,

$$\{\lambda_q, \mu_q\}_1^{n_p} \subset B(0, b_p) = \{\lambda : |\lambda| < b_p\}$$

$$\lambda_q, \mu_q \notin B[0, b_p] = \{\lambda : |\lambda| \leq b_p\} \quad (q \geq n_p + 1).$$

Therefore by (2.1), we have

$$\sum_{q=1}^{n_p}\Big(\lambda_q^2 - \mu_q^2\Big) = -\frac{1}{2\pi i}\int_{|\lambda|=b_p} \lambda^2 tr\Big(R_\lambda - R_\lambda^0\Big)d\lambda. \tag{2.2}$$

This is well known formula for the resolvents of the operators L_0 and L:

$$R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0 \qquad (\lambda \in \rho(L) \cap \rho(L_0)).$$

By using the last formula, we obtain

$$R_\lambda - R_\lambda^0 = \sum_{s=1}^{m}(-1)^s R_\lambda^0 (QR_\lambda^0)^s + (-1)^{m+1} R_\lambda (QR_\lambda^0)^{m+1},$$

for every positive integer m. By (2.2) and the last equality, we have

$$\sum_{q=1}^{n_p}\Big(\lambda_q^2 - \mu_q^2\Big) = \frac{1}{2\pi i}\int_{|\lambda|=b_p} \lambda^2 tr\Big(\sum_{s=1}^{m}(-1)^{s+1} R_\lambda^0 (QR_\lambda^0)^s + (-1)^m R_\lambda (QR_\lambda^0)^{m+1}\Big)d\lambda$$

or

$$\sum_{q=1}^{n_p}\Big(\lambda_q^2 - \mu_q^2\Big) = \sum_{s=1}^{m} D_{ps} + D_p^{(m)}. \tag{2.3}$$

Here,

$$D_{ps} = \frac{(-1)^{s+1}}{2\pi i}\int_{|\lambda|=b_p} \lambda^2 tr\Big(R_\lambda^0 (QR_\lambda^0)^s\Big)d\lambda, \quad (s = 1, 2, ...) \tag{2.4}$$

$$D_p^{(m)} = \frac{(-1)^m}{2\pi i}\int_{|\lambda|=b_p} \lambda^2 tr\Big(R_\lambda (QR_\lambda^0)^{m+1}\Big)d\lambda. \tag{2.5}$$

Theorem 2.1. *If* $\gamma_j \sim a j^{\alpha} \quad (0 < a, \alpha > \frac{2r}{2r-1})$ *as* $j \to \infty$ *, then*

$$D_{ps} = \frac{(-1)^s}{\pi i s} \int\limits_{|\lambda|=b_p} \lambda tr\Big((QR_\lambda^0)^s\Big) d\lambda \quad (s = 1, 2, ...).$$

Theorem 2.2. *If the operator function* $Q(x)$ *satisfies the conditions (Q1) and (Q2), then the series*

$$\sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \Big((k + \frac{1}{2})^{2r} + \gamma_j\Big) \int\limits_0^{\pi} (Q(x)\varphi_{j,}\varphi_j) \cos((2k+1)x) dx$$

is absolutely convergent.

We are at the position to give the main result:

Theorem 2.3. *If the operator function* $Q(x)$ *satisfies the conditions* $(Q1)$ *,* $(Q2)$ *, and* $\gamma_j \sim a j^{\alpha}$ *as* $j \to \infty \quad (a > 0, \ \frac{2r}{2r-1} < \alpha)$ *, then we have*

$$\lim_{p\to\infty} \sum_{q=1}^{n_p} \Big(\lambda_q^2 - \mu_q^2 - 2\sum_{s=2}^{m} (-1)^s s^{-1} Res_{\lambda=\mu_q} tr\Big(\lambda (QR_\lambda^0)^s\Big) - \frac{2\mu_q}{\pi} \int\limits_0^{\pi} (Q(x)\varphi_{j_q}, \varphi_{j_q}) dx\Big)$$

$$= (-1)^r 2^{-1-2r}\Big(trQ^{(2r)}(0) - trQ^{(2r)}(\pi)\Big) + \frac{1}{2}\Big(trAQ(0) - trAQ(\pi)\Big),$$

where $m = \left\lfloor \frac{2r\alpha+6r+3\alpha}{2r\alpha-2r-\alpha} \right\rfloor$*. Here,* $\lfloor . \rfloor$ *shows the greatest integer function whose value at any number x is the greatest integer less than or equal to x.*

3. Proofs

Proof of Theorem 1. We can show that the operator function $(QR_\lambda^0)^s$ is analytic with respect to the norm in the space $\sigma_1(H_1)$ in the region $\rho(L_0)$ and

$$tr\Big([(QR_\lambda^0)^s]'\Big) = str\Big((QR_\lambda^0)'(QR_\lambda^0)^{s-1}\Big), \quad (QR_\lambda^0)' = Q(R_\lambda^0)^2.$$

Therefore, we have

$$tr\Big([(QR_\lambda^0)^s]'\Big) = str\Big(R_\lambda^0 (QR_\lambda^0)^s\Big).$$

From (2.4) and the last equality we obtain

$$D_{ps} = \frac{(-1)^{s+1}}{2\pi i s} \int\limits_{|\lambda|=b_p} \lambda^2 tr\Big(\Big[(QR_\lambda^0)^s\Big]'\Big) d\lambda.$$

We can also write the last formula in the following form:

$$\begin{aligned} D_{ps} &= \frac{(-1)^{s+1}}{2\pi i s} \int\limits_{|\lambda|=b_p} tr\Big(\Big[\lambda^2 (QR_\lambda^0)^s\Big]' - 2\lambda (QR_\lambda^0)^s\Big) d\lambda \\ &= \frac{(-1)^s}{\pi i s} \int\limits_{|\lambda|=b_p} \lambda tr\Big((QR_\lambda^0)^s\Big) d\lambda + \frac{(-1)^{s+1}}{2\pi i s} \int\limits_{|\lambda|=b_p} tr\Big([\lambda^2 (QR_\lambda^0)^s]'\Big) d\lambda. \qquad (3.1) \end{aligned}$$

We can see

$$\int_{|\lambda|=b_p} tr\Big(\big[\lambda^2(QR_\lambda^0)^s\big]'\Big)d\lambda = \int_{|\lambda|=b_p} \Big[tr\Big(\lambda^2(QR_\lambda^0)^s\Big)\Big]'d\lambda. \tag{3.2}$$

We write the right hand side of above equality in the following way:

$$\int_{|\lambda|=b_p} \Big[tr\Big(\lambda^2(QR_\lambda^0)^s\Big)\Big]'d\lambda = \int_{\substack{|\lambda|=b_p\\ Im\lambda\geq 0}} \Big[tr\Big(\lambda^2(QR_\lambda^0)^s\Big)\Big]'d\lambda + \int_{\substack{|\lambda|=b_p\\ Im\lambda\leq 0}} \Big[tr\Big(\lambda^2(QR_\lambda^0)^s\Big)\Big]'d\lambda. \tag{3.3}$$

Let ε_0 be a positive number satisfying the inequality $b_p + \varepsilon_0 < \mu_{n_p+1}$.
We consider that the function $tr\Big(\lambda^2(QR_\lambda^0)^s\Big)$ is analytic in the simply connected regions:

$$G_1 = \{\lambda : b_p - \varepsilon_0 < |\lambda| < b_p + \varepsilon_0, \quad Im\lambda > -\varepsilon_0\},$$

$$G_2 = \{\lambda : b_p - \varepsilon_0 < |\lambda| < b_p + \varepsilon_0, \quad Im\lambda < \varepsilon_0\}$$

and

$$\{\lambda : |\lambda| = b_p, \quad Im\lambda \geq 0\} \subset G_1,$$

$$\{\lambda : |\lambda| = b_p, \quad Im\lambda \leq 0\} \subset G_2.$$

By using the Leibnitz Formula and (3.3), we get

$$\begin{aligned} &\int_{|\lambda|=b_p} \Big\{tr\Big(\lambda^2(QR_\lambda^0)^s\Big)\Big\}'d\lambda \\ = \;& tr\Big(b_p^2(QR_{-b_p}^0)^s\Big) - tr\Big(b_p^2(QR_{b_p}^0)^s\Big) + tr\Big(b_p^2(QR_{b_p}^0)^s\Big) - tr\Big(b_p^2(QR_{-b_p}^0)^s\Big) = 0. \end{aligned} \tag{3.4}$$

From (3.1), (3.2) and (3.4), we have

$$D_{ps} = \frac{(-1)^s}{\pi i s} \int_{|\lambda|=b_p} \lambda tr\Big((QR_\lambda^0)^s\Big)d\lambda. \square$$

Proof of Theorem 2. Let $h_j(x) = (Q(x)\varphi_j, \varphi_j)$. Using the integration by parts formula and the condition (Q1), we get

$$\int_0^\pi h_j(x)\cos((2k+1)x)dx = \int_0^\pi h_j(x)\Big(\frac{1}{2k+1}\sin(2k+1)x\Big)' dx$$

$$= \frac{1}{2k+1}\Big[h_j(x)\sin((2k+1)x)\Big|_0^\pi - \int_0^\pi h_j'(x)\sin((2k+1)x)dx\Big]$$

$$= \frac{1}{(2k+1)^2}\int_0^\pi h_j'(x)\Big(\cos(2k+1)x\Big)' dx$$

$$= \frac{1}{(2k+1)^2}\Big[h_j'(x)\cos((2k+1)x)\Big|_0^\pi - \int_0^\pi h_j''(x)\cos((2k+1)x)dx\Big]$$

$$= -\frac{1}{(2k+1)^3}\int_0^\pi h_j''(x)\Big(\sin(2k+1)x\Big)' dx$$

$$= .... = \frac{(-1)^{r+1}}{(2k+1)^{2r+2}}\int_0^\pi h_j^{(2r+2)}(x)\cos((2k+1)x)dx. \tag{3.5}$$

By (3.5), we find

$$\sum_{k=0}^\infty\sum_{j=1}^\infty\Big|\Big((k+\frac{1}{2})^{2r}+\gamma_j\Big)\int_0^\pi h_j(x)\cos((2k+1)x)dx\Big|$$

$$\leq \sum_{k=0}^\infty\sum_{j=1}^\infty(2k+1)^{-2}\int_0^\pi\Big(\Big|h_j^{(2r+2)}(x)\Big| + \gamma_j|h_j''(x)|\Big)dx$$

$$= \sum_{j=1}^\infty\int_0^\pi\Big(\Big|(Q^{(2r+2)}(x)\varphi_j,\varphi_j)\Big| + |(AQ''(x)\varphi_j,\varphi_j)|\Big)dx\sum_{k=0}^\infty(2k+1)^{-2}$$

$$\leq Const.\int_0^\pi\Big(\sum_{j=1}^\infty\Big|(Q^{(2r+2)}(x)\varphi_j,\varphi_j)\Big| + \sum_{j=1}^\infty|(AQ''(x)\varphi_j,\varphi_j)|\Big)dx$$

$$\leq Const.\int_0^\pi\Big(\Big\|Q^{(2r+2)}(x)\Big\|_{\sigma_1(H)} + \|AQ''(x)\|_{\sigma_1(H)}\Big)dx. \tag{3.6}$$

Since the functions $\|Q^{(2r+2)}(x)\|_{\sigma_1(H)}$ and $\|AQ''(x)\|_{\sigma_1(H)}$ in (3.6) are measurable and bounded in the interval $[0,\pi]$, we get

$$\sum_{k=0}^\infty\sum_{j=1}^\infty\Big|[(k+\frac{1}{2})^{2r}+\gamma_j]\int_0^\pi(Q(x)\varphi_j,\varphi_j)\cos((2k+1)x)dx\Big| < \infty.\square$$

Let $\{\psi_q\}_1^\infty$ be the orthornormal eigenvectors system corresponding to eigenvalues $\{\mu_q\}_1^\infty$ of the operator L_0, respectively. Since the orthornormal eigenvectors corresponding to eigenvalues $(k+\frac{1}{2})^{2r}+\gamma_j$ $(k =$

$0, 1, 2, \cdots; j = 1, 2, \cdots$) of the operator L_0 are $\sqrt{\frac{2}{\pi}}\cos((k+\frac{1}{2})x)\varphi_j$, respectively,

$$\mu_q = (k_q + \frac{1}{2})^{2r} + \gamma_{j_q} \qquad (q = 1, 2, \cdots)$$

and

$$\psi_q(x) = \sqrt{\frac{2}{\pi}}\cos((k_q + \frac{1}{2})x)\varphi_{j_q}. \tag{3.7}$$

We prove the main theorem of the paper.
Proof of Theorem 3.
By using the Theorem 1, one can write D_{ps} as follows:

$$\begin{aligned} D_{ps} &= 2(-1)^s s^{-1}\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p} tr\Big(\lambda(QR_\lambda^0)^s\Big)d\lambda \\ &= 2(-1)^s s^{-1}\sum_{q=1}^{n_p} Res_{\lambda=\mu_q} tr\Big(\lambda(QR_\lambda^0)^s\Big). \end{aligned}$$

By using last formula, we can rewrite (2.3) as follows:

$$\sum_{q=1}^{n_p}\left(\lambda_q^2 - \mu_q^2 - 2\sum_{s=2}^{m}(-1)^s s^{-1} Res_{\lambda=\mu_q} tr\Big(\lambda(QR_\lambda^0)^s\Big)\right) = D_{p1} + D_p^{(m)}, \tag{3.8}$$

$$D_{p1} = \frac{-1}{\pi i}\int\limits_{|\lambda|=b_p}\lambda tr(QR_\lambda^0)d\lambda. \tag{3.9}$$

Since (QR_λ^0) is a nuclear operator for every $\lambda \in \rho(L_0)$ and $\{\psi_q\}_1^\infty$ is an orthonormal basis in the space H_1, we have

$$tr(QR_\lambda^0) = \sum_{q=1}^{\infty}(QR_\lambda^0\psi_q, \psi_q)_{H_1}.$$

Here, $R_\lambda^0\psi_q = (\mu_q - \lambda I)^{-1}\psi_q$.
If we substitute the last two equalities into (3.9), then we get

$$\begin{aligned} D_{p1} &= \frac{-1}{\pi i}\int\limits_{|\lambda|=b_p}\lambda\sum_{q=1}^{\infty}(QR_\lambda^0\psi_q, \psi_q)_{H_1}d\lambda \\ &= \frac{-1}{\pi i}\int\limits_{|\lambda|=b_p}\lambda\sum_{q=1}^{\infty}\frac{1}{\mu_q - \lambda}(Q\psi_q, \psi_q)_{H_1}d\lambda \\ &= \frac{1}{\pi i}\sum_{q=1}^{\infty}(Q\psi_q, \psi_q)_{H_1}\int\limits_{|\lambda|=b_p}\frac{\lambda}{\lambda - \mu_q}d\lambda \end{aligned}$$

By using the Cauchy Integral Formula

$$\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\frac{\lambda}{\lambda - \mu_q}d\lambda = \begin{cases} \mu_q & , if \quad q \le n_p \\ 0 & , if \quad q > n_p \end{cases}$$

and by (3.7), we obtain

$$\begin{aligned}
D_{p1} &= 2\sum_{q=1}^{n_p}\mu_q(Q\psi_q,\psi_q)_{H_1} \\
&= 2\sum_{q=1}^{n_p}\mu_q\int_0^\pi (Q(x)\psi_q(x),\psi_q(x))dx \\
&= 2\sum_{q=1}^{n_p}\mu_q\int_0^\pi \Big(Q(x)\sqrt{\frac{2}{\pi}}\cos(k_q+\frac{1}{2})x\varphi_{j_q},\sqrt{\frac{2}{\pi}}\cos(k_q+\frac{1}{2})x\varphi_{j_q}\Big)dx \\
&= 2\sum_{q=1}^{n_p}\mu_q\frac{2}{\pi}\int_0^\pi \cos^2(k_q+\frac{1}{2})x(Q(x)\varphi_{j_q},\varphi_{j_q})dx \\
&= \sum_{q=1}^{n_p}\mu_q\frac{2}{\pi}\int_0^\pi (1+\cos(2k_q+1)x)(Q(x)\varphi_{j_q},\varphi_{j_q})dx \\
&= \frac{2}{\pi}\sum_{q=1}^{n_p}\mu_q\int_0^\pi \cos(2k_q+1)x(Q(x)\varphi_{j_q},\varphi_{j_q})dx+\frac{2}{\pi}\sum_{q=1}^{n_p}\mu_q\int_0^\pi (Q(x)\varphi_{j_q},\varphi_{j_q})dx. \qquad (3.10)
\end{aligned}$$

We substitude (3.10) in (3.8):

$$\begin{aligned}
&\sum_{q=1}^{n_p}\Big(\lambda_q^2-\mu_q^2-2\sum_{s=2}^{m}(-1)^s s^{-1}Res_{\lambda=\mu_q}tr\Big[\lambda(QR_\lambda^0)^s\Big]-\frac{2\mu_q}{\pi}\int_0^\pi h_{j_q}(x)dx\Big) \\
&= \frac{2}{\pi}\sum_{q=1}^{n_p}\mu_q\int_0^\pi h_{j_q}(x)\cos(2k_q+1)xdx+D_p^{(m)}. \qquad (3.11)
\end{aligned}$$

If we use Theorem 2, then we know that

$$\begin{aligned}
&\frac{2}{\pi}\lim_{p\to\infty}\sum_{q=1}^{n_p}\mu_q\int_0^\pi h_{j_q}(x)\cos(2k_q+1)xdx \\
&= \frac{2}{\pi}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\Big((k+\frac{1}{2})^{2r}+\gamma_j\Big)\int_0^\pi h_j(x)\cos(2k+1)xdx. \qquad (3.12)
\end{aligned}$$

If we substitude (3.5) in the right hand side of (3.12), then we get

$$\begin{aligned}
&\frac{2}{\pi}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\Big[(k+\frac{1}{2})^{2r}+\gamma_j\Big]\int_0^{\pi} h_j(x)\cos(2k+1)xdx \\
=\;& \sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{2}{\pi}\int_0^{\pi}\Big[(-\frac{1}{4})^r h_j^{(2r)}(x)+\gamma_j(x)h_j(x)\Big]\cos(2k+1)xdx \\
=\;& \frac{1}{\pi}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\Big(\int_0^{\pi}\Big[(-\frac{1}{4})^r h_j^{(2r)}(x)+\gamma_j(x)h_j(x)\Big]\cos(kx)dx \\
-\;& (-1)^k\int_0^{\pi}\Big[(-\frac{1}{4})^r h_j^{(2r)}(x)+\gamma_j(x)h_j(x)\Big]\cos(kx)dx\Big),
\end{aligned}$$

The sums according to the k on the right hand side of the last relation are the values at 0 and π of the Fourier Series of the function $(-\frac{1}{4})^r h_j^{(2r)}(x)+\gamma_j h_j(x)$ according to the functions $\{\cos kx\}_{k=0}^{\infty}$ on the interval $[0,\pi]$. Therefore,

$$\begin{aligned}
&\frac{2}{\pi}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\Big[(k+\frac{1}{2})^{2r}+\gamma_j\Big]\int_0^{\pi} h_j(x)\cos(2k+1)xdx \\
=\;& \frac{1}{2}\sum_{j=1}^{\infty}\Big[(-\frac{1}{4})^r(h_j^{(2r)}(0)-h_j^{(2r)}(\pi))+\gamma_j(h_j(0)-h_j(\pi))\Big] \\
=\;& (-1)^r 2^{-1-2r}\Big[trQ^{(2r)}(0)-trQ^{(2r)}(\pi)\Big]+\frac{1}{2}\Big[trAQ(0)-trAQ(\pi)\Big] \qquad (3.13)
\end{aligned}$$

From (3.12) and (3.13), we obtain

$$\begin{aligned}
&\frac{2}{\pi}\lim_{p\to\infty}\sum_{q=1}^{n_p}\mu_q\int_0^{\pi} h_{j_q}(x)\cos(2k_q+1)xdx \\
=\;& (-1)^r 2^{-1-2r}\Big[trQ^{(2r)}(0)-trQ^{(2r)}(\pi)\Big]+\frac{1}{2}\Big[trAQ(0)-trAQ(\pi)\Big] \qquad (3.14)
\end{aligned}$$

Let us estimate of $D_p^{(m)}$ for the large value of p. By using (2.5) we get

$$\begin{aligned}
\Big|D_p^{(m)}\Big| \;&\le\; \int_{|\lambda|=b_p}|\lambda|^2\Big|tr\Big(R_\lambda(QR_\lambda^0)^{m+1}\Big)\Big||d\lambda| \\
&\le\; b_p^2\int_{|\lambda|=b_p}\Big\|R_\lambda(QR_\lambda^0)^{m+1}\Big\|_{\sigma_1(H_1)}|d\lambda|
\end{aligned}$$

$$\leq b_p^2 \int\limits_{|\lambda|=b_p} \left\|R_\lambda\right\|_1 \left\|(QR_\lambda^0)^{m+1}\right\|_{\sigma_1(H_1)} |d\lambda|$$

$$\leq b_p^2 \int\limits_{|\lambda|=b_p} \left\|R_\lambda\right\|_1 \left\|(QR_\lambda^0)^{m}\right\|_1 \left\|QR_\lambda^0\right\|_{\sigma_1(H_1)} |d\lambda|$$

$$\leq b_p^2 \int\limits_{|\lambda|=b_p} \left\|R_\lambda\right\|_1 \left\|Q\right\|_1^m \left\|R_\lambda^0\right\|_1^m \left\|Q\right\|_1 \left\|R_\lambda^0\right\|_{\sigma_1(H_1)} |d\lambda|. \quad (3.15)$$

One can prove the following inequalities similarly in work [7]:

$$\left\|R_\lambda^0\right\|_{\sigma_1(H_1)} \leq const.{n_p}^{1-\delta},$$

$$\left\|R_\lambda\right\|_1 \leq const.{n_p}^{-\delta} \quad (\delta = \frac{2r\alpha}{2r+\alpha} - 1).$$

From last two inequalities and (3.15), we obtain

$$\left|D_p^{(m)}\right| \leq const.b_p^3 {n_p}^{-\delta m - 2\delta + 1}. \quad (3.16)$$

For large values of p

$$b_p = 2^{-1}\left(\mu_{n_p} + \mu_{n_p+1}\right) \leq const.n_p^{1+\delta}. \quad (3.17)$$

From (3.16) and (3.17), we obtain

$$\left|D_p^{(m)}\right| \leq const.\mu_p^{4-(m-1)\delta}.$$

Therefore, for $m = \left\lfloor \frac{2r\alpha+6r+3\alpha}{2r\alpha-2r-\alpha} \right\rfloor + 1$, we find

$$\lim_{p\to\infty} D_p^{(m)} = 0. \quad (3.18)$$

From (3.11), (3.14) and (3.18), we find the following formula for *second regularized trace formula of the operator L*

$$\lim_{p\to\infty} \sum_{q=1}^{n_p} \left(\lambda_q^2 - \mu_q^2 - 2\sum_{s=2}^{m} (-1)^s s^{-1} Res_{\lambda=\mu_q} tr\left(\lambda (QR_\lambda^0)^s\right) - \frac{2\mu_q}{\pi} \int\limits_0^\pi (Q(x)\varphi_{j_q}, \varphi_{j_q}) dx \right)$$

$$= (-1)^r 2^{-1-2r}\left[trQ^{(2r)}(0) - trQ^{(2r)}(\pi)\right] + \frac{1}{2}\left[trAQ(0) - trAQ(\pi)\right]. \square$$

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