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On Congruences with Binomial Coefficients and Harmonic Numbers

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Abstract. In this paper, we obtain super congruences

$$\sum_{k=1}^{[(p-1)/r]} (-1)^{k+1}\binom{\alpha p-1}{k}H_k^2 \pmod{p^2} \text{ and } \sum_{k=1}^{[(p-1)/r]} (-1)^{k} k\binom{\alpha p-1}{k}H_k^2 \pmod{p^2},$$

where $r \in \{1,2,3\}$ and α is a $p-$adic integer. Also, we give new congruences involving binomial coefficients and harmonic numbers.

1. Introduction

Harmonic numbers H_n are those rational numbers defined by

$$H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k}, \text{ for } n \in \mathbb{Z}^+,$$

and for $m \in \mathbb{Z}^+$, harmonic numbers of order m are those rational numbers given by

$$H_{0,m} = 0, \quad H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}, \quad \text{for } n \in \mathbb{Z}^+.$$

For a prime number p, let $\mathbb{Z}_p$ be the set of rational numbers whose denominator is not divisible by p. For $a \in \mathbb{Z}_p$, $q_p(a)$ is the Fermat quotient defined for a given prime number p by $q_p(a) = \left(a^{p-1}-1\right)/p$.
For an odd prime p and an integer a, the Legendre symbol is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a, \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p. \end{cases}$$

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Note that

$$\left(\frac{p}{3}\right)=\begin{cases} 1 & \text{if } p\equiv 1 \pmod 3,\\ -1 & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

The Bernoulli polynomial $B_n(x)$ is defined by

$$B_n(x)=\sum_{i=0}^{n}\binom{n}{i}B_i x^{n-i},\quad \text{for } n\geq 0,$$

where B_n are the Bernoulli numbers. In the special case, for $x=0$, $B_n(0)=B_n$.

In [12], J. Wolstenholme discovered that for any prime number $p>3$,

$$H_{p-1}\equiv 0 \pmod{p^2}. \tag{1}$$

In [11], Z.W. Sun gave that for an odd prime number p,

$$H_{p-1-k}\equiv H_k \pmod p,$$

$$\sum_{k=1}^{p-1}\frac{H_{k,2}}{k}\equiv -B_{p-3} \pmod p \text{ and } \sum_{k=1}^{p-1}\frac{H_k^2}{k}\equiv B_{p-3} \pmod p. \tag{2}$$

In [10], Z.H. Sun showed that for any prime number $p>3$,

$$H_{[p/3],2}\equiv \frac{1}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) \pmod p. \tag{3}$$

In [8], J. Spieβ obtained that for $n\in\mathbb{Z}^+$,

$$\sum_{k=1}^{n}\frac{H_k}{k}=\frac{H_{n,2}+H_n^2}{2}. \tag{4}$$

In [9], Z.H. Sun showed that for any prime number $p>3$ and an odd number $m\in\{1,3,\ldots,p-4\}$,

$$H_{p-1,m}\equiv \frac{m(m+1)}{2}\frac{B_{p-2-m}}{p-2-m}p^2 \pmod{p^3}, \tag{5}$$

and

$$H_{(p-1)/2,2}\equiv 0 \pmod p. \tag{6}$$

In [3], L. Elkhiri et al. gave that for an odd prime number p,

$$H_{[p/3]}\equiv -\frac{p}{6}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)-\frac{3}{2}q_p(3)+\frac{3}{4}pq_p^2(3) \pmod{p^2}. \tag{7}$$

In [5], E. Lehmer showed that for any prime number $p>3$,

$$H_{(p-1)/2}\equiv -2q_p(2)+pq_p^2(2) \pmod{p^2}, \tag{8}$$

and

$$H_{[p/4]}\equiv -3q_p(2) \pmod p. \tag{9}$$

In [1, 2, 4], it is given that for prime number $p\equiv 1 \pmod 4$,

$$\binom{(p-1)/2}{(p-1)/4}\equiv 2a \pmod p, \tag{10}$$

where $p = a^2 + b^2$ with $a \equiv 1 \pmod 4$ and $b \equiv 0 \pmod 2$.

It is well known that for $k = 0, 1, 2, ...,$

$$\binom{(p-1)/2}{k} \equiv \frac{1}{(-4)^k}\binom{2k}{k} \pmod p. \tag{11}$$

In [6, 7], the author obtained some congruences involving binomial coefficients, harmonic numbers and Bernoulli polynomials.

The main aim of this paper derive new congruences involving binomial coefficients and harmonic numbers by using combinatorial congruences. For example for any prime number $p > 3$,

$$\sum_{k=1}^{p-1} (-1)^{k+1}\binom{\alpha p-1}{k} H_k^2 \equiv 2 + p(6\alpha - 2) + p^2\left(\left(-\alpha^2 + \frac{1}{3}\alpha + \frac{1}{3}\right)B_{p-3} + 6\alpha(2\alpha - 1)\right) \pmod{p^3},$$

where α is a $p-$adic integer.

2. Main results

In this section, we will examine congruences involving binomial coefficients, harmonic numbers and Bernoulli numbers. We will start some auxiliary Lemmas:

Lemma 2.1. *Let p be a prime number and $i \in \{1, 2, ..., p-1\}$. Then, for any $p-$adic integer α, we have*

$$\binom{\alpha p-1}{i} \equiv (-1)^i\left(1 - \alpha p H_i + \frac{\alpha^2 p^2}{2}\left(H_i^2 - H_{i,2}\right)\right) \pmod{p^3}.$$

Proof. Observed that

$$\begin{aligned}
\binom{\alpha p-1}{i} &= \frac{(\alpha p-1)(\alpha p-2)\cdots(\alpha p-j)\cdots(\alpha p-i)}{1.2\cdots j\cdots i} \\
&= (\alpha p-1)\left(\frac{\alpha p}{2}-1\right)\cdots\left(\frac{\alpha p}{j}-1\right)\cdots\left(\frac{\alpha p}{i}-1\right) \\
&= (-1)^i(1-\alpha p)\left(1-\frac{\alpha p}{2}\right)\cdots\left(1-\frac{\alpha p}{j}\right)\cdots\left(1-\frac{\alpha p}{i}\right) \\
&\equiv (-1)^i\left(1 - \alpha p\sum_{k=1}^{i}\frac{1}{k} + \alpha^2 p^2 \sum_{1\le k<j\le i}\frac{1}{kj}\right) \\
&= (-1)^i\left(1 - \alpha p H_i + \frac{\alpha^2 p^2}{2}\left(H_i^2 - H_{i,2}\right)\right) \pmod{p^3}.
\end{aligned}$$

So Lemma 2.1 follows. □

Lemma 2.2. *For any prime number $p > 3$ and any $p-$adic integer α, we have*

$$\sum_{k=1}^{p-1}(-1)^k\frac{1}{k}\binom{\alpha p-1}{k} \equiv p^2\left(-\frac{1}{3} - \frac{1}{3}\alpha + \alpha^2\right)B_{p-3} \pmod{p^3}, \tag{12}$$

$$\sum_{k=1}^{(p-1)/2}(-1)^k\frac{1}{k}\binom{\alpha p-1}{k} \equiv -2q_p(2) + p(1-2\alpha)q_p^2(2) \pmod{p^2}, \tag{13}$$

$$\begin{aligned}
\sum_{k=1}^{[p/3]}(-1)^k\frac{1}{k}\binom{\alpha p-1}{k} &\equiv p\left(\frac{-1}{6} - \frac{\alpha}{4}\right)\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) \\
&\quad -\frac{3}{2}q_p(3) + p\left(\frac{3}{4} - \frac{9}{8}\alpha\right)q_p^2(3) \pmod{p^2}.
\end{aligned} \tag{14}$$

Proof. Firstly, we will give proof of (12). By Lemma 2.1, we have

$$\sum_{k=1}^{p-1}(-1)^k\frac{1}{k}\binom{\alpha p-1}{k} \equiv H_{p-1}-\alpha p\sum_{k=1}^{p-1}\frac{H_k}{k}+\frac{\alpha^2p^2}{2}\left(\sum_{k=1}^{p-1}\frac{H_k^2}{k}-\sum_{k=1}^{p-1}\frac{H_{k,2}}{k}\right) \pmod{p^3}.$$

Applying (2), (4) and (5), we get

$$\sum_{k=1}^{p-1}(-1)^k\frac{1}{k}\binom{\alpha p-1}{k} \equiv -\frac{1}{3}p^2B_{p-3}-\frac{1}{3}\alpha p^2B_{p-3}+\alpha^2p^2B_{p-3}=p^2\left(-\frac{1}{3}-\frac{1}{3}\alpha+\alpha^2\right)B_{p-3} \pmod{p^3}.$$

For the proof of (13), by Lemma 2.1, we have

$$\sum_{k=1}^{(p-1)/2}(-1)^k\frac{1}{k}\binom{\alpha p-1}{k}\equiv H_{(p-1)/2}-\alpha p\sum_{k=1}^{(p-1)/2}\frac{H_k}{k} \pmod{p^2}.$$

Hence, applying (8) and taking $n=(p-1)/2$ in (4), we write

$$\sum_{k=1}^{(p-1)/2}(-1)^k\frac{1}{k}\binom{\alpha p-1}{k} \equiv -2q_p(2)+pq_p^2(2)-\alpha p\frac{H_{(p-1)/2,2}+H_{(p-1)/2}^2}{2} \pmod{p^2}.$$

With the help of (6) and (8), the result is clearly complete.

Finally, for the proof of (14), from Lemma 2.1 and (7), we have

$$\sum_{k=1}^{[p/3]}(-1)^k\frac{1}{k}\binom{\alpha p-1}{k}$$
$$\equiv H_{[p/3]}-\alpha p\sum_{k=1}^{[p/3]}\frac{H_k}{k} \equiv \frac{-p}{6}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)-\frac{3}{2}q_3+\frac{3}{4}pq_3^2-\alpha p\sum_{k=1}^{[p/3]}\frac{H_k}{k} \pmod{p^2}.$$

Setting $n=[p/3]$ in (4) and from congruences (3) and (7), thus we complete the proof of Lemma 2.2 . □

Lemma 2.3. *For any prime number $p>3$ and any p–adic integer α, we have*

$$\sum_{k=1}^{p-1}(-1)^k\binom{\alpha p-1}{k}\equiv(1+\alpha p)(p-1)-\alpha^2p^2 \pmod{p^3}, \tag{15}$$

$$\sum_{k=1}^{(p-1)/2}(-1)^k\binom{\alpha p-1}{k}\equiv\frac{p-1}{2}-\frac{\alpha p}{2}+\alpha pq_p(2) \pmod{p^2}, \tag{16}$$

and

$$\sum_{k=1}^{[p/3]}(-1)^k\binom{\alpha p-1}{k}\equiv[p/3]+\frac{\alpha p}{12}\left(3\left(3+\left(\frac{p}{3}\right)\right)q_p(3)-2\left(3-\left(\frac{p}{3}\right)\right)\right) \pmod{p^2}. \tag{17}$$

Proof. With the help of Lemma 2.1, the proofs of these congruences are similar to the proof of Lemma 2.2. □

Lemma 2.4. *For $n\in\mathbb{Z}^+$ and $x\in\mathbb{R}\setminus\{0\}$, we have*

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}=(-1)^n\frac{x-n}{x}\binom{x}{n}, \tag{18}$$

and

$$\sum_{k=1}^{n}(-1)^k\binom{x}{k}H_k=(-1)^n\binom{x}{n}\frac{n-x}{x}\left(H_n+\frac{1}{x}\right)-\frac{1}{x}. \tag{19}$$

Proof. The proof is easily obtained by induction on n. □

Proposition 2.5. *For $n\in\mathbb{Z}^+$ and $x\in\mathbb{R}\setminus\{0\}$, we have*

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{x}{k}H_k^2 = (-1)^n\frac{n-x}{x}\binom{x}{n}\left(H_n^2+\frac{2}{x}H_n+\frac{2}{x^2}\right)+\frac{2}{x^2}+\frac{1}{x}\sum_{k=1}^{n}(-1)^k\frac{1}{k}\binom{x}{k}.$$

Proof. Consider that

$$\begin{aligned}
&(-1)^n nH_n^2\binom{x}{n}\\
=\ & \sum_{k=1}^{n}(-1)^k kH_k^2\binom{x}{k}-\sum_{k=0}^{n-1}(-1)^k k\binom{x}{k}H_k^2\\
=\ & \sum_{k=0}^{n-1}(-1)^{k+1}(k+1)\binom{x}{k+1}H_{k+1}^2-\sum_{k=0}^{n-1}(-1)^k k\binom{x}{k}H_k^2\\
=\ & \sum_{k=0}^{n-1}(-1)^{k+1}(x-k)\binom{x}{k}\left(H_k+\frac{1}{k+1}\right)^2-\sum_{k=0}^{n-1}(-1)^k k\binom{x}{k}H_k^2\\
=\ & \sum_{k=0}^{n-1}(-1)^{k+1}(x-k)\binom{x}{k}\left(H_k^2+\frac{2}{k+1}H_k+\frac{1}{(k+1)^2}\right)-\sum_{k=0}^{n-1}(-1)^k k\binom{x}{k}H_k^2\\
=\ & x\sum_{k=0}^{n-1}(-1)^{k+1}\binom{x}{k}H_k^2+\sum_{k=0}^{n-1}(-1)^{k+1}(x-k)\binom{x}{k}\left(\frac{2}{k+1}H_k+\frac{1}{(k+1)^2}\right).
\end{aligned}$$

Applying some elementary operations to aboving sum, we have

$$\begin{aligned}
&(-1)^n nH_n^2\binom{x}{n}\\
=\ & x\sum_{k=0}^{n-1}(-1)^{k+1}H_k^2\binom{x}{k}+\sum_{k=0}^{n-1}(-1)^{k+1}(k+1)\binom{x}{k+1}\left(\frac{2}{k+1}H_k+\frac{1}{(k+1)^2}\right)\\
=\ & x\sum_{k=0}^{n-1}(-1)^{k+1}H_k^2\binom{x}{k}+\sum_{k=0}^{n-1}(-1)^{k+1}\binom{x}{k+1}\left(2H_k+\frac{1}{k+1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= x\sum_{k=0}^{n-1}(-1)^{k+1}H_k^2\binom{x}{k}+\sum_{k=0}^{n-1}(-1)^{k+1}\binom{x}{k+1}\left(2\left(H_{k+1}-\frac{1}{k+1}\right)+\frac{1}{k+1}\right)\\
&= x\sum_{k=0}^{n-1}(-1)^{k+1}H_k^2\binom{x}{k}+\sum_{k=0}^{n-1}(-1)^{k+1}\binom{x}{k+1}\left(2H_{k+1}-\frac{1}{k+1}\right)\\
&= x\sum_{k=0}^{n-1}(-1)^{k+1}H_k^2\binom{x}{k}+\sum_{k=1}^{n}(-1)^{k}\binom{x}{k}\left(2H_{k}-\frac{1}{k}\right)\\
&= x\sum_{k=0}^{n-1}(-1)^{k+1}H_k^2\binom{x}{k}+2\sum_{k=1}^{n}(-1)^{k}\binom{x}{k}H_k-\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{x}{k},
\end{aligned}
$$

and with necessary arrangements, by (19), equals

$$
\begin{aligned}
&(-1)^n\frac{n-x}{x}\binom{x}{n}H_n^2-\frac{2}{x}\sum_{k=1}^{n}(-1)^kH_k\binom{x}{k}+\frac{1}{x}\sum_{k=1}^{n}(-1)^k\frac{1}{k}\binom{x}{k}\\
&= (-1)^n\frac{n-x}{x}\binom{x}{n}H_n^2+\frac{2}{x}\left[(-1)^n\binom{x}{n}\frac{n-x}{x}\left(H_n+\frac{1}{x}\right)-\frac{1}{x}\right]+\frac{1}{x}\sum_{k=1}^{n}(-1)^k\frac{1}{k}\binom{x}{k}\\
&= (-1)^n\frac{n-x}{x}\binom{x}{n}\left(H_n^2+\frac{2}{x}\left(H_n+\frac{1}{x}\right)\right)+\frac{2}{x^2}+\frac{1}{x}\sum_{k=1}^{n}(-1)^k\frac{1}{k}\binom{x}{k},
\end{aligned}
$$

as claimed. Thus we have the proof. □

Now, we will give Proposition 2.6 without the proof as follows:

Proposition 2.6. *For $n\in\mathbb{Z}^+$ and $x\in\mathbb{R}\setminus\{0,1\}$, we have*

$$
\begin{aligned}
&\sum_{k=1}^{n}(-1)^k k\binom{x}{k}H_k^2\\
&= (-1)^n\binom{x}{n}\frac{n(x-n)}{x-1}\left(H_{n-1}^2+2H_{n-1}\left(\frac{1}{x-1}+\frac{x-1}{nx}\right)+\frac{2}{(x-1)^2}\right)+\frac{2x}{(x-1)^2}-\frac{2}{x}\\
&\quad+(-1)^n\binom{x}{n}(x-n)\left(\frac{2}{x}\left(\frac{1}{n}+\frac{1}{x}\right)-\frac{1}{n(x-1)}\right)-\frac{1}{x-1}\sum_{k=1}^{n}(-1)^k\binom{x}{k}+\frac{1}{x-1}\sum_{k=1}^{n}(-1)^k\frac{1}{k}\binom{x}{k}.
\end{aligned}
$$

Theorem 2.7. *For any prime number $p>3$ and any $p-$adic integer α, we have*

$$
\sum_{k=0}^{(p-1)/2}(-1)^{k+1}\binom{\alpha p-1}{k}H_k^2\equiv 1-2q_p^2(2)+pT_1(p,\alpha)\pmod{p^2},
$$

where

$$
T_1(p,\alpha)=(2-4\alpha)\,q_p^3(2)-2q_p^2(2)-2q_p(2)+3\alpha-1.
$$

Proof. From Lemma 2.1 and (8), we have

$$
\binom{\alpha p-1}{(p-1)/2}\equiv(-1)^{(p-1)/2}\left(1+2\alpha p q_p(2)\right)\pmod{p^2}. \tag{20}
$$

Using $n = (p-1)/2$ and $x = \alpha p - 1$ in Proposition 2.5, we write

$$\sum_{k=0}^{(p-1)/2} (-1)^{k+1} \binom{\alpha p - 1}{k} H_k^2 = (-1)^{(p-1)/2} \frac{p - 2\alpha p + 1}{2(\alpha p - 1)} \binom{\alpha p - 1}{(p-1)/2} \left(H_{(p-1)/2}^2 + \frac{2}{\alpha p - 1} H_{(p-1)/2} + \frac{2}{(\alpha p - 1)^2} \right) + \frac{2}{(\alpha p - 1)^2} + \frac{1}{\alpha p - 1} \sum_{k=1}^{(p-1)/2} (-1)^k \frac{1}{k} \binom{\alpha p - 1}{k}.$$

Hence, from (8), (13) and (20), the proof is complete. □

Theorem 2.8. *For any prime number $p > 3$ and any $p-$adic integer α, we have*

$$\sum_{k=1}^{p-1} (-1)^{k+1} \binom{\alpha p - 1}{k} H_k^2 \equiv 2 + p(6\alpha - 2) + p^2 \left(\left(-\alpha^2 + \frac{1}{3}\alpha + \frac{1}{3} \right) B_{p-3} + 6\alpha(2\alpha - 1) \right) \pmod{p^3}.$$

Proof. Using $n = p - 1$ and $x = \alpha p - 1$ in Proposition 2.5, we write

$$\sum_{k=1}^{p-1} (-1)^{k+1} \binom{\alpha p - 1}{k} H_k^2 = \frac{(1-\alpha)p}{\alpha p - 1} \binom{\alpha p - 1}{p-1} \left(H_{p-1}^2 + \frac{2}{\alpha p - 1} H_{p-1} + \frac{2}{(\alpha p - 1)^2} \right) + \frac{2}{(\alpha p - 1)^2} + \frac{1}{\alpha p - 1} \sum_{k=1}^{p-1} (-1)^k \frac{1}{k} \binom{\alpha p - 1}{k}.$$

By taking $i = p - 1$ in Lemma 2.1 and by congruences (1) and (12), we can have

$$\begin{aligned}
&\sum_{k=1}^{p-1} (-1)^{k+1} \binom{\alpha p - 1}{k} H_k^2 \\
&\equiv \frac{2(1-\alpha)p}{(\alpha p - 1)^3} + \frac{2}{(\alpha p - 1)^2} + \frac{1}{\alpha p - 1} \sum_{k=1}^{p-1} (-1)^k \frac{1}{k} \binom{\alpha p - 1}{k} \\
&\equiv \frac{2(1-\alpha)p}{(\alpha p - 1)^3} + \frac{2}{(\alpha p - 1)^2} + \frac{p^2}{\alpha p - 1} \left(-\frac{1}{3} - \frac{1}{3}\alpha + \alpha^2 \right) B_{p-3} \\
&\equiv 2 + p(6\alpha - 2) + p^2 \left(\left(-\alpha^2 + \frac{1}{3}\alpha + \frac{1}{3} \right) B_{p-3} + 6\alpha(2\alpha - 1) \right) \pmod{p^3},
\end{aligned}$$

as claimed. □

Theorem 2.9. *For an odd prime number p and any $p-$adic integer α, we have*

$$\sum_{k=0}^{[p/3]} (-1)^{k+1} \binom{\alpha p - 1}{k} H_k^2 \equiv \begin{cases} -\frac{3}{2} q_p^2(3) - \frac{1}{2} q_p(3) + \frac{2}{3} + pT_2(p,\alpha) \pmod{p^2} & \text{if } p \equiv 1 \pmod 3, \\ -\frac{3}{4} q_p^2(3) + \frac{1}{2} q_p(3) + \frac{4}{3} + pT_3(p,\alpha) \pmod{p^2} & \text{if } p \equiv 2 \pmod 3, \end{cases}$$

where

$$T_2(p,\alpha) = \left(\frac{3}{2} - \frac{9}{4}\alpha \right) q_p^3(3) - \left(\frac{9}{8}\alpha + \frac{1}{2} \right) q_p^2(3) - \left(\frac{1}{3} B_{p-2}\left(\frac{1}{3}\right) + \frac{3}{2}\alpha + 1 \right) q_p(3) + B_{p-2}\left(\frac{1}{3}\right) \left(\frac{1}{4}\alpha - \frac{1}{18} \right) + 2\alpha,$$

and

$$T_3(p,\alpha) = \left(-\frac{9}{8}\alpha + \frac{3}{4} \right) q_p^3(3) + \left(\frac{9}{8}\alpha - 1 \right) q_p^2(3) + \left(\frac{1}{6} B_{p-2}\left(\frac{1}{3}\right) + \frac{3}{2}\alpha - 1 \right) q_p(3) - B_{p-2}\left(\frac{1}{3}\right) \left(\frac{1}{4}\alpha + \frac{1}{18} \right) + 4\alpha - \frac{2}{3}.$$

Proof. According to $i = [p/3]$ in Lemma 2.1 and (7), it is clear that

$$\binom{\alpha p-1}{[p/3]} \equiv (-1)^{[p/3]}\left(1+\frac{3}{2}\alpha p q_p(3)\right) \pmod{p^2}. \tag{21}$$

Setting $n = [p/3]$ and $x = \alpha p - 1$ in Proposition 2.5, we can write

$$\sum_{k=0}^{[p/3]}(-1)^{k+1}\binom{\alpha p-1}{k}H_k^2 = (-1)^{[p/3]}\frac{[p/3]-(\alpha p-1)}{\alpha p-1}\binom{\alpha p-1}{[p/3]}\left(H_{[p/3]}^2+\frac{2}{\alpha p-1}H_{[p/3]}+\frac{2}{(\alpha p-1)^2}\right)$$
$$+\frac{2}{(\alpha p-1)^2}+\frac{1}{\alpha p-1}\sum_{k=1}^{[p/3]}(-1)^k\frac{1}{k}\binom{\alpha p-1}{k}.$$

With the help of (7), (14) and (21), we have

$$\sum_{k=0}^{[p/3]}(-1)^{k+1}\binom{\alpha p-1}{k}H_k^2$$
$$\equiv \frac{[p/3]-(\alpha p-1)}{\alpha p-1}\left(1+\frac{3}{2}\alpha p q_p(3)\right)\left(\left(\frac{-p}{6}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)-\frac{3}{2}q_p(3)+\frac{3}{4}pq_p^2(3)\right)^2\right.$$
$$+\frac{2}{\alpha p-1}\left(\frac{-p}{6}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)-\frac{3}{2}q_p(3)+\frac{3}{4}pq_p^2(3)\right)+\left.\frac{2}{(\alpha p-1)^2}\right)+\frac{2}{(\alpha p-1)^2}$$
$$+\frac{1}{\alpha p-1}\left(\left(\frac{-1}{6}-\frac{\alpha}{4}\right)p\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)-\frac{3}{2}q_p(3)+\left(\frac{3}{4}-\frac{9}{8}\alpha\right)pq_p^2(3)\right) \pmod{p^2}.$$

For $p \equiv 1 \pmod 3$, we have

$$\sum_{k=0}^{[p/3]}(-1)^{k+1}\binom{\alpha p-1}{k}H_k^2 \equiv -\frac{3}{2}q_p^2(3)-\frac{1}{2}q_p(3)+\frac{2}{3}+pT_2(p,\alpha) \pmod{p^2}$$

where

$$T_2(p,\alpha) = \left(\frac{3}{2}-\frac{9}{4}\alpha\right)q_p^3(3)-\left(\frac{9}{8}\alpha+\frac{1}{2}\right)q_p^2(3)-\left(\frac{1}{3}B_{p-2}\left(\frac{1}{3}\right)+\frac{3}{2}\alpha+1\right)q_p(3)+B_{p-2}\left(\frac{1}{3}\right)\left(\frac{1}{4}\alpha-\frac{1}{18}\right)+2\alpha.$$

Similarly, for $p \equiv 2 \pmod 3$, the other congruence can be proved. Thus, the proof is finished. □

Theorem 2.10. *For any prime number $p > 3$ and any p–adic integer α, we have*

$$\sum_{k=1}^{p-1}(-1)^k k\binom{\alpha p-1}{k}H_k^2 \equiv 1+p\left(3\alpha-\tfrac{5}{4}\right)-p^2\left(-\tfrac{41}{8}\alpha^2+\tfrac{27}{8}\alpha-\tfrac{1}{4}+\left(-\tfrac{1}{6}-\tfrac{1}{6}\alpha+\tfrac{1}{2}\alpha^2\right)B_{p-3}\right) \pmod{p^3}.$$

Proof. Setting $n = p-1$ and $x = \alpha p - 1$ in Proposition 2.6, we have

$$\sum_{k=1}^{p-1}(-1)^k k\binom{\alpha p-1}{k}H_k^2 = \frac{(p-1)(\alpha p-p)}{\alpha p-2}\binom{\alpha p-1}{p-1}\left(H_{p-2}^2+2H_{p-2}\left(\frac{1}{\alpha p-2}+\frac{\alpha p-2}{(p-1)(\alpha p-1)}\right)+\frac{2}{(\alpha p-2)^2}\right)$$
$$+p\binom{\alpha p-1}{p-1}(\alpha-1)\left(\frac{2}{\alpha p-1}\left(\frac{1}{p-1}+\frac{1}{\alpha p-1}\right)-\frac{1}{(\alpha p-2)(p-1)}\right)+\frac{2(\alpha p-1)}{(\alpha p-2)^2}$$
$$-\frac{2}{\alpha p-1}-\frac{1}{\alpha p-2}\sum_{k=1}^{p-1}(-1)^k\binom{\alpha p-1}{k}+\frac{1}{\alpha p-2}\sum_{k=1}^{p-1}(-1)^k\frac{1}{k}\binom{\alpha p-1}{k}.$$

Taking $i = p-1$ in Lemma 2.1 and using (1), we have

$$\sum_{k=1}^{p-1} (-1)^k k \binom{\alpha p - 1}{k} H_k^2 \equiv \frac{3}{2} - p\left(\frac{7}{4} - \frac{15}{4}\alpha\right) + p^2\left(6\alpha^2 - \frac{33}{8}\alpha + \frac{1}{4}\right) - \frac{1}{\alpha p - 2}\sum_{k=1}^{p-1}(-1)^k \binom{\alpha p - 1}{k} + \frac{1}{\alpha p - 2}\sum_{k=1}^{p-1}(-1)^k \frac{1}{k}\binom{\alpha p - 1}{k} \pmod{p^3}.$$

From (12) and (15), hence the congruence holds. □

Theorem 2.11. *For any prime number $p > 3$ and p–adic integer α, we have*

$$\sum_{k=1}^{(p-1)/2} (-1)^k k \binom{\alpha p - 1}{k} H_k^2 \equiv -\frac{1}{2} q_p^2(2) - \frac{1}{4} q_p(2) + \frac{7}{16} - pT_4(p,\alpha) \pmod{p^2},$$

where

$$T_4(p,\alpha) = \frac{45}{32}\alpha - \frac{1}{2} + \left(\frac{1}{8}\alpha - 1\right) q_p(2) + \left(\frac{1}{8} - \frac{3}{4}\alpha\right) q_p^2(2) + \left(\frac{1}{2} - \alpha\right) q_p^3(2).$$

Proof. Setting $n = (p-1)/2$ and $x = \alpha p - 1$ in Proposition 2.6 and according to (8), (13), (16) and (20), the proof of the congruence is clearly obtained. □

Theorem 2.12. *For any prime number $p > 3$ and any p–adic integer α, we have*

$$\sum_{k=0}^{[p/3]} (-1)^k k \binom{\alpha p - 1}{k} H_k^2 \equiv \begin{cases} \frac{5}{18} - \frac{5}{12} q_p(3) - \frac{1}{4} q_p^2(3) - pT_5(p,\alpha) \pmod{p^2} & \text{if } p \equiv 1 \pmod{3}, \\ \frac{11}{18} + \frac{1}{12} q_p(3) - \frac{1}{4} q_p^2(3) - pT_6(p,\alpha) \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

where

$$T_5(p,\alpha) = \frac{11}{36} - \frac{11}{12}\alpha + \frac{5}{12} q_p(3)\left(1 + \frac{3}{2}\alpha\right) + q_p^2(3)\left(\frac{15}{16}\alpha - \frac{1}{3}\right) + \frac{1}{8} q_p^3(3)(3\alpha - 2) + \left(\frac{1}{18} q_p(3) + \frac{5}{108} - \frac{1}{8}\alpha\right) B_{p-2}\left(\frac{1}{3}\right),$$

and

$$T_6(p,\alpha) = -\frac{23}{12}\alpha + \frac{13}{36} + \frac{7}{24} q_p(3)(2 - 3\alpha) + q_p^2(3)\left(\frac{1}{6} - \frac{3}{16}\alpha\right) - \frac{1}{8} q_p^3(3)(2 - 3\alpha) + \left(\frac{1}{8}\alpha - \frac{1}{18} q_p(3) + \frac{1}{108}\right) B_{p-2}\left(\frac{1}{3}\right).$$

Proof. Using $n = [p/3]$ and $x = \alpha p - 1$ in Proposition 2.6, from (7), (14), (17) and (21), we have the proof of the desired result. □

Theorem 2.13. *Let p be a prime of the form $4k+1$ and $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$ and $a \equiv 1 \pmod 4$. Then,*

$$\sum_{k=0}^{(p-1)/4} \frac{1}{4^k}\binom{2k}{k}\left(\frac{2}{k} - H_k^2\right) \equiv -(-1)^{(p-1)/4}\left(9q_p^2(2) + 12q_p(2) + 8\right)a + 8 \pmod{p}.$$

Proof. Using $n = (p-1)/4$ and $x = (p-1)/2$ in Proposition 2.5, we write

$$\sum_{k=0}^{(p-1)/4} (-1)^{k+1} \binom{(p-1)/2}{k} H_k^2 = -(-1)^{(p-1)/4} \frac{1}{2} \binom{(p-1)/2}{(p-1)/4} \left(H_{(p-1)/4}^2 + \frac{4}{p-1} H_{(p-1)/4} + \frac{8}{(p-1)^2} \right) + \frac{8}{(p-1)^2} + \frac{2}{p-1} \sum_{k=1}^{(p-1)/4} (-1)^k \frac{1}{k} \binom{(p-1)/2}{k}.$$

Combining this with (9), (10) and (11), we have the result. □

Theorem 2.14. *Let p be a prime of the form $4k+1$ and $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$ and $a \equiv 1 \pmod 4$. Then,*

$$\sum_{k=0}^{(p-1)/4} \frac{1}{4^k} \binom{2k}{k} \left(\frac{2}{3k} + kH_k^2 \right) \equiv -(-1)^{(p-1)/4} \left(\frac{3}{8} q_p^2(2) + \frac{13}{6} q_p(2) + \frac{40}{27} \right) a + \frac{26}{9} \pmod p.$$

Proof. Using $n = (p-1)/4$ and $x = (p-1)/2$ in Proposition 2.6 and by taking (9), (10), (11) and (18), the proof is complete. □

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