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New Characterizations of the Generalized B-T Inverse

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Abstract. Characterizations and explicit expressions of the generalized B-T inverse are given, this generalized inverse exists for any square matrix and any integer. The relationships between the generalized B-T inverse and some well-known generalized inverses are investigated. Moreover, an explicit formula of the generalized B-T inverse is given by using Hartwig-Spindelböck decomposition.

1. Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ matrices over the complex field $\mathbb{C}$. Let A^*, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\mathrm{rank}(A)$ denote the conjugate transpose, column space, null space and rank of $A \in \mathbb{C}^{m\times n}$, respectively. For $A \in \mathbb{C}^{m\times n}$, if $X \in \mathbb{C}^{n\times m}$ satisfies $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$, then X is called a *Moore-Penrose inverse* of A [14, 15]. This matrix X is unique and denoted by $A^\dagger$.

The core inverse and the dual core inverse for a complex matrix were introduced by Baksalary and Trenkler [1]. Let $A \in \mathbb{C}^{n\times n}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called a core inverse of A, if it satisfies $AX = P_A$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, where $\mathcal{R}(A)$ denotes the column space of A, and P_A is the orthogonal projector onto $\mathcal{R}(A)$. And if such a matrix exists, then it is unique (and denoted by $A^{\textcircled{\#}}$). Baksalary and Trenkler gave several characterizations of the core inverse by using the decomposition of Hartwig and Spindelböck. Many existence criteria and properties of the core inverse can be found in [1, 2, 11, 12, 17, 18, 20] etc.

Let $A \in \mathbb{C}^{n\times n}$. A matrix $X \in \mathbb{C}^{n\times n}$ such that $XA^{k+1} = A^k$, $XAX = X$ and $AX = XA$ is called the *Drazin inverse* of A and denoted by A^D[5]. The Drazin inverse of a square matrix always exists and it is unique. The smallest such integer k is called the Drazin index of A, denoted by $\mathrm{ind}(A)$. If $\mathrm{ind}(A) \leq 1$, then the Drazin inverse of A is called the *group inverse* and denoted by $A^\#$.

The DMP-inverse for a complex matrix was introduced by Malik and Thome [10]. Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. A matrix $X \in \mathbb{C}^{n\times n}$ is called a *DMP-inverse* of A, if it satisfies $XAX = X$, $XA = A^DA$ and $A^kX = A^kA^\dagger$. It is unique and denoted by $A^{D,\dagger}$. Malik and Thome gave several characterizations of the DMP-inverse by using the decomposition of Hartwig and Spindelböck [9].

The notion of the core-EP inverse for a complex matrix was introduced by Manjunatha Prasad and Mohana [11]. A matrix $X \in \mathbb{C}^{n\times n}$ is a *core-EP inverse* of $A \in \mathbb{C}^{n\times n}$ if X is an outer inverse of A satisfying

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$\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$, where k is the index of A. The such matrix X always exists, it is unique and denoted by $A^{\oplus}$.

The $\langle i, m\rangle$-core inverse was introduced in [18] for a complex matrix. Let $A \in \mathbb{C}^{n\times n}$ and $m, i \in \mathbb{N}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called an *$\langle i, m\rangle$-core inverse* of A, if it satisfies $X = A^D AX$ and $A^m X = A^i (A^i)^{\dagger}$. If such X exists, then it is unique and denoted by $A^{\oplus}_{i,m}$.

In [6, Definition 1.3], Drazin introduced a new class of outer inverses in the setting of semigroups, namely, the (b, c)-inverse. Let R be a ring and $a, b, c \in R$. We say that $y \in R$ is the *(b, c)-inverse* of a if we have

$$y \in (bRy) \cap (yRc), \ yab = b \text{ and } cay = c. \tag{1}$$

If such $y \in R$ exists, then it is unique and denoted by $a^{\|(b,c)}$. The (b, c)-inverse is a generalization of the Moore-Penrose inverse, the Drazin inverse, the group inverse and the core inverse. Many existence criteria and properties of the (b, c)-inverse can be found in [3, 4, 6, 16, 19] etc.

The core inverse has several generalized forms, such as the DMP-inverse, the core-EP inverse and $\langle i, m\rangle$-core inverse. In the paper, we will investigate a generalization of the core inverse, namely, the generalized B-T inverse, which also is a generalization of B-T inverse [2], the B-T inverse also called generalized core inverse, let $A \in \mathbb{C}^{n\times n}$, a matrix $A^{\diamond}_{gn} \in \mathbb{C}^{n\times n}$ which is equal to $(A^2 A^{\dagger})^{\dagger}$ is called the generalized core inverse of A.

2. Characterizations and expressions of the generalized B-T inverse

In this section, we introduce a new inverse, which is a generalization of B-T inverse [2].

Definition 2.1. *Let $A \in \mathbb{C}^{m\times m}$ and $m, n \in \mathbf{N}$. A matrix $A^{\diamond}_{gn} \in \mathbb{C}^{m\times m}$ satisfying*

$$A^{\diamond}_{gn} = (A^{n+1} A^{\dagger})^{\dagger} \tag{2}$$

is called the generalized B-T inverse of A.

It is seen from Definition 2 that $A^{\diamond}_{gn}$ exists for every $A \in \mathbb{C}^{m\times m}$ for $m, n \in \mathbf{N}$ and is unique.

The next example says that the generalized B-T inverse of A is different from the generalized core inverse of A.

Example 2.2. Let R be the ring of all bi-finite real matrices with transpose as involution and let $e_{i,j}$ be the matrix in R with 1 in the (i, j) position and 0 elsewhere. Let $A = \sum_{i=1}^{\infty} e_{i+1,i}$ and $B = A^*$, now $AB = \sum_{i=2}^{\infty} e_{i,i}$, $BA = I$. It is easy to check that $A^2 A^{\dagger} = \sum_{i=3}^{\infty} e_{i+1,i}$ and $A^3 A^{\dagger} = \sum_{i=4}^{\infty} e_{i+1,i}$, that is $A^2 A^{\dagger} \neq A^3 A^{\dagger}$, which we can get that $A^{\diamond}_{gn} \neq A^{\diamond}$, thus the generalized B-T inverse of A is different from the generalized core inverse of A.

Example 2.3. *The generalized B-T inverse is different from the DMP-inverse, core-EP inverse and $\langle i, m\rangle$-core inverse, where i, m are arbitrary. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 3}$. Then it is easy to check that $A^{\diamond}_{g2} = \begin{bmatrix} 1/5 & 0 & 0 \\ 2/5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, but $A^{d,\dagger} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A^{\oplus} = A^{\oplus}_{i,m} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.*

Proposition 2.4. *Let $A \in \mathbb{C}^{m\times m}$ and $n \in \mathbf{N}$. Then $\mathcal{N}(A^{\diamond}_{gn}) = \mathcal{N}((A^{n+1})^*)$.*

Proof. Let $X \in \mathcal{N}(A^{\diamond}_{gn})$, then $A^{\diamond}_{gn} X = (A^{n+1} A^{\dagger})^{\dagger} X = 0$, which implies $(A^{n+1} A^{\dagger})^* X = 0$. Taking involution on $(A^{n+1} A^{\dagger})^* X = 0$, we can get $X^*((A^{n+1} A^{\dagger})^*)^* = 0$, that is $X^* A^{n+1} A^{\dagger} = 0$. Multiplying by A on the right side of $X^* A^{n+1} A^{\dagger} = 0$, we can get $X^* A^{n+1} = 0$, hence $X \in \mathcal{N}((A^{n+1})^*)$. Conversely, let $Y \in \mathcal{N}((A^{n+1})^*)$, that is $(A^{n+1})^* Y = 0$. Then $Y^* A^{n+1} = 0$ by taking involution on $(A^{n+1})^* Y = 0$. Multiplying by $A^{\dagger}$ on the right side of $Y^* A^{n+1} = 0$, we can get $Y^* A^{n+1} A^{\dagger} = 0$. Taking involution on $Y^* A^{n+1} A^{\dagger} = 0$, we have $(A^{n+1} A^{\dagger})^* Y = 0$, that is $A^{\diamond}_{gn} Y = 0$. Thus,$\mathcal{N}(A^{\diamond}_{gn}) = \mathcal{N}((A^{n+1})^*)$. $\square$

Remark 2.5. *Let $A \in \mathbb{C}^{m\times m}$ and $n \in \mathbf{N}$. Then, we have $A^{\diamond}_{gn} = 0 \Leftrightarrow (A^{n+1}A^{\dagger})^{\dagger} \Leftrightarrow A^{n+1}A^{\dagger} = 0 \Leftrightarrow A^{n+1} = 0$. Thus, $A^{\diamond}_{gn} = 0$ if and only if $A^{n+1} = 0$. That is, we can use the generalized B-T inverse of A to character the nilpotent matrix.*

Proposition 2.6. *Let $A, B, U \in \mathbb{C}^{n\times n}$ with $B = UAU^*$ and U is unitary matrix. Then for $n \in \mathbf{N}$, we have $B^{\diamond}_{gn} = UA^{\diamond}_{gn}U^*$.*

Proof. Since
$$B^{\diamond}_{gn} = (B^{n+1}B^{\dagger})^{\dagger} = ((UAU^*)^{n+1}(UAU^*)^{\dagger})^{\dagger} = (UA^{n+1}A^{\dagger}U^*)^{\dagger} = U(A^{n+1}A^{\dagger})^{\dagger}U^*,$$
so we have $B^{\diamond}_{gn} = UA^{\diamond}_{gn}U^*$. □

Theorem 2.7. *Let $A \in \mathbb{C}^{m\times m}$ and $n \in \mathbf{N}$. Then the generalized B-T inverse of A is the $(P_A(A^n)^*, P_A(A^n)^*)$-inverse of $P_A(A^n)^*$.*

Proof. Since the generalized B-T inverse of A is the Moore-Penrose inverse of $A^{n+1}A^{\dagger}$, and the Moore-Penrose inverse of $A^{n+1}A^{\dagger}$ is the $((A^{n+1}A^{\dagger})^*, (A^{n+1}A^{\dagger})^*)$-inverse of $A^{n+1}A^{\dagger}$, so we can get the generalized B-T inverse of A is the $(P_A(A^n)^*, P_A(A^n)^*)$-inverse of $P_A(A^n)^*$. □

By [1, Theorem 1], we have $(A^{\circledast})^n = (A^n)^{\circledast}$ for any $n \in \mathbf{N}$, but it is not true for the generalized B-T inverse of A, a counterexample can be found as follows: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 3}$. Then it is easy to check that $A^{\diamond}_{g2} = \begin{bmatrix} 1/5 & 0 & 0 \\ 2/5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $(A^{\diamond}_{g2})^2 = \begin{bmatrix} 1/25 & 0 & 0 \\ 2/25 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, it is trivial that $(A^{\diamond}_{g2})^2$ is not an idempotent matrix. By the definition of the generalized B-T inverse of A, we have $(A^2)^{\diamond}_{g2} = [(A^2)^3(A^2)^{\dagger}]^{\dagger} = [A^2(A^2)^{\dagger}]^{\dagger} = A^2(A^2)^{\dagger}$, which is an idempotent matrix, thus we have $(A^2)^{\diamond}_{g2} \neq (A^{\diamond}_{g2})^2$ by $(A^{\diamond}_{g2})^2$ is not an idempotent matrix.

3. How to compute the generalized B-T inverse of A

Every matrix $A \in \mathbb{C}^{n\times n}$ of rank r can be represented in the form
$$A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^*, \tag{3}$$
where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = \sigma_1 I_{r_1} \oplus \cdots \oplus \sigma_t I_{r_t}$ is the diagonal matrix of the nonzero singular values of A, where $\sigma_1 > \sigma_1 > \cdots > \sigma_t > 0$, $r_1 + \cdots + r_t = r$, and $K \in \mathbb{C}^{r\times r}$ and $L \in \mathbb{C}^{r\times(n-r)}$ satisfy
$$KK^* + LL^* = I_r.$$
The decomposition in (3) is known as the *Hartwig-Spindelböck decomposition* [9].

Theorem 3.1. *Let $A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^*$ be the Hartwig-Spindelböck decomposition of A as in* (3) *and $n \in \mathbf{N}$. Then, we have*
$$A^{\diamond}_{gn} = U\begin{bmatrix} ((\Sigma K)^n)^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{4}$$

Proof. Let $A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^*$ be the Hartwig-Spindelböck decomposition of A as in (3) and $n \in \mathbf{N}$. Then we have
$$A^{n+1} = U\begin{bmatrix} (\Sigma K)^{n+1} & (\Sigma K)^n \Sigma L \\ 0 & 0 \end{bmatrix}U^*, \tag{5}$$

and

$$A^{\dagger} = U\begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix}U^*, \tag{6}$$

By (5) and (6), we have

$$\begin{aligned} A^{n+1}A^{\dagger} &= U\begin{bmatrix} (\Sigma K)^{n+1} & (\Sigma K)^n\Sigma L \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (\Sigma K)^{n} & (\Sigma K)^{n-1}\Sigma L \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (\Sigma K)^{n} & (\Sigma K)^{n-1}\Sigma L \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (\Sigma K)^{n} & 0 \\ 0 & 0 \end{bmatrix}U^*. \end{aligned} \tag{7}$$

The rest is clear by the properties of the Moore-Penrose inverse. □

Theorem 3.2. *Let $A \in \mathbb{C}^{m\times m}$, $A^{\diamond}_{gn}$ is the generalized B-T inverse of A and $m, n \in \mathbf{N}$, then $A^{\diamond}_{gn}$ is a $\{2,3\}$-inverse of A^n.*

Proof. Let $A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^*$ be the Hartwig-Spindelböck decomposition of A as in (3) and $n \in \mathbf{N}$. Then we have

$$A^n = U\begin{bmatrix} (\Sigma K)^{n} & (\Sigma K)^{n-1}\Sigma L \\ 0 & 0 \end{bmatrix}U^*. \tag{8}$$

By Theorem 3.1, we have

$$A^{\diamond}_{gn} = U\begin{bmatrix} ((\Sigma K)^{n})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{9}$$

By (8) and (9), we have

$$\begin{aligned} A^nA^{\diamond}_{gn} &= U\begin{bmatrix} (\Sigma K)^{n} & (\Sigma K)^{n-1}\Sigma L \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} ((\Sigma K)^{n})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (\Sigma K)^{n}((\Sigma K)^{n})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^*, \end{aligned}$$

By (9), we have

$$\begin{aligned} A^{\diamond}_{gn}A^nA^{\diamond}_{gn} &= U\begin{bmatrix} ((\Sigma K)^{n})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} (\Sigma K)^{n}((\Sigma K)^{n})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} ((\Sigma K)^{n})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} (\Sigma K)^{n}((\Sigma K)^{n})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} ((\Sigma K)^{n})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^* = A^{\diamond}_{gn}, \end{aligned}$$

that is,

$$A^{\diamond}_{gn}A^nA^{\diamond}_{gn} = A^{\diamond}_{gn}. \tag{10}$$

By (10), we have $A^{\diamond}_{gn}$ is a $\{2,3\}$-inverse of A^n for $n \in \mathbf{N}$ □

In the following theorem, we will give the relationship between the generalized B-T inverse and Moore-Penrose inverse.

Theorem 3.3. *Let A has the Hartwig-Spindelböck decomposition of A as in* (3) *and $A^{\diamond}_{gn}$ be the generalized B-T inverse of A with $n \in \mathbf{N}$. If $((\Sigma K)^n)^*(\Sigma K)^{n-1}\Sigma L = 0$, then $A^{\diamond}_{gn}$ is the Moore-Penrose inverse of A^n.*

Proof. In the proof of Theorem 3.2, we have

$$\begin{aligned} A^nA^{\diamond}_{gn}A^n &= U\begin{bmatrix} (\Sigma K)^n((\Sigma K)^n)^\dagger & 0 \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} (\Sigma K)^n & (\Sigma K)^{n-1}\Sigma L \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (\Sigma K)^n((\Sigma K)^n)^\dagger & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} (\Sigma K)^n & (\Sigma K)^{n-1}\Sigma L \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (\Sigma K)^n((\Sigma K)^n)^\dagger(\Sigma K)^n & (\Sigma K)^n((\Sigma K)^n)^\dagger(\Sigma K)^{n-1}\Sigma L \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (\Sigma K)^n & (\Sigma K)^n((\Sigma K)^n)^\dagger(\Sigma K)^{n-1}\Sigma L \\ 0 & 0 \end{bmatrix}U^*, \end{aligned}$$

thus, the condition such that $A^nA^{\diamond}_{gn}A^n = A^n$ is $(\Sigma K)^n((\Sigma K)^n)^\dagger(\Sigma K)^{n-1}\Sigma L = 0$, which is equivalent to $((\Sigma K)^n)^\dagger(\Sigma K)^n((\Sigma K)^n)^\dagger(\Sigma K)^{n-1}\Sigma L = 0$, that is $((\Sigma K)^n)^\dagger(\Sigma K)^{n-1}\Sigma L = 0$, so if $((\Sigma K)^n)^*(\Sigma K)^{n-1}\Sigma L = 0$, then

$$A^nA^{\diamond}_{gn}A^n = A^n \tag{11}$$

By (8) and (9), we have

$$\begin{aligned} A^{\diamond}_{gn}A^n &= U\begin{bmatrix} ((\Sigma K)^n)^\dagger & 0 \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} (\Sigma K)^n & (\Sigma K)^{n-1}\Sigma L \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} ((\Sigma K)^n)^\dagger & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} (\Sigma K)^n & (\Sigma K)^{n-1}\Sigma L \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} ((\Sigma K)^n)^\dagger(\Sigma K)^n & ((\Sigma K)^n)^\dagger(\Sigma K)^{n-1}\Sigma L \\ 0 & 0 \end{bmatrix}U^*. \end{aligned} \tag{12}$$

If $((\Sigma K)^n)^\dagger\Sigma K(\Sigma L)^{n-1} = 0$, then by (12), we have

$$\begin{aligned} A^{\diamond}_{gn}A^n &= U\begin{bmatrix} ((\Sigma K)^n)^\dagger(\Sigma K)^n & ((\Sigma K)^n)^\dagger(\Sigma K)^{n-1}\Sigma L \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} ((\Sigma K)^n)^\dagger(\Sigma K)^n & 0 \\ 0 & 0 \end{bmatrix}U^*. \end{aligned}$$

So,

$$(A^{\diamond}_{gn}A^n)^* = A^{\diamond}_{gn}A^n. \tag{13}$$

By Theorem 3.2, (11) and (13), one can prove $A^{\diamond}_{gn}$ is the Moore-Penrose inverse of A^n by the definition of the Moore-Penrose inverse. □

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