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Strong Coupled Proximal Points of Cyclic Coupled Proximal Mappings Using C_k-Class Functions in S-Metric Spaces

G. V. R. Babu[a], P. Durga Sailaja[b], G. Srichandana[c]

[a]*Department of Mathematics, Andhra University, Visakhapatnam-530 003, INDIA.*
[b]*Department of Mathematics, Lendi Institute of Engineering and Technology, Vizianagaram-535 005, INDIA.*
[c]*Department of Mathematics, Satya Institute of Technology and Management, Vizianagaram-535 002, INDIA.*

Abstract. In this paper, we introduce cyclic coupled proximal mapping in S-metric spaces using C_k-class functions and prove the existence of strong coupled proximal points of such mappings in complete S-metric spaces. Also, we provide an example in support of our main result.

1. Introduction and Preliminaries

In 1969, Fan [11] introduced the notation of best proximity point. Later, in 2006, Eldred and Veeramani [10] established results on the existence and uniqueness of best proximity point in a uniformly convex Banach space. Let (X, d) be a metric space. Let A and B be two nonempty subsets of X and $T : A \to B$. A point $x \in A$ is called a best proximity point of T if $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(x, y) : (x, y) \in A \times B\}$. It is observed that best proximity point becomes a fixed point when the mapping T is a self-mapping. For more works on best proximity point results, we refer [3], [5], [10]. In 2009, Suzuki, Kikkawa and Vetro [18] extended Eldred and Veeramani [10] theorem to metric spaces by using UC property. Later, in 2012, coupled best proximity point in metric spaces was introduced by Sintunavarat and Kumam [17] and proved the existence of these points in metric spaces. For more works on coupled best proximity point results, we refer [2], [6], [12], [14], [18].

We recall the following definitions.

Definition 1.1. *[13] Let A and B be two nonempty subsets of X. A mapping $f : X \to X$ is cyclic with respect to A and B if $f(A) \subseteq B$ and $f(B) \subseteq A$.*

Definition 1.2. *[7] Let X be a nonempty set. Let $F : X \times X \to X$ be a mapping. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of F if $F(x, y) = x$ and $F(y, x) = y$.*

Definition 1.3. *[8] Let A and B be two nonempty subsets of X. A mapping $F : X \times X \to X$ is said to be cyclic with respect to A and B if $F(A, B) \subseteq B$ and $F(B, A) \subseteq A$.*

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Email addresses: gvr_babu@hotmail.com (G. V. R. Babu), sailajadurga@yahoo.com (P. Durga Sailaja), sri.chandan3@gmail.com (G. Srichandana)

Definition 1.4. *[8] Let X be a nonempty set. Let $F : X \times X \to X$ be a mapping. An element $(x, x) \in X \times X$ is said to be a strong coupled fixed point of F if $F(x, x) = x$.*

Definition 1.5. *[18] Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy the UC property if $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n\to\infty} d(x_n, y_n) = d(A, B)$ and $\lim_{n\to\infty} d(z_n, y_n) = d(A, B)$, then $\lim_{n\to\infty} d(x_n, z_n) = 0$.*

Lemma 1.6. *[18] Let A and B be subsets of a metric space (X, d). Assume that (A, B) has the UC property. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and B respectively, such that either of the following holds.*
$\lim_{m\to\infty} \sup_{n\geq m} d(x_m, y_n) = d(A, B)$ or $\lim_{n\to\infty} \sup_{m\geq n} d(x_m, y_n) = d(A, B)$. Then $\{x_n\}$ is Cauchy.

In 2014, Ansari [1] introduced C-class functions and proved fixed point theorems of generalized contractions involving C-class functions. Recently, in 2016, Ansari, Jacob, Marudai, Kumam [2] developed strong coupled proximity point results for generalized cyclic coupled proximal mappings using C-class functions.

Definition 1.7. *[1] A mapping $f : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and satisfies the following axioms:*

(1) $f(s, t) \leq s$

(2) $f(s, t) = s$ implies that either $s = 0$ or $t = 0$ for all $s, t \in [0, \infty)$.

We denote the set of all C-class functions by $\mathcal{C}$.

We denote
$\Psi = \{\psi : [0, \infty) \to [0, \infty)$ such that (i) ψ is lower semi-continuous, (ii) $\psi(0) \geq 0$, and (iii) $\psi(s) > 0$ for each $s > 0\}$.

Definition 1.8. *[2] Let A and B be two nonempty disjoint subsets of a meric space (X, d). A mapping $F : X\times X \to X$ is called a cyclic coupled proximal mapping of type $I_{f\psi}$ if F is cyclic with respect to A and B satisfying the inequality*
$$d(F(x, y), F(u, v)) \leq f(\max\{d(x, F(x, y)), d(u, F(u, v))\} - d(A, B), \psi(\max\{d(x, F(x, y)), d(u, F(u, v))\} - d(A, B)))$$
where $x, v \in A$ and $y, u \in B$ for some $\psi \in \Psi$, $f \in \mathcal{C}$.

Definition 1.9. *[2] Let A and B be two nonempty disjoint subsets of a meric space (X, d). A mapping $F : X\times X \to X$ is called a cyclic coupled proximal mapping of type $II_{f\psi}$ if F is cyclic with respect to A and B satisfying the inequality*
$$d(F(x, y), F(u, v)) \leq f(\tfrac{d(x,F(x,y))+d(u,F(u,v))}{2} - d(A, B), \psi(\tfrac{d(x,F(x,y))+d(u,F(u,v))}{2} - d(A, B)))$$
where $x, v \in A$ and $y, u \in B$ for some $\psi \in \Psi$, $f \in \mathcal{C}$.

Definition 1.10. *[2] Let (X, d) be a metric space. An element $(x, y) \in X \times X$ is said to be a strong coupled proximal point if $d(x, F(x, y)) = d(y, F(y, x)) = d(x, y) = d(A, B)$.*

Theorem 1.11. *[2] Let (X, d) be a complete metric space and A, B be two nonempty, closed, disjoint subsets of X. Let $F : X \times X \to X$ be a cyclic coupled proximal mapping of type $I_{f\psi}$. Then F has a strong coupled proximal point if (A, B) satisfies UC property.*

Theorem 1.12. *[2] Let (X, d) be a complete metric space and A, B be two nonempty, closed, disjoint subsets of X. Let $F : X \times X \to X$ be a cyclic coupled proximal mapping of type $II_{f\psi}$. Then F has a strong coupled proximal point if (A, B) satisfies UC property and $d(u, v) + d(A, B) \leq d(u, F(a, b))$ whenever $v = F(b, a)$ and $\{u, v\}$ belongs to the set A.*

We now define C_k-class functions as follows.

Definition 1.13. *[4] Let $f : [0, \infty)^2 \to \mathbb{R}$ be a function and $k \geq 1$. If f is continuous and*

(1) $f(s, t) \leq ks$

(2) $f(s,t) = ks$ *implies that either* $s = 0$ *or* $t = 0$ *for all* $s,t \in [0,\infty)$,

then we say that f *is a* C_k*-class function.*

Remark 1.14. *[4] Every C-class function is a* C_k*-class function with* $k = 1$*. But its converse is not true due to the following example.*

Example 1.15. *[4] A mapping* $f : [0,\infty)^2 \to \mathbb{R}$ *is a function such that* $f(s,t) = 2s$ *for all* $s,t \in [0,\infty)$*. Then clearly, F is a* C_k*-class function for any* $k > 2$*. But it is not a C-class function.*

Here onwards, we denote the class of all C_k-class functions by $\mathscr{C}_k$.

Example 1.16. *We define* $f : [0,\infty)^2 \to \mathbb{R}$ *as follows: for any* $s,t \in [0,\infty)$,

(1) $f(s,t) = ms$ *where* $0 < m < 1$.

(2) $f(s,t) = k(s - t)$ *where* $k \geq 1$.

(3) $f(s,t) = ks\beta(s)$ *where* $\beta : [0,\infty) \to [0,1)$ *is a continuous function and* $k \geq 1$.

(4) $f(s,t) = k\varphi_1(t)se^{-s}$ *where* $\varphi_1 : [0,\infty) \to [0,1]$ *is a bounded function and* $k \geq 1$.

(5) $f(s,t) = k(s - e^{-t})$ *where* $k \geq 1$.

(6) $f(s,t) = k(s - \frac{t}{1+t})$ *where* $k \geq 1$.

Then all of the above functions $f \in \mathscr{C}_k$.

In 2012, Sedghi, Shobe and Aliouche [15] introduced S-metric spaces and studied the existence of fixed points in these spaces. Recently, in 2016, Nantadilok [14] obtained best proximity point results in S-metric spaces and in 2017, Ansari and Nantadilok [3] established the best proximity point results for a certain class of proximal contractive mappings using C-class functions in S-metric spaces.

Definition 1.17. *[15] Let X be a nonempty set. An S-metric on X is a function* $S : X^3 \to [0,\infty)$ *that satisfies the following conditions: for each* $x,y,z,a \in X$

(S1) $S(x,y,z) = 0$ *if and only if* $x = y = z$, *and*

(S2) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) *is called an S-metric space.*

Throughout this paper, we denote the set of all real numbers by $\mathbb{R}$, the set of all natural numbers by $\mathbb{N}$.

Example 1.18. *[15] Let* (X,d) *be a metric space. We define* $S : X^3 \to [0,\infty)$ *by* $S(x,y,z) = d(x,y) + d(x,z) + d(y,z)$ *for all* $x,y,z \in X$*. Then S is an S-metric on X and S is called the S-metric induced by the metric d.*

Example 1.19. *[9] Let* $X = \mathbb{R}$*, the set of all real numbers and let* $S(x,y,z) = |y + z - 2x| + |y - z|$ *for all* $x,y,z \in X$*. Then* (X,S) *is an S-metric space.*

Example 1.20. *[16]* Let $X = \mathbb{R}$. Then $S(x,y,z) = |x - z| + |y - z|$ for all $x,y,z \in X$ is an S-metric on X. This S-metric is called the usual S-metric.

Lemma 1.21. *[15]* In an S-metric space, we have $S(x,x,y) = S(y,y,x)$.

Lemma 1.22. *[9]* Let (X,S) be an S-metric space. Then $S(x,x,z) \leq 2S(x,x,y) + S(y,y,z)$.

Definition 1.23. *[15] Let* (X,S) *be an S-metric space.*

(i) A sequence $\{x_n\} \subseteq X$ is said to converge to a point $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$ and we denote it by $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\} \subseteq X$ is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.

(iii) An S-metric space (X, S) is said to be complete if each Cauchy sequence in X is convergent.

Lemma 1.24. *[15]* Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma 1.25. *[15]* Let (X, S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Lemma 1.26. *[4] Let (X, S) be an S-metric space. Let $\{x_n\}$, $\{y_n\}$ be sequences in X and $\{x_n\}$ converges to x. Then $\lim_{n\to\infty} S(x_n, x_n, y_n) = \lim_{n\to\infty} S(x, x, y_n)$.*

Inspired by the works of Ansari, Jacob, Marudai and Kumam [2], in Section 2, we introduce cyclic coupled proximal mapping using C_k-class functions and we prove the existence of strong coupled proximal points in complete S-metric space for such mapping by using UC property. In Section 3, we draw some corollaries to our main result and present an illustrative example in support of our main result.

Throughout this paper, we denote
$\Phi = \{\varphi : [0, \infty) \to [0, \infty)$ such that (i) φ is continuous, and (ii) $\varphi(t) = 0$ if and only if $t = 0\}$.

2. Cyclic Coupled Proximal Mapping and Strong Coupled Proximal Points

Let A and B be nonempty subsets of an S-metric space (X, S), we define
$S(A, A, B) = inf\{S(x, x, y) : x \in A \text{ and } y \in B\}$.

We now define *UC* property in S-metric space setting as follows.

Definition 2.1. *Let A and B be nonempty subsets of an S-metric space (X, S). Then (A, B) is said to satisfy the UC property if $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(A, A, B)$ and $\lim_{n\to\infty} S(z_n, z_n, y_n) = S(A, A, B)$, then $\lim_{n\to\infty} S(x_n, x_n, z_n) = 0$.*

Motivated by the strong coupled proximal point defined in metric space setting by Ansari, Jacob, Marudai and Kumam [2], we define strong coupled proximal point in S-metric spaces as follows.

Definition 2.2. *Let (X, S) be an S-metric space. An element $(x, y) \in X \times X$ is said to be a strong coupled proximal point if $S(x, x, F(x, y)) = S(y, y, F(y, x)) = S(x, x, y) = S(A, A, B)$.*

In the following, we define cyclic coupled proximal mapping.

Definition 2.3. *Let (X, S) be an S-metric space. Let A and B be two nonempty subsets of X. Let $F : X \times X \to X$ be a mapping. If (i) F is cyclic with respect to A and B and (ii) there exist $\varphi \in \Phi$ and $f \in \mathscr{C}_k$, $k \geq 1$ such that*

$$\begin{aligned} kS(F(x,y),F(u,v),F(w,z)) \leq f(&\max\{S(x,x,F(x,y)),S(u,u,F(u,v)),\\ &S(w,w,F(w,z))\} - S(A,A,B),\\ &\varphi(\max\{S(x,x,F(x,y)),S(u,u,F(u,v)),\\ &S(w,w,F(w,z))\} - S(A,A,B))) + kS(A,A,B) \end{aligned} \tag{1}$$

where $x, u, z \in A$ and $y, v, w \in B$ then we say that F is a cyclic coupled proximal mapping.

The main result of this paper is the following.

Theorem 2.4. Let (X, S) be a complete S-metric space. Let A and B be two nonempty closed subsets of X. Let $F : X \times X \to X$ be a cyclic coupled proximal mapping. If (A, B) satisfies UC property then F has a strong coupled proximal point.

We prove the following lemmas to prove our main result.

Lemma 2.5. *Let (X, S) be an S-metric space. Let A and B be two nonempty subsets of X. Let $F : X \times X \to X$ be a cyclic coupled proximal mapping. Let $x_0 \in A$ and $y_0 \in B$. Then we have the following.*

(i) $S(x_{2n+1}, x_{2n+1}, y_{2n+2}) \leq S(y_{2n}, y_{2n}, x_{2n+1})$;

(ii) $S(y_{2n+1}, y_{2n+1}, x_{2n+2}) \leq S(x_{2n}, x_{2n}, y_{2n+1})$ *for all* $n = 0, 1, 2, ...$ *and*

(iii) $S(x_{2n}, x_{2n}, y_{2n+1}) \leq S(y_{2n-1}, y_{2n-1}, x_{2n})$;

(iv) $S(y_{2n}, y_{2n}, x_{2n+1}) \leq S(x_{2n-1}, x_{2n-1}, y_{2n})$, *for all* $n = 1, 2, ...$,

where the sequences $\{x_n\}$ in A and $\{y_n\}$ in B are defined by

$$x_{n+1} = F(y_n, x_n), \; y_{n+1} = F(x_n, y_n), \; n = 0, 1, 2, \dots . \tag{2}$$

Proof. Let $x_0 \in A$ and $y_0 \in B$ be arbitrary. We define sequences $\{x_n\}$ in A and $\{y_n\}$ in B by (2).
We consider

$$\begin{aligned} kS(x_1, x_1, y_2) &= kS(y_2, y_2, x_1) \\ &= kS(F(x_1, y_1), F(x_1, y_1), F(y_0, x_0)) \\ &\leq f(\max\{S(x_1, x_1, F(x_1, y_1)), S(y_0, y_0, F(y_0, x_0))\} - S(A, A, B), \\ &\quad \varphi(\max\{S(x_1, x_1, F(x_1, y_1)), S(y_0, y_0, F(y_0, x_0))\} - S(A, A, B))) \\ &\quad\quad + kS(A, A, B) \\ &= f(\max\{S(x_1, x_1, y_2), S(y_0, y_0, x_1)\} - S(A, A, B), \\ &\quad \varphi(\max\{S(x_1, x_1, y_2), S(y_0, y_0, x_1)\} - S(A, A, B))) + kS(A, A, B). \end{aligned}$$

If

$$S(x_1, x_1, y_2) > S(y_0, y_0, x_1) \tag{3}$$

then it follows that

$$\begin{aligned} &kS(x_1, x_1, y_2) - kS(A, A, B) \\ &\quad \leq f(S(x_1, x_1, y_2) - S(A, A, B), \varphi(S(x_1, x_1, y_2) - S(A, A, B))) \\ &\quad \leq kS(x_1, x_1, y_2) - kS(A, A, B) \text{ (by the first property of } f). \end{aligned}$$

Therefore $f(S(x_1, x_1, y_2) - S(A, A, B), \varphi(S(x_1, x_1, y_2) - S(A, A, B)))$
$= kS(x_1, x_1, y_2) - kS(A, A, B)$.
By the second property of f, we have either
$S(x_1, x_1, y_2) - S(A, A, B) = 0$ or $\varphi(S(x_1, x_1, y_2) - S(A, A, B)) = 0$ so that
$S(x_1, x_1, y_2) = S(A, A, B)$.
Now, we have $S(x_1, x_1, y_2) = S(A, A, B) \leq S(y_0, y_0, x_1)$, a contradiction to (3).
Therefore

$$S(x_1, x_1, y_2) \leq S(y_0, y_0, x_1). \tag{4}$$

Thus (i) holds for $n = 0$.
In general, for a fixed positive integer n, we consider

$$\begin{aligned} kS(x_{2n+1}, x_{2n+1}, y_{2n+2}) &= kS(y_{2n+2}, y_{2n+2}, x_{2n+1}) \\ &= kS(F(x_{2n+1}, y_{2n+1}), F(x_{2n+1}, y_{2n+1}), F(y_{2n}, x_{2n})) \\ &\leq f(\max\{S(x_{2n+1}, x_{2n+1}, F(x_{2n+1}, y_{2n+1})), \\ &\qquad S(y_{2n}, y_{2n}, F(y_{2n}, x_{2n}))\} - S(A, A, B), \\ &\quad \varphi(\max\{S(x_{2n+1}, x_{2n+1}, F(x_{2n+1}, y_{2n+1})), \\ &\qquad S(y_{2n}, y_{2n}, F(y_{2n}, x_{2n}))\} - S(A, A, B))) + kS(A, A, B). \end{aligned}$$

That is

$$\begin{aligned}kS(x_{2n+1},x_{2n+1},y_{2n+2}) \leq f(&\max\{S(x_{2n+1},x_{2n+1},y_{2n+2}),S(y_{2n},y_{2n},x_{2n+1})\}-S(A,A,B),\\ &\varphi(\max\{S(x_{2n+1},x_{2n+1},y_{2n+2}),S(y_{2n},y_{2n},x_{2n+1})\}-S(A,A,B)))\\ &+kS(A,A,B).\end{aligned} \tag{5}$$

If

$$S(y_{2n},y_{2n},x_{2n+1}) < S(x_{2n+1},x_{2n+1},y_{2n+2}) \tag{6}$$

then from (5), we have

$$\begin{aligned}kS(x_{2n+1},x_{2n+1},y_{2n+2})-kS(A,A,B) &\leq f(S(x_{2n+1},x_{2n+1},y_{2n+2})-S(A,A,B),\\ &\qquad \varphi(S(x_{2n+1},x_{2n+1},y_{2n+2})-S(A,A,B)))\\ &\leq k(S(x_{2n+1},x_{2n+1},y_{2n+2})-S(A,A,B)).\end{aligned}$$

Hence

$$\begin{aligned}f(S(x_{2n+1},x_{2n+1},y_{2n+2})-S(A,A,B),\varphi(S(x_{2n+1},x_{2n+1},y_{2n+2})-S(A,A,B)))\\ = k(S(x_{2n+1},x_{2n+1},y_{2n+2})-S(A,A,B)).\end{aligned}$$

By using the second property of f, we have
either $S(x_{2n+1},x_{2n+1},y_{2n+2})-S(A,A,B)=0$ or $\varphi(S(x_{2n+1},x_{2n+1},y_{2n+2})-S(A,A,B))=0$
so that $S(x_{2n+1},x_{2n+1},y_{2n+2})=S(A,A,B)$.
Also, $S(A,A,B)\leq S(y_{2n},y_{2n},x_{2n+1})$ which implies that
$S(x_{2n+1},x_{2n+1},y_{2n+2})\leq S(y_{2n},y_{2n},x_{2n+1})$, which is a contradiction to (6).
Therefore we have

$$S(x_{2n+1},x_{2n+1},y_{2n+2}) \leq S(y_{2n},y_{2n},x_{2n+1}), \quad \text{for} \quad n=1,2,\ldots. \tag{7}$$

Hence, from (4) and (7), we have (i) holds for $n=0,1,2,\ldots$.
In a similar way, we consider

$$\begin{aligned}kS(y_1,y_1,x_2) &= kS(F(x_0,y_0),F(x_0,y_0),F(y_1,x_1))\\ &\leq f(\max\{S(x_0,x_0,F(x_0,y_0)),S(y_1,y_1,F(y_1,x_1))\}-S(A,A,B),\\ &\qquad \varphi(\max\{S(x_0,x_0,F(x_0,y_0)),S(y_1,y_1,F(y_1,x_1))\}-S(A,A,B)))+kS(A,A,B).\end{aligned}$$

If

$$S(y_1,y_1,x_2) > S(x_0,x_0,y_1) \tag{8}$$

then we have

$$\begin{aligned}kS(y_1,y_1,x_2)-kS(A,A,B) &\leq f(S(y_1,y_1,x_2)-S(A,A,B),\varphi(S(y_1,y_1,x_2)-S(A,A,B)))\\ &\leq kS(y_1,y_1,x_2)-kS(A,A,B).\end{aligned}$$

Therefore $f(S(y_1,y_1,x_2)-S(A,A,B),\varphi(S(y_1,y_1,x_2)-S(A,A,B)))=kS(y_1,y_1,x_2)-kS(A,A,B)$.
By the second property of f, we get either
$S(y_1,y_1,x_2)-S(A,A,B)=0$ or $\varphi(S(y_1,y_1,x_2)-S(A,A,B))=0$ so that $S(y_1,y_1,x_2)=S(A,A,B)$.
Also, we have $S(A,A,B)=S(y_1,y_1,x_2)\leq S(x_0,x_0,y_1)$, a contradiction to (8).
Therefore

$$S(y_1,y_1,x_2) \leq S(x_0,x_0,y_1). \tag{9}$$

Similarly, for a fixed positive integer n, we consider

$$\begin{aligned}kS(y_{2n+1},y_{2n+1},x_{2n+2}) &= kS(F(x_{2n},y_{2n}),F(x_{2n},y_{2n}),F(y_{2n+1},x_{2n+1}))\\ &\leq f(\max\{S(x_{2n},x_{2n},F(x_{2n},y_{2n})),\\ &\qquad S(y_{2n+1},y_{2n+1},F(y_{2n+1},x_{2n+1}))\}-S(A,A,B),\\ &\qquad \varphi(\max\{S(x_{2n},x_{2n},F(x_{2n},y_{2n})),\\ &\qquad S(y_{2n+1},y_{2n+1},F(y_{2n+1},x_{2n+1}))\}-S(A,A,B)))+kS(A,A,B)\\ &= f(\max\{S(x_{2n},x_{2n},y_{2n+1}),S(y_{2n+1},y_{2n+1},x_{2n+2})\}-S(A,A,B),\\ &\qquad \varphi(\max\{S(x_{2n},x_{2n},y_{2n+1}),S(y_{2n+1},y_{2n+1},x_{2n+2})\}-S(A,A,B)))\\ &\qquad +kS(A,A,B).\end{aligned} \tag{10}$$

If

$$S(x_{2n}, x_{2n}, y_{2n+1}) < S(y_{2n+1}, y_{2n+1}, x_{2n+2}), \tag{11}$$

then from (10), we have

$kS(y_{2n+1}, y_{2n+1}, x_{2n+2}) - kS(A, A, B) \leq f(S(y_{2n+1}, y_{2n+1}, x_{2n+2}) - S(A, A, B),$
$\varphi(S(y_{2n+1}, y_{2n+1}, x_{2n+2}) - S(A, A, B)))$
$\leq k(S(y_{2n+1}, y_{2n+1}, x_{2n+2}) - S(A, A, B))$, so that
$f(S(y_{2n+1}, y_{2n+1}, x_{2n+2}) - S(A, A, B), \varphi(S(y_{2n+1}, y_{2n+1}, x_{2n+2}) - S(A, A, B)))$
$= k(S(y_{2n+1}, y_{2n+1}, x_{2n+2}) - S(A, A, B)).$

By using the second property of f, we have
either $S(y_{2n+1}, y_{2n+1}, x_{2n+2}) - S(A, A, B) = 0$ or $\varphi(S(y_{2n+1}, y_{2n+1}, x_{2n+2}) - S(A, A, B)) = 0$
which implies that
$S(y_{2n+1}, y_{2n+1}, x_{2n+2}) = S(A, A, B)$.
Also, since $S(A, A, B) \leq S(x_{2n}, x_{2n}, y_{2n+1})$. That is $S(y_{2n+1}, y_{2n+1}, x_{2n+2}) \leq S(x_{2n}, x_{2n}, y_{2n+1})$,
a contradiction to (11).
Therefore we have

$$S(y_{2n+1}, y_{2n+1}, x_{2n+2}) \leq S(x_{2n}, x_{2n}, y_{2n+1}), \quad \text{for} \quad n = 1, 2, \ldots . \tag{12}$$

Hence, from (9) and (12), we have (ii) hods for $n = 0, 1, 2, \ldots$.
Similarly, (iii) and (iv) also hold. □

Lemma 2.6. *Let (X, S) be an S-metric space. Let A and B be two nonempty subsets of X. Let $F : X \times X \to X$ be a cyclic coupled proximal mapping. Then for $x_0 \in A$ and $y_0 \in B$, the sequences $\{x_n\}$ and $\{y_n\}$ defined by* (2) *of Lemma* 2.5*, we have*
$\lim\limits_{n\to\infty} S(x_n, x_n, y_{n+1}) = \lim\limits_{n\to\infty} S(y_n, y_n, x_{n+1}) = S(A, A, B)$.

Proof. By using (i) and (iv) of Lemma 2.5 , we get

$$\begin{aligned} S(x_{2n+1}, x_{2n+1}, y_{2n+2}) &\leq S(y_{2n}, y_{2n}, x_{2n+1}) \\ &\leq S(x_{2n-1}, x_{2n-1}, y_{2n}) \\ &\leq S(y_{2n-2}, y_{2n-2}, x_{2n-1}) \end{aligned} \tag{13}$$

for $n = 1, 2, 3, \ldots$ and hence $\{S(x_{2n+1}, x_{2n+1}, y_{2n+2})\}$ and $\{S(y_{2n}, y_{2n}, x_{2n+1})\}$ are decreasing sequences and by using the inequality (13), it follows that these sequences converge to the same limit $r \geq 0$ (say). Therefore

$$\lim_{n\to\infty} S(x_{2n+1}, x_{2n+1}, y_{2n+2}) = \lim_{n\to\infty} S(y_{2n}, y_{2n}, x_{2n+1}) = r. \tag{14}$$

Now using (ii) and (iii) of Lemma 2.5,

$$\begin{aligned} S(y_{2n+1}, y_{2n+1}, x_{2n+2}) &\leq S(x_{2n}, x_{2n}, y_{2n+1}) \\ &\leq S(y_{2n-1}, y_{2n-1}, x_{2n}) \\ &\leq S(x_{2n-2}, x_{2n-2}, y_{2n-1}) \end{aligned} \tag{15}$$

for $n = 1, 2, 3, \ldots$ and hence $\{S(y_{2n+1}, y_{2n+1}, x_{2n+2})\}$ and $\{S(x_{2n}, x_{2n}, y_{2n+1})\}$ are decreasing sequences and by using the inequality (15), it follows that these sequences converge to the same limit $s \geq 0$ (say).
Therefore

$$\lim_{n\to\infty} S(y_{2n+1}, y_{2n+1}, x_{2n+2}) = \lim_{n\to\infty} S(x_{2n}, x_{2n}, y_{2n+1}) = s. \tag{16}$$

On letting $n \to \infty$ in the inequality (5) of Lemma 2.5 and using (14) we have
$kr - kS(A, A, B) \leq f(r - S(A, A, B), \varphi(r - S(A, A, B))) \leq k(r - S(A, A, B))$.
That is $f(r - S(A, A, B), \varphi(r - S(A, A, B))) = k(r - S(A, A, B))$.

By the second property of f, we get
$r - S(A, A, B) = 0$ or $\varphi(r - S(A, A, B)) = 0$.
Thus $r = S(A, A, B)$.
Similarly, on letting $n \to \infty$ in the inequality (10) of Lemma 2.5 and using (15) we have, we get that $s = S(A, A, B)$.
Hence we have $r = s = S(A, A, B)$.
Therefore from (14) and (16), we have

$$\lim_{n\to\infty} S(x_n, x_n, y_{n+1}) = \lim_{n\to\infty} S(y_n, y_n, x_{n+1}) = S(A, A, B).$$

□

Lemma 2.7. *Let (X, S) be an S-metric space. Let A and B be two nonempty subsets of X. Let $F : X \times X \to X$ be a cyclic coupled proximal mapping. If (A, B) satisfies UC property then for $x_0 \in A$ and $y_0 \in B$, the sequences $\{x_n\}$ and $\{y_n\}$ defined by* (2) *of Lemma* 2.5 *satisfies*

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = \lim_{n\to\infty} S(y_n, y_n, y_{n+1}) = 0.$$

Proof. From Lemma 2.6, we have

$$\lim_{n\to\infty} S(x_n, x_n, y_{n+1}) = \lim_{n\to\infty} S(y_n, y_n, x_{n+1}) = S(A, A, B). \tag{17}$$

We now consider

$$\begin{aligned} kS(x_{n+1}, x_{n+1}, y_{n+1}) &= kS(y_{n+1}, y_{n+1}, x_{n+1}) \\ &= kS(F(x_n, y_n), F(x_n, y_n), F(y_n, x_n)) \\ &\le f(\max\{S(x_n, x_n, F(x_n, y_n)), S(y_n, y_n, F(y_n, x_n))\} - S(A, A, B), \\ &\quad \varphi(\max\{S(x_n, x_n, F(x_n, y_n)), S(y_n, y_n, F(y_n, x_n))\} - S(A, A, B))) \\ &\quad + kS(A, A, B) \\ &= f(\max\{S(x_n, x_n, y_{n+1}), S(y_n, y_n, x_{n+1})\} - S(A, A, B), \\ &\quad \varphi(\max\{S(x_n, x_n, y_{n+1}), S(y_n, y_n, x_{n+1})\} - S(A, A, B))) + kS(A, A, B) \\ &\le k(\max\{S(x_n, x_n, y_{n+1}), S(y_n, y_n, x_{n+1})\} - S(A, A, B)) + kS(A, A, B). \end{aligned}$$

On taking limits as $n \to \infty$, we get $\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, y_{n+1}) \le S(A, A, B)$
which implies that

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, y_{n+1}) = S(A, A, B). \tag{18}$$

From (17), (18) and by using UC property, we get
$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0$ and $\lim_{n\to\infty} S(y_n, y_n, y_{n+1}) = 0$. □

Lemma 2.8. *Let (X, S) be an S-metric space. Let A and B be two nonempty subsets of X. Let $F : X \times X \to X$ be a cyclic coupled proximal mapping. If (A, B) satisfies UC property then for $x_0 \in A$ and $y_0 \in B$, the sequences $\{x_n\}$ and $\{y_n\}$ defined by* (2) *of Lemma* 2.5 *are Cauchy.*

Proof. First we prove that $\{x_n\}$ is a Cauchy sequence.

Assume that $\{x_n\}$ is not a Cauchy sequence. Then there exist an $\epsilon > 0$ and two sequences $\{l_k\}$ and $\{m_k\}$ such that $m_k > l_k$ and

$$S(x_{l_k}, x_{l_k}, x_{m_k}) > \epsilon \text{ for all } k \in \mathbb{N}. \tag{19}$$

We consider

$$\begin{aligned} kS(x_{l_k}, x_{l_k}, y_{l_k}) &= kS(y_{l_k}, y_{l_k}, x_{l_k}) \\ &= kS(F(x_{l_k-1}, y_{l_k-1}), F(x_{l_k-1}, y_{l_k-1}), F(y_{l_k-1}, x_{l_k-1})) \\ &\le f(\max\{S(x_{l_k-1}, x_{l_k-1}, F(x_{l_k-1}, y_{l_k-1})), S(y_{l_k-1}, y_{l_k-1}, F(y_{l_k-1}, x_{l_k-1}))\} - S(A, A, B), \\ &\quad \varphi(\max\{S(x_{l_k-1}, x_{l_k-1}, F(x_{l_k-1}, y_{l_k-1})), S(y_{l_k-1}, y_{l_k-1}, F(y_{l_k-1}, x_{l_k-1}))\} - S(A, A, B))) \end{aligned}$$

$$\begin{aligned}
&+ kS(A, A, B)\\
&\le f(\max\{S(x_{l_k-1}, x_{l_k-1}, y_{l_k}), S(y_{l_k-1}, y_{l_k-1}, x_{l_k})\} - S(A, A, B),\\
&\quad \varphi(\max\{S(x_{l_k-1}, x_{l_k-1}, y_{l_k}), S(y_{l_k-1}, y_{l_k-1}, x_{l_k})\} - S(A, A, B)))\\
&\quad + kS(A, A, B)\\
&\le k(\max\{S(x_{l_k-1}, x_{l_k-1}, y_{l_k}), S(y_{l_k-1}, y_{l_k-1}, x_{l_k})\}).
\end{aligned}$$

On taking limits as $k \to \infty$, from (17) of Lemma 2.7, we have
$\lim\limits_{k\to\infty} S(x_{l_k}, x_{l_k}, y_{l_k}) \le S(A, A, B)$.
Also, we have $S(A, A, B) \le \lim\limits_{k\to\infty} S(x_{l_k}, x_{l_k}, y_{l_k})$. Therefore

$$\lim_{k\to\infty} S(x_{l_k}, x_{l_k}, y_{l_k}) = S(A, A, B). \tag{20}$$

We now consider

$$\begin{aligned}
kS(x_{m_k}, x_{m_k}, y_{l_k}) &= kS(y_{l_k}, y_{l_k}, x_{m_k})\\
&= kS(F(x_{l_k-1}, y_{l_k-1}), F(x_{l_k-1}, y_{l_k-1}), F(y_{m_k-1}, x_{m_k-1}))\\
&\le f(\max\{S(x_{l_k-1}, x_{l_k-1}, F(x_{l_k-1}, y_{l_k-1})), S(y_{m_k-1}, y_{m_k-1}, F(y_{m_k-1}, x_{m_k-1}))\}\\
&\qquad - S(A, A, B),\\
&\qquad \varphi(\max\{S(x_{l_k-1}, x_{l_k-1}, F(x_{l_k-1}, y_{l_k-1})), S(y_{m_k-1}, y_{m_k-1}, F(y_{m_k-1}, x_{m_k-1}))\}\\
&\qquad\quad - S(A, A, B))) + kS(A, A, B)\\
&\le f(\max\{S(x_{l_k-1}, x_{l_k-1}, y_{l_k}), S(y_{m_k-1}, y_{m_k-1}, x_{m_k})\} - S(A, A, B),\\
&\qquad \varphi(\max\{S(x_{l_k-1}, x_{l_k-1}, y_{l_k}), S(y_{m_k-1}, y_{m_k-1}, x_{m_k})\} - S(A, A, B)))\\
&\qquad\quad + kS(A, A, B)\\
&\le k(\max\{S(x_{l_k-1}, x_{l_k-1}, y_{l_k}), S(y_{m_k-1}, y_{m_k-1}, x_{m_k})\}).
\end{aligned}$$

On taking limits as $k \to \infty$, we get

$$\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, y_{l_k}) = S(A, A, B). \tag{21}$$

Since (A,B) has the UC property, from (20) and (21) we obtain
$\lim\limits_{k\to\infty} S(x_{l_k}, x_{l_k}, x_{m_k}) = 0$, a contradiction to (19).

Therefore $\{x_n\}$ is a Cauchy sequence in A.

Similarly, by following the above procedure, it follows that $\{y_n\}$ is a Cauchy sequence in B. □

Now we give the proof of the main result.

Proof of Theorem 2.4. Let $x_0 \in A$ and $y_0 \in B$ be arbitrary. We define sequences $\{x_n\}$ in A and $\{y_n\}$ in B by

$$x_{n+1} = F(y_n, x_n),\ y_{n+1} = F(x_n, y_n),\ \ n = 0, 1, 2, \dots,$$

as in (2) of Lemma 2.5.
Then by Lemma 2.8, we have $\{x_n\}$ is a Cauchy sequence in A and $\{y_n\}$ is a Cauchy sequence in B. Since A and B are closed subsets of a complete S-metric space X, there exist $x \in A$ and $y \in B$ such that
$\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$.
Then from (18) of Lemma 2.7, we have $S(x, x, y) = S(A, A, B)$.
We consider

$$\begin{aligned}
kS(x, x, F(x, y)) &\le 2kS(x, x, x_{n+1}) + kS(x_{n+1}, x_{n+1}, F(x, y))\\
&= 2kS(x, x, x_{n+1}) + kS(F(x, y), F(x, y), F(y_n, x_n))\\
&\le 2kS(x, x, x_{n+1}) + f(\max\{S(x, x, F(x, y)), S(y_n, y_n, F(y_n, x_n))\} - S(A, A, B),\\
&\quad \varphi(\max\{S(x, x, F(x, y)), S(y_n, y_n, F(y_n, x_n))\} - S(A, A, B))) + kS(A, A, B)\\
&= 2kS(x, x, x_{n+1}) + f(\max\{S(x, x, F(x, y)), S(y_n, y_n, x_{n+1})\} - S(A, A, B),\\
&\quad \varphi(\max\{S(x, x, F(x, y)), S(y_n, y_n, x_{n+1})\} - S(A, A, B))) + kS(A, A, B).
\end{aligned}$$

On taking limits as $n \to \infty$, we get

$$\begin{aligned} kS(x,x,F(x,y)) \leq f(&\max\{S(x,x,F(x,y)),S(A,A,B)\} - S(A,A,B), \\ &\varphi(\max\{S(x,x,F(x,y)),S(A,A,B)\} - S(A,A,B))) \\ &+kS(A,A,B) \end{aligned} \tag{22}$$

$$\leq k(\max\{S(x,x,F(x,y)),S(A,A,B)\} - S(A,A,B)).$$

We have $S(A,A,B) \leq S(x,x,F(x,y))$ then from (22), we have
$kS(x,x,F(x,y)) - kS(A,A,B) \leq f(S(x,x,F(x,y)) - S(A,A,B), \varphi(S(x,x,F(x,y)) - S(A,A,B)))$
$\leq k(S(x,x,F(x,y)) - S(A,A,B))$.
That is $f(S(x,x,F(x,y)) - S(A,A,B), \varphi(S(x,x,F(x,y)) - S(A,A,B)))$
$= k(S(x,x,F(x,y)) - S(A,A,B))$.
By using the second property of f, we get
$S(x,x,F(x,y)) - S(A,A,B) = 0$ or $\varphi(S(x,x,F(x,y)) - S(A,A,B)) = 0$.
Thus $S(x,x,F(x,y)) = S(A,A,B)$.
Similarly, we can prove that $S(y,y,F(y,x)) = S(A,A,B)$.
Thus $S(x,x,F(x,y)) = S(y,y,F(y,x)) = S(x,x,y) = S(A,A,B)$.
Therefore $(x,y) \in X \times X$ is a strong coupled proximal point of F.

3. Corollaries and an Example

If $f(s,t) = k's$ for $0 < k' < 1$ in the inequality (1), then from Theorem 2.4 we obtain the following.

Corollary 3.1. Let (X,S) be a complete S-metric space. Let A and B be two nonempty closed subsets of X. Let $F : X \times X \to X$ be cyclic and F satisfies the following inequality: there exist $k' \in (0,1)$ and $k \geq 1$ such that

$$\begin{aligned} kS(F(x,y),F(u,v),F(w,z)) \leq k' \max\{&S(x,x,F(x,y)),S(u,u,F(u,v)), \\ &S(w,w,F(w,z))\} + (k-k')S(A,A,B) \end{aligned}$$

where $x,u,z \in A$ and $y,v,w \in B$. If (A,B) satisfies UC property then F has a strong coupled proximal point.

If $k = 1$ in Corollary 3.1, then we have the following.

Corollary 3.2. Let (X,S) be a complete S-metric space. Let A and B be two nonempty closed subsets of X. Let $F : X \times X \to X$ be cyclic and F satisfies the following inequality: there exists $k' \in (0,1)$ such that

$$\begin{aligned} S(F(x,y),F(u,v),F(w,z)) \leq k' \max\{&S(x,x,F(x,y)),S(u,u,F(u,v)), \\ &S(w,w,F(w,z))\} + (1-k')S(A,A,B) \end{aligned}$$

where $x,u,z \in A$ and $y,v,w \in B$. If (A,B) satisfies UC property then F has a strong coupled proximal point.

Remark 3.3. *Corollary 3.2 extends Theorem 2.4 of Kadwin and Marudai [12] to S-metric spaces.*

If we choose $f(s,t) = k(s-t)$ in the inequality (1), then from Theorem 2.4 we obtain the following.

Corollary 3.4. Let (X,S) be a complete S-metric space. Let A and B be two nonempty closed subsets of X. Let $F : X \times X \to X$ be cyclic and F satisfies the following inequality: there exists $\varphi \in \Phi$ such that
$S(F(x,y),F(u,v),F(w,z)) \leq \max\{S(x,x,F(x,y)),S(u,u,F(u,v)),S(w,w,F(w,z))\}$
$- \varphi(\max\{S(x,x,F(x,y)),S(u,u,F(u,v)),S(w,w,F(w,z))\} - S(A,A,B))$
where $x,u,z \in A$ and $y,v,w \in B$. If (A,B) satisfies UC property then F has a strong coupled proximal point.

Remark 3.5. *If $S(A,A,B) = 0$ in Theorem 2.4, then F has a strong coupled fixed point in $A \cap B$.*

The following example is in support of Theorem 2.4.

Example 3.6. *Let $X = [0, 4]$. We define $S : X^3 \to [0, \infty)$ by*

$$S(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x, y, z\} & \text{otherwise.} \end{cases}$$

Then (X, S) is a complete S-metric space.
Let $A = [2, 4]$ and $B = [0, 1]$. Clearly, $S(A, A, B) = 2$.
Also, (A, B) satisfies UC property. For, let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B such that $\lim_{n\to\infty} S(x_n, x_n, y_n) = 2$ and $\lim_{n\to\infty} S(z_n, z_n, y_n) = 2$, then
$\lim_{n\to\infty} x_n = 2$ and $\lim_{n\to\infty} z_n = 2$ so that
$\lim_{n\to\infty} S(x_n, x_n, z_n) = 0$.
We define $f : [0, \infty)^2 \to \mathbb{R}$ by $f(s, t) = k\varphi_1(t)se^{-s}$ where $\varphi_1 : [0, \infty) \to [0, 1]$ is such that $\varphi_1(t) = \frac{t^2}{1+t^2}$ then $f \in \mathscr{C}_k$ and $F : X \times X \to X$ be defined by

$$F(x, y) = \begin{cases} \frac{xy}{4} & \text{if } x \in A \quad \text{and} \quad y \in B \\ 2 & \text{otherwise.} \end{cases}$$

Then F is cyclic with respect to A and B. We define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \frac{t}{t+1}$.

We now verify the inequality (1).
Let $x, u, z \in A$ and $y, v, w \in B$. We have

$$\begin{aligned} &f(\max\{S(x, x, F(x, y)), S(u, u, F(u, v)), S(w, w, F(w, z))\} - S(A, A, B), \\ &\quad \varphi(\max\{S(x, x, F(x, y)), S(u, u, F(u, v)), S(w, w, F(w, z))\} - S(A, A, B))) \\ &\qquad + kS(A, A, B) = f(\max\{x, u, 2\} - 2, \varphi(\max\{x, u, 2\} - 2)) + 2k \\ &\qquad\qquad = f(\max\{x, u\} - 2, \varphi(\max\{x, u\} - 2)) + 2k \\ &\qquad\qquad = f(p, \varphi(p)) + 2k \text{ where } p = \max\{x, u\} - 2 \\ &\qquad\qquad = k\varphi_1(\varphi(p))pe^{-p} + 2k \\ &\qquad\qquad = k\left(\frac{(\varphi(p))^2}{1+(\varphi(p))^2}\right)pe^{-p} + 2k \\ &\qquad\qquad = k\left(\frac{(\frac{p}{1+p})^2}{1+(\frac{p}{1+p})^2}\right)pe^{-p} + 2k \\ &\qquad\qquad = k\left(\frac{p^2}{p^2+(1+p)^2}\right)pe^{-p} + 2k, \end{aligned}$$

where $p^2 > 0$ and $\frac{1}{p^2+(1+p)^2} > 0$.
We consider

$$\begin{aligned} &kS(F(x, y), F(u, v), F(w, z)) = 2k \\ &\qquad \leq k\left(\frac{p^2}{p^2+(1+p)^2}\right)pe^{-p} + 2k \\ &\qquad = f(\max\{S(x, x, F(x, y)), S(u, u, F(u, v)), S(w, w, F(w, z))\} - S(A, A, B), \\ &\qquad\quad \varphi(\max\{S(x, x, F(x, y)), S(u, u, F(u, v)), S(w, w, F(w, z))\} - S(A, A, B))) \\ &\qquad\quad + kS(A, A, B). \end{aligned}$$

Thus F is a cyclic coupled proximal mapping for any $k \geq 1$.
Also, for $x = 2$, $y \in [0, 1]$, we have $S(x, x, F(x, y)) = 2$, $S(y, y, F(y, x)) = 2$ and $S(x, x, y) = 2$.
For $x \in [0, 1]$, $y = 2$, we have $S(x, x, F(x, y)) = 2$, $S(y, y, F(y, x)) = 2$ and $S(x, x, y) = 2$.

Therefore F satisfies all the hypotheses of Theorem 2.4 for any $k \geq 1$ and
$\{(2, a) : a \in [0, 1]\} \cup \{(a, 2) : a \in [0, 1]\}$ is the set of all strong coupled proximal points of F.

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