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Domain of Padovan q-Difference Matrix in Sequence Spaces ℓ_p and ℓ_∞

Taja Yaying[a], Bipan Hazarika[b], S. A. Mohiuddine[c,d]

[a]*Department of Mathematics, Dera Natung Govt. College, Itanagar 791113, Arunachal Pradesh, India*
[b]*Department of Mathematics, Gauhati University, Gauhati 781014, Assam, India*
[c]*Department of General Required Courses, Mathematics, Faculty of Applied Studies, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[d]*Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia*

Abstract. In this study, we construct the difference sequence spaces $\ell_p\left(\mathcal{P}\nabla_q^2\right) = (\ell_p)_{\mathcal{P}\nabla_q^2}$, $1 \le p \le \infty$, where $\mathcal{P} = (\varrho_{rs})$ is an infinite matrix of Padovan numbers ϱ_s defined by

$$\varrho_{rs} = \begin{cases} \frac{\varrho_s}{\varrho_{r+5}-2} & 0 \le s \le r, \\ 0 & s > r. \end{cases}$$

and ∇_q^2 is a q-difference operator of second order. We obtain some inclusion relations, topological properties, Schauder basis and α-, β- and γ-duals of the newly defined space. We characterize certain matrix classes from the space $\ell_p\left(\mathcal{P}\nabla_q^2\right)$ to any one of the space ℓ_1, c_0, c or ℓ_∞. We examine some geometric properties and give certain estimation for von-Neumann Jordan constant and James constant of the space $\ell_p(\mathcal{P})$. Finally, we estimate upper bound for Hausdorff matrix as a mapping from ℓ_p to $\ell_p(\mathcal{P})$.

1. Introduction and Preliminaries

The construction of sequence space is an important study in the field of functional analysis. Sequence space is defined as the vector subspace of ω, the set of all real-valued sequences. The set of all p-absolutely summable sequences ℓ_p, bounded sequences ℓ_∞, null sequences c_0 and convergent sequences c, are some of the well-known examples of classical sequence spaces. A Banach space having continuous coordinates is said to be a BK-space. The spaces ℓ_p and ℓ_∞ are BK-spaces accompanied by the norms

$$\|z\|_{\ell_p} = \left(\sum_{r=0}^{\infty} |z_r|^p\right)^{1/p} \quad \text{and} \quad \|z\|_{\ell_\infty} = \sup_{r\in\mathbb{N}_0} |z_r|.$$

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Email addresses: tajayaying20@gmail.com (Taja Yaying), bh_rgu@yahoo.co.in; bh_gu@gauhati.ac.in (Bipan Hazarika), mohiuddine@gmail.com (S. A. Mohiuddine)

Here and in the rest of the paper, $1 \leq p < \infty$ unless stated otherwise, and $\mathbb{N}_0$ denote the set of all non-negative integers.

Let $A = (a_{rs})$ be an infinite matrix of real entries. Then A-transform of a sequence z is defined by the sequence $Az = \{(Az)_r\}_{r=0}^{\infty} = \left\{\sum_{s=0}^{\infty} a_{rs}z_s\right\}$. Furthermore, if for every sequence $z = (z_s)$ in the sequence space Z, the A-transform of z belongs to the space Z', then the matrix A is called a matrix mapping from sequence space Z to Z'. A matrix $A = (a_{rs})$ is called a triangle if $a_{rr} \neq 0$ and $a_{rs} = 0$ for $r < s$. It is well known that the set Z_A defined by

$$Z_A = \{z = (z_s) \in \omega : Az \in Z\} \tag{1}$$

is a sequence space and is called the domain of the matrix A in the space Z. Moreover, if A is a triangle and Z is a BK-space, then Z_A is also a BK-space accompanied by the norm $\|z\|_{Z_A} = \|Az\|_Z$. We refer the papers [2–6, 8–10, 14–17, 30, 34] and the monographs [7, 29] for studies related to theory of summability and construction of BK-spaces using the domain of triangles in the classical spaces.

The number sequence $1, 1, 1, 2, 2, 3, 4, 5, 7, 9, \ldots$ is called Padovan numbers. This number sequence was first discovered by Cordonnier in 1924 and independently rediscovered by Dom Hans Van der Laan [26] in 1928. Then Richard Padovan studied Padovan number and its applications in architectural studies in detail (cf. [31, 32]). In his honor, Stewart [36] designated this number sequence as "Padovan Sequence" and provided geometrical illustration of Padovan sequences by presenting a spiraling system of conjoined triangles in comparison to the spiraling system of golden rectangles as in the case of Fibonacci sequences. If (ϱ_r) denote the sequence of Padovan numbers, then

$$\varrho_r = \varrho_{r-2} + \varrho_{r-3} \ \text{ with } \ \varrho_0 = \varrho_1 = \varrho_2 = 1.$$

Padovan numbers exhibit following interesting properties:

$$\rho = \lim_{r\to\infty} \frac{\varrho_{r+1}}{\varrho_r} = 1.3247179572\ldots (\text{ Plastic number})$$

$$\sum_{s=0}^{r} \varrho_s = \varrho_{r+5} - 2.$$

$$\sum_{s=0}^{r} \varrho_{s+n} = \varrho_{r+n+5} - \varrho_{n+4} \ (n \in \mathbb{N}_0).$$

Padovan numbers have a great application in the field of architecture, engineering, music, etc. For more interesting research papers concerning Padovan numbers, we refer to [19, 27, 28, 35, 44].

The operators Δ and ∇ defined by $(\Delta z)_r = z_r - z_{r+1}$ and $(\nabla z)_r = z_r - z_{r-1}$ for all $r \in \mathbb{N}_0$, are known as forward and backward difference operators of first order, respectively, we assumed that $z_r = 0$ for $r < 0$. The domains $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are studied by Kızmaz [25]. Later on, the operators Δ and ∇ were extended to Δ^2 and ∇^2 defined by $(\Delta^2 z)_r = (\Delta z)_r - (\Delta z)_{r+1}$ and $(\nabla^2 z)_r = (\nabla z)_r - (\nabla z)_{r-1}$, respectively (cf. [13, 18]). Introduction of difference operators has a great impact in the study of summability theory. For instances, the sequence $z = (s)_{s=0}^{\infty}$ is not convergent in the ordinary sense. In fact it diverges to ∞. However, the sequences $\Delta z = (-1, -1, -1, \ldots)$ and $\Delta^2 z = (0, 0, 0, \ldots)$ converge to -1 and 0, respectively.

Recently, Yaying et al. [41] defined a generalized difference matrix $\mathcal{P}^\alpha = \left(\varrho_{rs}^{\alpha}\right)$ involving Padovan numbers by

$$\varrho_{rs}^{\alpha} = \begin{cases} \sum\limits_{v=r}^{s} (-1)^{v-s} \dfrac{\Gamma(\alpha+1)}{(v-s)!\Gamma(\alpha-v+s+1)} \dfrac{\varrho_v}{\varrho_{r+5}-2} & , \quad 0 \leq s \leq r, \\ 0 & , \quad s > r, \end{cases}$$

and studied its domain $\ell_p(\mathcal{P}^\alpha) = (\ell_p)_{\mathcal{P}^\alpha}$ and $\ell_\infty(\mathcal{P}^\alpha) = (\ell_\infty)_{\mathcal{P}^\alpha}$. It is clear that when $\alpha = 0, 1$ and 2, $\mathcal{P}^\alpha$ contracts to the matrices $\mathcal{P}$, $\mathcal{P}\nabla$ and $\mathcal{P}\nabla^2$, respectively, where $\mathcal{P} = \left(\varrho_{rs}\right)$ is defined by

$$\varrho_{rs} = \begin{cases} \dfrac{\varrho_s}{\varrho_{r+5} - 2} & , \quad 0 \le s \le r, \\ 0 & , \quad s > r, \end{cases}$$

In this paper, by the aid of a generalized difference matrix ∇_q^2, $(0 < q < 1)$, we construct Padovan difference matrix $\mathcal{P}\nabla_q^2$, and construct Padovan difference sequence spaces $\ell_p\left(\mathcal{P}\nabla_q^2\right)$ and $\ell_\infty\left(\mathcal{P}\nabla_q^2\right)$, investigate their topological properties, inclusion relations, and form Schauder basis of the space $\ell_p\left(\mathcal{P}\nabla_q^2\right)$. In section 3, we determine α-, β- and γ-duals of these spaces. In section 4, some characterization results concerning class of matrix mappings from the spaces $\ell_p\left(\mathcal{P}\nabla_q^2\right)$ and $\ell_\infty\left(\mathcal{P}\nabla_q^2\right)$ to anyone of the spaces ℓ_1, c_0, c or ℓ_∞ are examined. In section 5, we focus on the space $\ell_p(\mathcal{P})$ which is the special case of the space $\ell_p\left(\mathcal{P}^\alpha\right)$ when $\alpha = 0$, and exhibit certain geometric properties, compute von-Neumann Jordan constant and James constant of this space. In the final section, we provide an estimation for the upper bound of the Hausdorff matrix as a mapping from ℓ_p to $\ell_p(\mathcal{P})$.

2. Padovan Difference Sequence Spaces $\ell_p(\mathcal{P}\nabla_q^2)$ and $\ell_\infty(\mathcal{P}\nabla_q^2)$

We need the following notations and definitions for our study.

Let $0 < q < 1$. Then

$$r(q) = \begin{cases} \dfrac{1-q^r}{1-q} & , \quad r > 0, \\ 0 & , \quad r = 0. \end{cases}$$

Clearly $1(q) = 1$, $2(q) = 1 + q$, $3(q) = 1 + q + q^2$, and so on. We further emphasize that $r(q) = r$ when $q \to 1$. An interesting property of the q-numbers that differs it from ordinary numbers is that the sequence $(s)_{s=0}^\infty$ of ordinary numbers diverges to $+\infty$, whereas, on the contrary, the sequence $(s(q))_{s=0}^\infty$ of q-numbers converges to $\dfrac{1}{1-q}$.

Definition 2.1. ([21]) The q-binomial coefficient $\begin{pmatrix} r \\ s \end{pmatrix}_q$ is defined by

$$\begin{pmatrix} r \\ s \end{pmatrix}_q = \begin{cases} \dfrac{r(q)!}{(r-s)(q)!s(q)!} & , \quad r \ge s, \\ 0 & , \quad s > r, \end{cases}$$

where q-factorial $r(q)!$ of r is given by

$$r(q)! = r(q)(r-1)(q)\cdots 2(q)1(q).$$

In particular $\begin{pmatrix} 0 \\ 0 \end{pmatrix}_q = \begin{pmatrix} r \\ 0 \end{pmatrix}_q = \begin{pmatrix} r \\ r \end{pmatrix}_q = 1$ and $\begin{pmatrix} r \\ r-s \end{pmatrix}_q = \begin{pmatrix} r \\ s \end{pmatrix}_q$. We strictly refer to [21] for detailed studies in q-numbers and q-binomial coefficients and [42, 43] for sequence spaces involving q-numbers.

Define the difference operator $\nabla_q^2 : \omega \to \omega$ by

$$(\nabla_q^2 z)_r = z_r - (1+q)z_{r-1} + qz_{r-2},$$

where $r \in \mathbb{N}_0$. It is presumed that $z_r = 0$ for $r < 0$. The operator $\nabla_q^2 = (\delta_{rs}^{2;q})$ can be represented in the form of a triangle as

$$\delta_{rs}^{2;q} = \begin{cases} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{pmatrix} 2 \\ r-s \end{pmatrix}_q & , \quad 0 \le s \le r, \\ 0 & , \quad s > r, \end{cases}$$

for all $r,s\in\mathbb{N}_0$. With some elementary calculations, the inverse $\nabla_q^{-2}=\left(\delta_{rs}^{-2;q}\right)$ of the triangle ∇_q^2 is computed as

$$\delta_{rs}^{-2;q}=\begin{cases}\binom{r-s+1}{r-s}_q & , \quad 0\le s\le r,\\ 0 & , \quad s>r.\end{cases}$$

By combining Padovan matrix $\mathcal{P}$ and difference matrix ∇_q^2, we introduce generalized Padovan difference matrix $Q=\mathcal{P}\nabla_q^2=(\tilde{\varrho}_{rs})$ defined by

$$\tilde{\varrho}_{rs}=\begin{cases}\sum\limits_{v=s}^{r}(-1)^{v-s}q^{\binom{v-s}{2}}\binom{2}{v-s}_q\dfrac{\varrho_v}{\varrho_{r+5}-2} & , \quad 0\le s\le r,\\ 0 & , \quad s>r.\end{cases}$$

Since $\mathcal{P}$ and ∇_q^2 are triangles, their product Q is also a triangle. Hence the inverse $Q^{-1}=\left(\mathcal{P}\nabla_q^2\right)^{-1}=\nabla_q^{-2}\mathcal{P}^{-1}=\left(\tilde{\varrho}_{rs}^{-1}\right)$ exists (for $\mathcal{P}^{-1}$ see [41]) and is unique. It is computed as

$$\tilde{\varrho}_{rs}^{-1}=\begin{cases}\sum\limits_{v=s}^{s+1}(-1)^{v-s}\binom{r-v+1}{r-v}_q\dfrac{\varrho_{s+5}-2}{\varrho_v} & , \quad 0\le s\le r,\\ 0 & , \quad s>r,\end{cases}$$

The sequence $t=(t_r)$ defined by

$$t_r=(Qz)_r=\sum_{s=0}^{r}\sum_{v=s}^{r}(-1)^{v-s}q^{\binom{v-s}{2}}\binom{2}{v-s}_q\frac{\varrho_v}{\varrho_{r+5}-2}z_s,\ r\in\mathbb{N}_0 \tag{2}$$

is called Q-transform of the sequence $z=(z_r)$. Now in view of (1), we define Padovan difference sequence spaces $\ell_p\big(Q\big)$ and $\ell_\infty\big(Q\big)$ by

$$\ell_p\big(Q\big)=(\ell_p)_Q \text{ and } \ell_\infty\big(Q\big)=(\ell_\infty)_Q.$$

Here and in the sequel $1\le p<\infty$, unless stated. Equivalently

$$\begin{aligned}\ell_p(Q) &= \{z=(z_r)\in\omega : t=Qz\in\ell_p\},\\ \ell_\infty(Q) &= \{z=(z_r)\in\omega : t=Qz\in\ell_\infty\}.\end{aligned}$$

We emphasize that the spaces $\ell_p\big(Q\big)$ and $\ell_\infty\big(Q\big)$ contract to $\ell_p\big(\mathcal{P}\nabla^2\big)$ and $\ell_\infty\big(\mathcal{P}\nabla^2\big)$, respectively, when $q\to 1$.

In the rest of the paper, the sequences z and t are interrelated by (2). We begin with the following theorem:

Theorem 2.2. *The spaces $\ell_p\big(Q\big)$ and $\ell_\infty\big(Q\big)$ are BK-spaces equipped with the norms defined by*

$$\|z\|_{\ell_p(Q)}=\left(\sum_{r=0}^{\infty}|(Qz)_r|^p\right)^{1/p} \tag{3}$$

and

$$\|z\|_{\ell_\infty(Q)}=\sup_{r\in\mathbb{N}_0}|(Qz)_r|. \tag{4}$$

Proof. The proof is easy and hence details omitted. □

Theorem 2.3. *$\ell_p\big(Q\big)\cong\ell_p$ and $\ell_\infty\big(Q\big)\cong\ell_\infty$.*

Proof. We provide the proof for the space $\ell_p\big(Q\big)$. Define the mapping $\pi : \ell_p\big(Q\big) \to \ell_p$ by $z \mapsto t = \pi z = Qz$. We notice that π is linear and injective.
Let $t = (t_r) \in \ell_p$ and define the sequence $z = (z_s)$ by

$$z_r = \sum_{s=0}^{r}\sum_{v=s}^{s+1}(-1)^{s-v}\binom{r-v+1}{r-v}_q \frac{\varrho_{s+5}-2}{\varrho_v} t_s \ (s \in \mathbb{N}_0). \tag{5}$$

Then

$$\begin{aligned}
\|z\|_{\ell_p(Q)} &= \left(\sum_{r=0}^{\infty}\left|\big(Qz\big)_r\right|^p\right)^{1/p}\\
&= \left(\sum_{r=0}^{\infty}\left|\sum_{s=0}^{r}\sum_{v=s}^{r}(-1)^{v-s}q^{\binom{v-s}{2}}\binom{2}{v-s}_q \frac{\varrho_v}{\varrho_{r+5}-2} z_s\right|^p\right)^{1/p}\\
&= \left(\sum_{r=0}^{\infty}\left|\sum_{s=0}^{r}\sum_{v=s}^{r}(-1)^{v-s}q^{\binom{v-s}{2}}\binom{2}{v-s}_q \frac{\varrho_v}{\varrho_{r+5}-2}\right.\right.\\
&\quad \left.\left.\left(\sum_{u=0}^{s}\sum_{w=u}^{u+1}(-1)^{u-w}\binom{s-w+1}{s-w}_q \frac{\varrho_{u+5}-2}{\varrho_w} t_u\right)\right|^p\right)^{1/p}\\
&= \left(\sum_{r=0}^{\infty}|t_r|^p\right)^{1/p} = \|t\|_{\ell_p} < \infty.
\end{aligned}$$

Thus $z \in \ell_p\big(Q\big)$. Hence, π is surjective and norm preserving. Consequently $\ell_p\big(Q\big) \cong \ell_p$. The later part of the theorem can be established analogously. This completes the proof. □

Theorem 2.4. *The inclusion $\ell_p \subset \ell_p(Q)$ holds for $1 \le p \le \infty$.*

Proof. It is easy to see that the space $\ell_p \subset \ell_p\big(Q\big)$. To prove the strictness part, we consider the sequence $f = \big(q^s\big)_{s=0}^{\infty}$ for $0 < q < 1$. Clearly f is not a sequence in ℓ_p. However

$$Qf = \mathcal{P}\nabla_q^2 f = \Big(\frac{\varrho_0}{\varrho_5-2}, \frac{\varrho_0-\varrho_1}{\varrho_6-2}, \frac{q(\varrho_1-\varrho_2)}{\varrho_7-2}, \frac{q(\varrho_1-\varrho_2)}{\varrho_8-2}, \dots\Big) = (1,0,0,0,\dots)$$

is a sequence in ℓ_p.

Again, we take another sequence $g = (s)_{s=0}^{\infty}$. Then $g \notin \ell_\infty$. But

$$Qg = \mathcal{P}\nabla_q^2 g = \Big(0, \frac{\varrho_1}{\varrho_6-2}, \frac{\varrho_1+(1-q)\varrho_2}{\varrho_7-2}, \frac{\varrho_1+(1-q)\varrho_2+(1-q)\varrho_3}{\varrho_8-2}, \dots\Big).$$

Thus Qg is the sequence whose r^{th} $(r \ge 2)$ entry is given by $\sum_{s=2}^{r}\frac{\varrho_1+\varrho_s(1-q)}{\varrho_{r+5}-2}$ with the first two entries 0 and $\frac{1}{2}$. Since $\varrho_{r+5} > \sum_{s=2}^{r}\frac{\varrho_s}{\varrho_{r+5}-2}$ for $r \ge 2$, it is evident that the sequence Qg is convergent and so bounded. Thus the result is proved. □

Theorem 2.5. *The inclusion $\ell_p\big(Q\big) \subset \ell_\infty\big(Q\big)$ strictly holds.*

Proof. Since the inclusion $\ell_p \subset \ell_\infty$ holds strictly, the inclusion part is easy to prove. Consider a sequence $g = (g_s) \in \ell_\infty \setminus \ell_p$ and define a sequence $f = (f_s)$ in terms of the sequence $g = (g_s)$ by

$$f_r = \sum_{s=0}^{r}\sum_{v=s}^{s+1}(-1)^{s-v}\binom{r-v+1}{r-v}_q \frac{\varrho_{s+5}-2}{\varrho_v} g_s$$

for all $r = 0, 1, 2, \cdots$. Then, we deduce that $Qf = \mathcal{P}\nabla_q^2 f = g \in \ell_\infty \setminus \ell_p$. This confirms that $f \in \ell_\infty\big(Q\big) \setminus \ell_p\big(Q\big)$. □

Theorem 2.6. *If $1 \le p < m$, then $\ell_p(Q) \subset \ell_m(Q)$.*

Proof. This is similar to the proof of Theorem 2.5. So details are omitted. □

Before proceeding to the next result, we present the following definition.

Definition 2.7. A sequence $z = (z_s)$ is called a Schauder basis of a normed space $(Z, \|\cdot\|)$ if for every $z' \in Z$ there exists a unique sequence of scalars (a_s) such that

$$\lim_{r\to\infty} \left\| z' - \sum_{s=0}^{r} a_s z_s \right\| = 0.$$

We are well aware that a normed space Z has a Schauder basis if and only if the domain Z_A of the matrix A in the space Z has a Schauder basis. Thus we have the following result.

Theorem 2.8. *The sequence $\phi^{(s)} = (\phi_r^{(s)})$, $s \in \mathbb{N}_0$, defined by*

$$\phi_r^{(s)} = \begin{cases} \sum\limits_{v=s}^{s+1} (-1)^{v-s} \binom{r-v+1}{r-v}_q \dfrac{\varrho_{s+5}-2}{\varrho_v} & , \quad s \le r, \\ 0 & , \quad s > r, \end{cases}$$

forms a Schauder basis for the sequence space $\ell_p\big(Q\big)$ and every $z \in \ell_p\big(Q\big)$ can be expressed uniquely as $z = \sum_{s=0}^{\infty} t_s \phi^{(s)}$.

Corollary 2.9. *$\ell_p\big(Q\big)$ is a separable space.*

Proof. Theorems 2.2 and 2.8 immediately establish the result. □

3. α-, β- and γ-Duals

For the sequence spaces Z and Z', the set $\mu(Z, Z')$ is called the multiplier space of Z and Z' and is defined by

$$\mu(Z, Z') = \{d = (d_s) \in \omega : dz = (d_s z_s) \in Z', \ \forall z = (z_s) \in Z\}.$$

Then the α-, β- and γ-duals of Z are defined by

$$Z^\alpha = \mu(Z, \ell_1), \ Z^\beta = \mu(Z, cs), \ Z^\gamma = \mu(Z, bs)$$

respectively, where cs and bs represent the spaces of all convergent series and bounded series, respectively.

We need the following lemmas due to Stielglitz and Tietz [37] for our investigations. In the rest of the paper, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\mathcal{N}$ denote the family of all finite subsets of $\mathbb{N}_0$.

Lemma 3.1. *$A = (a_{rs}) \in (\ell_p, \ell_1)$ if and only if*

$$\sup_{s\in\mathbb{N}_0} \sum_{r=0}^{\infty} |a_{rs}| < \infty, \text{ in the case } p = 1; \tag{6}$$

$$\sup_{R\in\mathcal{N}} \sum_{s=0}^{\infty} \left| \sum_{r\in R} a_{rs} \right|^{p'} < \infty, \text{ in the case } 1 < p < \infty; \tag{7}$$

$$\sup_{R\in\mathcal{N}} \sum_{s=0}^{\infty} \left| \sum_{r\in R} a_{rs} \right| < \infty, \text{ in the case } p = \infty. \tag{8}$$

Lemma 3.2. *$A = (a_{rs}) \in (\ell_p, c)$ if and only if*

$$\exists \alpha_s \in \mathbb{C} \ni \lim_{r\to\infty} a_{rs} = \alpha_s \text{ for each } s \in \mathbb{N}_0; \tag{9}$$

$$\sup_{r,s} |a_{rs}| < \infty, \text{ in the case } p = 1; \tag{10}$$

$$\sup_{r\in\mathbb{N}_0} \sum_{s=0}^{\infty} |a_{rs}|^{p'} < \infty, \text{ in the case } 1 < p < \infty; \tag{11}$$

$$\lim_{r\to\infty} \sum_{s=0}^{\infty} |a_{rs}| = \sum_{s=0}^{\infty} \left|\lim_{r\to\infty} a_{rs}\right|, \text{ in the case } p = \infty. \tag{12}$$

Lemma 3.3. *$A = (a_{rs}) \in (\ell_p, \ell_\infty)$ if and only if (10) holds in the case $p = 1$; (11) holds in the case $1 < p < \infty$ and (11) holds with $p' = 1$ in the case $p = \infty$.*

Theorem 3.4. *Define the sets δ_1 and $\delta_1^{(p')}$ by*

$$\delta_1 = \left\{ d = (d_s) \in \omega : \sup_{s\in\mathbb{N}_0} \sum_{r=0}^{\infty} \left| \sum_{v=s}^{s+1} (-1)^{v-s} \binom{r-v+1}{r-v}_q \frac{\varrho_{s+5}-2}{\varrho_v} d_r \right| < \infty \right\};$$

$$\delta_1^{(p')} = \left\{ d = (d_s) \in \omega : \sup_{R\in\mathcal{N}} \sum_{s=0}^{\infty} \left| \sum_{r\in R} \sum_{v=s}^{s+1} (-1)^{v-s} \binom{r-v+1}{r-v}_q \frac{\varrho_{s+5}-2}{\varrho_v} d_r \right|^{p'} < \infty \right\}.$$

Then

$$\left[\ell_1(Q)\right]^\alpha = \delta_1, \ \left[\ell_p(Q)\right]^\alpha = \delta_1^{(p')}, \ 1 < p < \infty, \text{ and } \left[\ell_\infty(Q)\right]^\alpha = \delta_1^{(1)}.$$

Proof. Let $d = (d_s) \in \omega$ and $z = (z_s)$ is defined as in (5), then we have

$$d_r z_r = \sum_{s=0}^{r} \sum_{v=s}^{s+1} (-1)^{v-s} \binom{r-v+1}{r-v}_q \frac{\varrho_{s+5}-2}{\varrho_v} d_r t_s = (Ct)_r \tag{13}$$

for each $r \in \mathbb{N}_0$, where $C = (c_{rs})$ is defined by

$$c_{rs} = \begin{cases} \sum_{v=s}^{s+1} (-1)^{v-s} \binom{r-v+1}{r-v}_q \frac{\varrho_{s+5}-2}{\varrho_v} d_r & , \quad 0 \le s \le r, \\ 0 & , \quad s > r, \end{cases}$$

for all $r, s \in \mathbb{N}_0$.
Thus, we deduce from (13) that $dz = (d_r z_r) \in \ell_1$ whenever $z \in \ell_p(Q_q)$ if only if $Ct \in \ell_1$ whenever $t \in \ell_p$. This yields that $d = (d_r) \in \left[\ell_p(Q)\right]^\alpha$ if and only if $C \in (\ell_p, \ell_1)$.

Thus by using Lemma 3.1, we conclude that

$$\left[\ell_1(Q)\right]^\alpha = \delta_1, \ \left[\ell_p(Q)\right]^\alpha = \delta_1^{(p')}, \ 1 < p < \infty, \text{ and } \left[\ell_\infty(Q)\right]^\alpha = \delta_1^{(1)}.$$

□

Theorem 3.5. *Define the sets* $\delta_2^{(p')}$, δ_3 *and* δ_4 *by*

$$\delta_2 = \left\{ d = (d_s) \in \omega : \lim_{r\to\infty} \sum_{v=s}^{r} \left(\sum_{l=s}^{s+1} (-1)^{v-s} \binom{v-l+1}{v-l}_q \frac{\varrho_{s+5}-2}{\varrho_s} \right) d_v \text{ exists} \right\};$$

$$\delta_3 = \left\{ d = (d_s) \in \omega : \sup_{r,s\in\mathbb{N}_0} \left| \sum_{v=s}^{r} \left(\sum_{l=s}^{s+1} (-1)^{v-s} \binom{v-l+1}{v-l}_q \frac{\varrho_{s+5}-2}{\varrho_s} \right) d_v \right| < \infty \right\};$$

$$\delta_4^{(p')} = \left\{ d = (d_s) \in \omega : \sup_{r\in\mathbb{N}_0} \sum_{s=0}^{\infty} \left| \sum_{v=s}^{r} \left(\sum_{l=s}^{s+1} (-1)^{v-s} \binom{v-l+1}{v-l}_q \frac{\varrho_{s+5}-2}{\varrho_s} \right) d_v \right|^{p'} < \infty \right\};$$

$$\begin{aligned} \delta_5 = \Bigg\{ d = (d_s) \in \omega : &\lim_{r\to\infty} \sum_{s=0}^{\infty} \left| \sum_{v=s}^{r} \left(\sum_{l=s}^{s+1} (-1)^{v-s} \binom{v-l+1}{v-l}_q \frac{\varrho_{s+5}-2}{\varrho_s} \right) d_v \right| \\ = &\sum_{s=0}^{\infty} \left| \lim_{r\to\infty} \sum_{v=s}^{r} \left(\sum_{l=s}^{s+1} (-1)^{v-s} \binom{v-l+1}{v-l}_q \frac{\varrho_{s+5}-2}{\varrho_s} \right) d_v \right| \Bigg\}. \end{aligned}$$

Then, $\left[\ell_1\big(Q\big)\right]^\beta = \delta_2 \cap \delta_3$, $\left[\ell_p\big(Q\big)\right]^\beta = \delta_2 \cap \delta_4^{(p')}$ *and* $\left[\ell_\infty\big(Q\big)\right]^\beta = \delta_2 \cap \delta_5$, *where* $1 < p < \infty$.

Proof. Let $d = (d_s) \in \omega$ and $z = (z_s)$ is defined as in (5). Consider the equality

$$\begin{aligned} \sum_{s=0}^{r} d_s z_s &= \sum_{s=0}^{r} d_s \left[\sum_{v=0}^{s} \sum_{l=v}^{v+1} (-1)^{l-v} \binom{s-l+1}{s-l}_q \frac{\varrho_{v+5}-2}{\varrho_l} t_v \right] \\ &= \sum_{s=0}^{r} \left[\sum_{v=s}^{r} \left(\sum_{l=s}^{s+1} (-1)^{v-s} \binom{v-l+1}{v-l}_q \frac{\varrho_{s+5}-2}{\varrho_s} \right) d_v \right] t_s \\ &= (\tilde{C}t)_r, \text{ for each } r \in \mathbb{N}_0, \end{aligned} \tag{14}$$

where the matrix $\tilde{C} = (\tilde{c}_{rs})$ is defined by

$$\tilde{c}_{rs} = \begin{cases} \sum\limits_{v=s}^{r} \left(\sum\limits_{l=s}^{s+1} (-1)^{v-s} \binom{v-l+1}{v-l}_q \frac{\varrho_{s+5}-2}{\varrho_s} \right) d_v & , \quad 0 \le s \le r, \\ 0 & , \quad s > r, \end{cases}$$

for all $r, s \in \mathbb{N}_0$.
We noticed from (14) that $dz = (d_s z_s) \in cs$ whenever $z = (z_s) \in \ell_p\big(Q\big)$ if only if $\tilde{C}t \in c$ whenever $t = (t_s) \in \ell_p$. This implies that $d = (d_s) \in \left[\ell_p\big(Q\big)\right]^\beta$ if and only if $\tilde{C} \in (\ell_p, c)$.

Thus, by using Lemma 3.2, we deduce that

$$\lim_{r\to\infty} \tilde{c}_{rs} \text{ exists for all } s \in \mathbb{N}_0 \text{ and}$$

$$\sup_{r,s} |\tilde{c}_{rs}| < \infty, \text{ in the case } p = 1;$$

$$\sup_{r\in\mathbb{N}_0} \sum_{s=0}^{\infty} |\tilde{c}_{rs}|^{p'} < \infty, \text{ in the case } 1 < p < \infty;$$

$$\lim_{r\to\infty} \sum_{s=0}^{\infty} |\tilde{c}_{rs}| = \sum_{s=0}^{\infty} \left| \lim_{r\to\infty} \tilde{c}_{rs} \right|, \text{ in the case } p = \infty.$$

This gives us the required result. □

In the similar way, one can obtain the γ-dual of the space $\ell_p(Q)$, $1 \le p \le \infty$, by using Lemma 3.3. Hence we avoid the proof and state the result only.

Theorem 3.6. *Let* $1 < p < \infty$. *Then* $\left[\ell_1(Q)\right]^{\gamma} = \delta_3$, $\left[\ell_p(Q)\right]^{\gamma} = \delta_4^{(p')}$ *and* $\left[\ell_p(Q)\right]^{\gamma} = \delta_4^{(1)}$.

4. Matrix Mappings on the Sequence Space $\ell_p(Q)$

In this section certain results concerning characterization of matrix mappings from the space $\ell_p(Q)$ to any one of the space ℓ_1, c_0, c or ℓ_∞, are investigated.

The following result is the base for our examinations which is a direct consequence of [24, Theorem 4.1].

Theorem 4.1. *Let* $1 \le p \le \infty$ *and* Z *be any one of the sequence space* ℓ_1, c_0, c *or* ℓ_∞. *Then,* $A = (a_{rs}) \in (\ell_p(Q), Z)$ *if and only if* $B^{(r)} = \left(b_{ms}^{(r)}\right) \in (\ell_p, c)$ *for all* $r \in \mathbb{N}_0$ *and* $B = (b_{rs}) \in (\ell_p, Z)$, *where*

$$b_{ms}^{(r)} = \begin{cases} \sum\limits_{v=s}^{m} \left(\sum_{l=s}^{s+1} (-1)^{v-s} \binom{v-l+1}{v-l}_q \dfrac{\varrho_{s+5}-2}{\varrho_s} \right) a_{rv} & , \quad 0 \le s \le m \\ 0 & , \quad s > m, \end{cases}$$

$$b_{rs} = \sum_{v=s}^{\infty} \left(\sum_{l=s}^{s+1} (-1)^{v-s} \binom{v-l+1}{v-l}_q \frac{\varrho_{s+5}-2}{\varrho_s} \right) a_{rv}$$

for all $r, m, s = 0, 1, 2, 3, \cdots$.

Proof. Let $1 \le p \le \infty$ and $A \in (\ell_p(Q), Z)$. Assume that $z \in \ell_p(Q)$. Then, proceeding in the same way as in the proof of Theorem 3.5, we deduce the following equality

$$\begin{aligned} \sum_{s=0}^{m} a_{rs} z_s &= \sum_{s=0}^{m} \sum_{v=0}^{s} \sum_{l=v}^{v+1} (-1)^{l-v} \binom{s-l+1}{s-l}_q \frac{\varrho_{v+5}-2}{\varrho_l} t_v a_{rs} \\ &= \sum_{s=0}^{m} \left[\sum_{v=s}^{m} \left(\sum_{l=s}^{s+1} (-1)^{v-s} \binom{v-l+1}{v-l}_q \frac{\varrho_{s+5}-2}{\varrho_s} \right) a_{rv} \right] t_s \\ &= \sum_{s=0}^{m} b_{ms}^{(r)} t_s \end{aligned} \tag{15}$$

for all $m, r = 0, 1, 2, \cdots$. Since Az exists, so $B^{(r)} \in (\ell_p, c)$. Again passing the limit as $m \to \infty$ in (15), we obtain $Az = Bt$. Since $Az \in Z$, so $Bt \in Z$ which leads us to the fact that $B \in (\ell_p, Z)$.

Conversely, assume that $B^{(r)} = \left(b_{ms}^{(r)}\right) \in (\ell_p, c)$ for all $r = 0, 1, 2, \cdots$, and $B = (b_{rs}) \in (\ell_p, Z)$. Let $z \in \ell_p(Q)$. Then, for each $r \in \mathbb{N}_0$, $\left(b_{rs}\right)_{s \in \mathbb{N}_0} \in \ell_p^{\beta}$ which in turn implies the fact that $\left(a_{rs}\right)_{s \in \mathbb{N}_0} \in \left[\ell_p(Q)\right]^{\beta}$ for each $r \in \mathbb{N}$. Hence Az exists. Again from (15), $Az = Bt$ as $m \to \infty$. This implies that $A \in (\ell_p(Q), Z)$. This completes the proof. □

As a direct consequence of Theorem 4.1, we provide characterization of certain matrix classes from

$\ell_1(Q)$, $\ell_p(Q)$ and $\ell_\infty(Q)$ to $Z' \in \{\ell_1, c_0, c, \ell_\infty\}$. Before proceeding, we list certain conditions:

$$\lim_{m\to\infty} b_{ms}^{(r)} \text{ exists for each } r, s \in \mathbb{N}_0; \tag{16}$$

$$\sup_{m,s} \left|b_{ms}^{(r)}\right| < \infty; \tag{17}$$

$$\sup_{m} \sum_{s=0}^{m} \left|b_{ms}^{(r)}\right|^{p'} < \infty \text{ for each } r \in \mathbb{N}_0; \tag{18}$$

$$\lim_{m\to\infty} \sum_{s=0}^{m} b_{ms}^{(r)} \text{ exists for each } r \in \mathbb{N}_0; \tag{19}$$

$$\lim_{m\to\infty} \sum_{s=0}^{m} \left|b_{ms}^{(r)}\right| = \sum_{s=0}^{m} |b_{rs}| \text{ for each } r \in \mathbb{N}_0. \tag{20}$$

$$\lim_{r\to\infty} \sum_{s=0}^{\infty} |b_{rs}| = 0. \tag{21}$$

Corollary 4.2. *Assume that $1 < p < \infty$. Then, the necessary and sufficient condition that a matrix $A = (a_{rs}) \in (Z, Z')$, where $Z \in \left\{\ell_1(Q), \ell_p(Q), \ell_\infty(Q)\right\}$ and $Z' \in \{\ell_1, c_0, c, \ell_\infty\}$ can be read from Table 1, where*

A. (16) and (17)	**B.** (16) and (18)
C. (16) and (20)	**D.** (8) with z_{nk} instead of h_{nk}
E. (9) with $\alpha_s = 0\ \forall s$ and b_{rs} instead of a_{rs}	**F.** (9) with b_{rs} instead of a_{rs}
G. (11) with $p' = 1$ and b_{rs} instead of a_{rs}	**H.** (21)
I. (12) with b_{rs} instead of a_{rs}	**J.** (11) with b_{rs} instead of a_{rs}
K. (6) with b_{rs} instead of a_{rs}	**L.** (7) with b_{rs} instead of a_{rs}
M. (10) with b_{rs} instead of a_{rs}	

From\ To	ℓ_1	c_0	c	ℓ_∞
$\ell_1(Q)$	**A & K**	**A, E & M**	**A, F & M**	**A & M**
$\ell_p(Q)$	**B & L**	**B, E & J**	**B, F & J**	**B & J**
$\ell_\infty(Q)$	**C & D**	**C & H**	**C, F & I**	**C & G**

Table 1: Characterization of the matrix class (Z, Z'), where $Z \in \left\{\ell_1(Q), \ell_p(Q), \ell_\infty(Q)\right\}$ and $Z' \in \{\ell_1, c_0, c, \ell_\infty\}$

5. Some Geometric Properties of the Space $\ell_p(\mathcal{P})$

In this section, we consider the sequence space $\ell_p(\mathcal{P}) = (\ell_p)_{\mathcal{P}}$, which is a particular case of the sequence space $\ell_p(\mathcal{P}^\alpha)$ (when $\alpha = 0$) studied in [41]. We recall some geometric notions and definitions that are required for our examination.

Definition 5.1. ([12, Definition 1]) A Banach space Z is said to be uniformly convex if for each ς, $0 < \varsigma \leq 2$, there corresponds a $\delta(\varsigma) > 0$ such that $\|z\| = 1 = \|z'\|$, $\|z - z'\| \geq \varsigma$ imply $\left\|\frac{z+z'}{2}\right\| \leq 1 - \delta(\varsigma)$.

Definition 5.2. ([20]) A Banach space Z is said to be uniformly non-square if there exists a $\delta > 0$ such that $\left\|\frac{z+z'}{2}\right\| \leq 1 - \delta$ whenever $\left\|\frac{z-z'}{2}\right\| > 1 - \delta$, for $z, z' \in B(Z)$, where $B(Z)$ denotes the unit ball in Z.

Definition 5.3. ([23, pp. 92]) A normed linear space Z is said to be strictly convex if for $z, z' \in Z$ such that $z \neq z'$, $\|z\| = \|z'\| = 1$, we have $\left\|\frac{z+z'}{2}\right\| < 1$.

Remark 5.4. ([23, Proposition 7.1.1]) Every Hilbert space is uniformly convex.

Definition 5.5. ([33]) Let Z be a Banach space, $Z^{\dagger}$ be its adjoint, and $X^{\dagger\dagger}$ be the adjoint of $Z^{\dagger}$. Then the space Z is said to be reflexive if for each $Z_0 \in Z^{\dagger\dagger}$ there exists $z_0 \in Z$ such that $Z_0(z') = z'(z_0)$ holds for all $z' \in Z^{\dagger}$.

Lemma 5.6. ([33]) *Let Z be a uniformly convex Banach space. Then Z is reflexive.*

Definition 5.7. ([22]) The constant

$$J(Z) = \sup\{\min(\|z+z'\|, \|z-z'\|) : z, z' \in B(Z)\}$$

is called non-square or James constant of Z.

Definition 5.8. ([11]) The von-Neuman Jordan constant of a Banach space Z, denoted by $C_{NJ}Z$, is the smallest constant K satisfying

$$\frac{1}{K} \leq \frac{\|z+z'\|^2 + \|z-z'\|^2}{2(\|z\|^2 + \|z'\|^2)} \leq K$$

for all $z, z' \in B(Z)$ with $\|z\|^2 + \|z'\|^2 \neq 0$.

The following are some well known properties determined by James constant and von-Neumann Jordan constant:

(a) $\sqrt{2} \leq J(Z) \leq 2$; Z is a Hilbert space $\implies J(Z) = \sqrt{2}$.
(b) Z is uniformly non-square if and only if $J(Z) < 2$.
(c) $1 \leq C_{NJ}(Z) \leq 2$. Also Z is a Hilbert space if and only if $C_{NJ}(Z) = 1$.
(d) Z is uniformly non-square if and only if $C_{NJ}(Z) < 2$.
(e) Z is uniformly convex then $C_{NJ}(Z) < 2$.
(f) $C_{NJ}(Z) = 2$ if and only if Z is not uniformly non-square.
(g) $C_{NJ}(Z) = J(Z)$ if and only if Z is not uniformly non-square.

Lemma 5.9. ([40, Lemma 2.2]) *Let Z be a Banach space $\|z+z'\|^2 + \|z-z'\|^2 \leq 4 + J(Z)^2$ for any $z, z' \in Z$.*

Lemma 5.10. ([22, Theorem 1]) *For any Banach space Z*

$$\frac{1}{2}J(Z)^2 \leq C_{NJ}(Z) \leq \frac{J(Z)^2}{(J(Z)-1)^2 + 1}.$$

Theorem 5.11. *The space $\ell_p(\mathcal{P})$ does not exhibit absolute property.*

Proof. Let $z = (z_s) = (1, -1, 0, 0, \ldots)$. Then

$$\mathcal{P}z = (1, 0, 0, 0, \ldots) \text{ and } \mathcal{P}|z| = \left(1, 1, \frac{2}{3}, \frac{2}{5}, \ldots\right)$$

implying the fact that

$$\|z\|_{\ell_p(\mathcal{P})} \neq \||z|\|_{\ell_p(\mathcal{P})},$$

where $|z| = (|z_s|)$ for all $s \in \mathbb{N}_0$. $\square$

Theorem 5.12. *The spaces $\ell_1(\mathcal{P})$ and $\ell_\infty(\mathcal{P})$ are not uniformly convex. In fact, they are not even strictly convex.*

Proof. Consider $z = (z_r) = (1, -1, 0, 0, \dots)$ and $z' = (z'_r) = (0, -2, 2, 0, \dots)$. Clearly $z, z' \in \ell_1(\mathcal{P})$ and $\|z\|_{\ell_1(\mathcal{P})} = \|z'\|_{\ell_1(\mathcal{P})} = 1$, $\|z - z'\|_{\ell_1(\mathcal{P})} = 2$. But $\left\|\frac{z+z'}{2}\right\|_{\ell_1(\mathcal{P})} = 1$. Thus, $\ell_1(\mathcal{P})$ is not uniformly convex and strictly convex.

Similarly, $\ell_\infty(\mathcal{P})$ is not uniformly convex and strictly convex. □

Lemma 5.13. *$\ell_p(\mathcal{P})$ is not a Hilbert space, except for $p = 2$.*

Proof. We take the sequences $z = (z_s) = (1, 1, -2, 0, \dots)$ and $z' = (z'_s) = (1, -3, 2, 0, \dots)$. Then, we have $\mathcal{P}z = (1, 1, 0, 0, \dots)$ and $\mathcal{P}z' = (1, -1, 0, 0, \dots)$. Thus, one can easily verify that

$$\|z + z'\|^2_{\ell_p(\mathcal{P})} + \|z - z'\|^2_{\ell_p(\mathcal{P})} = 8 \neq 4\left(2^{2/p}\right) = 2\left(\|z\|^2_{\ell_p(\mathcal{P})} + \|z'\|^2_{\ell_p(\mathcal{P})}\right), \; p \neq 2.$$

Hence, the norm $\|z\|_{\ell_p(\mathcal{P})}$ with $p \neq 2$ violates the parallelogram identity. Consequently $\ell_p(\mathcal{P})$ is not a Hilbert space except for $p = 2$. □

Theorem 5.14. *The space $\ell_2(\mathcal{P})$ is uniformly convex. Also $C_{NJ}(\ell_2(\mathcal{P})) = 1$ and $J(\ell_2(\mathcal{P})) = \sqrt{2}$.*

Proof. The proof immediately follows from Lemma 5.13 and Remark 5.4. Since uniform convex implies strict convex, therefore, $\ell_2(\mathcal{P})$ is strictly convex as well. The second part is obvious. Moreover, $\ell_2(\mathcal{P})$ is uniformly non-square and super reflexive. □

Corollary 5.15. *The space $\ell_2(\mathcal{P})$ is strictly convex, reflexive and possesses uniform normal structure and fixed point property.*

Theorem 5.16. *The spaces $\ell_1(\mathcal{P})$ and $\ell_\infty(\mathcal{P})$ are not uniformly non-square.*

Proof. We shall establish the result for the space $\ell_1(\mathcal{P})$. Similar proof can be given for the space $\ell_\infty(\mathcal{P})$.

Consider the sequences $z = (1, -1, 0, 0, \dots)$ and $z' = (0, -2, 2, 0 \dots)$. It is clear that $z, z' \in \ell_1(\mathcal{P})$ and $\|z\|_{\ell_1(\mathcal{P})} = 1$, $\|z'\|_{\ell_1(\mathcal{P})} = 1$ such that $\|z\|^2_{\ell_1(\mathcal{P})} + \|z'\|^2_{\ell_1(\mathcal{P})} = 2$.

Therefore by applying Lemma 5.9, we obtain that $J(\ell_1(\mathcal{P})) \geq 2$. But $\sqrt{2} \leq J(Z) \leq 2$ for any Banach space Z. Thus, $J(\ell_1(\mathcal{P})) \geq 2$. Again, by using Lemma 5.10, we get that $C_{NJ}(\ell_1(\mathcal{P})) = 2$. That is $J(\ell_1(\mathcal{P})) = C_{NJ}(\ell_1(\mathcal{P}))$.

Thus by recalling [39, Corollary 7], we get that $\ell_1(\mathcal{P})$ is not uniformly non-square. □

6. Upper Bound for Hausdorff Matrix Operators

Let $d\nu$ be the Borel probability measure on $[0, 1]$ and $\mathsf{H}_\nu = \mathsf{H}_\nu(\alpha) = (h_{rs}(\alpha))$ be the Hausdorff matrix associated with $d\nu$ defined by

$$h_{rs} = \begin{cases} \binom{r}{s} \int_0^1 \alpha^p (1 - \alpha)^{r-s} d\nu(\alpha) & (s \leq r), \\ 0 & (s > r). \end{cases}$$

One can also express the entries of Hausdorff matrix by $h_{rs} = \binom{r}{s} \Delta^{r-s} \nu_s$ for $0 \leq s \leq r$, and $\nu = (\nu_0, \nu_1, \dots)$ is a sequence of real numbers with $\nu_0 = 1$ and

$$\nu_s = \int_0^1 \alpha^s d\nu(\alpha), \quad s = 0, 1, 2 \dots$$

and $\Delta \nu_s = \nu_s - \nu_{s+1}$.

The Hausdorff matrix is reduced to the following classes of matrices in the special case of $d\nu(\alpha)$:

(a) Taking $d\nu(\alpha) = a(1 - \alpha)^{a-1} d\alpha$ gives Cesàro matrix of order a.
(b) Taking $d\nu(\alpha) =$ point evaluation at $\alpha = a$ gives Euler matrix of order a.

(c) Taking $d\nu(\alpha) = \frac{|\log\alpha|^{a-1}}{\Gamma(a)} d\alpha$ gives Hölder matrix of order a.

(d) Taking $d\nu(\alpha) = a\alpha^{a-1} d\alpha$ gives Gamma matrix of order a.

When $a > 0$, then the Cesàro, Hölder and Gamma matrices have non negative entries, and when $0 < a < 1$ then the Euler matrix has non-negative entries.

Lemma 6.1. ([38, Lemma 3.1]) *Let $1 < p < \infty$ and z be a non-negative sequence. Then*

$$\sum_{r=0}^{\infty}\left(\sum_{s=0}^{r} h_{rs} z_s\right)^p < \left(\int_0^1 \alpha^{\frac{-1}{p}} d\nu(\alpha)\right)^p \sum_{s=0}^{\infty} z_s^p.$$

Theorem 6.2. *Let $1 < p < \infty$ and $z = (z_s)$ be a non-negative sequence of real numbers. Then the Hausdorff matrix H_ν maps ℓ_p to $\ell_p(\mathcal{P})$ and*

$$\|\mathsf{H}_\nu\|_{\ell_p,\ell_p\mathcal{P}} \le \left(\sup_{s\in\mathbb{N}_0}\sum_{r=s}^{\infty}\frac{\varrho_s}{\varrho_{r+5}-2}\right)^{1/p}\int_0^1 \alpha^{\frac{-1}{p}} d\nu(\alpha).$$

Proof. Let $z = (z_r)$ be a non negative sequence of real numbers in ℓ_p. By applying Hölder's inequality and Lemma 6.1, we have

$$\begin{aligned}
\|\mathsf{H}_\nu\|_{\ell_p,\ell_p(\mathcal{P})}^p &= \sum_{r=0}^{\infty}\left\{\sum_{s=0}^{r}\frac{\varrho_s}{\varrho_{r+5}-2}\sum_{v=0}^{s} h_{sv} z_v\right\}^p \\
&\le \sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{\varrho_s}{\varrho_{r+5}-2}\left(\sum_{v=0}^{s} h_{sv} z_v\right)^p\left(\sum_{s=0}^{r}\frac{\varrho_s}{\varrho_{r+5}-2}\right)^{k-1} \\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{\varrho_s}{\varrho_{r+5}-2}\left(\sum_{v=0}^{s} h_{sv} z_v\right)^p \\
&= \sum_{s=0}^{\infty}\left(\sum_{v=0}^{s} h_{sv} z_v\right)^p\sum_{r=s}^{\infty}\frac{\varrho_s}{\varrho_{r+5}-2} \\
&\le \sup_{s\in\mathbb{N}_0}\sum_{r=s}^{\infty}\frac{\varrho_s}{\varrho_{r+5}-2}\sum_{s=0}^{\infty}\left(\sum_{v=0}^{s} h_{sv} z_v\right)^p \\
&= \sup_{s\in\mathbb{N}_0}\sum_{r=s}^{\infty}\frac{\varrho_s}{\varrho_{r+5}-2}\left(\int_0^1 \alpha^{\frac{-1}{p}} d\nu(\alpha)\right)^p \|z\|_{\ell_p}^p .
\end{aligned}$$

This concludes that

$$\|\mathsf{H}_\nu\|_{\ell_p,\ell_p(\mathcal{P})} \le \left(\sup_{s\in\mathbb{N}_0}\sum_{r=s}^{\infty}\frac{\varrho_s}{\varrho_{r+5}-2}\right)^{1/p}\int_0^1 \alpha^{\frac{-1}{p}} d\nu(\alpha).$$

This completes the proof. □

Corollary 6.3. *Let $1 < p < \infty$, and $S = \sup_{s\in\mathbb{N}_0}\sum_{r=s}^{\infty}\frac{\varrho_s}{\varrho_{r+5}-2}$. Then Cesàro, Hölder, Gamma and Euler matrices map ℓ_p to*

$\ell_p(\mathcal{P})$ and

$$\|C(a)\|_{\ell_p,\ell_p(\mathcal{P})} \leq S^{1/p}\frac{\Gamma(a+1)\Gamma\left(\frac{1}{p'}\right)}{\Gamma\left(a+\frac{1}{p'}\right)}, \quad a>0;$$

$$\|H(a)\|_{\ell_p,\ell_p(\mathcal{P})} \leq \frac{S^{1/p}}{\Gamma(a)}\int_0^1 \alpha^{\frac{-1}{p}}\left|\log\alpha\right|^{a-1}d\alpha, \quad a>0;$$

$$\|\Gamma(a)\|_{\ell_p,\ell_p(\mathcal{P})} \leq S^{1/p}\frac{ap}{a-1}, \quad ap>1;$$

$$\|E(a)\|_{\ell_p,\ell_p(\mathcal{P})} \leq S^{1/p}a^{\frac{-1}{p}}, \quad 0<a<1.$$

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