



# The Common Solution of Twelve Matrix Equations over the Quaternions

Wei-Jie Yuan<sup>a</sup>, Qing-Wen Wang<sup>a,b</sup>

<sup>a</sup>Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China

<sup>b</sup>Collaborative Innovation Center for the Marine Artificial Intelligence, Shanghai 200444, China

**Abstract.** In this paper, we investigate a system of twelve quaternion matrix equations. Using the real representation of a quaternion matrix, we first derive the least-squares solution with the least norm to the system. Meanwhile, we establish the solvability conditions and an expression of the general solution to the system when it is consistent.

## 1. Introduction

Let  $\mathbf{R}^{m \times n}$ ,  $\mathbf{C}^{m \times n}$  and  $\mathbf{Q}^{m \times n}$  be the set of all  $m \times n$  real matrices, the set of all  $m \times n$  complex matrices and the set of all  $m \times n$  quaternion matrices, respectively. The set of all quaternions is denoted by

$$\mathbf{Q} = \{q_0 + q_1i + q_2j + q_3k \mid i^2 = j^2 = k^2 = ijk = -1, q_0, q_1, q_2, q_3 \in \mathbf{R}\}.$$

For any given  $A \in \mathbf{Q}^{m \times n}$ , the symbols  $\bar{A}$ ,  $A^T$ , and  $A^*$  stand for the conjugate of  $A$ , the transpose of  $A$  and the conjugate transpose of  $A$ , respectively. The identity matrix of order  $n$  is denoted by  $I_n$ . For  $A = (a_{ij}) \in \mathbf{R}^{m \times n}$ ,  $B \in \mathbf{R}^{p \times q}$ , the Kronecker product of  $A$  and  $B$  denoted by  $A \otimes B$ , is of the form

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

For  $A = A_0 + A_1i + A_2j + A_3k$ , we have  $\bar{A} = A_0 - A_1i - A_2j - A_3k$ ,  $A^T = (A_0)^T + (A_1)^T i + (A_2)^T j + (A_3)^T k$ ,  $A^* = (A_0)^T - (A_1)^T i - (A_2)^T j - (A_3)^T k$ . For any given  $A = (a_{ij}) \in \mathbf{Q}^{m \times n}$ , we define  $\text{vec}(A) = (a_1^T, a_2^T, \dots, a_n^T)^T$ , where  $a_i (i = 1, 2, \dots, n)$  is the  $i$ th column of  $A$ . We denote the Frobenius norm of  $A$  by  $\|A\|$  with given by

$$\|A\| = \sqrt{\sum_{i=1}^m \sum_{j=1}^n \|a_{ij}\|^2}.$$

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Corresponding author: Qing-Wen Wang

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Email addresses: weijievicky@163.com (Wei-Jie Yuan), wqw@t.shu.edu.cn (Qing-Wen Wang)

The Moore-Penrose inverse of  $A \in \mathbb{C}^{m \times n}$  is denoted by  $A^\dagger$ , which is the unique solution  $X$  to the following equations:

$$(1) AXA = A, (2) XAX = X, (3) (AX)^* = AX, (4) (XA)^* = XA.$$

In 1843, William Rowan Hamilton [2] discovered the quaternion. As is well known that quaternions form an associative division algebra. Quaternion matrices play an important role in quantum physics, mechanics, signal processing and so on [3, 9, 13, 16, 30].

Solving matrix equations has a relatively long history. Various solutions for matrix equations, like least square solution, symmetric solution, anti-symmetric solution, have been investigated (e.g. [5–8, 11, 14, 17, 18, 21, 24, 25, 29]). For example, Wang and He [4] gave a necessary and sufficient condition for the solvability to the system of matrix equations

$$A_1X_1 + Z_1B_1 = C_1, A_2Z_1 + X_2B_2 = C_2,$$

and established an expression of the general solution when it is consistent. The general solutions of some systems of mixed type generalized Sylvester matrix equations are also presented in [10, 15]. Rehman et al. [17] investigated the exact solution to the following system

$$\begin{aligned} A_3X_1 &= C_3, Y_1B_3 = C_5, \\ F_1Z_1 &= G_1, Z_1F_2 = G_2, \\ A_4X_2 &= C_4, Y_2B_4 = C_6, \\ A_1X_1 + Y_1B_1 + C_1Z_1D_1 &= E_1, \\ A_2X_2 + Y_2B_2 + C_2Z_1D_2 &= E_2. \end{aligned}$$

Wang et al. [23] derived the least norm of the general solution to the following system of quaternion matrix equations

$$\begin{aligned} A_1X_1 &= C_1, A_2X_2 = C_2, \\ A_3X_1B_1 + A_4X_2B_2 &= C_3, \end{aligned}$$

and then derived the least-squares solution with the least norm to the system of matrix equations in [26]. Yuan et al. [20, 22, 28] discussed the least-squares Hermitian solution of the matrix equation  $AXB + CXD = E$  with the least norm over the quaternions, the split quaternions, respectively, and derived the least-squares symmetric solution with the minimum norm of some matrix equations in [19]. To our best knowledge, there has been little information on expanding the matrix equations mentioned above. In this paper, we focus on considering the following system of matrix equations over the quaternion algebra:

$$\begin{aligned} A_1X_1 &= C_1, A_2X_2 = C_2, \\ Y_1B_1 &= C_3, Y_2B_2 = C_4, \\ A_3U_1 &= C_5, U_1B_3 = C_6, \\ A_4U_2 &= C_7, U_2B_4 = C_8, \\ A_{15}V_1 &= C_9, V_1B_{15} = C_{10}, \\ A_5X_1 + Y_1B_5 + D_1U_1E_1 + F_1V_1G_1 &= C_{11}, \\ A_6X_2 + Y_2B_6 + D_2U_2E_2 + F_2V_1G_2 &= C_{12} \end{aligned} \tag{1}$$

where  $X_i, Y_i, U_i (i = 1, 2)$ , and  $V_1$  are unknown quaternion matrices, and other matrices are given. Using the real representation of a quaternion matrix, we first derive the expression of the least-squares solution with the least norm to the system (1), then we give the solvability conditions and the general solution to the system (1).

We observe that the least-squares solution with the least norm of the system (1) can be formulated as follows:

**Problem 1.1.** Let  $A_t = A_{t0} + A_{t1}i + A_{t2}j + A_{t3}k \in \mathbf{Q}^{m \times n}$  ( $t = 1, 2, 3, 4, 15$ ),  $D_u, F_u, E_u, G_u \in \mathbf{Q}^{n \times n}$  ( $u = 1, 2$ ),  $A_i, B_i \in \mathbf{Q}^{n \times n}$  ( $i = 5, 6$ ),  $B_1, B_2, B_3, B_4, B_{15} \in \mathbf{Q}^{n \times k}$ ,  $C_1, C_2, C_5, C_7, C_9 \in \mathbf{Q}^{m \times n}$ ,  $C_3, C_4, C_6, C_8, C_{10} \in \mathbf{Q}^{n \times k}$ ,  $C_{11}, C_{12} \in \mathbf{Q}^{m \times n}$ , and

$$\begin{aligned}
 H_L &= \left\{ [X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \middle| X_1, X_2, Y_1, Y_2, U_1, U_2, V_1 \in \mathbf{Q}^{n \times n}, \right. \\
 &\quad \|A_1 X_1 - C_1\|^2 + \|A_2 X_2 - C_2\|^2 + \|Y_1 B_1 - C_3\|^2 + \|Y_2 B_2 - C_4\|^2 + \|A_3 U_1 - C_5\|^2 \\
 &\quad + \|U_1 B_3 - C_6\|^2 + \|A_4 U_2 - C_7\|^2 + \|U_2 B_4 - C_8\|^2 + \|A_{15} V_1 - C_9\|^2 \\
 &\quad + \|V_1 B_{15} - C_{10}\|^2 + \|A_5 X_1 + Y_1 B_5 + D_1 U_1 E_1 + F_1 V_1 G_1 - C_{11}\|^2 \\
 &\quad \left. + \|A_6 X_2 + Y_2 B_6 + D_2 U_2 E_2 + F_2 V_1 G_2 - C_{12}\|^2 \right\} \\
 &= \min_{[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \in \mathbf{Q}^{n \times n}} \left( \|A_1 X_{1_0} - C_1\|^2 + \|A_2 X_{2_0} - C_2\|^2 + \|Y_{1_0} B_1 - C_3\|^2 \right. \\
 &\quad + \|Y_{2_0} B_2 - C_4\|^2 + \|A_3 U_{1_0} - C_5\|^2 + \|U_{1_0} B_3 - C_6\|^2 + \|A_4 U_{2_0} - C_7\|^2 + \|U_{2_0} B_4 - C_8\|^2 \\
 &\quad + \|A_{15} V_{1_0} - C_9\|^2 + \|V_{1_0} B_{15} - C_{10}\|^2 + \|A_5 X_{1_0} + Y_{1_0} B_5 + D_1 U_{1_0} E_1 + F_1 V_{1_0} G_1 - C_{11}\|^2 \\
 &\quad \left. + \|A_6 X_{2_0} + Y_{2_0} B_6 + D_2 U_{2_0} E_2 + F_2 V_{1_0} G_2 - C_{12}\|^2 \right),
 \end{aligned}$$

find  $[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \in H_L$  such that

$$\begin{aligned}
 \|[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1]\|^2 &= \|X_1\|^2 + \|X_2\|^2 + \|Y_1\|^2 + \|Y_2\|^2 + \|U_1\|^2 \\
 &\quad + \|U_2\|^2 + \|V_1\|^2 \\
 &= \min_{[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \in H_L} (\|X_1\|^2 + \|X_2\|^2 + \|Y_1\|^2 \\
 &\quad + \|Y_2\|^2 + \|U_1\|^2 + \|U_2\|^2 + \|V_1\|^2).
 \end{aligned}$$

This article is organized as follows. In Section 2, we first recall some lemmas and the real representation of a quaternion matrix. In Section 3, we derive the expression of the least-squares solution with the least norm to the system (1), then give a necessary and sufficient condition for the solvability and an expression of the general solution to (1). Finally, we present an algorithm and a numerical example to illustrate the main results of this paper in Section 4.

## 2. Preliminaries

To study Problem I, we first recall some useful results which will be used in the sequel. Let  $A = A_0 + A_1i + A_2j + A_3k \in \mathbf{Q}^{m \times n}$ . We define the real representation of  $A$  as follows:

$$f(A) = \begin{bmatrix} A_0 & A_1 & A_2 & A_3 \\ -A_1 & A_0 & -A_3 & A_2 \\ -A_2 & A_3 & A_0 & -A_1 \\ -A_3 & -A_2 & A_1 & A_0 \end{bmatrix}.$$

It is easy to verify the following:

**Lemma 2.1.** [27] For  $A = A_0 + A_1i + A_2j + A_3k \in \mathbf{Q}^{m \times n}$ , we have that

- (1) if  $A \in \mathbf{Q}^{m \times n}$ ,  $B \in \mathbf{Q}^{n \times t}$ , then  $f(AB) = f(A)f(B)$ ;
- (2) if  $A, B \in \mathbf{Q}^{m \times n}$ ,  $s$  and  $t \in \mathbf{R}$ , then  $f(sA + tB) = sf(A) + tf(B)$ ;
- (3)  $f(A^*) = f(A)^T$ , for  $A \in \mathbf{Q}^{m \times n}$ .

**Lemma 2.2.** [1] The matrix equation  $Ax = b$ , with  $A \in \mathbf{R}^{m \times n}$  and  $b \in \mathbf{R}^n$ , has a solution  $x \in \mathbf{R}^n$  if and only if

$$AA^+b = b.$$

In this case, the general solution can be expressed as the following

$$x = A^\dagger b + (I - A^\dagger A)y,$$

where  $y \in \mathbf{R}^n$  is an arbitrary vector.

**Lemma 2.3.** [1] The least-squares solution to the matrix equation  $Ax = b$  can be expressed as

$$x = A^\dagger b + (I - A^\dagger A)y,$$

where  $y \in \mathbf{R}^n$  is an arbitrary vector. The least-squares solution with the least norm of the matrix equation  $Ax = b$  is  $x = A^\dagger b$ .

For any  $A = A_0 + A_1i + A_2j + A_3k \in \mathbf{Q}^{m \times n}$ , we define  $A \cong \Theta_A = (A_0, A_1, A_2, A_3)$ . Clearly,

$$\|A\| = \|\Theta_A\| = \sqrt{\|A_0\|^2 + \|A_1\|^2 + \|A_2\|^2 + \|A_3\|^2},$$

and

$$A + B \cong \Theta_A + \Theta_B.$$

Thus,

$$\|\Theta_{A+B}\| = \|\Theta_A + \Theta_B\|.$$

Let  $\vec{A} = (A_0, A_1, A_2, A_3)$  and

$$\text{vec}(\vec{A}) \cong \text{vec}(\Theta_A) = \begin{bmatrix} \text{vec}(A_0) \\ \text{vec}(A_1) \\ \text{vec}(A_2) \\ \text{vec}(A_3) \end{bmatrix}.$$

Then we easily have that

$$\|\text{vec}(\vec{A})\| = \|\text{vec}(\Theta_A)\| = \left\| \begin{bmatrix} \text{vec}(A_0) \\ \text{vec}(A_1) \\ \text{vec}(A_2) \\ \text{vec}(A_3) \end{bmatrix} \right\|.$$

In addition,  $\Theta_{AB}$  can be expressed as

$$\begin{aligned} \Theta_{AB} &= (A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3, A_0B_1 + A_1B_0 + A_2B_3 - A_3B_2, A_0B_2 - A_1B_3 \\ &\quad + A_2B_0 + A_3B_1, A_0B_3 + A_1B_2 - A_2B_1 + A_3B_0) \\ &= (A_0, A_1, A_2, A_3) \begin{bmatrix} B_0 & B_1 & B_2 & B_3 \\ -B_1 & B_0 & -B_3 & B_2 \\ -B_2 & B_3 & B_0 & -B_1 \\ -B_3 & -B_2 & B_1 & B_0 \end{bmatrix} \\ &= \Theta_A f(B). \end{aligned}$$

By the definition of  $A \otimes B$ , it follows that

$$(C^T \otimes A) \text{vec}(B) = \text{vec}(ABC). \tag{2}$$

**Proposition 2.4.** Let  $A = A_0 + A_1i + A_2j + A_3k \in \mathbf{Q}^{m \times n}$ ,  $B = B_0 + B_1i + B_2j + B_3k \in \mathbf{Q}^{n \times s}$ , and  $C = C_0 + C_1i + C_2j + C_3k \in \mathbf{Q}^{s \times t}$ . Then

$$\text{vec}(\Theta_{ABC}) = (f(C)^T \otimes A_0, f(C)^T \otimes A_1, f(C)^T \otimes A_2, f(C)^T \otimes A_3) \begin{bmatrix} \text{vec}(\Theta_B) \\ \text{vec}(\Theta_{iB}) \\ \text{vec}(\Theta_{jB}) \\ \text{vec}(\Theta_{kB}) \end{bmatrix},$$

$$\text{where } \text{vec}(\Theta_{iB}) = \begin{bmatrix} \text{vec}(-B_1) \\ \text{vec}(B_0) \\ \text{vec}(-B_3) \\ \text{vec}(B_2) \end{bmatrix}, \text{vec}(\Theta_{jB}) = \begin{bmatrix} \text{vec}(-B_2) \\ \text{vec}(B_3) \\ \text{vec}(B_0) \\ \text{vec}(-B_1) \end{bmatrix}, \text{vec}(\Theta_{kB}) = \begin{bmatrix} \text{vec}(-B_3) \\ \text{vec}(-B_2) \\ \text{vec}(B_1) \\ \text{vec}(B_0) \end{bmatrix}.$$

Proof. By using (2), we have

$$\begin{aligned} \Theta_{ABC} &= \Theta_A f(BC) \\ &= (A_0, A_1, A_2, A_3) \begin{bmatrix} B_0 & B_1 & B_2 & B_3 \\ -B_1 & B_0 & -B_3 & B_2 \\ -B_2 & B_3 & B_0 & -B_1 \\ -B_3 & -B_2 & B_1 & B_0 \end{bmatrix} \begin{bmatrix} C_0 & C_1 & C_2 & C_3 \\ -C_1 & C_0 & -C_3 & C_2 \\ -C_2 & C_3 & C_0 & -C_1 \\ -C_3 & -C_2 & C_1 & C_0 \end{bmatrix} \\ &= [(A_0B_0C_0 - A_1B_1C_0 - A_2B_2C_0 - A_3B_3C_0 - A_0B_1C_1 - A_1B_0C_1 - A_2B_3C_1 + A_3B_2C_1 \\ &\quad - A_0B_2C_2 + A_1B_3C_2 - A_2B_0C_2 - A_3B_1C_2 - A_0B_3C_3 - A_1B_2C_3 + A_2B_1C_3 - A_3B_0C_3) \\ &\quad (A_0B_0C_1 - A_1B_1C_1 - A_2B_2C_1 - A_3B_3C_1 + A_0B_1C_0 + A_1B_0C_0 + A_2B_3C_0 - A_3B_2C_0 \\ &\quad + A_0B_2C_3 - A_1B_3C_3 + A_2B_0C_3 + A_3B_1C_3 - A_0B_3C_2 - A_1B_2C_2 + A_2B_1C_2 - A_3B_0C_2) \\ &\quad (A_0B_1C_2 - A_1B_1C_2 - A_2B_2C_2 - A_3B_3C_2 - A_0B_1C_3 - A_1B_0C_3 - A_2B_3C_3 + A_3B_2C_3 \\ &\quad + A_0B_2C_0 - A_1B_3C_0 + A_2B_0C_0 + A_3B_1C_0 + A_0B_3C_1 + A_1B_2C_1 - A_2B_1C_1 + A_3B_0C_1) \\ &\quad (A_0B_0C_3 - A_1B_1C_3 - A_2B_2C_3 - A_3B_3C_3 + A_0B_1C_2 + A_1B_0C_2 + A_2B_3C_2 - A_3B_2C_2 \\ &\quad - A_0B_2C_1 + A_1B_3C_1 - A_2B_0C_1 - A_3B_1C_1 + A_0B_3C_0 + A_1B_2C_0 - A_2B_1C_0 + A_3B_0C_3)] \end{aligned}$$

Therefore,

$$\text{vec}(\Theta_{ABC}) = N_1 + N_2 + N_3 + N_4,$$

where

$$\begin{aligned} N_1 &= \begin{bmatrix} (C_0^T \otimes A_0) \text{vec}(B_0) + (-C_1^T \otimes A_0) \text{vec}(B_1) + (-C_2^T \otimes A_0) \text{vec}(B_2) + (-C_3^T \otimes A_0) \text{vec}(B_3) \\ (C_1^T \otimes A_0) \text{vec}(B_0) + (C_0^T \otimes A_0) \text{vec}(B_1) + (C_3^T \otimes A_0) \text{vec}(B_2) + (-C_2^T \otimes A_0) \text{vec}(B_3) \\ (C_2^T \otimes A_0) \text{vec}(B_0) + (-C_3^T \otimes A_0) \text{vec}(B_1) + (C_0^T \otimes A_0) \text{vec}(B_2) + (C_1^T \otimes A_0) \text{vec}(B_3) \\ (C_3^T \otimes A_0) \text{vec}(B_0) + (C_2^T \otimes A_0) \text{vec}(B_1) + (-C_1^T \otimes A_0) \text{vec}(B_2) + (C_0^T \otimes A_0) \text{vec}(B_3) \end{bmatrix} \\ &= \begin{bmatrix} C_0^T & -C_1^T & -C_2^T & -C_3^T \\ C_1^T & C_0^T & C_3^T & -C_2^T \\ C_2^T & -C_3^T & C_0^T & C_1^T \\ C_3^T & C_2^T & -C_1^T & C_0^T \end{bmatrix} \otimes A_0 \begin{bmatrix} \text{vec}(B_0) \\ \text{vec}(B_1) \\ \text{vec}(B_2) \\ \text{vec}(B_3) \end{bmatrix} \\ &= (f(C)^T \otimes A_0) \begin{bmatrix} \text{vec}(B_0) \\ \text{vec}(B_1) \\ \text{vec}(B_2) \\ \text{vec}(B_3) \end{bmatrix}. \end{aligned}$$

Similarly,

$$\begin{aligned} N_2 &= (f(C)^T \otimes A_1) \begin{bmatrix} \text{vec}(-B_1) \\ \text{vec}(B_0) \\ \text{vec}(-B_3) \\ \text{vec}(B_2) \end{bmatrix}, N_3 = (f(C)^T \otimes A_1) \begin{bmatrix} \text{vec}(-B_2) \\ \text{vec}(B_3) \\ \text{vec}(B_0) \\ \text{vec}(-B_1) \end{bmatrix}, \\ N_4 &= (f(C)^T \otimes A_1) \begin{bmatrix} \text{vec}(-B_3) \\ \text{vec}(-B_2) \\ \text{vec}(B_1) \\ \text{vec}(B_0) \end{bmatrix}. \end{aligned}$$

Hence,

$$\text{vec}(\Theta_{ABC}) = (f(C)^T \otimes A_0, f(C)^T \otimes A_1, f(C)^T \otimes A_2, f(C)^T \otimes A_3) \begin{bmatrix} \text{vec}(\Theta_B) \\ \text{vec}(\Theta_{iB}) \\ \text{vec}(\Theta_{jB}) \\ \text{vec}(\Theta_{kB}) \end{bmatrix}.$$

□

**Proposition 2.5.** Suppose that  $X = X_0 + X_1i + X_2j + X_3k \in \mathbf{Q}^{n \times k}$ , then

$$\begin{bmatrix} \text{vec}(\Theta_X) \\ \text{vec}(\Theta_{iX}) \\ \text{vec}(\Theta_{jX}) \\ \text{vec}(\Theta_{kX}) \end{bmatrix} = K_{nk} \text{vec}(\vec{X}),$$

where

$$K_{nk} = \begin{bmatrix} K_I \\ K_{il} \\ K_{jl} \\ K_{kl} \end{bmatrix}, K_I = \begin{bmatrix} I_{nk} & 0 & 0 & 0 \\ 0 & I_{nk} & 0 & 0 \\ 0 & 0 & I_{nk} & 0 \\ 0 & 0 & 0 & I_{nk} \end{bmatrix}, K_{il} = \begin{bmatrix} 0 & -I_{nk} & 0 & 0 \\ I_{nk} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{nk} \\ 0 & 0 & I_{nk} & 0 \end{bmatrix},$$

$$K_{jl} = \begin{bmatrix} 0 & 0 & -I_{nk} & 0 \\ 0 & 0 & 0 & I_{nk} \\ I_{nk} & 0 & 0 & 0 \\ 0 & -I_{nk} & 0 & 0 \end{bmatrix}, K_{kl} = \begin{bmatrix} 0 & 0 & 0 & -I_{nk} \\ 0 & 0 & -I_{nk} & 0 \\ 0 & I_{nk} & 0 & 0 \\ I_{nk} & 0 & 0 & 0 \end{bmatrix}.$$

*Proof.* For any given  $X = X_0 + X_1i + X_2j + X_3k \in \mathbf{Q}^{n \times k}$ , we have

$$\begin{aligned} \begin{bmatrix} \text{vec}(\Theta_X) \\ \text{vec}(\Theta_{iX}) \\ \text{vec}(\Theta_{jX}) \\ \text{vec}(\Theta_{kX}) \end{bmatrix} &= \begin{bmatrix} \text{vec}(X_0) + \text{vec}(X_1)i + \text{vec}(X_2)j + \text{vec}(X_3)k \\ \text{vec}(-X_1) + \text{vec}(X_0)i + \text{vec}(-X_3)j + \text{vec}(X_2)k \\ \text{vec}(-X_2) + \text{vec}(X_3)i + \text{vec}(X_0)j + \text{vec}(-X_1)k \\ \text{vec}(-X_3) + \text{vec}(-X_2)i + \text{vec}(X_1)j + \text{vec}(X_0)k \end{bmatrix} \\ &= \begin{bmatrix} K_I \\ K_{il} \\ K_{jl} \\ K_{kl} \end{bmatrix} \begin{bmatrix} \text{vec}(X_0) \\ \text{vec}(X_1) \\ \text{vec}(X_2) \\ \text{vec}(X_3) \end{bmatrix} \\ &= K_{nk} \text{vec}(\vec{X}). \end{aligned}$$

□

It follows from Propositions 2.1 and 2.2 that

$$\text{vec}(\Theta_{AXB}) = (f(B)^T \otimes A_0, f(B)^T \otimes A_1, f(B)^T \otimes A_2, f(B)^T \otimes A_3) K_{nk} \text{vec}(\vec{X}). \tag{3}$$

### 3. The solution of Problem I

From the above discussion, we now pay attention to solving Problem I and considering the system of quaternion matrix equations (1).

Let  $A_t = A_{i0} + A_{t1}i + A_{t2}j + A_{t3}k \in \mathbf{Q}^{m \times n} (t = 1, 2, 3, 4, 15)$ ,  $D_u, F_u, E_u, G_u \in \mathbf{Q}^{n \times n} (u = 1, 2)$ ,  $A_i, B_i \in \mathbf{Q}^{n \times n} (i = 5, 6)$ ,  $B_1, B_2, B_3, B_4, B_{15} \in \mathbf{Q}^{n \times k}$ ,  $C_1, C_2, C_5, C_7, C_9 \in \mathbf{Q}^{m \times n}$ ,  $C_3, C_4, C_6, C_8, C_{10} \in \mathbf{Q}^{n \times k}$ ,  $C_{11}, C_{12} \in \mathbf{Q}^{n \times n}$ . Then we have that

$$P = \begin{bmatrix} f(I_n)^T \otimes A_{10} & f(I_n)^T \otimes A_{11} & f(I_n)^T \otimes A_{12} & f(I_n)^T \otimes A_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f(I_n)^T \otimes A_{50} & f(I_n)^T \otimes A_{51} & f(I_n)^T \otimes A_{52} & f(I_n)^T \otimes A_{53} \\ 0 & 0 & 0 & 0 \end{bmatrix} K_n^2,$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ f(I_n)^T \otimes A_{20} & f(I_n)^T \otimes A_{21} & f(I_n)^T \otimes A_{22} & f(I_n)^T \otimes A_{23} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f(I_n)^T \otimes A_{60} & f(I_n)^T \otimes A_{61} & f(I_n)^T \otimes A_{62} & f(I_n)^T \otimes A_{63} \end{bmatrix} K_n^2$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f(B_1)^T \otimes I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f(B_5)^T \otimes I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} K_n^2$$

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f(B_2)^T \otimes I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f(B_6)^T \otimes I_n & 0 & 0 & 0 \end{bmatrix} K_n^2$$

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f(I_n)^T \otimes A_{30} & f(I_n)^T \otimes A_{31} & f(I_n)^T \otimes A_{32} & f(I_n)^T \otimes A_{33} \\ f(B_3)^T \otimes I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f(E_1)^T \otimes D_{10} & f(E_1)^T \otimes D_{11} & f(E_1)^T \otimes D_{12} & f(E_1)^T \otimes D_{13} \\ 0 & 0 & 0 & 0 \end{bmatrix} K_n^2$$

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f(I_n)^T \otimes A_{40} & f(I_n)^T \otimes A_{41} & f(I_n)^T \otimes A_{42} & f(I_n)^T \otimes A_{43} \\ f(B_4)^T \otimes I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f(E_2)^T \otimes D_{20} & f(E_2)^T \otimes D_{21} & f(E_2)^T \otimes D_{22} & f(E_2)^T \otimes D_{23} \end{bmatrix} K_n^2$$

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f(I_n)^T \otimes A_{150} & f(I_n)^T \otimes A_{151} & f(I_n)^T \otimes A_{152} & f(I_n)^T \otimes A_{153} \\ f(B_{15})^T \otimes I_n & 0 & 0 & 0 \\ f(G_1)^T \otimes F_{10} & f(G_1)^T \otimes F_{11} & f(G_1)^T \otimes F_{12} & f(G_2)^T \otimes F_{13} \\ f(G_2)^T \otimes F_{20} & f(G_2)^T \otimes F_{21} & f(G_2)^T \otimes F_{22} & f(G_2)^T \otimes F_{23} \end{bmatrix} K_n^2.$$

For  $P, Q, L, M, N, W$ , and  $S \in \mathbf{Q}^{m \times n}$ , we define

$$T_i = [P_i \ Q_i \ L_i \ M_i \ N_i \ W_i \ S_i] (i = 0, 1, 2, 3),$$

$$J_2 = \begin{bmatrix} T_0 \\ T_1 \\ T_2 \end{bmatrix}, e = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

$$e_i = \begin{bmatrix} \text{vec}(C_{1i}) \\ \text{vec}(C_{2i}) \\ \text{vec}(C_{3i}) \\ \text{vec}(C_{4i}) \\ \text{vec}(C_{5i}) \\ \text{vec}(C_{6i}) \\ \text{vec}(C_{7i}) \\ \text{vec}(C_{8i}) \\ \text{vec}(C_{9i}) \\ \text{vec}(C_{10i}) \\ \text{vec}(C_{11i}) \\ \text{vec}(C_{12i}) \end{bmatrix} (i = 0, 1, 2, 3), \tag{4}$$

and

$$\begin{aligned} R &= (I - J_2^\dagger J_2) T_3^T, \\ Z &= (I + (I - R^\dagger R) T_3 J_2^\dagger J_2^T T_3^T (I - R^\dagger R))^{-1}, \\ H &= R^\dagger + (I - R^\dagger R) Z T_3 J_2^\dagger J_2^T (I - T_3^T R^\dagger), \\ S_{11} &= I - J_2 J_2^\dagger + J_2^T T_3^T Z (I - R^\dagger R) T_3 J_2^\dagger, \\ S_{12} &= -J_2^T T_3^T (I - R^\dagger R) Z, \\ S_{22} &= (I - R^\dagger R) Z. \end{aligned}$$

From the results in [12], we have

$$\begin{aligned} \begin{bmatrix} J_2 \\ T_3 \end{bmatrix}^\dagger &= (J_2^\dagger - H^T T_3 J_2^\dagger, H^T), & \begin{bmatrix} J_2 \\ T_3 \end{bmatrix}^\dagger \begin{bmatrix} J_2 \\ T_3 \end{bmatrix} &= J_2^\dagger J_2 + R R^\dagger, \\ I - \begin{bmatrix} J_2 \\ T_3 \end{bmatrix} \begin{bmatrix} J_2 \\ T_3 \end{bmatrix}^\dagger &= \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}. \end{aligned}$$

**Theorem 3.1.** Let  $A_t = A_{t0} + A_{t1}i + A_{t2}j + A_{t3}k \in \mathbf{Q}^{m \times n}$ ,  $t = 1, 2, 3, 4, 15$ ,  $D_u = D_{u0} + D_{u1}i + D_{u2}j + D_{u3}k$ ,  $F_u = F_{u0} + F_{u1}i + F_{u2}j + F_{u3}k \in \mathbf{Q}^{n \times n}$ ,  $u = 1, 2, A_5, A_6, B_5, B_6, E_1, E_2, G_1, G_2 \in \mathbf{Q}^{n \times n}$ ,  $B_1, B_2, B_3, B_4, B_{15} \in \mathbf{Q}^{n \times k}$ ,  $C_1, C_2, C_5, C_7, C_9 \in$



$\mathbf{Q}^{m \times n}$ ,  $C_3, C_4, C_6, C_8, C_{10} \in \mathbf{Q}^{n \times k}$ ,  $C_{11}, C_{12} \in \mathbf{Q}^{n \times n}$  and  $T_0, T_1, T_2, T_3, e$  be defined as in (4). Then

$$H_L = \left\{ [X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \begin{bmatrix} \text{vec}(\vec{X}_1) \\ \text{vec}(\vec{X}_2) \\ \text{vec}(\vec{Y}_1) \\ \text{vec}(\vec{Y}_2) \\ \text{vec}(\vec{U}_1) \\ \text{vec}(\vec{U}_2) \\ \text{vec}(\vec{V}_1) \end{bmatrix} = (J_2^\dagger - H^T T_3 J_2^\dagger, H^T) e + (I - J_2^\dagger J_2 - RR^\dagger) z \right\}, \quad (5)$$

where  $z$  is an arbitrary vector of appropriate order.

*Proof.* It follows from Proposition 2.4 and (3) that

$$\begin{aligned} & \|A_1 X_1 - C_1\|^2 + \|A_2 X_2 - C_2\|^2 + \|Y_1 B_1 - C_3\|^2 + \|Y_2 B_2 - C_4\|^2 + \|A_3 U_1 - C_5\|^2 \\ & + \|U_1 B_3 - C_6\|^2 + \|A_4 U_2 - C_7\|^2 + \|U_2 B_4 - C_8\|^2 + \|A_{15} V_1 - C_9\|^2 \\ & + \|V_1 B_{15} - C_{10}\|^2 + \|A_5 X_1 + Y_1 B_5 + D_1 U_1 E_1 + F_1 V_1 G_1 - C_{11}\|^2 \\ & + \|A_6 X_2 + Y_2 B_6 + D_2 U_2 E_2 + F_2 V_1 G_2 - C_{12}\|^2 \\ & = \|\text{vec}(\Theta_{A_1 X_1 - C_1})\|^2 + \|\text{vec}(\Theta_{A_2 X_2 - C_2})\|^2 + \|\text{vec}(\Theta_{Y_1 B_1 - C_3})\|^2 + \|\text{vec}(\Theta_{Y_2 B_2 - C_4})\|^2 \\ & + \|\text{vec}(\Theta_{A_3 U_1 - C_5})\|^2 + \|\text{vec}(\Theta_{U_1 B_3 - C_6})\|^2 + \|\text{vec}(\Theta_{A_4 U_2 - C_7})\|^2 + \|\text{vec}(\Theta_{U_2 B_4 - C_8})\|^2 \\ & + \|\text{vec}(\Theta_{A_{15} V_1 - C_9})\|^2 + \|\text{vec}(\Theta_{V_1 B_{15} - C_{10}})\|^2 + \|\text{vec}(\Theta_{A_5 X_1 + Y_1 B_5 + D_1 U_1 E_1 + F_1 V_1 G_1 - C_{11}})\|^2 \\ & + \|\text{vec}(\Theta_{A_6 X_2 + Y_2 B_6 + D_2 U_2 E_2 + F_2 V_1 G_2 - C_{12}})\|^2 \\ & = \left\| \begin{bmatrix} P & Q & L & M & N & W & Z \end{bmatrix} \begin{bmatrix} \text{vec}(\vec{X}_1) \\ \text{vec}(\vec{X}_2) \\ \text{vec}(\vec{Y}_1) \\ \text{vec}(\vec{Y}_2) \\ \text{vec}(\vec{U}_1) \\ \text{vec}(\vec{U}_2) \\ \text{vec}(\vec{V}_1) \end{bmatrix} - \begin{bmatrix} \text{vec}(\Theta_{C_1}) \\ \text{vec}(\Theta_{C_2}) \\ \text{vec}(\Theta_{C_3}) \\ \text{vec}(\Theta_{C_4}) \\ \text{vec}(\Theta_{C_5}) \\ \text{vec}(\Theta_{C_6}) \\ \text{vec}(\Theta_{C_7}) \\ \text{vec}(\Theta_{C_8}) \\ \text{vec}(\Theta_{C_9}) \\ \text{vec}(\Theta_{C_{10}}) \\ \text{vec}(\Theta_{C_{11}}) \\ \text{vec}(\Theta_{C_{12}}) \end{bmatrix} \right\|^2 \\ & = \left\| \begin{bmatrix} J_2 \\ T_3 \end{bmatrix} \begin{bmatrix} \text{vec}(\vec{X}_1) \\ \text{vec}(\vec{X}_2) \\ \text{vec}(\vec{Y}_1) \\ \text{vec}(\vec{Y}_2) \\ \text{vec}(\vec{U}_1) \\ \text{vec}(\vec{U}_2) \\ \text{vec}(\vec{V}_1) \end{bmatrix} - e \right\|^2. \end{aligned}$$

By Lemma 2.2, it follows that

$$\begin{bmatrix} \text{vec}(\vec{X}_1) \\ \text{vec}(\vec{X}_2) \\ \text{vec}(\vec{Y}_1) \\ \text{vec}(\vec{Y}_2) \\ \text{vec}(\vec{U}_1) \\ \text{vec}(\vec{U}_2) \\ \text{vec}(\vec{V}_1) \end{bmatrix} = \begin{bmatrix} J_2 \\ T_3 \end{bmatrix}^\dagger e + \left[ I - \begin{bmatrix} J_2 \\ T_3 \end{bmatrix}^\dagger \begin{bmatrix} J_2 \\ T_3 \end{bmatrix} \right] z,$$

and thus

$$\begin{bmatrix} \text{vec}(\vec{X}_1) \\ \text{vec}(\vec{X}_2) \\ \text{vec}(\vec{Y}_1) \\ \text{vec}(\vec{Y}_2) \\ \text{vec}(\vec{U}_1) \\ \text{vec}(\vec{U}_2) \\ \text{vec}(\vec{V}_1) \end{bmatrix} = (J_2^\dagger - H^T T_3 J_2^\dagger, H^T) e + (I - J_2^\dagger J_2 - R R^\dagger) z.$$

□

**Corollary 3.2.** *The system of quaternion matrix equations (1) has a solution  $X_1, X_2, Y_1, Y_2, U_1, U_2, V_1 \in \mathbb{Q}^{n \times n}$  if and only if*

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} e = 0. \tag{6}$$

In this case, the solution set of the system (1) can be expressed as

$$H_E = \left\{ [X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \left| \begin{bmatrix} \text{vec}(\vec{X}_1) \\ \text{vec}(\vec{X}_2) \\ \text{vec}(\vec{Y}_1) \\ \text{vec}(\vec{Y}_2) \\ \text{vec}(\vec{U}_1) \\ \text{vec}(\vec{U}_2) \\ \text{vec}(\vec{V}_1) \end{bmatrix} = (J_2^\dagger - H^T T_3 J_2^\dagger, H^T) e + (I - J_2^\dagger J_2 - R R^\dagger) z \right. \right\}, \tag{7}$$

where  $z$  is an arbitrary vector of appropriate order. Furthermore, if (6) holds, then the system (1) has a unique solution  $[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \in H_E$  if and only if

$$\text{rank} \begin{bmatrix} J_2 \\ T_3 \end{bmatrix} = 28n^2. \tag{8}$$

In this case,

$$H_E = \left\{ [X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \left| \begin{bmatrix} \text{vec}(\vec{X}_1) \\ \text{vec}(\vec{X}_2) \\ \text{vec}(\vec{Y}_1) \\ \text{vec}(\vec{Y}_2) \\ \text{vec}(\vec{U}_1) \\ \text{vec}(\vec{U}_2) \\ \text{vec}(\vec{V}_1) \end{bmatrix} = (J_2^\dagger - H^T T_3 J_2^\dagger, H^T) e \right. \right\}. \tag{9}$$

**Theorem 3.3.** *Problem I has a unique solution  $[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \in H_L$ , which satisfies*

$$\begin{bmatrix} \text{vec}(\vec{X}_1) \\ \text{vec}(\vec{X}_2) \\ \text{vec}(\vec{Y}_1) \\ \text{vec}(\vec{Y}_2) \\ \text{vec}(\vec{U}_1) \\ \text{vec}(\vec{U}_2) \\ \text{vec}(\vec{V}_1) \end{bmatrix} = (J_2^\dagger - H^T T_3 J_2^\dagger, H^T) e. \tag{10}$$

*Proof.* From (5), if the conditions in Theorem 3.1 are satisfied, then there exists a unique solution  $[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \in H_L$  for Problem I, and it can be expressed as (10). It follows from Lemma 2.2 and (10) that

$$\begin{aligned} & \min_{[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \in H_L} (\| [X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \|^2) \\ &= \min_{[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \in H_L} (\|X_1\|^2 + \|X_2\|^2 + \|Y_1\|^2 + \|Y_2\|^2 + \|U_1\|^2 + \|U_2\|^2 + \|V_1\|^2) \\ &= \min_{[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \in H_L} (\| \text{vec}(\vec{X}_1) \|^2 + \| \text{vec}(\vec{X}_2) \|^2 + \| \text{vec}(\vec{Y}_1) \|^2 + \| \text{vec}(\vec{Y}_2) \|^2 \\ & \quad + \| \text{vec}(\vec{U}_1) \|^2 + \| \text{vec}(\vec{U}_2) \|^2 + \| \text{vec}(\vec{V}_1) \|^2) \\ &= \min_{[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \in H_L} \left\| \begin{bmatrix} \text{vec}(\vec{X}_1) \\ \text{vec}(\vec{X}_2) \\ \text{vec}(\vec{Y}_1) \\ \text{vec}(\vec{Y}_2) \\ \text{vec}(\vec{U}_1) \\ \text{vec}(\vec{U}_2) \\ \text{vec}(\vec{V}_1) \end{bmatrix} \right\|^2. \end{aligned}$$

By Lemma 2.2 and (5), we have

$$\begin{bmatrix} \text{vec}(\vec{X}_1) \\ \text{vec}(\vec{X}_2) \\ \text{vec}(\vec{Y}_1) \\ \text{vec}(\vec{Y}_2) \\ \text{vec}(\vec{U}_1) \\ \text{vec}(\vec{U}_2) \\ \text{vec}(\vec{V}_1) \end{bmatrix} = \begin{bmatrix} J_2 \\ T_3 \end{bmatrix}^\dagger e.$$

Therefore,

$$\begin{bmatrix} \text{vec}(\vec{X}_1) \\ \text{vec}(\vec{X}_2) \\ \text{vec}(\vec{Y}_1) \\ \text{vec}(\vec{Y}_2) \\ \text{vec}(\vec{U}_1) \\ \text{vec}(\vec{U}_2) \\ \text{vec}(\vec{V}_1) \end{bmatrix} = (J_2^\dagger - H^T T_3 J_2^\dagger, H^T) e.$$

□

**Corollary 3.4.** *The least norm problem*

$$\begin{aligned} \|[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1]\|^2 &= \|X_1\|^2 + \|X_2\|^2 + \|Y_1\|^2 + \|Y_2\|^2 + \|U_1\|^2 + \|U_2\|^2 + \|V_1\|^2 \\ &= \min_{[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \in H_E} \left( \|X_1\|^2 + \|X_2\|^2 + \|Y_1\|^2 + \|Y_2\|^2 \right. \\ &\quad \left. + \|U_1\|^2 + \|U_2\|^2 + \|V_1\|^2 \right) \end{aligned}$$

has a unique solution  $[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \in H_E$ , which can be expressed as (10).

In particular, we can derive the expression of the least-squares solution for the following system of quaternion matrix equations

$$\begin{aligned} A_1 X_1 + Y_1 B_1 + F_1 V_1 G_1 &= C_1, \\ A_2 X_2 + Y_2 B_2 + F_2 V_1 G_2 &= C_2. \end{aligned}$$

**Corollary 3.5.** *Let  $A_1 = A_{10} + A_{11}i + A_{12}j + A_{13}k, A_2 = A_{20} + A_{21}i + A_{22}j + A_{23}k \in \mathbb{Q}^{n \times n}, B_1, B_2 \in \mathbb{Q}^{n \times n}, F_1 = F_{10} + F_{11}i + F_{12}j + F_{13}k, F_2 = F_{20} + F_{21}i + F_{22}j + F_{23}k \in \mathbb{Q}^{n \times n}, G_1, G_2, C_1, C_2 \in \mathbb{Q}^{n \times n}$ . Then*

$$\begin{aligned} H_{L1} &= \left\{ [X_1, X_2, Y_1, Y_2, V_1] \mid X_1, X_2, Y_1, Y_2, V_1 \in \mathbb{Q}^{n \times n}, \|A_1 X_1 + Y_1 B_1 + F_1 V_1 G_1 - C_1\|^2 \right. \\ &\quad \left. + \|A_2 X_2 + Y_2 B_2 + F_2 V_1 G_2 - C_2\|^2 \right. \\ &= \min_{[X_1, X_2, Y_1, Y_2, V_1] \in \mathbb{Q}^{n \times n}} \left( \|A_1 X_1 + Y_1 B_1 + F_1 V_1 G_1 - C_1\|^2 \right. \\ &\quad \left. + \|A_2 X_2 + Y_2 B_2 + F_2 V_1 G_2 - C_2\|^2 \right) \left. \right\} \\ &= \left\{ [X_1, X_2, Y_1, Y_2, V_1] \mid \begin{bmatrix} \text{vec}(X_1) \\ \text{vec}(X_2) \\ \text{vec}(Y_1) \\ \text{vec}(Y_2) \\ \text{vec}(V_1) \end{bmatrix} = (J_{2_i}^\dagger - H_i^T T_3 J_{2_i}^\dagger, H_i^T) e_i + (I - J_{2_i}^\dagger J_{2_i} - R_i R_i^\dagger) z_i \right\}, \end{aligned}$$

where  $z_i$  is an arbitrary vector of appropriate order,

$$\begin{aligned} P_i &= \begin{bmatrix} f(I_n)^T \otimes A_{10} & f(I_n)^T \otimes A_{11} & f(I_n)^T \otimes A_{12} & f(I_n)^T \otimes A_{13} \\ 0 & 0 & 0 & 0 \end{bmatrix} K_{n^2}, \\ Q_i &= \begin{bmatrix} f(I_n)^T \otimes A_{20} & f(I_n)^T \otimes A_{21} & f(I_n)^T \otimes A_{22} & f(I_n)^T \otimes A_{23} \\ 0 & 0 & 0 & 0 \end{bmatrix} K_{n^2}, \\ M_i &= \begin{bmatrix} f(B_1)^T \otimes I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} K_{n^2}, \quad N_i = \begin{bmatrix} f(B_2)^T \otimes I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} K_{n^2}, \\ W_i &= \begin{bmatrix} f(G_1)^T \otimes F_{10} & f(G_1)^T \otimes F_{11} & f(G_1)^T \otimes F_{12} & f(G_1)^T \otimes F_{13} \\ f(G_2)^T \otimes F_{20} & f(G_2)^T \otimes F_{21} & f(G_2)^T \otimes F_{22} & f(G_2)^T \otimes F_{23} \end{bmatrix} K_{n^2}, \\ e_i &= \begin{bmatrix} \text{vec } \Theta_{C_1} \\ \text{vec } \Theta_{C_2} \end{bmatrix}, \quad J_{2_i} = \begin{bmatrix} P_{i0} & Q_{i0} & M_{i0} & N_{i0} & W_{i0} \\ P_{i1} & Q_{i1} & M_{i1} & N_{i1} & W_{i1} \\ P_{i2} & Q_{i2} & M_{i2} & N_{i2} & W_{i2} \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} R_i &= (I - J_{2_i}^\dagger J_{2_i}) T_{13}^T, \\ Z_i &= (I + (I - R_i^\dagger R_i) T_{13} J_{2_i}^\dagger J_{2_i}^T T_{13} (I - R_i^\dagger R_i))^{-1}, \\ H_i &= R_i^\dagger + (I - R_i^\dagger R_i) Z_i T_{13} J_{2_i}^\dagger J_{2_i}^T (I - T_{13}^T R_i^\dagger). \end{aligned}$$

In this case, the least norm problem

$$\| [X_{1_0}, X_{2_0}, Y_{1_0}, Y_{2_0}, V_{1_0}] \|^2 = \min_{[X_1, X_2, Y_1, Y_2, V_1] \in H_{L1}} (\|X_1\|^2 + \|X_2\|^2 + \|Y_1\|^2 + \|Y_2\|^2 + \|V_1\|^2)$$

has a unique solution  $[X_{1_0}, X_{2_0}, Y_{1_0}, Y_{2_0}, V_{1_0}] \in H_{L1}$  and it can be expressed as

$$\begin{bmatrix} \text{vec}(X_{1_0}^{\vec{}}) \\ \text{vec}(X_{2_0}^{\vec{}}) \\ \text{vec}(Y_{1_0}^{\vec{}}) \\ \text{vec}(Y_{2_0}^{\vec{}}) \\ \text{vec}(V_{1_0}^{\vec{}}) \end{bmatrix} = (J_{2_l}^{\dagger} - H_l^T T_3 J_{2_l}^{\dagger}, H_l^T) e_l.$$

#### 4. An algorithm and a numerical example

In this Section, we give an algorithm and a numerical example to illustrate the results of this paper. The following Algorithm 4.1 provides a method to find the solution of Problems I. If the conditions (6) for the system of matrix equations (1) holds, the following algorithm presents the numerical solution of Problem I for  $X_1, X_2, Y_1, Y_2, U_1, U_2, V_1 \in \mathbf{Q}^{n \times n}$ .

- 
- Algorithm 4.1.** 1. Input  $A_t = A_{t0} + A_{t1}i + A_{t2}j + A_{t3}k \in \mathbf{Q}^{m \times n}, t = 1, 2, 3, 4, 15, D_u = D_{u0} + D_{u1}i + D_{u2}j + D_{u3}k, F_u = F_{u0} + F_{u1}i + F_{u2}j + F_{u3}k \in \mathbf{Q}^{n \times n}, u = 1, 2, A_5, A_6, B_5, B_6, E_1, E_2, G_1, G_2 \in \mathbf{Q}^{n \times n}, B_1, B_2, B_3, B_4, B_{15} \in \mathbf{Q}^{n \times k}, C_1, C_2, C_5, C_7, C_9 \in \mathbf{Q}^{m \times n}, C_3, C_4, C_6, C_8, C_{10} \in \mathbf{Q}^{n \times k}, C_{11}, C_{12} \in \mathbf{Q}^{n \times n}$ .
2. Compute  $J_2, T_3, R, Z, H, S_{11}, S_{12}, S_{22}, e$ .
3. If (6) holds and  $\begin{bmatrix} J_2 \\ T_3 \end{bmatrix}$  has full column rank, then calculate  $[X_{1_l}, X_{2_l}, Y_{1_l}, Y_{2_l}, U_{1_l}, U_{2_l}, V_{1_l}] \in H_E$  according to (9). Otherwise go to the next step.
4. If (6) holds, then calculate  $[X_{1_l}, X_{2_l}, Y_{1_l}, Y_{2_l}, U_{1_l}, U_{2_l}, V_{1_l}] \in H_E$  according to (7). Otherwise go to the next step.
5. Calculate  $[X_{1_l}, X_{2_l}, Y_{1_l}, Y_{2_l}, U_{1_l}, U_{2_l}, V_{1_l}] \in H_L$  according to (10).
- 

To make sure that Problem I has a solution, for Algorithm 4.1, we always assume that the quaternion matrix equations (1) is consistent. Thus the norms

$$N_1 = \left\| \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} e \right\|, \quad N_2 = \left\| \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^{\dagger} e - e \right\|,$$

$$N_3 = \left\| I - \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^{\dagger} - \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \right\|$$

are small.

**Example 4.2.** Let  $m = 8, n = 5, k = 6$ , and  $A_u = A_{u0} + A_{u1}i + A_{u2}j + A_{u3}k, B_u = B_{u0} + B_{u1}i + B_{u2}j + B_{u3}k, u = 1, 2, \dots, 6, 15, D_v = D_{v0} + D_{v1}i + D_{v2}j + D_{v3}k, E_v = E_{v0} + E_{v1}i + E_{v2}j + E_{v3}k, F_v = F_{v0} + F_{v1}i + F_{v2}j + F_{v3}k, G_v = G_{v0} + G_{v1}i + G_{v2}j + G_{v3}k, v = 1, 2$ .

Taking

$$\begin{aligned}
 A_{10} &= \begin{bmatrix} 0 & -2 & -1 & 0 & 1 \\ -1 & 1 & 0 & -2 & 0 \\ 2 & 2 & -2 & 0 & 0 \\ 1 & -2 & 1 & 2 & -1 \\ 1 & -2 & 0 & 2 & -2 \\ -2 & -2 & 1 & -2 & 2 \\ 0 & 0 & -2 & 2 & 0 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}, A_{20} = \begin{bmatrix} 1 & -2 & 2 & -1 & 0 \\ 0 & 2 & 2 & 2 & 1 \\ 1 & 0 & 1 & 1 & -2 \\ -1 & -2 & 1 & -1 & -1 \\ 0 & -2 & -2 & -1 & -2 \\ 2 & -1 & 0 & -1 & -1 \\ 2 & -2 & -2 & 0 & -1 \\ 2 & 1 & 2 & 0 & -2 \end{bmatrix}, \\
 A_{30} &= \begin{bmatrix} 1 & 1 & 2 & 0 & -1 \\ 1 & -2 & -1 & -1 & 2 \\ 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 1 & -2 & -2 & 0 & -2 \\ 1 & 2 & -1 & -2 & -1 \\ -1 & 0 & 1 & -2 & 1 \\ -1 & 2 & -1 & -1 & 0 \end{bmatrix}, A_{40} = \begin{bmatrix} 2 & -1 & -1 & 2 & -2 \\ 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 & 2 \\ 2 & 0 & 1 & -2 & 2 \\ 0 & 2 & 2 & 0 & -2 \\ 2 & 1 & -2 & -1 & 2 \\ 1 & 0 & 1 & -2 & 2 \\ -2 & 0 & -2 & -1 & -2 \end{bmatrix}, \\
 A_{150} &= \begin{bmatrix} -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -2 & -2 \\ 2 & -2 & -1 & -1 & 2 \\ 0 & 1 & -2 & 1 & 0 \\ -1 & -2 & 1 & 2 & 2 \\ -1 & 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 2 & -1 & -1 & -2 & -1 \end{bmatrix}, B_{10} = \begin{bmatrix} 2 & -1 & 0 & 0 & -2 & -1 \\ 0 & -2 & 2 & 0 & 0 & -1 \\ 1 & -2 & 1 & -1 & 2 & -1 \\ 0 & 2 & -1 & 2 & -1 & 1 \\ 2 & -1 & 2 & 2 & 0 & 0 \end{bmatrix}, \\
 B_{20} &= \begin{bmatrix} 2 & 1 & -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 2 \\ 1 & 2 & -1 & 0 & -1 & -2 \\ 0 & 2 & 0 & -2 & 1 & -2 \\ -1 & 0 & -2 & -2 & 2 & 2 \end{bmatrix}, B_{30} = \begin{bmatrix} 2 & -2 & -1 & 2 & -2 & 1 \\ 2 & -2 & -2 & 2 & 2 & -2 \\ -1 & 2 & 0 & 2 & -1 & -2 \\ 0 & 2 & -2 & 0 & 2 & -2 \\ -1 & 1 & 2 & -1 & 1 & -2 \end{bmatrix}, \\
 B_{40} &= \begin{bmatrix} -2 & -1 & 2 & 0 & 1 & -1 \\ -1 & 2 & 2 & 2 & 0 & 2 \\ -1 & 2 & 0 & 2 & 0 & -1 \\ 0 & -2 & 2 & 1 & 1 & 1 \\ 0 & 2 & 2 & -1 & 1 & 0 \end{bmatrix}, B_{150} = \begin{bmatrix} -2 & 0 & -2 & 2 & -1 & 0 \\ 1 & -2 & 1 & 2 & 0 & -2 \\ 1 & 2 & -1 & -2 & -2 & 1 \\ -1 & 2 & -2 & -1 & 2 & 2 \\ -1 & -1 & 2 & -1 & -2 & -1 \end{bmatrix}, \\
 A_{50} &= \begin{bmatrix} -1 & -2 & 1 & 2 & 0 \\ 1 & 0 & -1 & 0 & -1 \\ -2 & -2 & -1 & -1 & 0 \\ -1 & 2 & 0 & 1 & -1 \\ -2 & 2 & -2 & -1 & -1 \end{bmatrix}, A_{60} = \begin{bmatrix} 2 & 1 & 1 & -1 & -1 \\ 2 & 2 & 1 & 2 & -2 \\ 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 2 & -2 \\ -2 & 0 & 0 & 0 & 2 \end{bmatrix}, \\
 B_{50} &= \begin{bmatrix} -1 & 1 & 1 & -1 & 1 \\ -2 & -2 & 1 & 1 & 2 \\ 0 & 0 & 2 & -2 & 1 \\ -1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 2 & 2 & 1 \end{bmatrix}, B_{60} = \begin{bmatrix} -1 & 1 & 1 & -1 & -2 \\ -2 & -2 & 1 & 1 & 1 \\ 0 & 0 & 2 & -2 & 2 \\ -1 & 1 & 1 & 0 & 1 \\ -2 & -1 & 0 & -2 & -1 \end{bmatrix}, \\
 D_{10} &= \begin{bmatrix} 0 & 2 & 1 & 2 & -2 \\ 0 & 1 & -2 & 2 & 0 \\ -2 & 2 & -1 & -2 & 0 \\ 1 & 1 & 2 & -1 & -2 \\ -1 & 1 & 2 & -1 & -2 \end{bmatrix}, D_{20} = \begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & -1 & -1 & 2 & 2 \\ 2 & -2 & -1 & -2 & 1 \\ 1 & -2 & -1 & -1 & -2 \\ -1 & 1 & -2 & -2 & 1 \end{bmatrix},
 \end{aligned}$$

$$\begin{aligned}
 E_{10} &= \begin{bmatrix} 2 & 2 & 1 & 2 & 2 \\ -2 & -1 & 0 & 2 & -2 \\ 1 & -1 & 1 & 0 & 1 \\ 1 & 0 & -2 & 0 & 1 \\ -2 & -1 & -1 & 2 & 1 \end{bmatrix}, E_{20} = \begin{bmatrix} -2 & -1 & 2 & 0 & 2 \\ 2 & -1 & 1 & 2 & 2 \\ -2 & -1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 2 & -2 \\ -2 & 1 & -1 & 2 & -1 \end{bmatrix}, \\
 F_{10} &= \begin{bmatrix} 1 & -2 & 1 & -2 & -2 \\ -2 & -1 & 2 & -1 & 0 \\ 1 & -2 & -2 & 1 & -2 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 2 & 0 & 1 & -1 \end{bmatrix}, F_{20} = \begin{bmatrix} 1 & -2 & 1 & -2 & -2 \\ -2 & -1 & 2 & -1 & 0 \\ 1 & -2 & -2 & 1 & -2 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 2 & 0 & 1 & -1 \end{bmatrix}, \\
 G_{10} &= \begin{bmatrix} -1 & 0 & -1 & 2 & -2 \\ 0 & 1 & 2 & -1 & 0 \\ 1 & 2 & 2 & 0 & -2 \\ -1 & 1 & -1 & 2 & 0 \\ 0 & 2 & 1 & 1 & -1 \end{bmatrix}, G_{20} = \begin{bmatrix} 2 & 1 & -1 & 0 & -2 \\ 0 & 1 & 2 & 2 & 0 \\ -1 & 2 & 0 & 2 & -2 \\ 1 & 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 2 & 1 \end{bmatrix}.
 \end{aligned}$$

Let

$$\begin{aligned}
 X_{10} &= \begin{bmatrix} 1 & -1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}, X_{11} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}, \\
 X_{12} &= \begin{bmatrix} -1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 & 1 \end{bmatrix}, X_{13} = \begin{bmatrix} 0 & 0 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \\
 X_{20} &= \begin{bmatrix} -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & -1 & 0 & 1 \\ -1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 & 1 \end{bmatrix}, X_{21} = \begin{bmatrix} 1 & 0 & -1 & 1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 \\ 1 & -1 & -1 & 0 & -1 \end{bmatrix}, \\
 X_{22} &= \begin{bmatrix} 0 & 0 & 1 & -1 & -1 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 \end{bmatrix}, X_{23} = \begin{bmatrix} -1 & 0 & -1 & 1 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix}, \\
 Y_{10} &= \begin{bmatrix} 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}, Y_{11} = \begin{bmatrix} 1 & 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & -1 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix}, \\
 Y_{12} &= \begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 \end{bmatrix}, Y_{13} = \begin{bmatrix} 1 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix},
 \end{aligned}$$

$$\begin{aligned}
 Y_{20} &= \begin{bmatrix} -1 & -1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 & -1 \end{bmatrix}, Y_{21} = \begin{bmatrix} 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \end{bmatrix}, \\
 Y_{22} &= \begin{bmatrix} -1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 \end{bmatrix}, Y_{23} = \begin{bmatrix} -1 & -1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix}, \\
 U_{10} &= \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 \end{bmatrix}, U_{11} = \begin{bmatrix} -1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 \end{bmatrix}, \\
 U_{12} &= \begin{bmatrix} 1 & 1 & -1 & 0 & -1 \\ -1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & -1 & 0 & 1 & -1 \end{bmatrix}, U_{13} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \\
 U_{20} &= \begin{bmatrix} 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 1 & -1 & 1 \\ -1 & 0 & 0 & -1 & 0 \end{bmatrix}, U_{21} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & -1 \\ 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & -1 & 0 \end{bmatrix}, \\
 U_{22} &= \begin{bmatrix} 1 & -1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 \end{bmatrix}, U_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 \\ -1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 \end{bmatrix}, \\
 V_{10} &= \begin{bmatrix} 1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 \end{bmatrix}, V_{11} = \begin{bmatrix} 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 & -1 \end{bmatrix}, \\
 V_{12} &= \begin{bmatrix} -1 & 0 & -1 & -1 & 1 \\ -1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & -1 & -1 \end{bmatrix}, V_{13} = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 \\ 0 & -1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 \end{bmatrix},
 \end{aligned}$$

By using Matlab and Algorithm 4.1, we obtain  $N_1 = 3.4078e^{-11}$ . According to Step 4 of Algorithm 4.1, we know that the system (1.4) is consistent. We also compute that  $N_2 = 1.8907e^{-10}, N_3 = 1.5100e^{-10}$ . Moreover, since  $\text{rank} \begin{bmatrix} J_2 \\ T_3 \end{bmatrix} = 700$ , we can see that the system (1) has a unique solution with the least  $\text{norm}[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] \in H_E$ . We can compute  $\|[X_1, X_2, Y_1, Y_2, U_1, U_2, V_1] - [X_1, X_2, Y_1, Y_2, U_1, U_2, V_1]\| = 3.1508e^{-11}$ .



## 5. Conclusion

In this paper, we have investigated the system of quaternion matrix equations (1) by the Kronecker product, the Moore-Penrose inverse, and the real representation of a quaternion matrix. We derive the least-squares solution with the least norm to the system. Meanwhile, we have also obtained the solvability conditions and the formula of the general solution to (1). A numerical example has also been given to illustrate the results of this paper.

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