Filomat 36:3 (2022), 881–886
https://doi.org/10.2298/FIL2203881L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Remarks on R-Separability of Pixley–Roy Hyperspaces

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Abstract.
Let PR(X) denote the hyperspace of nonempty finite subsets of a topological space X with the Pixley–Roy topology. In this paper, motivated by [4], we introduced cf-covers and rcf-covers of X to establish the R-selective separability and the M-selective separability in PR(X) under the Pixley–Roy topology. We proved that the following statements are equivalent for a space X:

(1) PR(X) is R-separable (resp., M-separable);
(2) X satisfies $\mathsf{S}_1(\mathcal{C}_{rcf}, \mathcal{C}_{rcf})$ (*resp.*, $\mathsf{S}_{\mathsf{fin}}(\mathcal{C}_{rcf}, \mathcal{C}_{rcf})$);
(3) X is countable and each co-finite subset of X satisfies $\mathsf{S}_1(\mathcal{C}_{cf}, \mathcal{C}_{cf})$ (*resp.*, $\mathsf{S}_{\mathsf{fin}}(\mathcal{C}_{cf}, \mathcal{C}_{cf})$);
(4) X is countable and PR(X) has countable strong fan tightness (resp., PR(X) has countable fan tightness).

1. Introduction

Throughout the paper all spaces are assumed to be infinite and T_1. $\mathbb{N}$ denotes the set of natural numbers. ω is the first infinite ordinal.

Let PR(X) be the family of all nonempty finite subsets of a space X. For $A \in \mathrm{PR}(X)$ and an open set $U \subset X$, let

$$[A, U] = \{B \in \mathrm{PR}(X) : A \subset B \subset U\}.$$

The family

$$\{[A, U] : A \in \mathrm{PR}(X), U \text{ is open in } X\}$$

is a base of PR(X) for the *Pixley–Roy topology* [11] on PR(X).

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of sets of an infinite set X.

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle: for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{b_n : n \in \mathbb{N}\}$ such that $b_n \in A_n$ for each $n \in \mathbb{N}$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{\mathsf{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection principle: for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{B_n : n \in \mathbb{N}\}$ such that B_n is a finite subset of A_n for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

An open cover $\mathcal{U}$ of a space X is called an *ω-cover* of X if every finite subset of X is contained in a member of $\mathcal{U}$ and X is not a member of $\mathcal{U}$. A family ξ of subsets of a space X is called a *π-network of* X if for each open set U of X, there exists $M \in \xi$ such that $M \subset U$.

2020 *Mathematics Subject Classification*. Primary 54B20; Secondary 54D20
Keywords. Pixley–Roy topology, R-separable, M-separable, co-finite set, cf-cover, rcf-cover
Received: 17 March 2021; Revised: 11 July 2021; Accepted: 12 July 2021
Communicated by Ljubiša D.R. Kočinac
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For a space X, we write

- Ω : the collection of ω-covers of X;
- Π_ω : the collection of π-networks of X;
- $\mathcal{D}$: the collection of dense subsets of X.

A space X is *R-separable* [1, 6] if for every sequence $\{D_n : n \in \mathbb{N}\}$ of dense subsets of X, there exists $x_n \in D_n$ such that $\{x_n : n \in \mathbb{N}\}$ is dense in X. So a space X is R-separable if and only if X satisfies $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$. A space X is *M-separable* [1, 2, 14] if for every sequence $\{D_n : n \in \mathbb{N}\}$ of dense subsets of X, there exists a finite subset $F_n \subset D_n$ for each $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} F_n$ is dense in X. So a space X is M-separable if only if X satisfies $\mathsf{S}_{\mathsf{fin}}(\mathcal{D}, \mathcal{D})$.

In the theory of selection principles, ω-covers play notable roles since they were introduced by J. Gerlits, Zs. Nagy [5]. Some well-known dualities of selection principles between X and PR(X) were established in terms of ω-covers. M. Scheepers obtained that [14, Corollaries 11 and 24] for a subset X of the real line, PR(X) satisfies $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$ (resp., $\mathsf{S}_{\mathsf{fin}}(\mathcal{D}, \mathcal{D})$) if and only if X satisfies $\mathsf{S}_1(\Omega, \Omega)$ (resp., $\mathsf{S}_{\mathsf{fin}}(\Omega, \Omega)$). By results of M. Sakai [13], M. Bonanzinga, F. Cammaroto, B.A. Pansera, B. Tsaban [3, Theorem 2.21] pointed out that for a countable space X, PR(X) satisfies $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$ (resp., $\mathsf{S}_{\mathsf{fin}}(\mathcal{D}, \mathcal{D})$) if and only if all finite powers of X satisfy $\mathsf{S}_1(\Pi_\omega, \Pi_\omega)$ (resp., $\mathsf{S}_{\mathsf{fin}}(\Pi_\omega, \Pi_\omega)$).

On the other hand, from several examples [Examples 2.9, 2.14 and Remark 2.15] of this paper, it shows that ω-covers don't characterize the dual properties of selection principles between a general space X and its hyperspace PR(X). We should introduce new covers different from ω-covers of X [Examples 2.5 and 2.6] to be dual to selection principles in the hyperspace PR(X). Thus the following question arises.

Problem 1.1. *For a space X, find the collections $\mathcal{A}$ and $\mathcal{B}$ of X such that:*

PR(X) is R-separable $\Longleftrightarrow$ X satisfies S_1 and PR(X) is M-separable $\Longleftrightarrow$ X satisfies S_{fin}

G.Di Maio, Lj.D.R. Kočinac and E. Meccariello [4] investigated $\mathsf{S}_1(\mathcal{A}, \mathcal{A})$ and $\mathsf{S}_{\mathsf{fin}}(\mathcal{B}, \mathcal{B})$ in 2^X under co-compact topology F^+ and co-finite topology Z^+ when $\mathcal{A}$ is the collection of ω-covers and $\mathcal{B}$ is the collection of k-covers of X, and Lj.D.R. Kočinac [8] studied selection principles of Pixley–Roy topology. More information about Fell topology, the Vietoris topology and function spaces issues can be found in [9, 10].

In this paper, motivated by these co-subset ideas, we introduced cf-covers and rcf-covers of X to study $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$ and $\mathsf{S}_{\mathsf{fin}}(\mathcal{D}, \mathcal{D})$ in PR(X).

2. Main results

Definition 2.1. A subset U of X is called a co-finite subset of X if $0 < |X - U| < \omega$. A family $\mathcal{U}$ consisting of co-finite subsets of X is said to be a co-finite family of X.

Definition 2.2. Let X be a topological space. A co-finite family $\mathcal{U}$ of X is called a regular co-finite cover (briefly, rcf-cover) of X, if for any closed set C and any nonempty finite set F with $F \cap C = \emptyset$, there exists $U \in \mathcal{U}$ such that $C \subset U$ and $F \cap U = \emptyset$.

Definition 2.3. Let $Y \subsetneq X$. A subset U of Y is called a co-finite subset of Y if $0 \leq |Y - U| < \omega$. A family $\mathcal{U}$ consisting of co-finite subsets of Y is called a co-finite family of Y.

Definition 2.4. Let $Y \subsetneq X$. A co-finite family $\mathcal{U}$ of Y is called a co-finite cover (briefly, cf-cover) of Y, if for every $C \subset Y$ closed in X, there exists $U \in \mathcal{U}$ such that $C \subset U$.

For a space X, we write

- $\mathcal{C}_{rcf}$: the collection of rcf-covers of X;
- $\mathcal{C}_{cf}$: the collection of cf-covers of each $Y \subsetneq X$.

Obviously, we have

$$\mathcal{C}_{rcf} \subsetneq \Omega \text{ and } \mathcal{C}_{cf} \subsetneq \Omega \text{ for each } Y \subsetneq X.$$

Example 2.5. Let $\mathbb{R}$ be the set of real numbers. Put $\mathcal{U} = \{\mathbb{R} - \{\frac{1}{n}\} : n \in \mathbb{N}\}$. Then $\mathcal{U}$ is a co-finite ω-cover of $\mathbb{R}$. Let $C = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and $F = \{2\}$, then C is closed and F is nonempty finite with $F \cap C = \emptyset$. There is no $U \in \mathcal{U}$ such that $C \subset U$ and $F \cap U = \emptyset$. So $\mathcal{U}$ is not an rcf-cover of $\mathbb{R}$.

Example 2.6. Let $Y = [0, 1] \subset \mathbb{R}$ and C the Cantor set of Y closed in $\mathbb{R}$. Put $\mathcal{U} = \{Y - \{x\} : x \in C\}$. Then $\mathcal{U}$ is a co-finite ω-cover of Y. There is no $U \in \mathcal{U}$ such that $C \subset U$. So $\mathcal{U}$ is not a cf-cover of Y.

For a subset $U \subset X$ and a family $\mathcal{U}$ of subsets of X, we write:

$$U^c = X - U \text{ and } \mathcal{U}^c = \{U^c : U \in \mathcal{U}\}.$$

Theorem 2.7. *For a space X, the following are equivalent:*
(1) $\mathrm{PR}(X)$ *satisfies* $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$*;*
(2) X *satisfies* $\mathsf{S}_1(\mathcal{C}_{rcf}, \mathcal{C}_{rcf})$*.*

Proof. (1)⇒(2) Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of rcf-covers of X. For each $n \in \mathbb{N}$, $\mathcal{U}_n^c$ is a dense subset of $\mathrm{PR}(X)$. Indeed, let $[A, V]$ be a basic open subset of $\mathrm{PR}(X)$, then $X - V$ is closed and A is nonempty finite with $A \cap (X - V) = \emptyset$. There exists $U \in \mathcal{U}_n$ such that $X - V \subset U$ and $A \cap U = \emptyset$. Thus $A \subset U^c \subset V$, i.e., $U^c \in [A, V]$. Since $\mathrm{PR}(X)$ satisfies $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$, there exists $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $\overline{\{U_n^c : n \in \mathbb{N}\}} = \mathrm{PR}(X)$. We show that $\{U_n : n \in \mathbb{N}\}$ is an rcf-cover of X. In fact, let C be a closed set and let F be a nonempty finite set with $C \cap F = \emptyset$. Then $[F, X - C]$ is an open subset of $\mathrm{PR}(X)$. Thus $[F, X - C] \cap \{U_n^c : n \in \mathbb{N}\} \neq \emptyset$. Let $U_k^c \in [F, X - C]$. Then $C \subset U_k$ and $F \cap U_k = \emptyset$. So X satisfies $\mathsf{S}_1(\mathcal{C}_{rcf}, \mathcal{C}_{rcf})$.

(2)⇒(1) Let $\{\mathcal{D}_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of $\mathrm{PR}(X)$. For each $n \in \mathbb{N}$, $\mathcal{D}_n^c$ is an rcf-cover of X. Indeed, fix $n \in \mathbb{N}$. For a closed set C and a nonempty finite set F with $C \cap F = \emptyset$, then $[F, X - C]$ is an open subset of $\mathrm{PR}(X)$. There exists $D \in \mathcal{D}_n$ such that $D \in [F, X - C]$. Thus $C \subset D^c$ and $F \cap D^c = \emptyset$. By (2), there exists $D_n \in \mathcal{D}_n$ for each $n \in \mathbb{N}$ such that $\{D_n^c : n \in \mathbb{N}\}$ is an rcf-cover of X. Hence $\overline{\{D_n : n \in \mathbb{N}\}} = \mathrm{PR}(X)$. So $\mathrm{PR}(X)$ satisfies $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$. □

Theorem 2.8. *For a space X, the following are equivalent:*
(1) X satisfies $\mathsf{S}_1(\mathcal{C}_{rcf}, \mathcal{C}_{rcf})$*;*
(2) X is countable and each co-finite subset of X satisfies $\mathsf{S}_1(\mathcal{C}_{cf}, \mathcal{C}_{cf})$*.*

Proof. (1)⇒(2) Suppose that $|X| > \omega$. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{X - A : A \in [X]^{<\omega} \backslash \{\emptyset\}\}$, where $[X]^{<\omega} = \{A \subset X : A \text{ is finite}\}$. Then $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of rcf-covers of X. There exists $X - A_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $\{X - A_n : n \in \mathbb{N}\}$ is an rcf-cover of X. Take $x \in X - \bigcup_{n \in \mathbb{N}} A_n$ and put $F = \{x\}$. There is no $X - A_k \in \{X - A_n : n \in \mathbb{N}\}$ such that $F \cap (X - A_k) = \emptyset$, a contradiction.

Let Y be a co-finite subset of X and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ a sequence of cf-covers of Y. For each $n \in \mathbb{N}$, put $\mathcal{W}_n' = \{U - B : U \in \mathcal{U}_n, B \in [Y]^{<\omega}\}$. Let

$$\mathcal{W}_n = \mathcal{W}_n' \bigcup \{(Y - F^{(1)}) \cup F^{(2)} : F^{(1)} \in [Y]^{<\omega},\ F^{(2)} \in [Y^c]^{<\omega} \backslash \{\emptyset\}\} \backslash \{X\}.$$

Then $\mathcal{W}_n$ is an rcf-cover of X. Indeed, let C be a closed set and F a nonempty finite set with $F \cap C = \emptyset$. Let $F = F_1 \cup F_2$, where $F_1 \subset Y$ and $F_2 \subset Y^c$.

Case 1. If $C \subset Y$, take $U \in \mathcal{U}_n$ such that $C \subset U$ since $\mathcal{U}_n$ is a cf-cover of Y. Then $U - F_1 \in \mathcal{W}_n' \subset \mathcal{W}_n$ such that

$$C \subset U - F_1 \text{ and } F \cap (U - F_1) = \emptyset.$$

Case 2. If $C - Y \neq \emptyset$, then $Y^c - F_2 \neq \emptyset$ since $F_2 \cap C = \emptyset$. Let $F^{(1)} = F_1$, $F^{(2)} = Y^c - F_2$, then $(Y - F^{(1)}) \cup F^{(2)} \in \mathcal{W}_n$ such that

$$C \subset (Y - F^{(1)}) \cup F^{(2)} \text{ and } F \cap [(Y - F^{(1)}) \cup F^{(2)}] = \emptyset.$$

By (1), there exists $W_n \in \mathcal{W}_n$ for each $n \in \mathbb{N}$ such that $\{W_n : n \in \mathbb{N}\}$ is an rcf-cover of X. Arrange $\{W_n : n \in \mathbb{N}\}$ into

$$\{W_n' : n \in \mathbb{N}_1\} \cup \{W_n'' : n \in \mathbb{N}_2\},$$

where

$$W'_n = U_n - B_n \in \mathcal{W}'_n\ (n \in \mathbb{N}_1),\ W''_n = (Y - F_n^{(1)}) \cup F_n^{(2)}\ (n \in \mathbb{N}_2),\ \mathbb{N}_1 \cup \mathbb{N}_2 = \mathbb{N} \text{ and } \mathbb{N}_1 \cap \mathbb{N}_2 = \emptyset.$$

For each $n \in \mathbb{N}$, if $n \in \mathbb{N}_1$, take $U_n \in \mathcal{U}_n$ such that $U_n - B_n = W'_n \in \mathcal{W}'_n$; if $n \in \mathbb{N}_2$, take any $U_n \in \mathcal{U}_n$. Then $\{U_n : n \in \mathbb{N}\}$ is a cf-cover of Y. In fact, let $C \subset Y$ be closed in X, there exists $k \in \mathbb{N}_1$ such that $C \subset W'_k$ and $W'_k \cap Y^c = \emptyset$ since $\{W'_n : n \in \mathbb{N}_1\} \cup \{W''_n : n \in \mathbb{N}_2\}$ is an rcf-cover of X and $W''_n \cap Y^c \neq \emptyset$ for any $n \in \mathbb{N}_2$. Thus $C \subset W'_k = U_k - B_k \subset U_k$.

(2)⇒(1) Put $[X]^{<\omega} \backslash \{\emptyset\} = \{A_m : m \in \mathbb{N}\}$ since X is countable. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of rcf-covers of X. Rearrange $\{\mathcal{U}_n : n \in \mathbb{N}\}$ as $\{\mathcal{U}_{n,m} : n, m \in \mathbb{N}\}$. For each $n, m \in \mathbb{N}$, let

$$\mathcal{W}_{n,m} = \{U \in \mathcal{U}_{n,m} : U \cap A_m = \emptyset\}.$$

Then $\{\mathcal{W}_{n,m} : n \in \mathbb{N}\}$ is a sequence of cf-covers of $X - A_m$. There exists $U_{n,m} \in \mathcal{W}_{n,m}$ such that $\{U_{n,m} : n \in \mathbb{N}\}$ is a cf-cover of $X - A_m$. It is clear that each $U_{n,m} \in \mathcal{U}_{n,m}$. We show that $\{U_{n,m} : n, m \in \mathbb{N}\}$ is an rcf-cover of X. Let C be a closed set and F a nonempty finite set with $C \cap F = \emptyset$, there is $A_m \in [X]^{<\omega} \backslash \{\emptyset\}$ such that $F = A_m$; moreover, $C \subset X - A_m$. There exists $U_{k,m} \in \{U_{n,m} : n \in \mathbb{N}\} \subset \{U_{n,m} : n, m \in \mathbb{N}\}$ such that $C \subset U_{k,m}$. Thus $C \subset U_{k,m}$ and $A_m \cap U_{k,m} = \emptyset$. So X satisfies $\mathsf{S}_1(\mathcal{C}_{rcf}, \mathcal{C}_{rcf})$. □

Example 2.9. The following two examples show that $\mathsf{S}_1(\Omega, \Omega) \neq \mathsf{S}_1(\mathcal{C}_{rcf}, \mathcal{C}_{rcf})$.

1. Let $S_\omega = \mathbb{N}^2 \bigcup \{0\}$ be the sequential fan, where each $(n, m) \in \mathbb{N}^2$ is isolated in S_ω and a basic open neighbourhood of 0 is the form $\bigcup_{n \in \mathbb{N}} \{(n, k) : k \geq m_n\} \bigcup \{0\}$, where $m_n \in \mathbb{N}$. Obviously, S_ω satisfies $\mathsf{S}_1(\Omega, \Omega)$ since S_ω is countable. Let $Y = \mathbb{N}^2$, then Y is a co-finite subset of S_ω. Let $\mathcal{U}_n = \{Y - \{(n, m)\} : m \in \mathbb{N}\}$, then $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of cf-covers of Y. Take any $Y - \{(n, m_n)\} \in \mathcal{U}_n$ and put $C = \{(n, m_n) : n \in \mathbb{N}\}$, then $C \subset Y$ is closed in X. There is no $Y - \{(k, m_k)\} \in \{Y - \{(n, m_n)\} : n \in \mathbb{N}\}$ such that $C \subset Y - \{(k, m_k)\}$. So Y does not satisfy $\mathsf{S}_1(\mathcal{C}_{cf}, \mathcal{C}_{cf})$. By Theorem 2.8, S_ω does not satisfy $\mathsf{S}_1(\mathcal{C}_{rcf}, \mathcal{C}_{rcf})$.

2. Assuming CH, let L be the Lusin set of Theorem 2.13 in [7]. Then L satisfies $\mathsf{S}_1(\Omega, \Omega)$. By Theorem 2.8, L does not satisfy $\mathsf{S}_1(\mathcal{C}_{rcf}, \mathcal{C}_{rcf})$ since it is uncountable. So $\mathsf{S}_1(\Omega, \Omega) \neq \mathsf{S}_1(\mathcal{C}_{rcf}, \mathcal{C}_{rcf})$.

A space X is said to have *countable strong fan tightness* [13], if for each $x \in X$ and each sequence $\{A_n\}_{n \in \mathbb{N}}$ of X with $x \in \bigcap_{n \in \mathbb{N}} \overline{A_n}$, there exists $x_n \in A_n$ such that $x \in \overline{\{x_n : n \in \mathbb{N}\}}$.

Theorem 2.10. *For a space X, the following are equivalent:*
(1) PR(X) *has countable strong fan tightness;*
(2) Each co-finite subset of X satisfies $\mathsf{S}_1(\mathcal{C}_{cf}, \mathcal{C}_{cf})$*.*

Proof. (1)⇒(2) Let Y be a co-finite subset of X and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ a sequence of cf-covers of Y, then $\{\mathcal{U}_n^c : n \in \mathbb{N}\}$ is a sequence of subsets of PR(X) and $Y^c \in \overline{\mathcal{U}_n^c}$ for each $n \in \mathbb{N}$. In fact, let $[A, V]$ be a neighbourhood of Y^c, then $X - V \subset Y \subset X - A$. There exists $U \in \mathcal{U}_n$ such that $X - V \subset U \subset Y$. Thus $U^c \in [A, V]$. Since PR(X) has countable strong fan tightness, there exists $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $Y^c \in \overline{\{U_n^c : n \in \mathbb{N}\}}$. Let $C \subset Y$ be a closed set in X, then $[Y^c, X - C]$ is a neighbourhood of Y^c. Thus $[Y^c, X - C] \cap \{U_n^c : n \in \mathbb{N}\} \neq \emptyset$. Take $U_k^c \in [Y^c, X - C]$, then $C \subset U_k$. So $\{U_n : n \in \mathbb{N}\}$ is a cf-cover of Y.

(2)⇒(1) Let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be a sequence of subsets of PR(X) and $A \in \overline{\mathcal{A}_n} \cap$ PR(X) for each $n \in \mathbb{N}$. Denote $\mathcal{B}_n = \{B \in \mathcal{A}_n : A \subset B\}$, then $\mathcal{B}_n^c$ is a cf-cover of A^c. Indeed, let $C \subset A^c$ be closed in X, then $[A, X - C]$ is a neighbourhood of A. There exists $B \in \mathcal{A}_n$ such that $B \in [A, X - C]$. Thus $B \in \mathcal{B}_n$ such that $C \subset B^c \subset A^c$. By (2), there exists $B_n \in \mathcal{B}_n$ such that $\{B_n^c : n \in \mathbb{N}\}$ is a cf-cover of A^c. Hence $A \in \overline{\{B_n : n \in \mathbb{N}\}}$. So PR($X$) has countable strong fan tightness. □

From Theorems 2.7, 2.8 and 2.10, we have

Corollary 2.11. *For a space X, the following are equivalent:*
(1) PR(X) *is R-separable;*
(2) X satisfies $\mathsf{S}_1(\mathcal{C}_{rcf}, \mathcal{C}_{rcf})$*;*
(3) X is countable and each co-finite subset of X satisfies $\mathsf{S}_1(\mathcal{C}_{cf}, \mathcal{C}_{cf})$*;*
(4) X is countable and PR(X) *has countable strong fan tightness.*

Theorem 2.12. *For a space X, the following are equivalent:*
(1) PR(X) *satisfies* $S_{fin}(\mathcal{D},\mathcal{D})$*;*
(2) X satisfies $S_{fin}(C_{rcf}, C_{rcf})$*.*

Proof. (1)⇒(2) Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of rcf-covers of X. For each $n \in \mathbb{N}$, $\mathcal{U}_n^c$ is a dense subset of PR(X). By (1), there is a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\overline{\bigcup_{n\in\mathbb{N}} \mathcal{V}_n^c} = \mathrm{PR}(X)$. Let C be a closed set, F a nonempty finite set with $F \cap C = \emptyset$. Then $[F, X-C]$ is an open subset of PR(X). Thus $[F, X-C] \cap (\bigcup_{n\in\mathbb{N}} \mathcal{V}_n^c) \neq \emptyset$. Let $U^c \in [F, X-C] \cap (\bigcup_{n\in\mathbb{N}} \mathcal{V}_n^c)$. Then $U \in \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ such that $C \subset U$ and $F \cap U = \emptyset$. So $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is an rcf-cover of X.

(2)⇒(1) Let $\{\mathcal{D}_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of PR(X), then $\{\mathcal{D}_n^c : n \in \mathbb{N}\}$ is a sequence of rcf-covers of X. There is a finite subset $\mathcal{B}_n \subset \mathcal{D}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n^c$ is an rcf-cover of X. Thus $\overline{\bigcup_{n\in\mathbb{N}} \mathcal{B}_n} = \mathrm{PR}(X)$. So PR($X$) satisfies $S_{fin}(\mathcal{D},\mathcal{D})$. □

In a similar way as in Theorem 2.8, we obtain

Theorem 2.13. *For a space X, the following are equivalent:*
(1) X satisfies $S_{fin}(C_{rcf}, C_{rcf})$*;*
(2) X is countable and each co-finite subset of X satisfies $S_{fin}(C_{cf}, C_{cf})$*.*

We shall now give two examples of $S_{fin}(\Omega,\Omega) \neq S_{fin}(C_{rcf}, C_{rcf})$.

Example 2.14. $\mathbb{R}$ is σ-compact and $\beta\mathbb{N}$ is compact. By Theorem 2.2 in [7], $\mathbb{R}$ and $\beta\mathbb{N}$ satisfy $S_{fin}(\Omega,\Omega)$. By Theorem 2.13, $\mathbb{R}$ and $\beta\mathbb{N}$ do not satisfy $S_{fin}(C_{rcf}, C_{rcf})$ since they are uncountable. So $S_{fin}(\Omega,\Omega) \neq S_{fin}(C_{rcf}, C_{rcf})$.

Remark 2.15. From Theorems 2.7, 2.8, 2.12 and 2.13, it implies that "X has property $S_1(\Omega,\Omega)$ (resp., $S_{fin}(\Omega,\Omega)$)" and "PR(X) has property $S_1(\mathcal{D},\mathcal{D})$ (resp., $S_{fin}(\mathcal{D},\mathcal{D})$)" are not equivalent. So (2) and (9) of Corollaries 11 and 24 in [14] are not equivalent for general spaces.

A space X is said to have *countable fan tightness* if for each sequence $\{A_n\}_{n\in\mathbb{N}}$ of subsets of X and $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, there exists a finite subset $B_n \subset A_n$ such that $x \in \overline{\bigcup_{n\in\mathbb{N}} B_n}$.

Theorem 2.16. *For a space X, the following are equivalent:*
(1) PR(X) *has countable fan tightness;*
(2) Each co-finite subset of X satisfies $S_{fin}(C_{cf}, C_{cf})$*.*

Proof. (1)⇒(2) Let Y be a co-finite subset of X and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ a sequence of cf-covers of Y. Then $\{\mathcal{U}_n^c : n \in \mathbb{N}\}$ is a sequence of subsets of PR(X) and $Y^c \in \overline{\mathcal{U}_n^c}$ for each $n \in \mathbb{N}$. Since PR(X) has countable fan tightness, there is a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ such that $Y^c \in \overline{\bigcup_{n\in\mathbb{N}} \mathcal{V}_n^c}$. Let $C \subset Y$ be closed in X, then $[Y^c, X-C]$ is a neighbourhood of Y^c. Thus $[Y^c, X-C] \cap (\bigcup_{n\in\mathbb{N}} \mathcal{V}_n^c) \neq \emptyset$. Let $U^c \in [Y^c, X-C] \cap (\bigcup_{n\in\mathbb{N}} \mathcal{V}_n^c)$, then $U \in \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ such that $C \subset U$. So $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cf-cover of Y.

(2)⇒(1) Let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be a sequence of subsets of PR(X) and $A \in \overline{\mathcal{A}_n} \cap \mathrm{PR}(X)$ for all $n \in \mathbb{N}$. Denote $\mathcal{B}_n = \{B \in \mathcal{A}_n : A \subset B\}$, then $\{\mathcal{B}_n^c : n \in \mathbb{N}\}$ is a sequence of cf-covers of A^c. By (2), there exists a finite subset $C_n \subset \mathcal{B}_n \subset \mathcal{A}_n$ such that $\bigcup_{n\in\mathbb{N}} C_n^c$ is a cf-cover of A^c. Thus $A \in \overline{\bigcup_{n\in\mathbb{N}} C_n}$. So PR($X$) has countable fan tightness. □

Combining Theorems 2.12, 2.13 and 2.16, we have

Corollary 2.17. *For a space X, the following are equivalent:*
(1) PR(X) *is M-separable;*
(2) X satisfies $S_{fin}(C_{rcf}, C_{rcf})$*;*
(3) X is countable and each co-finite subset of X satisfies $S_{fin}(C_{cf}, C_{cf})$*;*
(4) X is countable and PR(X) *has countable fan tightness.*

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