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Some Results in Cauchy-Stieltjes Kernel Families

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Abstract. In this paper we present two different results in the theory of Cauchy-Stieltjes Kernel (CSK) families. We firstly provide the construction of free Sheffer systems with the theory of CSK families. We associate a free additive convolution semigroup of probability measures to any free Sheffer systems and we prove that this is the only one that leads to an orthogonal free Sheffer systems. We also show that the orthogonality of free Sheffer systems occurs if and only if the associated free additive convolution semigroup of probability measures generates CSK families with quadratic variance function. Secondly, we are interested in the study of boolean additive convolution. Based on the criteria of convergence for a sequence of variance functions we give an approximation of elements of the CSK family generated by the boolean Gaussian distribution and an approximation of elements of the CSK family generated by the boolean Poisson distribution.

1. Introduction

The theory of natural exponential families has received a great deal of attention in the classical probability and statistical literature and it remains a very interesting topic. This is in particular due to the fact that the most common distribution belong either to natural exponential families or to general exponential families. It is well known that the definition of a real natural exponential families is based on the kernel $(\theta, x) \longmapsto \exp(\theta x)$. In the framework of free probability theory and in analogy with the theory of natural exponential families, a theory of Cauchy-Stieltjes Kernel (CSK) families has been recently introduced, it is based on the Cauchy-Stieltjes kernel $1/(1-\theta x)$. The study of CSK families is initiated in [3] for compactly supported probability measures ν and in [4] the authors have extended the results established in [3] to allow probability measures with unbounded support. In the present paper we continue the study of CSK families. In section 2, we provide the construction of free Sheffer system with the theory of CSK families. We associate a free additive convolution semigroup of probability measures to any free Sheffer system and we prove that this is the only one that leads to an orthogonal free Sheffer system. We also show that the orthogonality of free Sheffer system occurs if and only if the associated free additive convolution semigroup of probability measures generates CSK families with quadratic variance function. In section 3, we approximate elements of the CSK family generated by the centered boolean Gaussian distribution and elements of the CSK family generated by the boolean Poisson distribution. In the rest of this section, we recall a few features about

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CSK families. Our notations are the ones used in [5], [8],[10], [12] and [13].
Let ν be a non-degenerate probability measure with support bounded from above. Then

$$M_\nu(\theta) = \int \frac{1}{1-\theta x}\nu(dx) \tag{1}$$

is defined for all $\theta \in [0, \theta_+)$ with $1/\theta_+ = \max\{0, \sup \text{supp}(\nu)\}$.
For $\theta \in [0, \theta_+)$, we set

$$P_{(\theta,\nu)}(dx) = \frac{1}{M_\nu(\theta)(1-\theta x)}\nu(dx).$$

The set

$$\mathcal{K}_+(\nu) = \{P_{(\theta,\nu)}(dx); \theta \in (0, \theta_+)\}$$

is called the one-sided CSK family generated by ν.
Let $k_\nu(\theta) = \int x P_{(\theta,\nu)}(dx)$ denote the mean of $P_{(\theta,\nu)}$. According to [4, page 579-580] the map $\theta \mapsto k_\nu(\theta)$ is strictly increasing on $(0, \theta_+)$, it is given by the formula

$$k_\nu(\theta) = \frac{M_\nu(\theta) - 1}{\theta M_\nu(\theta)}. \tag{2}$$

The image of $(0, \theta_+)$ by k_ν is called the (one sided) domain of the mean of the family $\mathcal{K}_+(\nu)$, it is denoted $(m_0(\nu), m_+(\nu))$. This leads to a parametrization of the family $\mathcal{K}_+(\nu)$ by the mean. In fact, denoting by ψ_ν the reciprocal of k_ν, and writing for $m \in (m_0(\nu), m_+(\nu))$, $Q_{(m,\nu)}(dx) = P_{(\psi_\nu(m),\nu)}(dx)$, we have that

$$\mathcal{K}_+(\nu) = \{Q_{(m,\nu)}(dx); m \in (m_0(\nu), m_+(\nu))\}.$$

Now let $B = B(\nu) = \max\{0, \sup \text{supp}(\nu)\} = 1/\theta_+ \in [0, \infty)$. It is shown in [4] that the bounds $m_0(\nu)$ and $m_+(\nu)$ of the one-sided domain of means $(m_0(\nu), m_+(\nu))$ are given by

$$m_0(\nu) = \lim_{\theta \to 0^+} k_\nu(\theta) \qquad \text{and} \qquad m_+(\nu) = B - \lim_{z \to B^+} \frac{1}{G_\nu(z)},$$

where $G_\nu(z)$ is the Cauchy transform of ν given by

$$G_\nu(z) = \int \frac{1}{z-x}\nu(dx). \tag{3}$$

It is worth mentioning here that one may define the one-sided CSK family for a measure ν with support bounded from below. This family is usually denoted $\mathcal{K}_-(\nu)$ and parameterized by θ such that $\theta_- < \theta < 0$, where θ_- is either $1/A(\nu)$ or $-\infty$ with $A = A(\nu) = \min\{0, \inf supp(\nu)\}$. The domain of the means for $\mathcal{K}_-(\nu)$ is the interval $(m_-(\nu), m_0(\nu))$ with $m_-(\nu) = A - 1/G_\nu(A)$.
If ν has compact support, the natural domain for the parameter θ of the two-sided CSK family $\mathcal{K}(\nu) = \mathcal{K}_+(\nu) \cup \mathcal{K}_-(\nu) \cup \{\nu\}$ is $\theta_- < \theta < \theta_+$.
We come now to the notions of variance and pseudo-variance functions. The variance function

$$m \mapsto V_\nu(m) = \int (x-m)^2 Q_{(m,\nu)}(dx) \tag{4}$$

is a fundamental concept both in the theory of natural exponential families and in the theory of CSK families as presented in [3]. Unfortunately, if ν hasn't a first moment which is for example the case for a 1/2-stable law, all the distributions in the CSK family generated by ν have infinite variance. This fact has led the authors in [4] to introduce a notion of pseudo-variance function defined by

$$\mathbb{V}_\nu(m) = m\left(\frac{1}{\psi_\nu(m)} - m\right), \tag{5}$$

If $m_0(\nu) = \int x d\nu$ is finite, then (see [4]) the pseudo-variance function is related to the variance function by

$$\frac{\mathbb{V}_\nu(m)}{m} = \frac{V_\nu(m)}{m - m_0}. \tag{6}$$

In particular, $\mathbb{V}_\nu = V_\nu$ when $m_0(\nu) = 0$.
The generating measure ν is uniquely determined by the pseudo-variance function $\mathbb{V}_\nu$. In fact, if we set

$$z = z(m) = m + \frac{\mathbb{V}_\nu(m)}{m}, \tag{7}$$

then the Cauchy transform satisfies

$$G_\nu(z) = \frac{m}{\mathbb{V}_\nu(m)}. \tag{8}$$

Also the distribution $Q_{(m,\nu)}(dx)$ may be written as $Q_{(m,\nu)}(dx) = f_\nu(x, m)\nu(dx)$ with

$$f_\nu(x, m) := \begin{cases} \frac{\mathbb{V}_\nu(m)}{\mathbb{V}_\nu(m)+m(m-x)}, & m \neq 0 \quad ; \\ 1, & m = 0, \ \mathbb{V}_\nu(0) \neq 0 \quad ; \\ \frac{\mathbb{V}'_\nu(0)}{\mathbb{V}'_\nu(0)-x}, & m = 0, \ \mathbb{V}_\nu(0) = 0 \quad . \end{cases} \tag{9}$$

To close this section we recall the effect on a CSK family of applying an affine transformation to the generating measure. Consider the affine transformation $\varphi : x \longmapsto (x - \lambda)/\beta$, where $\beta \neq 0$ and $\lambda \in \mathbb{R}$ and let $\varphi(\nu)$ be the image of ν by φ. In other words, if X is a random variable with law ν, then $\varphi(\nu)$ is the law of $(X - \lambda)/\beta$, or $\varphi(\nu) = D_{1/\beta}(\nu \boxplus \delta_{-\lambda})$, where $D_r(\mu)$ denotes the dilation of measure μ by a number $r \neq 0$, that is $D_r(\mu)(U) = \mu(U/r)$. The point m_0 is transformed to $(m_0 - \lambda)/\beta$. In particular, if $\beta < 0$ the support of the measure $\varphi(\nu)$ is bounded from below so that it generates the left-sided family $\mathcal{K}_-(\varphi(\nu))$. For m close enough to $(m_0 - \lambda)/\beta$, the pseudo-variance function is

$$\mathbb{V}_{\varphi(\nu)}(m) = \frac{m}{\beta(m\beta + \lambda)}\mathbb{V}_\nu(\beta m + \lambda). \tag{10}$$

In particular, if the variance function exists, then $V_{\varphi(\nu)}(m) = \frac{1}{\beta^2}V_\nu(\beta m + \lambda)$.
Note that using the special case where φ is the reflection $\varphi(x) = -x$, on can transform a right-sided CSK family to a left-sided family. If ν has support bounded from above and its right-sided CSK family $\mathcal{K}_+(\nu)$ has domain of means (m_0, m_+) and pseudo-variance function $\mathbb{V}_\nu(m)$, then $\varphi(\nu)$ generates the left-sided CSK family $\mathcal{K}_-(\varphi(\nu))$ with domain of means $(-m_+, -m_0)$ and pseudo-variance function $\mathbb{V}_{\varphi(\nu)}(m) = \mathbb{V}_\nu(-m)$.

2. Free Sheffer systems

In this section, we associate a free additive convolution semigroup of probability measures to any free Sheffer systems. We also characterize free Sheffer system based on orthogonality condition. We first recall what we call free additive convolution semigroup of probability measures. Let ν be a probability measure on $\mathbb{R}$, and consider its Stieltjes transform G_ν given by (3). It was proved in [7] that the inverse G_ν^{-1} of G_ν is defined on a domain of the form

$$\{z \in \mathbb{C} \ : \ \Re z > c, \ |z| < M\},$$

where c and M are two positive constants.
The $\mathcal{R}$-transform of ν is defined in the same domain by

$$\mathcal{R}_\nu(z) = G_\nu^{-1}(z) - 1/z.$$

The free additive convolution $\mu \boxplus \nu$ of the probability measures μ, ν on Borel sets of the real line is a uniquely defined probability measure $\mu \boxplus \nu$ such that

$$\mathcal{R}_{\mu\boxplus\nu}(z) = \mathcal{R}_\mu(z) + \mathcal{R}_\nu(z)$$

for all z in an appropriate domain (see [7, Sect. 5] for details).
A probability measure ν on the real line is $\boxplus-$infinitely divisible, if for each $n \in \mathbb{N}$, there exists probability measure μ_n on the real line such that

$$\nu = \underbrace{\mu_n \boxplus ..... \boxplus \mu_n}_{n \text{ times}}.$$

For $\alpha > 0$, we denote by $= \nu^{\boxplus\alpha}$ the free convolution power of a probability measure ν, which is given by $\mathcal{R}_{\nu^{\boxplus\alpha}}(z) = \alpha\mathcal{R}_\nu(z)$. Convolution power of order $\alpha \in [1, \infty)$ exists by [2, Sect. 2]. Convolution power of order $\alpha > 0$ exists for $\boxplus$-infinitely divisible laws.

In other words, if ν is $\boxplus-$infinitely divisible distribution, this means that there exists a free additive convolution semigroup $(\nu^{\boxplus t})_{t\geq 0}$ of probability measures, characterized by the properties that $\nu^{\boxplus 0} = \delta_0$, $\nu^{\boxplus t} \boxplus \nu^{\boxplus s} = \nu^{\boxplus t+s}$, $\nu^{\boxplus 1} = \nu$ and

$$\mathcal{R}_{\nu^{\boxplus t}}(z) = t\mathcal{R}_\nu(z). \tag{11}$$

Now, we specify what we call free Sheffer system (see [1] for more details).

Definition 2.1. *A polynomial set $\{T_n(x, t) : n \in \mathbb{N}, t > 0\}$ is called a semigroup-free Sheffer systems if it is defined by a generating function of the form*

$$H(x, t, z) = \sum_{n=0}^{+\infty} T_n(x, t)z^n = \frac{1}{1 + tu(z)\mathcal{R}_\nu(u(z)) - xu(z)}, \tag{12}$$

where $z \longmapsto u(z)$ can be expanded in a formal power series such that $u(0) = 0$, $u'(0) = 1$ and ν is an $\boxplus$-infinitely divisible probability measure on $\mathbb{R}$.

One sees via (11) that a free Sheffer systems is connected to a free additive convolution semigroup of probability measures $(\nu^{\boxplus t})_{t>0}$. We are interested in finding the correspondence between such a free Sheffer systems and the families of associated probability distributions.
Before presenting our results concerning free Sheffer systems, we introduce the following result given by [4, Proposition 3.8] which lists properties of the $\mathcal{R}$-transform that we need.

Proposition 2.2. *Suppose $\mathbb{V}_\nu$ is a pseudo-variance function of the CSK family $\mathcal{K}_+(\nu)$ generated by a probability measure ν with $b = \sup\ supp(\nu) < \infty$. Then*

(i) $\mathcal{R}_\nu$ *is strictly increasing on* $(0, G_\nu(b))$.

(ii) For $m \in (m_0, m_+)$

$$\mathcal{R}_\nu\left(\frac{m}{\mathbb{V}_\nu(m)}\right) = m. \tag{13}$$

(iii) $\lim_{z\searrow 0} \mathcal{R}_\nu(z) = m_0 \geq -\infty$.

(iv) $\lim_{z\searrow 0} z\mathcal{R}_\nu(z) = 0$. *(The only new contribution is the case $m_0 = -\infty$).*

Remark 2.3. *According to [4, Corollary 3.9], the function $m \longmapsto m/\mathbb{V}_\nu(m)$ is strictly increasing and smooth function on (m_0, m_+). Furthermore, $m/\mathbb{V}_\nu(m) \longrightarrow 0$ as $m \longrightarrow m_0$.*

2.1. Construction of free Sheffer systems

We provide the construction of free Sheffer systems with the theory of CSK families. We associate a free additive convolution semigroup of probability measures to any free Sheffer system. Let $\{\nu^{\boxplus t} : t \geq 0\}$ be a family of centered compactly supported probability measures associated to a free Lévy process $(X_t)_{t\geq 0}$. To such a family of probability measures we associate a family of polynomials from the Taylor expansion of the function $f_{\nu^{\boxplus t}}(x, m)$ of the form (9). The following result provides the construction of free Sheffer systems from a free additive convolution semigroup of probability measures. We will see that this is the unique one leading to an orthogonal free Sheffer systems.

Theorem 2.4. *Let $n \in \mathbb{N}$ and let $\{\nu^{\boxplus t} : t > 0\}$ be a free additive convolution semigroup of non degenerate centered probability measures with compact support generating a type of CSK family $\mathcal{K}(\nu)$. Define*

$$P_n(x, t) = \frac{1}{n!} \frac{\partial^n}{\partial m^n} f_{\nu^{\boxplus t}}(x, m)\Big|_{m=0}. \tag{14}$$

Then $\{T_n(x, t) = t^n P_n(x, t) : n \in \mathbb{N},\ t > 0\}$ form a semigroup-free Sheffer systems.

Proof. Using the same reasoning in [11, Proposition 2.1], one see that $P_n(x, t)$ given by (14) are polynomials in x of degree n. Hence, it is the same for $T_n(x, t)$. From [11, Lemma 2.2], there exists $r > 0$ such that for all $z \in]-r, r[$,

$$\sum_{n=0}^{+\infty} T_n(x, t) z^n = f_{\nu^{\boxplus t}}(x, tz) = \frac{\mathbb{V}_{\nu^{\boxplus t}}(tz)}{\mathbb{V}_{\nu^{\boxplus t}}(tz) + tz(tz - x)}$$

On the other hand the effect on the pseudo-variance function of the free additive convolution power is given by [4, Proposition 3.10]. More precisely, it was shown that for $\alpha > 0$ and for m close enough to $m_0(\nu^{\boxplus \alpha}) = \alpha m_0(\nu)$,

$$\mathbb{V}_{\nu^{\boxplus \alpha}}(m) = \alpha \mathbb{V}_\nu(m/\alpha). \tag{15}$$

This implies that

$$\sum_{n=0}^{+\infty} T_n(x, t) z^n = \frac{t\mathbb{V}_\nu(z)}{t\mathbb{V}_\nu(z) + tz(tz - x)}.$$

That is

$$\sum_{n=0}^{+\infty} T_n(x, t) z^n = \frac{1}{1 + tz^2/\mathbb{V}_\nu(z) - xz/\mathbb{V}_\nu(z)}.$$

Hence equation (12) occurs with $u(z) = z/\mathbb{V}_\nu(z)$ and formula (13). □

Next, we link all semigroup-free Sheffer systems to a unique free additive convolution semigroup of probability measures following the classical orthogonality.

Theorem 2.5. *Let $\{S_n(x, t) : n \in \mathbb{N},\ t > 0\}$ be a semigroup-free Sheffer system. Then there exists a unique free additive convolution semigroup of probability measures $\{\nu^{\boxplus t},\ t > 0\}$ such that $\{S_n(x, t) : n \in \mathbb{N}\}$ is $\nu^{\boxplus t}$–orthogonal.*

Proof. By taking generating function in

$$\int_{\mathbb{R}} S_n(x, t) S_p(x, t) \nu^{\boxplus t}(dx) = \delta_{np} c_n,$$

($\delta_{np} = 1$ when $n = p$ and 0 for $n \neq p$) and setting $n = 0$ we obtain

$$\int_{\mathbb{R}} \frac{1}{1 + tu(z)\mathcal{R}_\nu(u(z)) - xu(z)} \nu^{\boxplus t}(dx) = c_0 = 1.$$

This implies that

$$G_{\nu^{\boxplus t}}(1/u(z) + t\mathcal{R}_\nu(u(z))) = u(z).$$

That is

$$(1/u(z) + t\mathcal{R}_\nu(u(z)))G_{\nu^{\boxplus t}}(1/u(z) + t\mathcal{R}_\nu(u(z))) = 1 + tu(z)\mathcal{R}_\nu(u(z)). \tag{16}$$

Put $y = 1/u(z) + t\mathcal{R}_\nu(u(z))$. We have that $y \longrightarrow \infty$, when $z \longrightarrow 0$. This is due to the fact that $u(0) = 0$ and $\lim_{z\searrow 0}\mathcal{R}_\nu(z) = m_0(\nu) = 0$. This together with (16) implies that $\lim_{y\longrightarrow\infty} yG_{\nu^{\boxplus t}}(y) = 1$. According to [7, Proposition 5.1], $G_{\nu^{\boxplus t}}(.)$ is indeed a Cauchy-Stieltjes transform. It characterizes of unique manner each $\nu^{\boxplus t}$ of $\{\nu^{\boxplus t} : t > 0\}$. □

2.2. *Characterization of free Sheffer systems*

We prove that the orthogonality of the free Sheffer polynomials occurs if and only if the corresponding semigroup of probability measures generates quadratic CSK families. We restrict our attention to $\boxplus-$infinitely divisible probability measures. The following theorem shows an intrinsic construction of the semigroup-free Sheffer systems.

Theorem 2.6. *Consider $\{\nu^{\boxplus t} : t > 0\}$ a free additive convolution semigroup of non degenerate centered probability measures with compact support. For all $t > 0$ let a polynomial sequence $\{S_n(x,t) : n \in \mathbb{N}\}$ be $\nu^{\boxplus t}-$orthogonal. Then the following statements are equivalents:*

(i) *$\{S_n(x,t) : n \in \mathbb{N}\}$ form a semigroup-free Sheffer systems.*

(ii) *There exists $\gamma \in \mathbb{R}\backslash\{0\}$ such that $S_n(x,t) = (\gamma t)^n P_n(x,t)$ for all $(n,t) \in \mathbb{N}\times(0,+\infty)$, where polynomials $P_n(x,t)$ are given by* (14).

Proof. $(i) \Leftarrow (ii)$ According to Theorem 2.4 the family $\{t^n P_n(x,t) : n \in \mathbb{N}, t > 0\}$ form a semigroup-free Sheffer systems. We have that, for $|z|$ small enough,

$$\sum_{n=0}^{+\infty} S_n(x,t)z^n = \sum_{n=0}^{+\infty} t^n P_n(x,t)(\gamma z)^n = \frac{1}{1 + tu(\gamma z)\mathcal{R}_\nu(u(\gamma z)) - xu(\gamma z)}.$$

For $z' = \gamma z$, we get the desired result in (i).

$(i) \Rightarrow (ii)$ The desired result is obtained in the same spirit as [11, Theorem 3.5]. From [11, Lemma 2.2], there exists $r > 0$ such that for all $z \in]-r, r[$

$$\int\left(\sum_{n=0}^{+\infty} S_n(x,t)z^n\right)\nu^{\boxplus t}(dx) = \sum_{n=0}^{+\infty} z^n \int S_n(x,t)\nu^{\boxplus t}(dx) = \int S_0(x,t)\nu^{\boxplus t}(dx) = 1.$$

In addition, by writing the generating function of $S_n(x,t)$ as in (12), we have

$$\int\left(\sum_{n\in\mathbb{N}} S_n(x,t)z^n\right)\nu^{\boxplus t}(dx) = \int \frac{1}{1 + u(z)\mathcal{R}_{\nu^{\boxplus t}}(u(z)) - xu(z)}\nu^{\boxplus t}(dx) = \frac{1}{u(z)}G_{\nu^{\boxplus t}}(\mathcal{R}_{\nu^{\boxplus t}}(u(z)) + 1/u(z)).$$

Hence,

$$u(z) = G_{\nu^{\boxplus t}}(\mathcal{R}_{\nu^{\boxplus t}}(u(z)) + 1/u(z)). \tag{17}$$

Proceeding in a similar manner, we have

$$\int\left(\sum_{n\in\mathbb{N}} S_n(x,t)S_1(x,t)z^n\right)\nu^{\boxplus t}(dx) = \left(\int (S_1(x,t))^2\nu^{\boxplus t}(dx)\right)z. \tag{18}$$

It is well known (see [1]) that if $\{S_n(x,t):\ n\in\mathbb{N}\}$ is a free Sheffer system then $S_n(x,t)$ is a polynomial of degree n in x and of degree n in t. Then $\alpha\in\mathbb{R}\backslash\{0\}$ and $\beta\in\mathbb{R}$ exists such that

$$S_1(x,t)=\alpha tx+\beta. \tag{19}$$

Since

$$\int S_1(x,t)S_0(x,t)\nu^{\boxplus t}(dx)=\int S_1(x,t)\nu^{\boxplus t}(dx)=0$$

and $\nu^{\boxplus t}$ is a centered probability measure, then we obtain $\beta=0$. In addition

$$\int (S_1(x,t))^2\nu^{\boxplus t}(dx)=\int(\alpha t)^2x^2\nu^{\boxplus t}(dx)=\alpha^2t^2V_{\nu^{\boxplus t}}(0).$$

Furthermore, by using (17)-(19), we obtain

$$\begin{aligned}
\left(\int (S_1(x,t))^2\nu^{\boxplus t}(dx)\right)z &= \int\frac{1}{1+u(z)\mathcal{R}_{\nu^{\boxplus t}}(u(z))-xu(z)}S_1(x,t)\nu^{\boxplus t}(dx)\\
&= \int\frac{\alpha tx}{G_{\nu^{\boxplus t}}(\mathcal{R}_{\nu^{\boxplus t}}(u(z))+1/u(z))[\mathcal{R}_{\nu^{\boxplus t}}(u(z))+1/u(z)-x]}\nu^{\boxplus t}(dx)\\
&= \int\frac{\alpha tx}{M_{\nu^{\boxplus t}}\left(\frac{1}{\mathcal{R}_{\nu^{\boxplus t}}(u(z))+1/u(z)}\right)\left[1-\frac{x}{\mathcal{R}_{\nu^{\boxplus t}}(u(z))+1/u(z)}\right]}\nu^{\boxplus t}(dx)\\
&= \int\alpha tx P_{\left(\frac{1}{\mathcal{R}_{\nu^{\boxplus t}}(u(z))+1/u(z)},\nu^{\boxplus t}\right)}(dx)\\
&= \alpha t k_{\nu^{\boxplus t}}\left(\frac{1}{\mathcal{R}_{\nu^{\boxplus t}}(u(z))+1/u(z)}\right).
\end{aligned}$$

We deduce that

$$\alpha^2t^2V_{\nu^{\boxplus t}}(0)z=\alpha t k_{\nu^{\boxplus t}}\left(\frac{1}{\mathcal{R}_{\nu^{\boxplus t}}(u(z))+1/u(z)}\right).$$

Therefore, with $\gamma=\alpha V_{\nu^{\boxplus t}}(0)$, we have

$$\psi_{\nu^{\boxplus t}}(\gamma tz)=\frac{1}{\mathcal{R}_{\nu^{\boxplus t}}(u(z))+1/u(z)}.$$

Finally, we obtain

$$\sum_{n\in\mathbb{N}}S_n(x,t)z^n=\frac{1}{G_{\nu^{\boxplus t}}(1/\psi_{\nu^{\boxplus t}}(\gamma tz))[1/\psi_{\nu^{\boxplus t}}(\gamma tz)-x]}=f_{\nu^{\boxplus t}}(x,\gamma tz).$$

This ends the proof of (*ii*).
□

Now, we provide the CSK -version of Pommeret's results given in [16]. We give the characterization of free Sheffer systems by classical orthogonality condition.

Theorem 2.7. *Let ν be a non degenerate centered compactly supported and $\boxplus$-infinitely divisible probability measure. Let $\{S_n(x,t):\ n\in\mathbb{N},\ t>0\}$ be a semigroup-free Sheffer systems associated to ν. Then the $\nu^{\boxplus t}$−orthogonality of the semigroup-free Sheffer system occurs if and only if $\mathcal{K}(\nu)$ is a quadratic CSK family.*

Proof. Assume that $(S_n(x,t))_{n\in\mathbb{N}}$ are $\nu^{\boxplus t}$−orthogonal. From Theorem 2.6, there exists $\gamma\in\mathbb{R}\backslash\{0\}$ such that $S_n(x,t)=(\gamma t)^nP_n(x,t)$ for all $(n,t)\in\mathbb{N}\times(0,+\infty)$, where polynomials $P_n(x,t)$ are given by (14). To show that the CSK family $\mathcal{K}(\nu)$ is quadratic, we may fix $t=1$. The remainder is easily obtained from [11, Theorem 3.2].

Conversely, suppose that CSK family $\mathcal{K}(\nu)$ is quadratic. From the formula of variance functions, under power of free additive convolution, given by (15), it is easy to see that if the variance function $\mathbb{V}_\nu(.)$ is quadratic, the variance function $\mathbb{V}_{\nu^{\boxplus t}}(.)$ is also quadratic. Thus we content to show that polynomials $(S_n(x,1))_{n\in\mathbb{N}}$ are $\nu-$orthogonal. From the fact that $\{S_n(x,t):\ n\in\mathbb{N},\ t>0\}$ is a semigroup-free Seffer systems associated to ν it is easy to see (according to Theorem 2.5) that $u(z)=z/\mathbb{V}_\nu(z)$, and

$$\sum_{n=0}^{+\infty} S_n(x,1)z^n = \frac{1}{1+u(z)\mathcal{R}_\nu(u(z))-xu(z)} = f_\nu(x,z). \tag{20}$$

The $\nu-$orthogonality of polynomials $(S_n(x,1))_{n\in\mathbb{N}}$ follows from [11, Theorem 3.2].
□

It is worth mentioning that the authors in [6] describe several operations that allow us to construct additional variance functions from known ones. They construct a class of examples which exhausts all cubic variance functions, and provide examples of polynomial variance functions of arbitrary degree. They also relate CSK families with polynomial variance functions to generalized orthogonality. There is a substantial literature on generalized orthogonality and finite-step recursions for polynomials. Reference [6] introduce the following generalized orthogonality condition.

Definition 2.8. *Fix $d\in\mathbb{N}$ and a probability measure ν with moments of all orders. We say that polynomials $\{P_n\}$ are $(\nu;d)$-orthogonal if $\int P_n(x)\nu(dx)=0$ for all $n\geq 1$, and*

$$\int P_n(x)P_k(x)\nu(dx)=0 \text{ for all } n\geq 2+(k-1)d, \quad k=1,2\ldots.$$

It is clear that for measures with infinite support, $(\nu;1)$-orthogonality is just the standard orthogonality. For $d=2$, we recover [9, Definition 3.1]. The concept of d-orthogonality introduced in [18] is different as even for $d=1$ it has no positivity requirements for the functional/measure. When $d>2$, condition of pseudo-orthogonality in [14, 15] is also different.

Basing on the notion of $(\nu;d)$-orthogonality for a sequence of polynomials, authors in [6] gives the generalization of [11, Theorem 3.2] to $d>1$, see [6, Theorem 3.2] for more details.

Now, we state the characterization of free Sheffer systems by generalized orthogonality. We omit the proof because it similar to the proof of Theorem 2.7.

Theorem 2.9. *Let ν be a non degenerate centered compactly supported and $\boxplus$-infinitely divisible probability measure. Let $\{S_n(x,t):\ n\in\mathbb{N},\ t>0\}$ be a semigroup-free Sheffer systems associated to ν. Then the semigroup-free Sheffer systems $\{S_n(x,t)\}$ is $(\nu^{\boxplus t};d)-$orthogonal if and only if the variance function of the CSK family $\mathcal{K}(\nu)$ is a polynomial function in the mean m of degree at most $d+1$.*

3. Approximations in CSK families

Our results in this section are related to boolean additive convolution of probability measures. We give an approximation of elements of the CSK family generated by the centered boolean Gaussian distribution. We also approximate elements of the CSK family generated by the boolean Poisson distribution. The calculation of the limiting distributions is based on the corresponding variance functions. This is due to a technical result using convergence of a sequence of variance functions. For completeness, we state the following result:

Proposition 3.1. *[3, Proposition 4.2] Suppose V_{ν_n} is a family of analytic functions which are variance functions of a sequence of CSK families $\{\mathcal{K}(\nu_n):\ n\geq 1\}$. If $V_{\nu_n}\longrightarrow V$ uniformly in a (complex) neighborhood of $m_0\in\mathbb{R}$ and $V(m_0)>0$, then there is $\delta>0$ such that $V=V_\nu$ is a variance function of a CSK family $\mathcal{K}(\nu)$, generated by a probability measure ν parameterized by the mean $m\in(m_0-\delta,m_0+\delta)$. Moreover, if a sequence of measures $\mu_n\in\mathcal{K}(\nu_n)$ such that $m_1=\int x\mu_n(dx)\in(m_0-\delta,m_0+\delta)$ does not depends on n, then $\mu_n\xrightarrow{n\to+\infty}\mu$ in distribution, where $\mu\in\mathcal{K}(\nu)$ has the same mean $\int x\mu(dx)=m_1$.*

3.1. Boolean additive convolution.

For a probability measure ν on $\mathbb{R}$, its Cauchy transform G_ν is defined by (3). The boolean additive convolution is determined by the K-transform K_ν of ν which is defined as

$$K_\nu(z) = z - \frac{1}{G_\nu(z)}, \quad \text{for } z \in \mathbb{C}^+. \tag{21}$$

The function K_ν is usually called self energy and it represent the analytic backbone of boolean additive convolution. For two probabilities measures μ and ν, the boolean additive convolution $\mu \uplus \nu$ is determined by

$$K_{\mu\uplus\nu}(z) = K_\mu(z) + K_\nu(z), \quad \text{for } z \in \mathbb{C}^+, \tag{22}$$

and $\mu \uplus \nu$ is again a probability measure.
A probability measure ν on $\mathbb{R}$ is infinitely divisible in the boolean sense if for each $n \in \mathbb{N}$, there exists a probability measure ν_n on $\mathbb{R}$ such that

$$\nu = \underbrace{\nu_n \uplus \ldots\ldots \uplus \nu_n}_{n \text{ times}}.$$

Note that all probability measure ν on $\mathbb{R}$ are $\uplus$-infinitely divisible, see [17, Theorem 3.6].
In [12], the author study boolean additive convolution from the perspective of CSK families. A formula is given for pseudo-variance function (or variance function V_ν in case of existence) under boolean additive convolution power. In particular , it is shown in [12, Theorem 2.3] that if ν is a real probability measure with $m_0 < +\infty$, then for $\alpha > 0$ we have

$$V_{\nu^{\uplus\alpha}}(m) = \alpha V_\nu(m/\alpha) + m(m - \alpha m_0)(1/\alpha - 1). \tag{23}$$

In this section the aim is to give some approximations in CSK families.

3.2. Approximation of Boolean Gaussian CSK family.

According to [17], the centered Boolean Gaussian distribution μ_{0,σ^2} with variance σ^2 (or symmetric Bernoulli distribution)

$$\mu_{0,\sigma^2} = \frac{1}{2}(\delta_{-\sigma} + \delta_\sigma),$$

has a self energy or a Cauchy transform

$$K_{\mu_{0,\sigma^2}}(z) = \frac{\sigma^2}{z} \quad \text{or} \quad G_{\mu_{0,\sigma^2}}(z) = \frac{1}{z - \sigma^2/z},$$

respectively. We have, for all $\theta \in (-1/\sigma, 1/\sigma)$

$$M_{\mu_{0,\sigma^2}}(\theta) = \frac{1}{1 - \theta^2\sigma^2} \quad \text{and} \quad k_{\mu_{0,\sigma^2}}(\theta) = \theta\sigma^2.$$

The inverse of the function $k_{\mu_{0,\sigma^2}}(.)$ is $\psi_{\mu_{0,\sigma^2}}(m) = m/\sigma^2$ for all $m \in (-\sigma, \sigma) = k_{\mu_{0,\sigma^2}}((-1/\sigma, 1/\sigma))$. With $m_0 = 0$, the variance function of the CSK family generated by μ_{0,σ^2} is

$$V_{\mu_{0,\sigma^2}}(m) = \mathbb{V}_{\mu_{0,\sigma^2}}(m) = \sigma^2 - m^2.$$

The two sided CSK family generated by μ_{0,σ^2} is given by

$$\mathcal{K}(\mu_{0,\sigma^2}) = \left\{Q_{(m,\mu_{0,\sigma^2})}(dx) = \mu_{m,\sigma^2}(dx) = \frac{1}{2\sigma}\left[(\sigma - m)\delta_{-\sigma} + (\sigma + m)\delta_\sigma\right] : \; m \in (-\sigma, \sigma)\right\}.$$

The family $\mathcal{K}(\mu_{0,\sigma^2})$ consists of Boolean Gaussian distributions with mean $m \in (-\sigma, \sigma)$. The following result gives an approximation of elements of the CSK family $\mathcal{K}(\mu_{0,\sigma^2})$.

Theorem 3.2. *Suppose the variance function V_ν of a CSK family $\mathcal{K}(\nu)$ is analytic and strictly positive in a neighborhood of $m_0 = 0$. Then there is $\delta > 0$ such that if, for $\alpha > 0$, $\mathcal{L}(Y_\alpha) \in \mathcal{K}(\nu_\alpha)$, with $\nu_\alpha = D_{1/\alpha}(\nu^{\uplus\alpha})$, has mean $\mathbb{E}(Y_\alpha) = m/\sqrt{\alpha}$ with $|m| < \delta$, then*

$$\sqrt{\alpha}Y_\alpha \xrightarrow{\alpha\to+\infty} \mu_{m,\sigma^2} \qquad \textit{in distribution,}$$

where $\sigma^2 = V_\nu(0)$.

Proof. Since $\mathcal{L}(Y_\alpha)$ is in the CSK family $\mathcal{K}(\nu_\alpha)$ having variance function of the form

$$V_{\nu_\alpha}(m) = V_\nu(m)/\alpha + (1/\alpha - 1)m^2,$$

then $\mathcal{L}(\sqrt{\alpha}Y_\alpha)$ is in the CSK family having variance function of the form

$$V_\alpha(m) = V_\nu(m/\sqrt{\alpha}) + (1/\alpha - 1)m^2.$$

We use Proposition 3.1 to the sequence of variance functions

$$V_\alpha(m) \xrightarrow{\alpha\to+\infty} V_\nu(0) - m^2.$$

From proposition 3.1, we deduce that there is $\delta > 0$ such that if $|m| < \delta$ and $\mathbb{E}(Y_\alpha) = m/\sqrt{\alpha}$, then with $\sigma^2 = V_\nu(0)$,

$$\mathcal{L}(\sqrt{\alpha}Y_\alpha) \xrightarrow{\alpha\to+\infty} \mu_{m,\sigma^2} \in \mathcal{K}(\mu_{0,\sigma^2}) \text{ in distribution.}$$

□

3.3. Approximation of Boolean Poisson CSK family.

For $N \in \mathbb{N}$, $s > 0$ and $0 < \lambda < N$, consider

$$\mu_N = (1 - \frac{\lambda}{N})\delta_0 + \frac{\lambda}{N}\delta_s.$$

We have that for all $\theta \in (-\infty, \frac{1}{s})$,

$$M_{\mu_N}(\theta) = 1 - \frac{\lambda}{N} + \frac{\lambda/N}{1-\theta s} \quad \text{and} \quad k_{\mu_N}(\theta) = \frac{\lambda s}{N - N\theta s + \lambda\theta s}.$$

As the inverse of the function $k_{\mu_N}(.)$, we have that for all $m \in (0, s) = k_{\mu_N}((-\infty, \frac{1}{s}))$,

$$\psi_{\mu_N}(m) = \frac{\lambda s - Nm}{sm(\lambda - N)}.$$

Formula (5) implies that the pseudo-variance function of the two sided CSK family $\mathcal{K}(\mu_N)$ is

$$\mathbb{V}_{\mu_N}(m) = \frac{Nm^2(m-s)}{\lambda s - Nm}.$$

With $m_0(\mu_N) = \lambda s/N$, we see from (6) that the variance function of the two sided CSK family $\mathcal{K}(\mu_N)$ is

$$V_{\mu_N}(m) = m(s - m).$$

The CSK family generated by μ_N is given by

$$\mathcal{K}(\mu_N) = \left\{ Q_{(m,\mu_N)}(dx) = \frac{s-m}{s}\delta_0 + \frac{m}{s}\delta_s : \ m \in (0, s) \right\}.$$

The boolean Poisson distribution $\pi_\lambda^{(s)}$ with jump size s and parameter λ ($s,\lambda \geq 0$) is given by

$$\pi_\lambda^{(s)} = \frac{1}{\lambda+1}[\delta_0 + \lambda\delta_{s(\lambda+1)}].$$

We have for all $\theta \in (-\infty, \frac{1}{s(\lambda+1)})$

$$M_{\pi_\lambda^{(s)}}(\theta) = \frac{1-\theta s}{1-\theta s(1+\lambda)} \quad \text{and} \quad k_{\pi_\lambda^{(s)}}(\theta) = \frac{\lambda s}{1-\theta s}.$$

As the inverse of the function $k_{\pi_\lambda^{(s)}}(.)$, we have that for all $m \in (0, s(1+\lambda)) = k_{\pi_\lambda^{(s)}}((-\infty, \frac{1}{s(\lambda+1)}))$,

$$\psi_{\pi_\lambda^{(s)}}(m) = \frac{m-\lambda s}{sm}.$$

Formula (5) implies that the pseudo-variance function of the two sided CSK family $\mathcal{K}(\pi_\lambda^{(s)})$ is

$$\mathbb{V}_{\pi_\lambda^{(s)}}(m) = \frac{m^2(s(\lambda+1)-m)}{m-\lambda s}.$$

With $m_0(\pi_\lambda^{(s)}) = \lambda s$, we see from (6) that the variance function of the two sided CSK family $\mathcal{K}(\pi_\lambda^{(s)})$ is

$$V_{\pi_\lambda^{(s)}}(m) = m(s(\lambda+1)-m).$$

The CSK family generated by $\pi_\lambda^{(s)}$ is given by

$$\mathcal{K}(\pi_\lambda^{(s)}) = \left\{ Q_{(m,\pi_\lambda^{(s)})}(dx) = \frac{s(\lambda+1)-m}{(\lambda+1)s}\delta_0 + \frac{m(s(\lambda+1)-m)}{(\lambda+1)s^2}\delta_{s(\lambda+1)} : \ m \in (0, s(\lambda+1)) \right\}.$$

Theorem 3.3. *For $N \in \mathbb{N}$, $s > 0$ and $0 < \lambda < N$, let*

$$\mu_N = (1-\frac{\lambda}{N})\delta_0 + \frac{\lambda}{N}\delta_s,$$

and consider the CSK family generated by $\mu_N^{\uplus N}$, with mean $m_0(\mu_N^{\uplus N}) = \lambda s$ and variance function $V_{\mu_N^{\uplus N}}(.)$. We have that

$$Q_{(m,\mu_N^{\uplus N})} \xrightarrow{N\to+\infty} Q_{(m,\pi_\lambda^{(s)})}, \quad \text{in distribution.}$$

for all m in a neighborhood of $m_0 = \lambda s$. In particular we get the boolean Poisson limit theorem

$$\mu_N^{\uplus N} \xrightarrow{N\to+\infty} \pi_\lambda^{(s)}, \quad \text{in distribution.}$$

Proof. We have that $m_0(\mu_N^{\uplus N}) = \lambda s = m_0(\pi_\lambda^{(s)})$. There exists $\varepsilon > 0$ such that $(m_-(\mu_N^{\uplus N}), m_+(\mu_N^{\uplus N})) \cap (m_-(\pi_\lambda^{(s)}), m_+(\pi_\lambda^{(s)})) = (\lambda s - \varepsilon, \lambda s + \varepsilon)$. For all $m \in (\lambda s - \varepsilon, \lambda s + \varepsilon)$

$$\int xQ_{(m,\mu_N^{\uplus N})}(dx) = m = \int xQ_{(m,\pi_\lambda^{(s)})}(dx).$$

Using variance functions and formula (23), we have for all $m \in (\lambda s - \varepsilon, \lambda s + \varepsilon)$

$$\begin{aligned} V_{\mu_N^{\uplus N}}(m) &= NV_{\mu_N}(m/N) + m(m - Nm_0(\mu_N))(1/N-1) \\ &= m(s - m/N) + m(m - s\lambda)(1/N - 1) \\ &\xrightarrow{N\to+\infty} ms(\lambda+1) - m^2 = V_{\pi_\lambda^{(s)}}(m). \end{aligned}$$

This together with Proposition 3.1 applied to the sequence of measure $Q_{(\mu_N^{\uplus N},m)}$ gives that

$$Q_{(m,\mu_N^{\uplus N})} \xrightarrow{N\to+\infty} Q_{(m,\pi_\lambda^{(s)})}, \quad \text{in distribution,}$$

for all $m \in (\lambda s - \varepsilon, \lambda s + \varepsilon)$. In particular for $m = \lambda s$ we get the boolean Poisson limit theorem

$$\mu_N^{\uplus N} \xrightarrow{N\to+\infty} \pi_\lambda^{(s)}, \quad \text{in distribution.}$$

□

References

[1] Anshelevich, M. Free martingale polynomials. J. Funct. Anal. 201, 228–261 (2003).

[2] Belinschi, S.T., Bercovici, H. Partially defined semigroups relative to multiplicative free convolution. Int. Math. Res. Not. 2, 65–101 (2005)

[3] Bryc, W. (2009). Free exponential families as kernel families. *Demonstr. Math.*, XLII(3):657–672. arxiv.org:math.PR:0601273.

[4] Bryc, W. and Hassairi, A. (2011). One-sided Cauchy-Stieltjes kernel families. *Journ. Theoret. Probab.*, 24(2):577–594. arxiv.org/abs/0906.4073.

[5] W. Bryc, R. Fakhfakh and A. Hassairi. On Cauchy-Stieltjes kernel families. *Journ. Multivariate. Analysis.*. 124: 295-312, 2014

[6] W. Bryc, R. Fakhfakh and W. Mlotkowski. Cauchy-Stieltjes families with polynomial variance funtions and generalized orthogonality. Probability and Mathematical Statistics Vol. 39, Fasc. 2 (2019), pp. 237–258 doi:10.19195/0208-4147.39.2.1.

[7] Bercovici, H., Voiculescu, D. Free convolution of measures with unbounded support. Indiana Univ. Math. J. 42(3), 733–773 (1993).

[8] Hassairi, A. Fakhfakh, R. Cauchy-Stieltjes kernel families and free multiplicative convolution. arXiv:2004.07191 [math.PR] (2020).

[9] Hassairi, A., Zarai, M. Characterization of the cubic exponential families by orthogonality of polynomials, The Annals of Probability, 2004.

[10] Fakhfakh, R. The mean of the reciprocal in a Cauchy-Stieltjes family. Statistics and Probability Letters 129 (2017) 1–11.

[11] Fakhfakh, R. Characterization of quadratic Cauchy-Stieltjes Kernels families based on the orthogonality of polynomials. J. Math.Anal.Appl. 459(2018) 577–589.

[12] Fakhfakh, R. Variance function of boolean additive convolution. https://arxiv.org/abs/2003.09703. Statistics and Probability letters Volume 163, August 2020, 108777 https://doi.org/10.1016/j.spl.2020.108777

[13] Fakhfakh, R. Boolean multiplicative convolution and Cauchy-Stieltjes Kernels families. Bull. Korean Math. Soc. 58 (2021), No. 2, pp. 515–526 https://doi.org/10.4134/BKMS.b200380.

[14] Kokonendji, C. C. Characterizations of some polynomial variance functions by d-pseudo-orthogonality. *Journal of Applied Mathematics and Computing 19*, 1-2 (2005), 427–438.

[15] Kokonendji, C. C. On d-orthogonality of the Sheffer systems associated to a convolution semigroup. *Journal of computational and applied mathematics 181*, 1 (2005), 83–91.

[16] Pommeret, D. Orthogonality of the Sheffer system associated to a Lévy process. J. Statist. Plann. Inference 86 (2000) 1–10.

[17] R. Speicher, R. Woroudi. Boolean convolution. Fields Inst. Commun 12 (1997) 267–279.

[18] Van Iseghem, J. *Approximants de Pade vectoriels*. PhD thesis, Université des sciences et techniques de Lille-Flandres-Artois, 1987.