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Discontinuous Linear Hamiltonian Systems

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Abstract. We study a discontinuous linear Hamiltonian system. We obtain a result on the existence and uniqueness of solutions. Later, we introduce the corresponding maximal and minimal operators for this problem and the self-adjoint extensions of such a minimal operator are established. Finally, we obtain an eigenfunction expansion.

1. Introduction

Discontinuous problems have been of growing interest in recent years because these problems describe processes that experience a sudden change of their state at certain moments. Such processes arise in some problems of the theory of the mass and heat transfer, population dynamics, control theory, radio science, and medicine (see [4–21, 28–34]).

On the other hand, the Hamiltonian system has been used as a powerful tool for modeling and analysis of some physical systems, e.g., electromechanical, electrical, and complex network systems with negligible dissipation (see [36]). While the theories of Hamiltonian systems are well-developed (see e.g. [1–3, 22–29]), the literature concerning discontinuous Hamiltonian systems is scarce. The present paper deals with discontinuous linear Hamiltonian systems. In the analysis that follows, we will largely follow the development of the theory in [1–3, 35]

The present paper is organized as follows. In the second section, an existence and uniqueness theorem is proved. The corresponding maximal and minimal operators for discontinuous linear Hamiltonian systems are constructed. Finally, the self-adjoint extensions of such a minimal operator are established and an eigenfunction expansion is constructed in the third section.

2. Discontinuous linear Hamiltonian system

Consider the following discontinuous linear Hamiltonian system:

$$\Omega(\mathcal{X}) := J\mathcal{X}'(t) - A_2(t)\mathcal{X}(t) = \lambda A_1(x)\mathcal{X}(t), \ t \in (a,c) \cup (c,b), \tag{1}$$

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with $\lambda \in \mathbb{C}$, where $-\infty < a < c < b < +\infty$; $A_1(t)$ and $A_2(t)$ are $2n \times 2n$ complex Hermitian matrix-valued functions, defined on $[a,c) \cup (c,b]$, and are Lebesgue measurable and integrable functions on $[a,c) \cup (c,b]$; $A_1(t)$ is nonnegative definite and

$$J = \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix}.$$

Let

$$L^2_{A_1}[(a,c) \cup (c,b); E] = \left\{ \mathcal{X} : \int_a^c (A_1\mathcal{X}, \mathcal{X})_E \, dt + \int_c^b (A_1\mathcal{X}, \mathcal{X})_E \, dt < \infty \right\}$$

with the inner product

$$(\mathcal{X}, \mathcal{Y}) := \int_a^c (A_1\mathcal{X}, \mathcal{Y})_E \, dt + \int_c^b (A_1\mathcal{X}, \mathcal{Y})_E \, dt$$

$$= \int_a^c \mathcal{Y}^* A_1 \mathcal{X} dt + \int_c^b \mathcal{Y}^* A_1 \mathcal{X} dt,$$

where $E := \mathbb{C}^{2n}$ is the $2n$-dimensional Euclidean space.

Now, we will obtain the existence and uniqueness of solutions of a discontinuous linear Hamiltonian system.

Theorem 2.1. *Let C is the $2n \times 2n$ matrix with entries from $\mathbb{R}$ such that $CJC^* = J$. Then Eq. (1) has a unique solution such that*

$$\mathcal{X}(a,\lambda) = K, \ \mathcal{X}(c+,\lambda) = C\mathcal{X}(c-,\lambda), \tag{2}$$

where $K \in \mathbb{C}^{2n}$ and $\lambda \in \mathbb{C}$.

Proof. It follows from (1) that

$$\mathcal{X}(t,\lambda) = K - \int_a^c J[\lambda A_1(x,\lambda) + A_2(x,\lambda)] \mathcal{X}(x,\lambda) \, dx + \int_c^t J[\lambda A_1(x,\lambda) + A_2(x,\lambda)] \mathcal{X}(x,\lambda) \, dx, \tag{3}$$

where $t \in [a,c) \cup (c,b]$. Conversely, every solution of Eq. (3) is also a solution of Eq. (1).

Define the sequence $\{\mathcal{X}_m\}_{m \in \mathbb{N}}$ ($\mathbb{N} := \{1,2,3,...\}$ by

$$\mathcal{X}_0(t,\lambda) = K,$$

$$\mathcal{X}_{m+1}(t,\lambda) = K - \int_a^c J[\lambda A_1(x,\lambda) + A_2(x,\lambda)] \mathcal{X}_m(x,\lambda) \, dx$$

$$+ \int_c^t J[\lambda A_1(x,\lambda) + A_2(x,\lambda)] \mathcal{X}_m(x,\lambda) \, dx, \tag{4}$$

where $t \in [a,c) \cup (c,b]$ and $m = 0,1,2,....$ Now, we will prove that $\{\mathcal{X}_m\}_{m \in \mathbb{N}}$ converges to a function $\mathcal{X}$ uniformly on each compact subset of $[a,c) \cup (c,b]$. There exist positive numbers $\eta(\lambda)$ and $\xi(\lambda)$ such that

$$\| J[\lambda A_1(t,\lambda) + A_2(t,\lambda)] \| \leq \eta(\lambda),$$

$$\|\mathcal{X}_1(t,\lambda)\| \leq \xi(\lambda),\ t \in [a,c) \cup (c,b].$$

An explicit calculation shows that

$$\|\mathcal{X}_{m+1}(t,\lambda) - \mathcal{X}_m(t,\lambda)\| \leq \eta(\lambda) \frac{(\xi(\lambda)(t-a))^m}{m!},$$

where $m \in \mathbb{N}$. It follows from the Weierstrass M-test that the sequence $\{\mathcal{X}_m\}_{m\in\mathbb{N}}$ converges to a function $\mathcal{X}$ uniformly on each compact subset of $[a,c) \cup (c,b]$. It is clear that $\mathcal{X}$ satisfies (2).

Now, we prove that Eq. (1) has a unique solution assume $\mathcal{Y}$ is another one. Since $\mathcal{Y}$ is continuous, there exists a positive number $\mathcal{N}$ such that $\left\|\mathcal{X} - \mathcal{Y}\right\| \leq \mathcal{N}$. Proceeding as above we see that

$$\left\|\mathcal{X}(t,\lambda) - \mathcal{Y}(t,\lambda)\right\| \leq \mathcal{N}\eta(\lambda) \frac{(t-a)^m}{m!},$$

where $m \in \mathbb{N}$. Since

$$\lim_{m\to\infty} \mathcal{N}\eta(\lambda) \frac{(x-a)^m}{m!} = 0.$$

it follows that $\mathcal{X} = \mathcal{Y}$ on the interval $[a,c) \cup (c,b]$. □

Now, we introduce maximal and minimal operators associated with the discontinuous linear Hamiltonian system.

Define

$$\mathcal{D}om_{\max} := \left\{ \begin{array}{c} \mathcal{X} \in L^2_{A_1}\left[(a,c)\cup(c,b);E\right] : \\ \mathcal{X} \text{ is absolutely continuous on} \\ [a,c)\cup(c,b],\ \text{one-sided limits } \mathcal{X}(c\pm) \text{ exist} \\ \text{and finite, } J\mathcal{X}'(t) - A_2(t)\mathcal{X}(t) = A_1(t)F(t) \\ \text{exists in } (a,c)\cup(c,b),\ F \in L^2_{A_1}\left[(a,c)\cup(c,b);E\right], \\ \mathcal{X}(c+) = C\mathcal{X}(c-),\ CJC^* = J \end{array} \right\},$$

$$\mathcal{D}om_{\min} := \left\{ \begin{array}{c} \mathcal{X} \in L^2_{A_1}\left[(a,c)\cup(c,b);E\right] : \\ \mathcal{X} \text{ is absolutely continuous on} \\ [a,c)\cup(c,b],\ \text{one-sided limits } \mathcal{X}(c\pm) \text{ exist} \\ \text{and finite, } J\mathcal{X}'(t) - A_2(t)\mathcal{X}(t) = A_1(t)F(t) \\ \text{exists in } (a,c)\cup(c,b),\ F \in L^2_{A_1}\left[(a,c)\cup(c,b);E\right], \\ \mathcal{X}(c+) = C\mathcal{X}(c-),\ CJC^* = J \\ \text{and } \mathcal{X}(a) = \mathcal{X}(b) = 0 \end{array} \right\}. \tag{5}$$

The operator $\mathcal{T}_{\max}$ defined by

$$\begin{aligned} &\mathcal{T}_{\max} : \mathcal{D}om_{\max} \to L^2_{A_1}\left[(a,c)\cup(c,b);E\right], \\ &\quad \mathcal{X} \to \mathcal{T}_{\max}\mathcal{X} = F \text{ if and only if } \Omega(\mathcal{X}) = A_1F. \end{aligned}$$

is called the maximal operator for the discontinuous Hamiltonian system. Similarly, the operator $\mathcal{T}_{\min}$ defined by

$$\begin{aligned} &\mathcal{T}_{\min} : \mathcal{D}om_{\min} \to L^2_{A_1}\left[(a,c)\cup(c,b);E\right], \\ &\quad \mathcal{X} \to \mathcal{T}_{\min}\mathcal{X} = F \text{ if and only if } \Omega(\mathcal{X}) = A_1F. \end{aligned}$$

is called the minimal operator generated by Eq. (1).

Now, we will give the following Green's formula.

Theorem 2.2 (Green's formula). *Let $\mathcal{X},\mathcal{Y}\in\mathcal{D}om_{\max}$. Then we have the following relation*

$$(\mathcal{T}_{\max}\mathcal{X},\mathcal{Y})-(\mathcal{X},\mathcal{T}_{\max}\mathcal{Y})=[\mathcal{X},\mathcal{Y}]_b+[\mathcal{X},\mathcal{Y}]_{c-}-[\mathcal{X},\mathcal{Y}]_a-[\mathcal{X},\mathcal{Y}]_{c+} \tag{6}$$

where $[\mathcal{X},\mathcal{Y}]_t:=\mathcal{Y}^*(t)J\mathcal{X}(t),\ t\in[a,c)\cup(c,b]$.

Lemma 2.3. *The operator $\mathcal{T}_{\min}$ is Hermitian.*

Proof. Let $\mathcal{X},\mathcal{Y}\in\mathcal{D}om_{\min}$. Then there exist $F,G\in L^2_{A_1}[(a,c)\cup(c,b);E]$ such that $\Omega(\mathcal{X})=A_1F$ and $\Omega(\mathcal{Y})=A_1G$. From (5) and (6), we conclude that

$$(\mathcal{T}_{\min}\mathcal{X},\mathcal{Y})-(\mathcal{X},\mathcal{T}_{\min}\mathcal{Y})=(F,\mathcal{Y})-(\mathcal{X},G)$$

$$=\int_a^c[\mathcal{Y}^*(t)A_1F-G^*(t)A_1\mathcal{X}(t)]\,dt+\int_c^b[\mathcal{Y}^*(t)A_1F-G^*(t)A_1\mathcal{X}(t)]\,dt$$

$$=\int_a^c[\mathcal{Y}^*(t)\Omega(\mathcal{X})-\Omega^*(\mathcal{Y})\mathcal{X}(t)]\,dt+\int_c^b[\mathcal{Y}^*(t)\Omega(\mathcal{X})-\Omega^*(\mathcal{Y})\mathcal{X}(t)]\,dt$$

$$=[\mathcal{X},\mathcal{Y}]_b+[\mathcal{X},\mathcal{Y}]_{c-}-[\mathcal{X},\mathcal{Y}]_a-[\mathcal{X},\mathcal{Y}]_{c+}=0.$$

The following lemma has a similar proof of Lemma 2.3. □

Lemma 2.4. *Let $\mathcal{Y}\in\mathcal{D}om_{\max}$ and $\mathcal{X}\in\mathcal{D}om_{\min}$. Then we have the following relation*

$$(\mathcal{T}_{\min}\mathcal{X},\mathcal{Y})=(\mathcal{X},\mathcal{T}_{\max}\mathcal{Y}).$$

Lemma 2.5. *Let us denote by $Nul(\mathcal{T})$ and $Ran(\mathcal{T})$ the null space and the range of an operator $\mathcal{T}$, respectively. Then we have*

$$Ran(\mathcal{T}_{\min})=Nul(\mathcal{T}_{\max})^{\perp}.$$

Proof. Let $\xi\in Ran(\mathcal{T}_{\min})$. There exists $\mathcal{X}\in\mathcal{D}om_{\min}$ such that $\mathcal{T}_{\min}\mathcal{X}=\xi$. It follows from Lemma 2.4 that for each $\mathcal{Y}\in Nul(\mathcal{T}_{\max})$,

$$(\xi,\mathcal{Y})=(\mathcal{T}_{\min}\mathcal{X},\mathcal{Y})=(\mathcal{X},\mathcal{T}_{\max}\mathcal{Y})=0,$$

i.e., $Ran(\mathcal{T}_{\min})\subset Nul(\mathcal{T}_{\max})^{\perp}$.

For any given $\xi\in Nul(\mathcal{T}_{\max})^{\perp}$ and for all $\mathcal{Y}\in Nul(\mathcal{T}_{\max})$, we have $(\xi,\mathcal{Y})=0$. Let us consider the following problem:

$$\begin{array}{c}J\mathcal{X}'(t)-A_2(t)\mathcal{X}(t)=A_1(t)\xi(t),\ t\in(a,c)\cup(c,b),\\ \mathcal{X}(a)=0,\ \mathcal{X}(c+)=C\mathcal{X}(c-).\end{array} \tag{7}$$

It follows from Theorem 2.1 that the problem (7) has a unique solution on $(a,c)\cup(c,b)$. Let

$$\Psi(t)=(\psi_1,\psi_2,...,\psi_{2n})$$

be the fundamental solution of the system

$$\begin{array}{c}J\mathcal{X}'(t)-A_2(t)\mathcal{X}(t)=0,\ t\in(a,c)\cup(c,b),\\ \Psi(b)=J,\ \mathcal{X}(c+)=C\mathcal{X}(c-).\end{array}$$

It is clear that $\psi_i \in Nul(\mathcal{T}_{\max})$ for $1 \leq i \leq 2n$. By Theorem 2.2, for $1 \leq i \leq 2n$, we have

$$0 = (\xi, \psi_i) = \int_a^c \psi_i^* (t) A_1 (t) \xi (t) dt + \int_c^b \psi_i^* (t) A_1 (t) \xi (t) dt$$

$$= \int_a^c \psi_i^* (t) \Omega (\mathcal{X}) (t) dt + \int_c^b \psi_i^* (t) \Omega (\mathcal{X}) (t) dt$$

$$= \int_a^c \psi_i^* (t) \Omega (\mathcal{X}) (t) dt + \int_c^b \psi_i^* (t) \Omega (\mathcal{X}) (t) dt$$

$$- \int_a^c \Omega (\psi_i)^* (t) \mathcal{X} (t) dt - \int_c^b \Omega (\psi_i)^* (t) \mathcal{X} (t) dt$$

$$= [\mathcal{X}, \psi_i]_b + [\mathcal{X}, \psi_i]_{c-} - [\mathcal{X}, \psi_i]_{c+} - [\mathcal{X}, \psi_i]_a$$

$$= [\mathcal{X}, \psi_i]_b = \psi_i(b) J \mathcal{X}(b).$$

This implies that

$$\Psi^*(b) J \mathcal{X}(b) = \mathcal{X}(b) = 0,$$

i.e., $\xi \in Ran(\mathcal{T}_{\min})$. □

Theorem 2.6. *The operator $\mathcal{T}_{\min}$ is a densely defined operator and the operator $\mathcal{T}_{\min}$ is symmetric. Furthermore $\mathcal{T}_{\min}^* = \mathcal{T}_{\max}$.*

Proof. Let $\xi \in \mathcal{D}om_{\min}^{\perp}$. Then, we have $(\xi, \mathcal{Y}) = 0$ for all $\mathcal{Y} \in \mathcal{D}om_{\min}$. Define $\mathcal{T}_{\min}\mathcal{Y}(t) = \phi(t)$.

Let $\mathcal{X}(t)$ be any solution of the following problem

$$J\mathcal{X}'(t) - A_2(t)\mathcal{X}(t) = A_1(t)\xi(t), \ t \in (a,c) \cup (c,b),$$
$$\mathcal{X}(c+) = C\mathcal{X}(c-).$$

From Theorem 2.2, we conclude that

$$\left(\mathcal{X}, \phi\right) - (\xi, \mathcal{Y})$$

$$= \int_a^c \phi^* (t) A_1 (t) \mathcal{X} (t) dt + \int_c^b \phi^* (t) A_1 (t) \mathcal{X} (t) dt$$

$$- \int_a^c \mathcal{Y}^* (t) A_1 (t) \xi (t) dt - \int_c^b \mathcal{Y}^* (t) A_1 (t) \xi (t) dt$$

$$= \int_a^c \Omega (\mathcal{Y})^* (t) \mathcal{X} (t) dt + \int_c^b \Omega (\mathcal{Y})^* (t) \mathcal{X} (t) dt$$

$$- \int_a^c \mathcal{Y}^* (t) \Omega (\mathcal{X}) (t) dt - \int_c^b \mathcal{Y}^* (t) \Omega (\mathcal{X}) (t) dt$$

$$= [\mathcal{X}, \mathcal{Y}]_a - [\mathcal{X}, \mathcal{Y}]_{c-} + [\mathcal{X}, \mathcal{Y}]_{c+} - [\mathcal{X}, \mathcal{Y}]_b = 0.$$

It follows from Lemma 2.5 that

$$\mathcal{X} \in Ran\left(\mathcal{T}_{\min}\right)^{\perp} = Nul\left(\mathcal{T}_{\max}\right).$$

Thus $\xi = 0$, i.e., $\mathcal{D}om_{\min}^{\perp} = \{0\}$.

Let us denote by $\mathcal{D}om_{\min}^{*}$ the domain of the operator $\mathcal{T}_{\min}^{*}$. Now, we will prove that $\mathcal{D}om_{\min}^{*} = \mathcal{D}om_{\max}$, and $\mathcal{T}_{\min}^{*}\mathcal{X} = \mathcal{T}_{\max}\mathcal{X}$ for all $\mathcal{X} \in \mathcal{D}om_{\min}^{*}$. It follows from Lemma 2.4 that

$$(\mathcal{X}, \mathcal{T}_{\min}\mathcal{Y}) = (\mathcal{T}_{\max}\mathcal{X}, \mathcal{Y}),$$

where $\mathcal{X} \in \mathcal{D}om_{\min}$ and $\mathcal{Y} \in \mathcal{D}om_{\max}$. Hence, the functional $(\mathcal{X}, \mathcal{T}_{\min}(.))$ is continuous on $\mathcal{D}om_{\min}$ and $\mathcal{X} \in \mathcal{D}om_{\min}^{*}$, i.e., $\mathcal{D}om_{\max} \subset \mathcal{D}om_{\min}^{*}$.

Now, we will show that $\mathcal{D}om_{\min}^{*} \subset \mathcal{D}om_{\max}$.

If $\mathcal{X} \in \mathcal{D}om_{\min}^{*}$, then $\mathcal{X}, \phi \in L_{A_1}^{2}\left[(a,c) \cup (c,b); E\right]$, where $\phi := \mathcal{T}_{\min}^{*}\mathcal{X}$. Assume that $\mathcal{U}$ is a solution of the problem

$$J\mathcal{U}'(t) - A_2(t)\,\mathcal{U}(t) = A_1(t)\,\phi(t),\ \mathcal{U}(c+) = C\mathcal{U}(c-). \tag{8}$$

It follows from Lemma 2.4 that

$$\left(\phi, \mathcal{Y}\right) = (\mathcal{T}_{\max}\mathcal{U}, \mathcal{Y}) = (\mathcal{U}, \mathcal{T}_{\min}\mathcal{Y}).$$

Hence

$$\begin{aligned}(\mathcal{X} - \mathcal{U}, \mathcal{T}_{\min}\mathcal{Y}) &= (\mathcal{X}, \mathcal{T}_{\min}\mathcal{Y}) - (\mathcal{U}, \mathcal{T}_{\min}\mathcal{Y}) \\ &= \left(\mathcal{T}_{\min}^{*}\mathcal{X}, \mathcal{Y}\right) - \left(\phi, \mathcal{Y}\right) = 0,\end{aligned}$$

i.e.,

$$\mathcal{X} - \mathcal{U} \in Ran\left(\mathcal{T}_{\min}\right)^{\perp}.$$

By Lemma 2.5, we deduce that $\mathcal{X} - \mathcal{U} \in Nul\left(\mathcal{T}_{\max}\right)$.

From (8), we conclude that

$$\begin{aligned}&J\mathcal{X}'(t) - A_2(t)\,\mathcal{X}(t) \\ &= J\mathcal{U}'(t) - A_2(t)\,\mathcal{U}(t) = A_1(t)\,\phi(t),\end{aligned}$$

where $t \in (a,c) \cup (c,b)$. Then, we get

$$\mathcal{T}_{\max}\mathcal{X} = \phi = \mathcal{T}_{\min}^{*}\mathcal{X},$$

and $\mathcal{X} \in \mathcal{D}om_{\max}$ since $\mathcal{X}, \phi \in L_{A_1}^{2}\left[(a,c) \cup (c,b); E\right]$. □

3. Eigenfunction expansion

Firstly, we will obtain a criterion under which discontinuous linear Hamiltonian system is self-adjoint. Later, we will construct the Green function and will give an eigenfunction expansion.

Let Θ, Λ are $m \times 2n$ matrices such that $rank\,(\Theta : \Lambda) = m$. Then we define the operator $\mathcal{T}$ by

$$\mathcal{T} : \mathcal{D}om \to L_{A_1}^{2}\left[(a,c) \cup (c,b); E\right], \tag{9}$$

$$\mathcal{X} \to \mathcal{T}\mathcal{X} = F \text{ if and only if } \Omega\left(\mathcal{X}\right) = A_1 F, \tag{10}$$

where

$$\mathcal{D}om := \left\{ \begin{array}{c} \mathcal{X} \in L^2_{A_1}\left[(a,c) \cup (c,b)\,; E\right] : \\ \mathcal{X} \text{ is absolutely continuous on } [a,c) \cup (c,b], \\ J\mathcal{X}'(t) - A_2(t)\,\mathcal{X}(t) = A_1(t)\,F(t) \text{ exists in} \\ (a,c) \cup (c,b)\,, \text{ one-sided limits } \mathcal{X}(c\pm) \text{ exist and} \\ \text{finite, } F \in L^2_{A_1}\left[(a,c) \cup (c,b)\,; E\right], \ \mathcal{X}(c+) = C\mathcal{X}(c-), \\ CJC^* = J \text{ and } \Theta\mathcal{X}(a) + \Lambda\mathcal{X}(b) = 0 \end{array} \right\}. \tag{11}$$

Let

$$\begin{pmatrix} \Theta & \Lambda \\ \Gamma & \Upsilon \end{pmatrix}$$

is a nonsingular matrix, where Γ and Υ are $(4n-m) \times 2n$ matrices such that $rank\,(\Gamma : \Upsilon) = 4n - m$. Let

$$\begin{pmatrix} \widetilde{\Theta} & \widetilde{\Lambda} \\ \widetilde{\Gamma} & \widetilde{\Upsilon} \end{pmatrix}$$

be a matrix such that the following relation holds

$$\begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix} = \begin{pmatrix} \widetilde{\Theta} & \widetilde{\Lambda} \\ \widetilde{\Gamma} & \widetilde{\Upsilon} \end{pmatrix}^* \begin{pmatrix} \Theta & \Lambda \\ \Gamma & \Upsilon \end{pmatrix}. \tag{12}$$

Then we have a

Theorem 3.1. *Let $\mathcal{X}, \mathcal{Y} \in \mathcal{D}om_{\max}$. Then the following relation holds*

$$(\mathcal{T}_{\max}\mathcal{X}, \mathcal{Y}) - (\mathcal{X}, \mathcal{T}_{\max}\mathcal{Y}) = \left[\widetilde{\Theta}\mathcal{Y}(a) + \widetilde{\Lambda}\mathcal{Y}(b)\right]^* \left[\Theta\mathcal{X}(a) + \Lambda\mathcal{X}(b)\right] + \left[\widetilde{\Gamma}\mathcal{Y}(a) + \widetilde{\Upsilon}\mathcal{Y}(b)\right]^* \left[\Gamma\mathcal{X}(a) + \Upsilon\mathcal{X}(b)\right].$$

Proof. From (6) and (12), we see that

$$(\mathcal{T}_{\max}\mathcal{X}, \mathcal{Y}) - (\mathcal{X}, \mathcal{T}_{\max}\mathcal{Y})$$

$$= [\mathcal{X}, \mathcal{Y}]_b + [\mathcal{X}, \mathcal{Y}]_{c-} - [\mathcal{X}, \mathcal{Y}]_a - [\mathcal{X}, \mathcal{Y}]_{c+}$$

$$= \begin{pmatrix} \mathcal{Y}^*(a) & \mathcal{Y}^*(b) \end{pmatrix} \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} \mathcal{X}(a) \\ \mathcal{X}(b) \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{Y}^*(a) & \mathcal{Y}^*(b) \end{pmatrix} \begin{pmatrix} \widetilde{\Theta} & \widetilde{\Lambda} \\ \widetilde{\Gamma} & \widetilde{\Upsilon} \end{pmatrix}^* \begin{pmatrix} \Theta & \Lambda \\ \Gamma & \Upsilon \end{pmatrix} \begin{pmatrix} \mathcal{X}(a) \\ \mathcal{X}(b) \end{pmatrix}$$

$$= \left[\begin{pmatrix} \widetilde{\Theta} & \widetilde{\Lambda} \\ \widetilde{\Gamma} & \widetilde{\Upsilon} \end{pmatrix} \begin{pmatrix} \mathcal{Y}(a) \\ \mathcal{Y}(b) \end{pmatrix}\right]^* \left[\begin{pmatrix} \Theta & \Lambda \\ \Gamma & \Upsilon \end{pmatrix} \begin{pmatrix} \mathcal{X}(a) \\ \mathcal{X}(b) \end{pmatrix}\right]$$

$$= \begin{pmatrix} \widetilde{\Theta}\mathcal{Y}(a) + \widetilde{\Lambda}\mathcal{Y}(b) \\ \widetilde{\Gamma}\mathcal{Y}(a) + \widetilde{\Upsilon}\mathcal{Y}(b) \end{pmatrix}^* \begin{pmatrix} \Theta\mathcal{X}(a) + \Lambda\mathcal{X}(b) \\ \Gamma\mathcal{X}(a) + \Upsilon\mathcal{X}(b) \end{pmatrix}.$$

■

Now, we will give the adjoint of $\mathcal{T}$.

Theorem 3.2. *Let*

$$\mathcal{D}om^{*} := \left\{ \begin{array}{c} \mathcal{Y} \in L^2_{A_1}\left[(a,c)\cup(c,b);E\right]: \\ \mathcal{Y} \text{ is absolutely continuous on } [a,c)\cup(c,b], \\ J\mathcal{Y}' - A_2(t)\mathcal{Y}(t) = A_1(t)F_1(t) \text{ exists in} \\ (a,c)\cup(c,b), \text{ one-sided limits } \mathcal{Y}(c\pm) \text{ exist and} \\ \text{finite, } F_1 \in L^2_{A_1}\left[(a,c)\cup(c,b);E\right],\ \mathcal{Y}(c+) = C\mathcal{Y}(c-), \\ CJC^{*} = J \text{ and } \widetilde{\Gamma}\mathcal{Y}(a) + \widetilde{\Upsilon}\mathcal{Y}(b) = 0 \end{array} \right\}.$$

Then we define the operator $\mathcal{T}^{*}$ by

$$\mathcal{T}^{*}: \mathcal{D}om^{*} \to L^2_{A_1}\left[(a,c)\cup(c,b);E\right].$$

For $\mathcal{Y} \in \mathcal{D}om^{*}$, $\mathcal{T}^{*}\mathcal{Y} = F_1$ if and only if

$$J\mathcal{Y}'(t) - A_2(t)\mathcal{Y}(t) = A_1(t)F_1(t). \tag{13}$$

Proof. Since $\mathcal{T}_{\min} \subset \mathcal{T} \subset \mathcal{T}_{\max}$, we have $\mathcal{T}_{\min} \subset \mathcal{T}^{*} \subset \mathcal{T}_{\max}$. Let $\mathcal{X} \in \mathcal{D}om$ and $\mathcal{Y} \in \mathcal{D}om^{*}$. From Theorem 3.1, we get

$$\begin{aligned}(\mathcal{T}\mathcal{X},\mathcal{Y}) - (\mathcal{X},\mathcal{T}^{*}\mathcal{Y}) &= \left[\widetilde{\Theta}\mathcal{Y}(a) + \widetilde{\Lambda}\mathcal{Y}(b)\right]^{*}\left[\Theta\mathcal{X}(a) + \Lambda\mathcal{X}(b)\right] \\ &+ \left[\widetilde{\Gamma}\mathcal{Y}(a) + \widetilde{\Upsilon}\mathcal{Y}(b)\right]^{*}\left[\Gamma\mathcal{X}(a) + \Upsilon\mathcal{X}(b)\right].\end{aligned}$$

Hence

$$0 = \left[\widetilde{\Gamma}\mathcal{Y}(a) + \widetilde{\Upsilon}\mathcal{Y}(b)\right]^{*}\left[\Gamma\mathcal{X}(a) + \Upsilon\mathcal{X}(b)\right].$$

Therefore we deduce that

$$\widetilde{\Gamma}\mathcal{Y}(a) + \widetilde{\Upsilon}\mathcal{Y}(b) = 0,$$

since $\Gamma\mathcal{X}(a) + \Upsilon\mathcal{X}(b)$ is arbitrary.

Conversely, if $\mathcal{Y}$ satisfies the criteria listed above, then $\mathcal{Y} \in \mathcal{D}om^{*}$. ■

We will find parametric boundary conditions for $\mathcal{D}om$ and $\mathcal{D}om^{*}$. Recall that

$$\Gamma\mathcal{X}(a) + \Upsilon\mathcal{X}(b) = F_2,\ \Theta\mathcal{X}(a) + \Lambda\mathcal{X}(b) = 0, \tag{14}$$

where F_2 is arbitrary. Then we get

$$\begin{pmatrix} \Theta & \Lambda \\ \Gamma & \Upsilon \end{pmatrix}\begin{pmatrix} \mathcal{X}(a) \\ \mathcal{X}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ F_2 \end{pmatrix}. \tag{15}$$

If we multiply both sides of (15) by

$$\begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}\begin{pmatrix} \widetilde{\Theta} & \widetilde{\Lambda} \\ \widetilde{\Gamma} & \widetilde{\Upsilon} \end{pmatrix}^{*}$$

then we conclude that

$$\begin{pmatrix} \mathcal{X}(a) \\ \mathcal{X}(b) \end{pmatrix} = \begin{pmatrix} J\widetilde{\Gamma}^{*}F_2 \\ -J\widetilde{\Upsilon}^{*}F_2 \end{pmatrix}. \tag{16}$$

Similarly, one can find parametric boundary conditions for $\mathcal{D}om^{*}$. Since

$$\widetilde{\Gamma}\mathcal{Y}(a) + \widetilde{\Upsilon}\mathcal{Y}(b) = 0,\ \widetilde{\Theta}\mathcal{Y}(a) + \widetilde{\Lambda}\mathcal{Y}(b) = F_3,$$

where F_3 is arbitrary, we get

$$\begin{pmatrix} \mathcal{Y}^*(a) & \mathcal{Y}^*(b) \end{pmatrix} \begin{pmatrix} \widetilde{\Theta} & \widetilde{\Lambda} \\ \widetilde{\Gamma} & \widetilde{\Upsilon} \end{pmatrix}^* = \begin{pmatrix} F_3^* & 0 \end{pmatrix}. \tag{17}$$

Multiplying both sides of (17) by

$$\begin{pmatrix} \Theta & \Lambda \\ \Gamma & \Upsilon \end{pmatrix} \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}$$

it follows that

$$\mathcal{Y}(a) = -J\Theta^* F_3, \ \mathcal{Y}(b) = J\Lambda^* F_3. \tag{18}$$

Now, we have the following theorem.

Theorem 3.3. *$\Theta J \Theta^* = \Lambda J \Lambda^*$ and $m = 2n$ if and only if $\mathcal{T}$ is the self-adjoint operator.*

Proof. Let $\Theta J \Theta^* = \Lambda J \Lambda^*$. Then we deduce that

$$\begin{pmatrix} -\Theta J & \Lambda J \end{pmatrix} \begin{pmatrix} \Theta^* \\ \Lambda^* \end{pmatrix} = 0.$$

That is, the columns of $\begin{pmatrix} \Theta^* \\ \Lambda^* \end{pmatrix}$ for $2n$ independent solutions to the equation

$$\begin{pmatrix} -\Theta J & \Lambda J \end{pmatrix} X = 0.$$

It follows from (14) and (16) that

$$\begin{pmatrix} -\Theta J & \Lambda J \end{pmatrix} \begin{pmatrix} \widetilde{\Gamma}^* \\ \widetilde{\Upsilon}^* \end{pmatrix} = 0.$$

Thus, we get

$$\begin{pmatrix} \Theta & \Lambda \end{pmatrix} = K \begin{pmatrix} \widetilde{\Gamma} & \widetilde{\Upsilon} \end{pmatrix}$$

or

$$\begin{pmatrix} \widetilde{\Gamma}^* \\ \widetilde{\Upsilon}^* \end{pmatrix} K^* = \begin{pmatrix} \Theta^* \\ \Lambda^* \end{pmatrix},$$

where K is a constant, nonsingular matrix.

The conditions $\Theta\mathcal{X}(a) + \Lambda\mathcal{X}(b) = 0$ and $\Gamma\mathcal{X}(a) + \Upsilon\mathcal{X}(b) = 0$ are equivalent. Since the forms of $\mathcal{T}$ and $\mathcal{T}^*$ are the same, we see that $\mathcal{T} = \mathcal{T}^*$.

Conversely, let $\mathcal{T}$ be a self-adjoint operator. Then $\mathcal{X}$ satisfies the boundary conditions for $\mathcal{D}om$, i.e.,

$$\Theta\mathcal{X}(a) + \Lambda\mathcal{X}(b) = 0.$$

By (18), we obtain

$$\begin{aligned} \Theta(-J\Theta^* F_3) + \Lambda(J\Lambda^* F_3) &= 0 \\ [\Theta J\Theta^* - \Lambda J \Lambda^*] F_3 &= 0. \end{aligned}$$

Then we get

$$\Theta J \Theta^* = \Lambda J \Lambda^*,$$

since F_3 is arbitrary. □

Let us define the operator $\mathcal{T}_1$ by

$$\mathcal{T}_1 : \mathcal{D}om_1 \to L^2_{A_1}\left[(a,c)\cup(c,b);E\right], \tag{19}$$

$$\mathcal{X} \to \mathcal{T}_1\mathcal{X} = F \Leftrightarrow J\mathcal{X}' - A_2\mathcal{X} = A_1F, \tag{20}$$

where

$$\mathcal{D}om_1 := \{\mathcal{X} \in \mathcal{D}om : \Theta J\Theta^* = \Lambda J\Lambda^*\}. \tag{21}$$

Let $X(t,\lambda)$ be a fundamental matrix solution of the system $\Omega(\mathcal{X}) = \lambda A_1\mathcal{X}$ satisfying $X(a,\lambda) = I$ and $X(c+,\lambda) = CX(c-,\lambda)$. It is clear that

$$X^*(t,\lambda)JX(t,\lambda) = J \tag{22}$$

for all $t \in [a,c)\cup(c,b]$ ([3]).

Theorem 3.4. *The resolvent operator of $\mathcal{T}_1$ is given by the formula*

$$R_1(\lambda)F(t) = (\mathcal{T}_1 - \lambda I)^{-1}F(t)$$
$$= \int_a^c G(t,u,\lambda)A_1(u)F(u)\,du + \int_c^b G(t,u,\lambda)A_1(u)F(u)\,du,$$

where $G(t,y,\lambda)$ is the matrix Green function defined as

$$G(t,u,\lambda) =$$
$$\begin{cases} X(t,\lambda)\left[\Theta + \Lambda X(b,\lambda)\right]^{-1}\Theta JX^*\left(u,\overline{\lambda}\right), & a \le u \le t \le b,\ u \ne c,\ t \ne c \\ -X(t,\lambda)\left[\Theta + \Lambda X(b,\lambda)\right]^{-1}\Lambda JX^*\left(u,\overline{\lambda}\right), & a \le t \le u \le b,\ u \ne c,\ t \ne c. \end{cases}$$

Proof. Let $\mathcal{X}$ satisfies the equation $\Omega(\mathcal{X}) = A_1F$ and let $K(t,\lambda)$ is $2n \times 1$ vector function. We seek a solution of the form

$$\mathcal{X}(t,\lambda) = X(t,\lambda)K(t,\lambda),$$

by using the method of variation of constants. Then we get

$$J\mathcal{X}' = JX'K + JXK',$$
$$(\lambda A_1 + A_2)\mathcal{X} = (\lambda A_1 + A_2)XK.$$

Hence

$$\begin{aligned} A_1F &= J\mathcal{X}' - (\lambda A_1 + A_2)\mathcal{X} \\ &= JX'K + JXK' - (\lambda A_1 + A_2)XK \\ &= \left[JX' - (\lambda A_1 + A_2)X\right]K + JXK' = JXK' \end{aligned}$$

i.e.,

$$K' = [JX]^{-1}A_1F.$$

It follows from (22) that

$$K' = -JX^*\left(t,\overline{\lambda}\right)A_1F.$$

Then, we see that

$$\begin{aligned}\mathcal{X}(t,\lambda) &= -X(t,\lambda)\int_a^c JX^*\left(t,\overline{\lambda}\right)A_1(u)F(u)\,du\\&\quad - X(t,\lambda)\int_c^t JX^*\left(t,\overline{\lambda}\right)A_1(u)F(u)\,du + X(t,\lambda)K_1.\end{aligned}$$

By the condition $\Theta X(a) + \Lambda X(b) = 0$, we deduce that

$$\begin{aligned}\mathcal{X}(a) &= K_1,\\ \mathcal{X}(b) &= -X(b,\lambda)\int_a^c JX^*\left(t,\overline{\lambda}\right)A_1(u)F(u)\,du\\&\quad - X(b,\lambda)\int_c^b JX^*\left(t,\overline{\lambda}\right)A_1(u)F(u)\,du + Z(b,\lambda)K_1.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{X}(t,\lambda) &= -X(t,\lambda)\left[\Theta + \Lambda X(b)\right]^{-1}\Theta\int_a^c JX^*\left(t,\overline{\lambda}\right)A_1(u)F(u)\,du\\&\quad - X(t,\lambda)\left[\Theta + \Lambda X(b)\right]^{-1}\Theta\int_c^t JX^*\left(t,\overline{\lambda}\right)A_1(u)F(u)\,du\\&\quad + X(t,\lambda)\left[\Theta + \Lambda X(b)\right]^{-1}\Lambda\int_t^b JX^*\left(t,\overline{\lambda}\right)A_1(u)F(u)\,du.\end{aligned}$$

■

Theorem 3.5. *Let $\lambda \notin \mathbb{R}$.Then $R(\lambda)$ exists and is a bounded operator. It exists also for all real λ for which* $\det[\Theta + \Lambda X(b)] \neq 0$ *as a bounded operator. The spectrum of $\mathcal{T}_1$ consists entirely of isolated eigenvalues, zeros of* $\det[\Theta + \Lambda X(b)] = 0$. *Furthermore, eigenfunctions associated with different eigenvalues are mutually orthogonal.*

Proof. It is clear that the operator $R(\lambda)$ exists for all real λ except the zeros of $\det[\Theta + \Lambda X(b)] = 0$. Since $\mathcal{T}_1$ is self-adjoint operator, it follows that the operator $R(\lambda)$ exists for all nonreal λ. Eigenfunctions associated with different eigenvalues are mutually orthogonal. The spectrum of $\mathcal{T}_1$ consists entirely of isolated eigenvalues, zeros of $\det[\Theta + \Lambda X(b)] = 0$ because $\det[\Theta + \Lambda X(b)]$ is analytic in λ and is not identically zero. These zeros can accumulate only at $\pm\infty$.

Now, we will show that the operator $R(\lambda)$ is a bounded operator. Let

$$W(t,\eta,\lambda) = A_1^{1/2}(\eta)\,G(t,u,\lambda)\,A_1^{1/2}(t),$$

and

$$f(\eta) = A_1^{1/2}(\eta)\,F(\eta)$$

where $A_1^{1/2}$ is a square root of the matrix A_1. Then, we have

$$\|R(\lambda)F\|^2 = \|\mathcal{X}\|^2 = \int_a^c \mathcal{X}^* A_1 \mathcal{X}\,dt + \int_c^b \mathcal{X}^* A_1 \mathcal{X}\,dt$$

$$= \int_a^c \left[\int_a^c G(t,\eta,\lambda) A_1(\eta) F(\eta) d\eta \right]^* A_1(t) \left[\int_a^c G(t,\eta,\lambda) A_1(\eta) F(\eta) d\eta \right] dt$$
$$+ \int_c^b \left[\int_c^b G(t,\eta,\lambda) A_1(\eta) F(\eta) d\eta \right]^* A_1(t) \left[\int_c^b G(t,\eta,\lambda) A_1(\eta) F(\eta) d\eta \right] dt$$
$$= \int_a^c \left[\int_a^c f^*(\eta) W^*(t,\eta,\lambda) d\eta \right] \left[\int_a^c W(t,u,\lambda) f(u) du \right] dt$$
$$+ \int_c^b \left[\int_c^b f^*(\eta) W^*(t,\eta,\lambda) d\eta \right] \left[\int_c^b W(t,u,\lambda) f(u) du \right] dt.$$

By using Cauchy-Schwarz's inequality, we get

$$\|\mathcal{X}\|^2 \leq \|W\|^2 \left\|f\right\|^2,$$

where

$$\|W\|^2 = \int_a^c \int_a^c \sum_{i=1}^{2n} \sum_{j=1}^{2n} \left|W_{ij}(t,\eta,\lambda)\right|^2 d\eta dt + \int_c^b \int_c^b \sum_{i=1}^{2n} \sum_{j=1}^{2n} \left|W_{ij}(t,\eta,\lambda)\right|^2 d\eta dt.$$

■

There is no loss of generality in assuming that zero is not an eigenvalue. Then, the solution of the following problem

$$J\mathcal{X}' - A_2\mathcal{X} = A_1 F,$$
$$\mathcal{X}(c+) = C\mathcal{X}(c-),$$
$$\Theta\mathcal{X}(a) + \Lambda\mathcal{X}(b) = 0,$$

is given by

$$\mathcal{X}(x) = \int_a^c G(x,t) A_1(t) F(t) dt + \int_c^b G(x,t) A_1(t) F(t) dt,$$

where $G(x,t) = G(x,t,0)$.

Now we have the following theorems.

Theorem 3.6. *Let*

$$\mathcal{T}_1^{-1} F = \mathcal{T}_2 F = \mathcal{X}.$$

Then $\mathcal{T}_2$ is a bounded operator and

$$\|\mathcal{T}_2\| = \sup\left\{\left|\lambda_m^{-1}\right| : \lambda_m \in \sigma(\mathcal{T}_1)\right\},$$

where $\sigma(\mathcal{T}_1)$ denotes the spectrum of the operator $\mathcal{T}_1$.

Proof. If $\mathcal{T}_1\varphi_m = \lambda_m\varphi_m$, $\left\|\varphi_m\right\| = 1$ $(m \in \mathbb{N})$, then $\mathcal{T}_2\varphi_m = \tau_m\varphi_m$, where $\tau_m = \frac{1}{\lambda_m}$. Then, we have

$$\|\mathcal{T}_2\| = \sup_{\substack{\chi \in L^2_{A_1}[(a,c)\cup(c,b);E] \\ \|\chi\|=1}} |(\mathcal{T}_2\chi, \chi)|$$
$$= \sup\{|\tau_m| : \tau_m \in \sigma(\mathcal{T}_2)\} = \sup\{\left|\lambda_m^{-1}\right| : \lambda_m \in \sigma(\mathcal{T}_1)\}.$$

■

Now, we shall order the eigenvalues of $\mathcal{T}_2$ such that $|\tau_1| \geq |\tau_2| \geq ... \geq |\tau_m| \geq ...$, where

$$\lim_{m\to\infty} |\tau_m| = 0. \tag{23}$$

Theorem 3.7. *Let* $\{\mathcal{T}_{2,m}\}_{m=1}^{\infty}$ *be a sequence defined by*

$$\mathcal{T}_{2,m}F = \mathcal{T}_2F - \sum_{i=1}^{m-1} \tau_i\varphi_i\left(F,\varphi_i\right).$$

Then we have

$$\lim_{m\to\infty} \mathcal{T}_{2,m} = 0, \tag{24}$$

and

$$\left\|\mathcal{T}_{2,m}\right\| = |\tau_m|,$$

where $m \in \mathbb{N}$.

Proof. It is clear that

$$\mathcal{T}_{2,m}\varphi_j = \begin{cases} \tau_j\varphi_j, & \text{if } m \le j < \infty \\ 0, & \text{if } 1 \le j \le m-1. \end{cases}$$

Further, the operator $\mathcal{T}_{2,m}$ is bounded and self-adjoint. Then, we obtain

$$\left\|\mathcal{T}_{2,m}\right\| = \sup_{\substack{\chi\in L^2_{A_1}[(a,c)\cup(c,b);E] \\ \|\varphi\|=1}} \left|(\mathcal{T}_{2,m}\varphi,\varphi)\right|$$

$$= \sup_{\substack{\chi\in L^2_{A_1}[(a,c)\cup(c,b);E] \\ \|\varphi\|=1 \\ \varphi\neq\varphi_1,\dots,\varphi_{m-1}}} \left|(\mathcal{T}_{2,m}\varphi,\varphi)\right| = |\tau_m|.$$

For a finite m, we can stop this process such that $\mathcal{T}_{2,m} = 0$. Hence, for all $f \in L^2_{A_1}\left[(a,c)\cup(c,b);E\right]$, we see that

$$\mathcal{T}_2F = \sum_{i=1}^{m-1} \tau_i\varphi_i\left(F,\varphi_i\right). \tag{25}$$

If we apply $\mathcal{T}_1$ to the equality (25), then we conclude that

$$F = \sum_{i=1}^{m-1} \varphi_i\left(F,\varphi_i\right),$$

i.e., F is differentiable. Since there are F's which are not, the process cannot stop. It follows from (23) that

$$\lim_{m\to\infty} \mathcal{T}_{2,m} = 0.$$

■

Theorem 3.8. *Let* $F \in L^2_{A_1}\left[(a,c)\cup(c,b);E\right]$ *and* $\mathcal{X} \in \mathcal{D}om_1$. *Then we have*

$$F = \sum_{i=1}^{\infty} \varphi_i\left(F,\varphi_i\right),$$

$$\mathcal{T}_2F = \sum_{i=1}^{\infty} \tau_i\varphi_i\left(F,\varphi_i\right),$$

$$\mathcal{T}_1\mathcal{X} = \sum_{i=1}^{\infty} \lambda_i\varphi_i\left(\mathcal{X},\varphi_i\right).$$

Proof. It follows from (24) that

$$\mathcal{T}_2 F = \sum_{i=1}^{\infty} \tau_i \varphi_i (F, \varphi_i). \tag{26}$$

Applying $\mathcal{T}_1$ to the equality (26), we conclude that

$$F = \sum_{i=1}^{\infty} \varphi_i (F, \varphi_i).$$

Further,

$$(F, \varphi_i) = (\mathcal{T}_1 \mathcal{X}, \varphi_i) = (\mathcal{X}, \mathcal{T}_1 \varphi_i) = \lambda_i (\mathcal{X}, \varphi_i).$$

Hence we get

$$\mathcal{T}_1 \mathcal{X} = \sum_{i=1}^{\infty} \lambda_i \varphi_i (\mathcal{X}, \varphi_i).$$

■

Theorem 3.9. *There exists a collection of projection operators* $\{E(\lambda)\}$ *satisfying*
(a) $\lim_{\lambda\to\infty} E(\lambda) = I$, $\lim_{\lambda\to-\infty} E(\lambda) = 0$,
(b) $E(\lambda_1) \leq E(\lambda_2)$ *when* $\lambda_1 \leq \lambda_2$,
(c) $E(\lambda)$ *is continuous from above,*
(d) For all $F \in L^2_{A_1}[(a,c) \cup (c,b); E]$ *and* $\mathcal{X} \in \mathcal{D}om_1$, *we have*

$$F = \int_{-\infty}^{\infty} dE(\lambda) F,$$

$$\mathcal{T}_1 \mathcal{X} = \int_{-\infty}^{\infty} \lambda dE(\lambda) \mathcal{X},$$

$$\mathcal{T}_2 F = \int_{-\infty}^{\infty} \frac{1}{\lambda} dE(\lambda) F.$$

Proof. Define

$$P_i F = \varphi_i (F, \varphi_i),$$

where P_i is a projection operator. Let us define

$$E(\lambda) F = \sum_{\lambda_i \leq \lambda} P_i F,$$

then $E(\lambda)$ is a Stieltjes measure. The integrals in (d) are obtained from this series. ■

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