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Second Degree Linear Forms and Semiclassical Forms of Class One. A Case Study

Imed Ben Salah[a], Francisco Marcellán[b], Mohamed Khalfallah[a]

[a] *University of Monastir, Faculty of Sciences of Monastir, Department of Mathematics, 5019 Monastir, Tunisia*
[b] *Departamento de Matemáticas, Universidad Carlos III de Madrid, Leganés, Spain*

Abstract. Based on their second degree character, in this contribution we study new characterizations of families of symmetric and quasi-symmetric semiclassical linear forms of class one. In fact, by using the Stieltjes function and the moments of those forms, we give necessary and sufficient conditions for a regular form to be at the same time of the second degree, symmetric (resp. quasi-symmetric) and semiclassical of class one. We focus our attention on the link between these forms and the Jacobi forms $\mathcal{T}_{p,q} = \mathcal{J}(p-1/2, q-1/2),\ p, q \in \mathbb{Z},\ p+q \geq 0$. All of them are rational transformations of the first kind Chebychev form $\mathcal{T} = \mathcal{J}(-1/2, -1/2)$. Finally, we study a family of second degree linear forms which are semiclassical of class one and are not included in the above families.

1. Introduction

Semiclassical orthogonal polynomials (OP) have been introduced in the seminal paper by J. Shohat [34] and they arise as a natural extension of the well-known classical OP of Hermite, Laguerre, Jacobi and Bessel. Semiclassical OP attracted the interest of many researchers from 1980 taking into account their applications in several domains. More precisely, this theory has been developed, from an algebraic aspect and a distributional one by P. Maroni and extensively studied during the last three decades (see [22] as a nice survey on this topic, as well as [15] with the applications in the framework of Sobolev inner products). In particular, the classification of semiclassical forms (linear functionals) according to some criteria of optimal information from the so-called Pearson equation, i.e. a first order linear differential equation satisfied by the form, plays a central role in the construction of such forms. In [6], S. Belmehdi makes use of this approach to provide a full description of all semiclassical forms of class $s = 1$. Moreover, in [3], M. Bachène established the non linear system satisfied by the coefficients of the recurrence relation of semiclassical orthogonal sequences of class $s = 1$ which is difficult to solve in general. In particular case (see [22, 26]) the authors are able to find such a nonlinear system. In the framework of general semiclassical linear forms these equations are called in the literature (see[7]) Laguerre-Freud equations.

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Email addresses: imed.bensalha@gmail.com (Imed Ben Salah), pacomarc@ing.uc3m.es (Francisco Marcellán), khalfallah.mohamed91@gmail.com (Mohamed Khalfallah)

Taking into account the difficulties to solve the Laguerre-Freud equations, it is more important to use other tools for the construction and characterizations of some semiclassical forms [2, 4, 5, 9, 10, 31, 32] based either on the moments, the corresponding Stieltjes function or their integral representation. As an illustrative example of semiclassical linear forms in [24], second degree forms have been introduced. These forms are characterized by the fact that their formal Stieltjes function $S(w)(z) = -\sum_{n\geq 0} \langle w, x^n \rangle / z^{n+1}$ satisfies a quadratic equation

$$BS^2(w) + CS(w) + D = 0,$$

where B, C, D are polynomials such that $B \neq 0, C^2 - 4BD \neq 0$ and $D \neq 0$. The most famous and elementary examples of second degree forms are the Tchebychev forms of the first and second kind and, more generally. the Bernstein-Szegö forms and their generalizations [16, 27, 29] as well as [17]. It is worthy to mention that the unique classical second degree forms are the Jacobi forms $\mathcal{J}(k - \frac{1}{2}, l - \frac{1}{2})$, where k, l are integer numbers with $k + l \geq 0$ (see [4]).

Our work is focused on the analysis of semiclassical forms of class $s = 1$ which are second degree forms. In particular, we are interested in description, by using the second degree character, of a large family of forms such that their corresponding sequences of orthogonal polynomials $\{W_n\}_{n\geq 0}$ satisfy the three-term recurrence relation (TTRR)

$$W_{n+2}(x) = \left(x - (-1)^{n+1}\beta_0\right) W_{n+1}(x) - \gamma_{n+1} W_n(x),\ n \geq 0,$$

with initial conditions $W_0(x) = 1,\ W_1(x) = x - \beta_0$, where $\beta_0 \in \mathbb{C}$. These forms are general enough to accommodate all the symmetric forms as well as some particular non-symmetric ones. The sequence $\{W_n\}_{n\geq 0}$ has been the subject of several works [1, 2, 11, 12, 21, 25, 26, 30–32, 35]. Indeed, if we take $\beta_0 = 0$, we get the symmetric forms. When $\beta_0 \neq 0$, we will say that corresponding form of $\{W_n\}_{n\geq 0}$ is quasi-symmetric. Later on, the authors of [2, 4, 31] determined all the symmetric and quasi-symmetric second degree semiclassical forms of class one. But unfortunately, the link between all these forms and the classical one has not been studied until now. The above families of orthogonal polynomials appear when you deal with quadratic decompositions of sequences of orthogonal polynomials (see [14], [21]).

The aim of the present contribution is not only to describe all the semiclassical forms of class one which are of second degree and symmetric (resp. quasi-symmetric), but also we go further by giving a new identification of this family of forms by presenting their link with the above classical forms which are of the second degree. Our approach is quite different of those presented in [4] (for the symmetric case) and [2] (for the quasi-symmetric case), respectively, since our main tool is the representation of the corresponding Stieltjes functions and, as a consequence, the moments of the forms are deduced in a straightforward way. Thus we have the explicit expression of these Stieltjes functions as well as we deduce the quadratic equation as well the first order ordinary linear differential equation they satisfy.

This paper is organized as follows. In Section 2 the notations and basic background which will be used in the forthcoming sections. In Section 3, we first recall the definitions as well as the main proprieties of second degree forms. Afterwards, we will give some results concerning second degree classical forms, denoted by $\mathcal{T}_{p,q} = \mathcal{J}(p - 1/2, q - 1/2),\ p, q \in \mathbb{Z},\ p + q \geq 0$, which are needed in the sequel. In Sections 4 and 5, we state our main results. Through the second degree forms, we give an identification of the family of symmetric (resp. quasi-symmetric) regular forms. Indeed, we give necessary and sufficient conditions for a regular form to be at the same time a symmetric (resp. quasi-symmetric), second degree and semiclassical form of class one. Thus, we establish the connection between all these forms and the Jacobi forms $\mathcal{T}_{p,q} = \mathcal{J}(p - 1/2, q - 1/2),\ p, q \in \mathbb{Z},\ p + q \geq 0$, and also we show that all these forms are rational transformations of the Tchebychev form of the first kind $\mathcal{T} = \mathcal{J}(-1/2, -1/2)$. Finally, by using a canonical Christoffel transformation of classical linear forms of second degree, in Section 6 we deal with a large family of second degree linear forms which are semiclassical of class one and, as a consequence, do not belong to the families analyzed in the previous sections.

2. Notations and basic background

Let $\mathcal{P}$ be the linear space of algebraic polynomials with complex coefficients. $\langle w, p\rangle$ will denote the action of the form (linear functional) $w \in \mathcal{P}'$ over the polynomial $p \in \mathcal{P}$, where $\mathcal{P}'$ denotes the algebraic dual of the linear space $\mathcal{P}$. In particular, $\langle w, x^n\rangle := (w)_n,\ n \geq 0$, represent the moments of w.
Let us define the following operations in the algebraic dual space of the polynomials: the left product of w by a polynomial, defined as $\langle fw, p\rangle = \langle w, fp\rangle, p \in \mathcal{P}$; the derivative Dw of the linear form w is defined as $\langle Dw, p\rangle = -\langle w, p'\rangle, p \in \mathcal{P}$; the dilations and shifted forms $h_a w$ and $\tau_b w$ are defined, respectively, as $\langle h_a w, p\rangle = \langle w, h_a p\rangle = \langle w, p(ax)\rangle$, $\langle \tau_b w, p\rangle = \langle w, \tau_{-b} p\rangle = \langle w, p(x+b)\rangle$; the form $x^{-1}u$ defined as $\langle x^{-1}w, p\rangle = \langle w, \theta_0 p\rangle = \left\langle w, \frac{p(x)-p(0)}{x}\right\rangle$; the Cauchy product of two forms, v and w, defined as $\langle vw, f\rangle := \langle v, wf\rangle, f \in \mathcal{P}$, where the right product of a linear form by a polynomial is given by $(wp)(x) := \left\langle w, \frac{xp(x)-\zeta p(\zeta)}{x-\zeta}\right\rangle = \sum_{i=0}^{n}\left(\sum_{j=i}^{n}(w)_{j-i}a_j\right)x^i$, being $p(x) = \sum_{i=0}^{n} a_i x^i$.
In $\mathcal{P}'$, we have the well-known formula

$$\tau_b \circ h_a = h_a \circ \tau_{a^{-1}b},\ a \in \mathbb{C} - \{0\},\ b \in \mathbb{C}. \tag{1}$$

The linear form $w \in \mathcal{P}'$ is said to be a rational perturbation of $v \in \mathcal{P}'$, if there exist polynomials p and q, such that

$$q(x)w = p(x)v.$$

The even part of a form w is given by

$$\langle \sigma(w), p\rangle = \langle w, \sigma(p)\rangle,\ \ p \in \mathcal{P}.$$

where the linear operator $\sigma : \mathcal{P} \to \mathcal{P}$ is defined by $\sigma(p)(x) := p(x^2)$ for every $p \in \mathcal{P}$.
We introduce the so-called anti-symmetrization operator $\boldsymbol{\alpha} : \mathcal{P}' \to \mathcal{P}'$ defined by, for $\varpi \in \mathcal{P}'$ [22]

$$\Big(\boldsymbol{\alpha}(\varpi)\Big)_{2n} = 0, \quad \Big(\boldsymbol{\alpha}(\varpi)\Big)_{2n+1} = (\varpi)_n,\ \ n \geq 0. \tag{2}$$

We will also use the so-called formal Stieltjes function associated with $w \in \mathcal{P}'$ that is defined by [13, 22]

$$S(w)(z) = -\sum_{n\geq 0} \frac{(w)_n}{z^{n+1}}.$$

Remark 2.1. *For any $p \in \mathcal{P}$ and $w \in \mathcal{P}'$, $S(w)(z) = p(z)$ if and only if $w = 0$ and $f = 0$.*

For any $p \in \mathcal{P}$ and $u, v \in \mathcal{P}'$, the following properties hold [22]

$$S(uv)(z) = -zS(u)(z)S(v)(z), \tag{3}$$
$$S(pu)(z) = p(z)S(u)(z) + (u\theta_0 p)(z). \tag{4}$$

Let us recall that a form w is called regular (quasi-definite) if there exists a monic polynomial sequence $\{W_n\}_{n\geq 0}$ with $\deg W_n = n$ such that [13]

$$\langle w, W_n W_m\rangle = r_n \delta_{n,m},\ \ n, m \geq 0,$$

where $\{r_n\}_{n\geq 0}$ is a sequence of nonzero complex numbers and $\delta_{n,m}$ is the Kronecker symbol.
$\{W_n\}_{n\geq 0}$ is called a monic orthogonal polynomial sequence (MOPS, in short) with respect to the form w. It is characterized by the following three-term recurrence relation

$$\begin{aligned} &W_0(x) = 1,\ \ W_1(x) = x - \beta_0,\\ &W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \gamma_{n+1}W_n(x),\ n \geq 0. \end{aligned} \tag{5}$$

Here $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$ are sequences of complex numbers such that $\gamma_{n+1} \neq 0$ for all n. This is the so called Favard's theorem (see [13, 22, 23]). The form w is said to be normalized if $(w)_0 = 1$. In the sequel, we only consider normalized forms.

In this work, we will consider a (MOPS) $\{W_n\}_{n\geq 0}$ with respect to the form w fulfilling a second-order recurrence relation (5) with coefficients

$$\beta_n = \beta_0(-1)^n, \ \ n \geq 0, \ \ \beta_0 \in \mathbb{C}.$$

In this case, we will say that $\{W_n\}_{n\geq 0}$ is quasi-symmetric (respectively, symmetric) if $\beta_0 \neq 0$ (respectively, $\beta_0 = 0$). The corresponding form w is quasi-symmetric (respectively, symmetric).

In the sequel, the following results will be useful.

Lemma 2.2. *[1, 11, 13, 21] Let $\{W_n\}_{n\geq 0}$ be a (MOPS) with respect to the form w. The following statements are equivalent:*

(1) $\{W_n\}_{n\geq 0}$ is symmetric.
(2) $W_n(-x) = (-1)^n W_n(x), \ n \geq 0$.
(3) $(w)_{2n+1} = 0, \ n \geq 0$.
(4) The sequence $\{W_n\}_{n\geq 0}$ has the following quadratic decomposition

$$W_{2n}(x) = P_n\left(x^2\right), \quad W_{2n+1}(x) = xR_n\left(x^2\right), \quad n \geq 0,$$

where $\{P_n\}_{n\geq 0}$ is orthogonal with respect to the form $u = \sigma(w)$ and $\{R_n\}_{n\geq 0}$ is orthogonal with respect to the form $v = \gamma_1^{-1}x\sigma(w)$.

Thus,

$$S\big(w\big)(z) = zS\big(u\big)(z^2), \tag{6}$$
$$S\big(v\big)(z) = \gamma_1^{-1}zS\big(u\big)(z) + \gamma_1^{-1}, \tag{7}$$

On the other hand, if $\beta_0 \neq 0$, by using a suitable dilation of the form we can assume that $\beta_0 = 1$, i. e.

$$\beta_n = (-1)^n, \ \ n \geq 0. \tag{8}$$

Next result concerns the quasi-symmetric case.

Lemma 2.3. *Let $\{W_n\}_{n\geq 0}$ be a (MOPS) with respect to the form w. The following statements are equivalent:*

(1) $\{W_n\}_{n\geq 0}$ satisfies (5)-(8).
(2) $(w)_{2n+1} = (w)_{2n}, \ n \geq 0$.
(3) The sequence $\{W_n\}_{n\geq 0}$ has the following quadratic decomposition

$$W_{2n}(x) = P_n(x^2), \quad W_{2n+1}(x) = (x-1)R_n(x^2), \quad n \geq 0,$$

where $\{P_n\}_{n\geq 0}$ is a (MOPS) with respect to the form $u = \sigma(w)$ and $\{R_n\}_{n\geq 0}$ is a (MOPS) with respect to the form $v = \gamma_1^{-1}(x-1)\sigma(w)$.

Notice that

$$S\big(w\big)(z) = (z+1)S\big(u\big)(z^2), \tag{9}$$
$$S\big(v\big)(z) = \gamma_1^{-1}(z-1)S\big(u\big)(z) + \gamma_1^{-1}. \tag{10}$$

A form w is called semiclassical when it is regular and there exist two polynomials ϕ and ψ, ϕ monic, $\deg\phi \geq 0$, $\deg\psi \geq 1$, such that w satisfies a Pearson's equation

$$D(\phi w) + \psi w = 0. \tag{11}$$

Equivalently, the formal Stieltjes function of w satisfies a nonhomogeneous first order linear differential equation with polynomial coefficients

$$A_0(z)S'(w)(z) = C_0(z)S(w)(z) + D_0(z), \tag{12}$$

where

$$A_0 = \phi, \;\; C_0 = -\phi' - \psi, \;\; D_0 = -(u\theta_0\phi)' - (u\theta_0\psi). \tag{13}$$

Furthermore, if the polynomials A_0, C_0, and D_0 appearing in (13) are coprime, then the class of w is defined by

$$s = \max\{\deg C_0 - 1, \deg D_0\}.$$

If $\{W_n\}_{n\geq 0}$ is an (OPS) with respect to a semiclassical form w of class s then, $\{W_n\}_{n\geq 0}$ is called a semiclassical (OPS) of class s. In particular, when $s = 0$ (so that deg $\phi \leq 2$ and deg $\psi = 1$) one obtains, up to an affine change of variables, the four well-known families of classical forms : Hermite, $\mathcal{H}$; Laguerre, $\mathcal{L}(\alpha)$; Jacobi, $\mathcal{J}(\alpha, \beta)$ and Bessel, $\mathcal{B}(\alpha)$ (see[23]). Taking into account Jacobi linear forms $\mathcal{J}(\alpha, \beta)$ will be used in the sequel, we point out that $\phi(x) = x^2 - 1, \psi(x) = -(\alpha + \beta + 2)x + (\alpha - \beta)$.

The semiclassical character of a form is preserved by an affine transformation. Indeed, the shifted form $\widehat{w} = (h_{a^{-1}} \circ \tau_{-b})w$, $a \in \mathbb{C} - \{0\}$, $b \in \mathbb{C}$, is also semiclassical and has the same class as w. It satisfies

$$D\Big(a^{-\deg\phi}\phi(ax+b)\widehat{w}\Big) + a^{1-\deg\phi}\psi(ax+b)\widehat{w} = 0.$$

The sequence $\{\widehat{W}_n\}_{n\geq 0}$, where $\widehat{W}_n(x) = a^{-n}W_n(ax+b)$, $n \geq 0$, is orthogonal with respect to $\widehat{w}$. The recurrence coefficients are given by [22]

$$\widehat{\beta_n} = \frac{\beta_n - b}{a}, \quad \widehat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n \geq 0.$$

The formal Stieltjes function of $\widehat{w} = (h_{a^{-1}} \circ \tau_{-b})w$, $a \in \mathbb{C} - \{0\}$, $b \in \mathbb{C}$, satisfies [9]

$$S\Big(\widehat{w}\Big)(z) = aS(w)(az+b). \tag{14}$$

On the other hand, it is easy to check that for any $a \in \mathbb{C} - \{0\}$, $b \in \mathbb{C}$, and $w \in \mathcal{P}'$ we have

$$\Big((h_{a^{-1}} \circ \tau_{-b})w\Big)_n = n!a^{-n}\sum_{\nu+\mu=n}\frac{(-b)^\nu}{\nu!\mu!}(w)_\mu, \quad n \geq 0. \tag{15}$$

3. Second degree forms

In this section we recall the definition and some basic properties of the second degree regular forms which we will need later.

Definition 3.1. *[24] A regular form w is said to be a second degree form if there exist two polynomials B, monic, and C such that*

$$B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0, \tag{16}$$

where D depends on B, C and w.

Remark 3.2. *[24]*

1. *The regularity of w means that $B \neq 0$, $C^2 - 4BD \neq 0$, and $D \neq 0$.*
2. *$D(z) = (w\theta_0 C)(z) - (w^2\theta_0^2 B)(z)$.*

3. *It is a well-known result that, in terms of the form w* (16) *is equivalent to [24]*

$$B(x)w^2 = xC(x)w, \ \ \langle w^2, \theta_0 B\rangle = \langle w, C\rangle. \tag{17}$$

4. *The polynomials B and C, given in* (16) *and* (17)*, are not unique, because B and C can be multiplied by an arbitrary polynomial. If in* (16) *the polynomials B, C and D are coprime, then the pair* (B, C) *is called a primitive pair. The primitive pair is unique.*

It is well known that the Chebychev form of first kind $\mathcal{T} := \mathcal{J}\Big(-\frac{1}{2}, -\frac{1}{2}\Big)$ is a second degree form. Indeed, its Stieltjes function is $S(\mathcal{T})(z) = -(z^2-1)^{-\frac{1}{2}}$ and satisfies the quadratic equation $(z^2-1)S^2(\mathcal{T})(z) - 1 = 0$.

If w is a second degree form, then w is a semiclassical form and satisfies (11) with [24]

$$\lambda\phi(x) = B(x)\Big(C^2(x) - 4B(x)D(x)\Big), \ \ \lambda\psi(x) = -\tfrac{3}{2}B(x)\Big(C^2(x) - 4B(x)D(x)\Big)',$$

where λ is a normalization constant chosen in order to make $\phi(x)$ monic.

The second degree character is preserved by an affine transformation. Indeed, if w is a second degree form satisfying (16), then $\widehat{w}$ is also a second degree form [24]. Indeed,

$$\widehat{B}(z)S^2(\widehat{w})(z) + \widehat{C}(z)S(\widehat{w})(z) + \widehat{D}(z) = 0,$$

with $\widehat{B}(x) = a^{-r}B(ax+b)$, $\widehat{C}(x) = a^{1-r}C(ax+b)$, $\widehat{D}(x) = a^{2-r}D(ax+b)$, $r = \deg B$.
Elementary transformations of forms as linear and rational spectral transformations (see [37]), association (see [36]), antiassociation (see [28]) and inversion (see [22] preserve the family of linear forms of second degree [4, 5, 19, 24]. Moreover, we have

Lemma 3.3. *[5] Let u and v be two regular forms satisfying $q(x)u = p(x)v$, where q and p are polynomials. If one of the two forms u and v is a second degree form then the other one is also a second degree form. Indeed, if u is a second degree form and $B_uS^2(v) + C_uS(v) + D_u = 0$, then v is also a second degree form such*

$$B_vS^2(v) + C_vS(v) + D_v = 0,$$

with

$$\begin{aligned} B_v &= B_u p^2, \\ C_v &= p\Big\{2B_u\Big((v\theta_0 p) - (u\theta_0 q)\Big) + qC_u\Big\}, \\ D_v &= B_u\Big((v\theta_0 p) - (u\theta_0 q)\Big)^2 + qC_u\Big((v\theta_0 p) - (u\theta_0 q)\Big) + q^2 D_u. \end{aligned}$$

3.1. Second degree classical forms

In [5] the author analyze the classical forms which are of second degree.

Theorem 3.4. *[5] Among the classical forms, only the Jacobi forms $\mathcal{J}(p-1/2, q-1/2)$ are second degree forms, assuming $p+q \geq 0$, $p, q \in \mathbb{Z}$.*

Remark 3.5. *In the sequel, we denote $\mathcal{T}_{p,q} := \mathcal{J}\Big(p - \frac{1}{2}, q - \frac{1}{2}\Big)$, with $p+q \geq 0$, $p, q \in \mathbb{Z}$.*

We begin with a lemma that is stated in [5], which gives a relation between the forms $\mathcal{T}_{p,q}, p, q \in \mathbb{Z}, p+q \geq 0$ and the Tchebychev form of first kind $\mathcal{T}$. We report it here in a simpler version which is needed for our purposes. More precisely, the following lemma illustrates that all the forms $\mathcal{T}_{p,q}$ are rational perturbations of $\mathcal{T}$.

Lemma 3.6. *[5, Lemma 3.4]*
Let $p, q \in \mathbb{Z}$ with $p+q \geq 0$. The forms $\mathcal{T}_{p,q}$ and $\mathcal{T}$ are related by

$$f_{p,q}(x)\mathcal{T}_{p,q} = g_{p,q}(x)\mathcal{T}, \tag{18}$$

where $f_{p,q}(x)$ and $g_{p,q}(x)$ are polynomials defined by

$$f_{p,q}(x) = \Big\langle \mathcal{T}, (x+1)^{\frac{|p|+p}{2}}(x-1)^{\frac{|q|+q}{2}} \Big\rangle (x+1)^{\frac{|p|-p}{2}}(x-1)^{\frac{|q|-q}{2}}, \tag{19}$$

$$g_{p,q}(x) = \Big\langle \mathcal{T}_{p,q}, (x+1)^{\frac{|p|-p}{2}}(x-1)^{\frac{|q|-q}{2}} \Big\rangle (x+1)^{\frac{|p|+p}{2}}(x-1)^{\frac{|q|+q}{2}}. \tag{20}$$

The following remarks concerning the forms $\mathcal{T}_{p,q}$ defined as above will be very useful later on.

Remark 3.7. *1. From Lemma 3.3, taking into account the expression of the first kind Chebychev form and* (18)*, $\mathcal{T}_{p,q}$ is a second degree form since*

$$B_{p,q}(z)S^2\big(\mathcal{T}_{p,q}\big)(z) + C_{p,q}(z)S\big(\mathcal{T}_{p,q}\big)(z) + D_{p,q}(z) = 0, \tag{21}$$

with

$$\begin{aligned} B_{p,q}(z) &= (z^2-1)f^2_{p,q}(z), \\ C_{p,q}(z) &= 2(z^2-1)f_{p,q}(z)\Big((\mathcal{T}_{p,q}\theta_0 f_{p,q})(z) - (\mathcal{T}\theta_0 g_{p,q})(z)\Big), \\ D_{p,q}(z) &= (z^2-1)\Big((\mathcal{T}_{p,q}\theta_0 f_{p,q})(z) - (\mathcal{T}\theta_0 g_{p,q})(z)\Big)^2 - g^2_{p,q}(z). \end{aligned} \tag{22}$$

2. Using the first order linear differential equation satisfied by the Stieltjes function of the Jacobi form [22], it is a straightforward exercise to prove that $S(\mathcal{T}_{p,q})(z)$ satisfies

$$\Phi(z)S'(\mathcal{T}_{p,q})(z) = C_0^{p,q}(z)S(\mathcal{T}_{p,q})(z) + D_0^{p,q}(z), \tag{23}$$

where $\Phi(z)$, $C_0^{p,q}(z)$, and $D_0^{p,q}(z)$ are polynomials given by

$$\Phi(z) = z^2-1, \;\; C_0^{p,q}(z) = (p+q-1)z + q - p, \;\; D_0^{p,q}(z) = p+q. \tag{24}$$

3. Let us recall that the moments of the Jacobi form $\mathcal{T}_{p,q}$, where $p+q \geq 0,\ p,q \in \mathbb{Z}$, are given by [23]

$$\big(\mathcal{T}_{p,q}\big)_n = \sum_{\nu=0}^{n} \binom{n}{\nu} 2^{\nu-1} \frac{\Gamma(p+q+1)}{\Gamma(\nu+p+q+1)} F_{n,\nu}\big(p-\tfrac{1}{2}, q-\tfrac{1}{2}\big), \;\; n \geq 0, \tag{25}$$

where

$$F_{n,\nu}\big(p-\tfrac{1}{2}, q-\tfrac{1}{2}\big) = (-1)^{n-\nu}\frac{\Gamma(\nu+p+\frac{1}{2})}{\Gamma(p+\frac{1}{2})} + (-1)^{\nu}\frac{\Gamma(\nu+q+\frac{1}{2})}{\Gamma(q+\frac{1}{2})}, \tag{26}$$

and Γ is the gamma function [23].

Remark 3.8. *In the sequel, we denote $\widehat{\mathcal{T}}_{p,q} := (h_{(-1/2} \circ \tau_{-1})\mathcal{T}_{p,q}$, with $p+q \geq 0,\ p,q \in \mathbb{Z}$.*

In the sequel, we need the following lemma:

Corollary 3.9. *One has*

$$S(\widehat{\mathcal{T}})(z^2) = z^{-1}S\big(\mathcal{T}\big)(z). \tag{27}$$

Proof. From Remarks 3.5 and 3.8 we get $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}_{0,0} = (h_{(-2)^{-1}} \circ \tau_{-1})\mathcal{T}$. As a consequence,

$$\begin{aligned} S(\widehat{\mathcal{T}})(z^2) &\overset{by\ (14)}{=} -2S(\mathcal{T})(-2z^2+1) \\ &= 2\left(4z^4 - 4z^2\right)^{-\frac{1}{2}} \\ &= -z^{-1}\left(z^2-1\right)^{-\frac{1}{2}} \\ &= z^{-1}S\left(\mathcal{T}\right)(z). \end{aligned}$$

□

4. Second degree symmetric semiclassical forms of class one

In this section, we obtain several characterizations for the symmetric semiclassical forms of class one which are of second degree, by pointing out the connection with the forms $\mathcal{T}_{p,q}$, their corresponding Stieltjes function and their moments.

In [4] all the second degree symmetric semiclassical forms of class one are determined. Indeed, we keep the same notation in [4], let $\mathcal{I}(\mu,\nu)$ which is regular if $\mu \neq -n,\ \nu \neq -n,\ \mu+\nu \neq -n-1,\ n \in \mathbb{N}^*$, and satisfies the following distributional equation:

$$\left(x(x^2-1)\mathcal{I}(\mu,\nu)\right)' + \left(-2(\mu+\nu+2)x^2 + 2(\nu+1)\right)\mathcal{I}(\mu,\nu) = 0.$$

Moreover, the linear form $\sigma\left(\mathcal{I}(\mu,\nu)\right)$ is classical fulfilling

$$\left(h_{(-1/2)^{-1}} \circ \tau_{-1/2}\right)\sigma\left(\mathcal{I}(\mu,\nu)\right) = \mathcal{J}(\mu,\nu). \tag{28}$$

Only one solution appears, up to affine transformation. Indeed,

Theorem 4.1. *[4] Among the symmetric semiclassical forms of class $s=1$, only the forms $\mathcal{I}\left(p-\frac{1}{2}, q-\frac{1}{2}\right)$ are second degree forms, provided $p+q \geq 0,\ q \neq 0,\ p,q \in \mathbb{Z}$.*

The corresponding orthogonal polynomials are known in the literature as Generalized Gegenbauer ([8], [13]). The coefficients of the three term recurrence relation they satisfy are explicitly given therein.

The integral representation of the linear form is given. In particular, a positive symmetric semiclassical form of class $s=1$ is of second degree if and only if the weight function is a Christoffel perturbation $x^{2q}(1-x^2)^p$, p a nonnegative integer number and q a positive integer number, of the weight of Chebychev polynomials of first kind.

The main result of this section provides a characterization of the second degree symmetric semiclassical forms of class one in terms of their formal Stieltjes function (that is explicitly given) and, as consequence, the moments are deduced.

Proposition 4.2. *Let w be a regular form. The following statements are equivalent.*

(a) w is a second degree symmetric semiclassical form of class one.

(b) There exists $(p,q) \in \mathbb{Z} \times \mathbb{Z}^*, p+q \geq 0$, such that

$$f_{p,q}(-2x^2+1)w = g_{p,q}(-2x^2+1)\mathcal{T}, \tag{29}$$

and

$$\left(w\theta_0(f_{p,q}(-2x^2+1))\right)(z) = 2z\left(\left(\mathcal{T}\theta_0 g_{p,q}\right) - \left(\mathcal{T}_{p,q}\theta_0 f_{p,q}\right)\right)(-2z^2+1) + \left(\mathcal{T}\theta_0(g_{p,q}(-2x^2+1))\right)(z), \tag{30}$$

where $f_{p,q}$ and $g_{p,q}$ are polynomials defined by (19) *and* (20)*, respectively.*

(c) There exists $(p,q)\in\mathbb{Z}\times\mathbb{Z}^*$ *with* $p+q\geq 0$ *such that*

$$S(w)(z) = zS\big(\widehat{\mathcal{T}}_{p,q}\big)(z^2). \tag{31}$$

(d) There exists $(p,q)\in\mathbb{Z}\times\mathbb{Z}^*$ *with* $p+q\geq 0$ *such that*

$$w = x\boldsymbol{\alpha}\big(\widehat{\mathcal{T}}_{p,q}\big).$$

(e) There exists $(p,q)\in\mathbb{Z}\times\mathbb{Z}^*$ *with* $p+q\geq 0$ *such that*

$$(w)_{2n+1} = 0, \tag{32}$$

$$(w)_{2n} = n!(-2)^{-n}\sum_{\nu+\mu=n}\frac{1}{\nu!\mu!}\sum_{i=0}^{\mu}\binom{\mu}{i}2^{i-1}\frac{\Gamma(p+q+1)}{\Gamma(i+p+q+1)}F_{\mu,i}\big(p-\tfrac{1}{2},q-\tfrac{1}{2}\big),\quad n\geq 0, \tag{33}$$

where $F_{\mu,i}\big(p-\frac{1}{2},q-\frac{1}{2}\big)$ *is defined by* (26).

Proof. $(a)\Rightarrow(b)$ Let w be a second degree symmetric semiclassical form of class one. Taking into account Theorem 4.1, there exists $(p,q)\in\mathbb{Z}\times\mathbb{Z}^*$ with $p+q\geq 0$ such that

$$w = \mathcal{I}\left(p-\frac{1}{2},q-\frac{1}{2}\right).$$

From (1), (28) becomes

$$u = (h_{(-2)^{-1}}\circ\tau_{-1})\mathcal{T}_{p,q}. \tag{34}$$

According to (6), (14) and (34), we get

$$S(w)(z) = -2zS(\mathcal{T}_{p,q})(-2z^2+1).$$

Multiplying both sides of last equation by $f_{p,q}(-2z^2+1)$, from (4) we deduce

$$\begin{aligned}
f_{p,q}(-2z^2+1)S(w)(z) &= -2zS\big(f_{p,q}\mathcal{T}_{p,q}\big)(-2z^2+1) + 2z\big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\big)(-2z^2+1)\\
&\overset{by\ (18)}{=} -2zS\big(g_{p,q}\mathcal{T}\big)(-2z^2+1) + 2z\big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\big)(-2z^2+1)\\
&\overset{by\ (4)}{=} -2zg_{p,q}(-2z^2+1)S(\mathcal{T})(-2z^2+1) - 2z\Big(\big(\mathcal{T}\theta_0 g_{p,q}\big) - \big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\big)\Big)(-2z^2+1)\\
&\overset{by\ (27)}{=} g_{p,q}(-2z^2+1)S\big(\mathcal{T}\big)(z) - 2z\Big(\big(\mathcal{T}\theta_0 g_{p,q}\big) - \big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\big)\Big)(-2z^2+1).
\end{aligned}$$

Using (4), the above relation reads as

$$S\big(f_{p,q}(-2x^2+1)w\big)(z) = S\big(g_{p,q}(-2x^2+1)\mathcal{T}\big)(z) + P(z),$$

with

$$\begin{aligned}
P(z) = &-2z\Big(\big(\mathcal{T}\theta_0 g_{p,q}\big) - \big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\big)\Big)(-2z^2+1)\\
&+\big(w\theta_0(f_{p,q}(-2x^2+1))\big)(z) - \big(\mathcal{T}\theta_0(g_{p,q}(-2x^2+1))\big)(z),
\end{aligned}$$

or equivalently,

$$S\big(f_{p,q}(-2x^2+1)w - g_{p,q}(-2x^2+1)\mathcal{T}\big)(z) = P(z)\in\mathcal{P}.$$

Thus, taking into Remark 2.1 we get

$$f_{p,q}(-2x^2+1)w - g_{p,q}(-2x^2+1)\mathcal{T} = 0 \quad \text{in } \mathcal{P}',$$

and

$$P(z) = 0.$$

Thus the result follows.
$(b) \Rightarrow (c)$ Applying the operator S to (29) and taking into account (4) we get

$$\begin{aligned} f_{p,q}(-2z^2+1)S(w)(z) =& g_{p,q}(-2z^2+1)S\Big(\mathcal{T}\Big)(z) - \Big(w\theta_0(f_{p,q}(-2x^2+1))\Big)(z) \\ &+ \Big(\mathcal{T}\theta_0(g_{p,q}(-2x^2+1))\Big)(z). \end{aligned}$$

Thus, from (27)

$$\begin{aligned} f_{p,q}(-2z^2+1)S(w)(z) =& -2zg_{p,q}(-2z^2+1)S\Big(\mathcal{T}\Big)(-2z^2+1) - \Big(w\theta_0(f_{p,q}(-2x^2+1))\Big)(z) \\ &+ \Big(\mathcal{T}\theta_0(g_{p,q}(-2x^2+1))\Big)(z) \\ \overset{by\ (4)-(18)}{=}& -2zS\Big(f_{p,q}\mathcal{T}_{p,q}\Big)(-2z^2+1) - \Big(w\theta_0(f_{p,q}(-2x^2+1))\Big)(z) \\ &+ \Big(\mathcal{T}\theta_0(g_{p,q}(-2x^2+1))\Big)(z) + 2z\Big(\mathcal{T}\theta_0 g_{p,q}\Big)(-2z^2+1) \\ \overset{by\ (4)}{=}& -2zf_{p,q}(-2z^2+1)S\Big(\mathcal{T}_{p,q}\Big)(-2z^2+1) - \Big(w\theta_0(f_{p,q}(-2x^2+1))\Big)(z) \\ &+ \Big(\mathcal{T}\theta_0(g_{p,q}(-2x^2+1))\Big)(z) + 2z\Big(\mathcal{T}\theta_0 g_{p,q}\Big)(-2z^2+1) - 2z\Big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\Big)(-2z^2+1). \end{aligned}$$

Therefore, by using (30), the last equation becomes

$$f_{p,q}(-2z^2+1)S(w)(z) = -2zf_{p,q}(-2z^2+1)S\Big(\mathcal{T}_{p,q}\Big)(-2z^2+1).$$

As a consequence,

$$S(w)(z) = zS\Big(\widehat{\mathcal{T}}_{p,q}\Big)(z^2).$$

The statement (c) holds.
$(c) \Rightarrow (d)$ First, observe that

$$S\left(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\right)(z) = -\sum_{n\geq 0}\frac{\Big(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_n}{z^{n+1}} \overset{by\ (2)}{=} -\sum_{n\geq 0}\frac{\Big(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n+1}}{z^{2n+2}} \overset{by\ (2)}{=} -\sum_{n\geq 0}\frac{\Big(\widehat{\mathcal{T}}_{p,q}\Big)_n}{z^{2(n+1)}} = S\Big(\widehat{\mathcal{T}}_{p,q}\Big)(z^2). \tag{35}$$

Together with (31) we have

$$S(w)(z) = zS\left(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\right)(z).$$

Therefore, using (4), the last equation becomes

$$S(w)(z) = S\left(x\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\right)(z) - 1.$$

By using Remark 2.1 the desired relation holds.
$(d) \Rightarrow (e)$

$$(w)_{2n+1} = \Big(x\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n+1} = \Big(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n+2} \overset{by\ (2)}{=} 0, \quad n \geq 0. \tag{36}$$

$$(w)_{2n} = \Big(x\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n} = \Big(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n+1} \overset{by\ (2)}{=} \Big(\widehat{\mathcal{T}}_{p,q}\Big)_n, \quad n \geq 0. \tag{37}$$

Using (15) and taking into account (25)-(26) (32)-(33) follow in a straightforward way.
$(e) \Rightarrow (a)$ By hypothesis we have

$$(w)_{2n+1} = 0, \quad (w)_{2n} = \Big(\widehat{\mathcal{T}}_{p,q}\Big)_n, \quad n \geq 0.$$

It remains to show that

$$S(w)(z) = -\sum_{n\geq 0} \frac{(w)_{2n}}{z^{2n+1}} - \sum_{n\geq 0} \frac{(w)_{2n+1}}{z^{2n+2}} = -\sum_{n\geq 0} \frac{\left(\widehat{\mathcal{T}}_{p,q}\right)_n}{z^{2(n+1)}} = zS\left(\widehat{\mathcal{T}}_{p,q}\right)(z^2). \tag{38}$$

Moreover, the fact that the affine transformation of a second degree form is also a second degree form yields $\widehat{\mathcal{T}}_{p,q}$ is a second degree form such that

$$\widehat{B}_{p,q}(z)S^2\left(\widehat{\mathcal{T}}_{p,q}\right)(z) + \widehat{C}_{p,q}(z)S\left(\widehat{\mathcal{T}}_{p,q}\right)(z) + \widehat{D}_{p,q}(z) = 0, \tag{39}$$

with

$$\widehat{B}_{p,q}(z) = (-2)^{-t}B_{p,q}(-2z+1),\ \ \widehat{C}_{p,q}(z) = (-2)^{1-t}C_{p,q}(-2z+1),\ \ \widehat{D}_{p,q}(z) = (-2)^{2-t}D_{p,q}(-2z+1),$$

where $B_{p,q}$, $C_{p,q}$ and $D_{p,q}$ are polynomials given in (22) and $t = \frac{|p|-p+|q|-q}{2}$.
Making $z \leftarrow z^2$ in (39), multiplying this equation by z^2 and taking into account (38) we get

$$\widehat{B}_{p,q}(z^2)S^2(w)(z) + z\widehat{C}_{p,q}(z^2)S(w)(z) + z^2\widehat{D}_{p,q}(z^2) = 0.$$

As a consequence, w is a second degree form.
To finish the proof, it remains to prove that the class of w is one. Based on (14), the relation (38) becomes

$$S(w)(z) = -2zS(\mathcal{T}_{p,q})(-2z^2+1). \tag{40}$$

Taking formal derivatives in the last equation we get

$$S'(w)(z) = 8z^2S'(\mathcal{T}_{p,q})(-2z^2+1) - 2S(\mathcal{T}_{p,q})(-2z^2+1).$$

From the above two expressions we obtain

$$S'(\mathcal{T}_{p,q})\left(-2z^2+1\right) = \frac{zS'(w)(z) - S(w)(z)}{8z^3}. \tag{41}$$

In (23) the change of variable $z \leftarrow -2z^2+1$ yields

$$\Phi(-2z^2+1)S'(\mathcal{T}_{p,q})(-2z^2+1) = C_0^{p,q}(-2z^2+1)S(\mathcal{T}_{p,q})(-2z^2+1) + D_0^{p,q}(-2z^2+1). \tag{42}$$

Replacing (40) and (41) in (42), and multiplying both sides of the resulting equation by $8z^3$, one obtains

$$\phi_w(z)S'(w)(z) = C_w(z)S(w)(z) + D_w(z), \tag{43}$$

where the polynomials ϕ_w, C_w and D_w are

$$\begin{aligned}
&\phi_w(z) = z\Phi(-2z^2+1),\\
&C_w(z) = \Phi(-2z^2+1) - 4z^2C_0^{p,q}(-2z^2+1),\\
&D_w(z) = 8z^3D_0^{p,q}(-2z^2+1).
\end{aligned}$$

Therefore, from (24) $S(w)(z)$ fulfils (43) with

$$\begin{aligned}
&\phi_w(z) = 4z^3(z^2-1),\\
&C_w(z) = 4z^2(z^2-1) - 4z^2\Big[(p+q-1)(-2z^2+1) + q - p\Big],\\
&D_w(z) = 8(p+q)z^3.
\end{aligned} \tag{44}$$

Therefore, the polynomials ϕ_w, C_w, and D_w given by (44) have $4z^2$ as a common factor, and so dividing these polynomials by $4z^2$ we obtain

$$\phi_w(z) = z(z^2-1),$$
$$C_w(z) = (2p+2q-1)z^2 - 2q,$$
$$D_w(z) = 2(p+q)z.$$

Now, taking into account that $q \neq 0$ and $2p+2q-1 \neq 0$ hold, then ϕ_w, C_w, and D_w are coprime. As a consequence, since $\deg D_w = 1$ and $\deg C_w = 2$, the class of w is one. □

5. Second degree quasi-symmetric semiclassical forms of class one

In this section we establish several characterizations of the semiclassical forms of class one which are of second degree such that their corresponding (MOPS) verifies (5)-(8). In particular, we focus our attention on the link between these forms and the classical forms. Notice that all the quasi-symmetric semiclassical forms of class one are determined by several authors with different methods [12, 26, 30]. The unique solution, up to affine transformation, is the linear form $\mathcal{L}(\alpha,\beta)$ that is regular if $\alpha \neq -n, \beta \neq -n, \alpha+\beta \neq -n-1, n \in \mathbb{N}$, and satisfies the Pearson equation

$$\left(x\left(x^2-1\right)\mathcal{L}(\alpha,\beta)\right)' + \left(-2(\alpha+\beta+2)x^2 + x + 2\beta + 3\right)\mathcal{L}(\alpha,\beta) = 0.$$

In [2, 31], the authors give all the second degree quasi-symmetric semiclassical forms of class one. In the sequel, we keep the same notation in [2]. Let $\mathcal{K}(k,l)$ be the form satisfying the following distributional equation

$$D\left(x\left(x^2-1\right)\mathcal{K}(k,l)\right) + \left(-2(k+l+1)x^2 + x + 2l\right)\mathcal{K}(k,l) = 0.$$

Moreover, the linear form $\sigma(\mathcal{K}(k,l))$ is classical and

$$\left(h_{(-1/2)^{-1}} \circ \tau_{-1/2}\right)\sigma(\mathcal{K}(k,l)) = \mathcal{J}\left(k-\frac{1}{2}, l-\frac{1}{2}\right). \tag{45}$$

Theorem 5.1. *[2] Among the semiclassical forms of class $s=1$ such that the corresponding (MOPS) $\{W_n\}_{n\geq 0}$ satisfies* (5) *with $\beta_0 = 1$, only the forms $\mathcal{K}(k,l)$ with $k+l \geq 1$, $k,l \in \mathbb{Z}$ are second degree forms.*

We are now ready to state the main result of this section.

Proposition 5.2. *Let w be a regular form. The following statements are equivalent.*

(a) The form w is a second degree quasi-symmetric semiclassical form of class one.

(b) There exists $(p,q) \in \mathbb{Z}^2$ with $p+q \geq 1$ such that

$$x f_{p,q}(-2x^2+1)w = (x+1)g_{p,q}(-2x^2+1)\mathcal{T}, \tag{46}$$

and

$$\begin{aligned}\left(w(f_{p,q}(-2x^2+1))\right)(z) =& 2z(z+1)\left(\left(\mathcal{T}\theta_0 g_{p,q}\right) - \left(\mathcal{T}_{p,q}\theta_0 f_{p,q}\right)\right)(-2z^2+1) \\ &+ \left(\mathcal{T}\theta_0((x+1)g_{p,q}(-2x^2+1))\right)(z),\end{aligned} \tag{47}$$

where $f_{p,q}$ and $g_{p,q}$ are polynomials defined by (19) *and* (20), *respectively.*

(c) There exists $(p,q) \in \mathbb{Z}^2$ with $p+q \geq 1$ such that

$$S(w)(z) = (z+1)S\left(\widehat{\mathcal{T}}_{p,q}\right)(z^2). \tag{48}$$

(d) *There exists* $(p,q) \in \mathbb{Z}^2$ *with* $p+q \geq 1$ *such that*

$$w = (x+1)\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}_{p,q}}\Big).$$

(e) *There exists* $(p,q) \in \mathbb{Z}^2$ *with* $p+q \geq 1$ *such that*

$$(w)_{2n} = (w)_{2n+1} = n!(-2)^{-n} \sum_{\nu+\mu=n} \frac{1}{\nu!\mu!} \sum_{i=0}^{\mu} \binom{\mu}{i} 2^{i-1} \frac{\Gamma(p+q+1)}{\Gamma(i+p+q+1)} F_{\mu,i}\Big(p-\tfrac{1}{2}, q-\tfrac{1}{2}\Big), \quad n \geq 0, \tag{49}$$

where $F_{\mu,i}\Big(p-\frac{1}{2}, q-\frac{1}{2}\Big)$ *is defined by* (26).

Proof. $(a) \Rightarrow (b)$ Let w be a second degree quasi-symmetric semiclassical form of class one. According to Theorem 5.1, there exists $(p,q) \in \mathbb{Z}^2$ with $p+q \geq 1$ such that

$$w = \mathcal{K}(p,q).$$

From (1), (45) becomes

$$u = \sigma(w) = (h_{(-2)^{-1}} \circ \tau_{-1})\mathcal{T}_{p,q}. \tag{50}$$

Taking into account (9), (14) and (50), we get

$$S(w)(z) = -2(z+1)S(\mathcal{T}_{p,q})(-2z^2+1).$$

Multiplying both sides of last equation by $f_{p,q}(-2z^2+1)$, from (4) we deduce

$$\begin{aligned} f_{p,q}(-2z^2+1)S(w)(z) &= -2(z+1)S\Big(f_{p,q}\mathcal{T}_{p,q}\Big)(-2z^2+1) + 2(z+1)\Big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\Big)(-2z^2+1) \\ &\overset{by\ (18)}{=} -2(z+1)S\Big(g_{p,q}\mathcal{T}\Big)(-2z^2+1) + 2(z+1)\Big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\Big)(-2z^2+1) \\ &\overset{by\ (4)}{=} -2(z+1)g_{p,q}(-2z^2+1)S(\mathcal{T})(-2z^2+1) - 2(z+1)\Big(\Big(\mathcal{T}\theta_0 g_{p,q}\Big) - \Big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\Big)\Big)(-2z^2+1) \\ &\overset{by\ (27)}{=} (z+1)g_{p,q}(-2z^2+1)z^{-1}S\Big(\mathcal{T}\Big)(z) - 2(z+1)\Big(\Big(\mathcal{T}\theta_0 g_{p,q}\Big) - \Big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\Big)\Big)(-2z^2+1). \end{aligned}$$

Multiplying by z and using (4), the latter becomes

$$S\Big(xf_{p,q}(-2x^2+1)w\Big)(z) = S\Big((x+1)g_{p,q}(-2x^2+1)\mathcal{T}\Big)(z) + R(z),$$

with

$$\begin{aligned} R(z) = &-2z(z+1)\Big(\Big(\mathcal{T}\theta_0 g_{p,q}\Big) - \Big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\Big)\Big)(-2z^2+1) \\ &+ \Big(w(f_{p,q}(-2x^2+1))\Big)(z) - \Big(\mathcal{T}\theta_0((x+1)g_{p,q}(-2x^2+1))\Big)(z), \end{aligned}$$

or equivalently,

$$S\Big(xf_{p,q}(-2x^2+1)w - (x+1)g_{p,q}(-2x^2+1)\mathcal{T}\Big)(z) = R(z) \in \mathcal{P}.$$

According to Remark 2.1

$$xf_{p,q}(-2x^2+1)w - (x+1)g_{p,q}(-2x^2+1)\mathcal{T} = 0 \quad \text{in } \mathcal{P}',$$

and

$$R(z) = 0.$$

The statement is proved.
$(b) \Rightarrow (c)$ Applying the operator S to (46) and taking into account formula (4) we get

$$\begin{aligned} zf_{p,q}(-2z^2+1)S(w)(z) =& (z+1)g_{p,q}(-2z^2+1)S\Big(\mathcal{T}\Big)(z) - \Big(w(f_{p,q}(-2x^2+1))\Big)(z) \\ &+ \Big(\mathcal{T}\theta_0((x+1)g_{p,q}(-2x^2+1))\Big)(z). \end{aligned}$$

From (27)

$$\begin{aligned} zf_{p,q}(-2z^2+1)S(w)(z) =& -2z(z+1)g_{p,q}(-2z^2+1)S\Big(\mathcal{T}\Big)(-2z^2+1) - \Big(w(f_{p,q}(-2x^2+1))\Big)(z) \\ &+ \Big(\mathcal{T}\theta_0((x+1)g_{p,q}(-2x^2+1))\Big)(z) \\ \overset{by\ (4)-(18)}{=}& -2z(z+1)S\Big(f_{p,q}\mathcal{T}_{p,q}\Big)(-2z^2+1) - \Big(w(f_{p,q}(-2x^2+1))\Big)(z) \\ &+ \Big(\mathcal{T}\theta_0((x+1)g_{p,q}(-2x^2+1))\Big)(z) + 2z(z+1)\Big(\mathcal{T}\theta_0 g_{p,q}\Big)(-2z^2+1) \\ \overset{by\ (4)}{=}& -2z(z+1)f_{p,q}(-2z^2+1)S\Big(\mathcal{T}_{p,q}\Big)(-2z^2+1) - \Big(w(f_{p,q}(-2x^2+1))\Big)(z) \\ &+ \Big(\mathcal{T}\theta_0((x+1)g_{p,q}(-2x^2+1))\Big)(z) + 2z(z+1)\Big(\Big(\mathcal{T}\theta_0 g_{p,q}\Big) - \Big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\Big)\Big)(-2z^2+1). \end{aligned}$$

Therefore, using (47), the last equation becomes

$$zf_{p,q}(-2z^2+1)S(w)(z) = -2z(z+1)f_{p,q}(-2z^2+1)S\Big(\mathcal{T}_{p,q}\Big)(-2z^2+1),$$

which readily gives

$$S(w)(z) = (z+1)S\Big(\widehat{\mathcal{T}}_{p,q}\Big)(z^2).$$

This yields the statement (c).
$(c) \Rightarrow (d)$ From the definition of the anti-symmetrization operator $\boldsymbol{\alpha}$ we have

$$S\left(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\right)(z) = -\sum_{n\geq 0} \frac{\Big(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_n}{z^{n+1}} \overset{by\ (2)}{=} -\sum_{n\geq 0} \frac{\Big(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n+1}}{z^{2n+2}} \overset{by\ (2)}{=} -\sum_{n\geq 0} \frac{\Big(\widehat{\mathcal{T}}_{p,q}\Big)_n}{z^{2(n+1)}} = S\Big(\widehat{\mathcal{T}}_{p,q}\Big)(z^2). \tag{51}$$

Combined with (48) we then have

$$S(w)(z) = (z+1)S\left(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\right)(z).$$

Therefore, using (4), the last equation becomes

$$S(w)(z) = S\left((x+1)\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\right)(z) - 1.$$

Then, using Remark 2.1 we have the desired relation.
$(d) \Rightarrow (e)$

$$\begin{aligned} (w)_{2n+1} &= \Big((x+1)\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n+1} = \Big(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n+2} + \Big(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n+1} \overset{by\ (2)}{=} \Big(\widehat{\mathcal{T}}_{p,q}\Big)_n, \quad n \geq 0, \\ (w)_{2n} &= \Big((x+1)\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n} = \Big(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n+1} + \Big(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n} \overset{by\ (2)}{=} \Big(\widehat{\mathcal{T}}_{p,q}\Big)_n, \quad n \geq 0. \end{aligned}$$

Using (15) and taking into account (25)-(26) we deduce the statement (e).
$(e) \Rightarrow (a)$ By hypothesis we have

$$(w)_{2n} = (w)_{2n+1} = \Big(\widehat{\mathcal{T}}_{p,q}\Big)_n,\ n \geq 0.$$

Then,

$$S(w)(z) = -\sum_{n\geq 0} \frac{(w)_{2n}}{z^{2n+1}} - \sum_{n\geq 0} \frac{(w)_{2n+1}}{z^{2n+2}} = -z\sum_{n\geq 0} \frac{\left(\widehat{\mathcal{T}}_{p,q}\right)_n}{z^{2(n+1)}} - \sum_{n\geq 0} \frac{\left(\widehat{\mathcal{T}}_{p,q}\right)_n}{z^{2(n+1)}} = (z+1)S\left(\widehat{\mathcal{T}}_{p,q}\right)(z^2). \tag{52}$$

On the other hand, the fact that the affine transformation of a second degree form is also a second degree form yields $\widehat{\mathcal{T}}_{p,q}$ is a second degree form such that

$$\widehat{B}_{p,q}(z)S^2\left(\widehat{\mathcal{T}}_{p,q}\right)(z) + \widehat{C}_{p,q}(z)S\left(\widehat{\mathcal{T}}_{p,q}\right)(z) + \widehat{D}_{p,q}(z) = 0, \tag{53}$$

with

$$\widehat{B}_{p,q}(z) = (-2)^{-t}B_{p,q}(-2z+1),\ \ \widehat{C}_{p,q}(z) = (-2)^{1-t}C_{p,q}(-2z+1),\ \ \widehat{D}_{p,q}(z) = (-2)^{2-t}D_{p,q}(-2z+1),$$

where $B_{p,q}$, $C_{p,q}$ and $D_{p,q}$ are polynomials given in (22), and $t = \frac{|p|-p+|q|-q}{2}$.
Making $z \leftarrow z^2$ in (53), multiplying this equation by $(z+1)^2$ and on account of (52) we get

$$\widehat{B}_{p,q}(z^2)S^2(w)(z) + (z+1)\widehat{C}_{p,q}(z^2)S(w)(z) + (z+1)^2\widehat{D}_{p,q}(z^2) = 0.$$

As a consequence, we conclude that w is a second degree form and, thus, w is a semiclassical form.
Based on (14), (52) becomes

$$S(w)(z) = -2(z+1)S(\mathcal{T}_{p,q})(-2z^2+1). \tag{54}$$

The formal derivative of the last formula gives

$$S'(w)(z) = 8z(z+1)S'(\mathcal{T}_{p,q})(-2z^2+1) - 2S(\mathcal{T}_{p,q})(-2z^2+1).$$

From the above two expressions we obtain

$$S'(\mathcal{T}_{p,q})\left(-2z^2+1\right) = \frac{(z+1)S'(w)(z) - S(w)(z)}{8z(z+1)^2}. \tag{55}$$

In (23) the change of variable $z \leftarrow -2z^2+1$ yields

$$\Phi(-2z^2+1)S'(\mathcal{T}_{p,q})(-2z^2+1) = C_0^{p,q}(-2z^2+1)S(\mathcal{T}_{p,q})(-2z^2+1) + D_0^{p,q}(-2z^2+1). \tag{56}$$

Replacing (54) and (55) in (56), and multiplying both sides of the resulting equation by $8z(z+1)^2$, one obtains

$$\phi_w(z)S'(w)(z) = C_w(z)S(w)(z) + D_w(z), \tag{57}$$

where the polynomials ϕ_w, C_w and D_w are

$$\begin{aligned}
\phi_w(z) &= (z+1)\Phi(-2z^2+1),\\
C_w(z) &= \Phi(-2z^2+1) - 4z(z+1)C_0^{p,q}(-2z^2+1),\\
D_w(z) &= 8z(z+1)^2D_0^{p,q}(-2z^2+1).
\end{aligned}$$

Therefore, it follows from (24) that $S(w)(z)$ fulfils (57) with

$$\begin{aligned}
\phi_w(z) &= 4(z+1)z^2(z^2-1),\\
C_w(z) &= 4z^2(z^2-1) - 4z(z+1)\Big[(p+q-1)(-2z^2+1) + q - p\Big],\\
D_w(z) &= 8(p+q)z(z+1)^2.
\end{aligned} \tag{58}$$

Therefore, the polynomials ϕ_w, C_w, and D_w given by (58) have $4z(z+1)$ as a common factor, and so dividing these polynomials by $4z(z+1)$ we obtain

$$\phi_w(z) = z(z^2-1),$$
$$C_w(z) = (2p+2q-1)z^2 - z - 2q + 1,$$
$$D_w(z) = 2(p+q)(z+1).$$

Now, we see that the conditions $2p+2q-1 \neq 0$ and $-2q+1 \neq 0$ hold. Thus ϕ_w, C_w, and D_w are coprime and since $\deg D_w \leq 1$ and $\deg C_w = 2$ the class of w is one. Thus, the proof is finished. □

6. A new family of second degree semiclassical forms of class one

Proposition 6.1. *Let w be a regular form. The following statements are equivalent.*

(a) There exist $c \in \mathbb{C}$ with $|c| > 1$ and $(p,q) \in \mathbb{Z}^2$ with $p+q \geq 0$ such that

$$w = (x-c)\mathcal{T}_{p,q}. \tag{59}$$

(b) There exist $c \in \mathbb{C}$ with $|c| > 1$ and $(p,q) \in \mathbb{Z}^2$ with $p+q \geq 0$ such that

$$S(w)(z) = (z-c)S\big(\mathcal{T}_{p,q}\big)(z) + 1. \tag{60}$$

(c) There exist $c \in \mathbb{C}$ with $|c| > 1$ and $(p,q) \in \mathbb{Z}^2$ with $p+q \geq 0$ such that

$$f_{p,q}(x)w = (x-c)g_{p,q}(x)\mathcal{T}, \tag{61}$$

and

$$\big(w\theta_0 f_{p,q}\big)(z) = -(z-c)\Big(\big(\mathcal{T}\theta_0 g_{p,q}\big) - \big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\big)\Big)(z) + \Big(\mathcal{T}\theta_0((x-c)g_{p,q}(x))\Big)(z) - f_{p,q}(z), \tag{62}$$

where $f_{p,q}$ and $g_{p,q}$ are polynomials defined by (19) *and* (20)*, respectively.*

(d) There exist $c \in \mathbb{C}$ with $|c| > 1$ and $(p,q) \in \mathbb{Z}^2$ with $p+q \geq 0$ such that

$$(w)_n = \sum_{\nu=0}^{n+1}\binom{n+1}{\nu}2^{\nu-1}\frac{\Gamma(p+q+1)}{\Gamma(\nu+p+q+1)}F_{n+1,\nu}\Big(p-\tfrac{1}{2}, q-\tfrac{1}{2}\Big)$$
$$- c\sum_{\nu=0}^{n}\binom{n}{\nu}2^{\nu-1}\frac{\Gamma(p+q+1)}{\Gamma(\nu+p+q+1)}F_{n,\nu}\Big(p-\tfrac{1}{2}, q-\tfrac{1}{2}\Big),\quad n \geq 0,$$

where $F_{n,\nu}\big(p-\frac{1}{2}, q-\frac{1}{2}\big)$ is defined by (26).

(e) There exist $c \in \mathbb{C}$ with $|c| > 1$ and $(p,q) \in \mathbb{Z}^2$ with $p+q \geq 0$ such that the form w is a second degree semiclassical form of class one satisfying

$$\phi_w(z)S'(w)(z) = C_w(z)S(w)(z) + D_w(z),$$

with

$$\phi_w(z) = (x-c)(x^2-1),$$
$$C_w(z) = (p+q)z^2 + \big(q-p-c(p+q-1)\big)z - c(q-p) - 1,$$
$$D_w(z) = -\big(q-p+c(p+q+1)\big)z + c^2(p+q) + c(q-p) + 1.$$

where $(w)_0 = \frac{p-q}{p+q+1} - c$ *and* $(w)_1 = \frac{2(p+\frac{1}{2})(p+\frac{3}{2})+(q+\frac{1}{2})(q+\frac{3}{2})}{(p+q+2)(p+q+1)} - 1 - c\frac{p-q}{p+q+1}$.

For the proof we need the following lemma:

Lemma 6.2. *[18] Let ϖ_1 and ϖ_2 be two semiclassical forms satisfying* (11) *with* $\deg\phi = \deg\psi + 1 = t$. *If* $(\varpi_1)_i = (\varpi_2)_i$, $0 \le i \le t-2$, *then* $\varpi_1 = \varpi_2$.

Proof. $(a) \Rightarrow (b)$ Applying the operator S to (59) and taking into account (4) we obtain the desired relation.
$(b) \Rightarrow (c)$ Multiplying both sides of (60) by $f_{p,q}(z)$, from (4) we deduce

$$\begin{aligned} f_{p,q}(z)S(w)(z) &= (z-c)S\Big(f_{p,q}\mathcal{T}_{p,q}\Big)(z) - (z-c)\Big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\Big)(z) + f_{p,q}(z) \\ &\overset{by\ (18)}{=} (z-c)S\Big(g_{p,q}\mathcal{T}\Big)(z) - (z-c)\Big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\Big)(z) + f_{p,q}(z) \\ &\overset{by\ (4)}{=} (z-c)g_{p,q}(z)S(\mathcal{T})(z) + (z-c)\Big(\Big(\mathcal{T}\theta_0 g_{p,q}\Big) - \Big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\Big)\Big)(z) + f_{p,q}(z). \end{aligned}$$

i. e.

$$S\Big(f_{p,q}(x)w\Big)(z) = S\Big((x-c)g_{p,q}(x)\mathcal{T}\Big)(z) + Q(z),$$

with

$$\begin{aligned} Q(z) =& (z-c)\Big(\Big(\mathcal{T}\theta_0 g_{p,q}\Big) - \Big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\Big)\Big)(z) \\ &+ \Big(w\theta_0 f_{p,q}(x)\Big)(z) - \Big(\mathcal{T}\theta_0((x-c)g_{p,q}(x))\Big)(z) + f_{p,q}(z), \end{aligned}$$

or equivalently,

$$S\Big(f_{p,q}(x)w - (x-c)g_{p,q}(x)\mathcal{T}\Big)(z) = Q(z) \in \mathcal{P}.$$

We may now invoke Remark 2.1 to argue that

$$f_{p,q}(x)w - (x-c)g_{p,q}(x)\mathcal{T} = 0 \quad \text{in } \mathcal{P}',$$

and

$$Q(z) = 0.$$

Thus the result follows.
$(c) \Rightarrow (d)$ Applying the operator S to (61) and taking into account (4) we get

$$f_{p,q}(z)S(w)(z) = (z-c)g_{p,q}(z)S\Big(\mathcal{T}\Big)(z) - \Big(w\theta_0 f_{p,q}\Big)(z) + \Big(\mathcal{T}\theta_0((x-c)g_{p,q}(x))\Big)(z).$$

Then, from (27), one has

$$\begin{aligned} f_{p,q}(z)S(w)(z) =& (z-c)g_{p,q}(z)S\Big(\mathcal{T}\Big)(z) - \Big(w\theta_0 f_{p,q}\Big)(z) + \Big(\mathcal{T}\theta_0((x-c)g_{p,q}(x))\Big)(z) \\ \overset{by\ (4)-(18)}{=}& (z-c)S\Big(f_{p,q}\mathcal{T}_{p,q}\Big)(z) - \Big(w\theta_0 f_{p,q}\Big)(z) \\ &+ \Big(\mathcal{T}\theta_0((x-c)g_{p,q}(x))\Big)(z) - (z-c)\Big(\mathcal{T}\theta_0 g_{p,q}\Big)(z) \\ \overset{by\ (4)}{=}& (z-c)f_{p,q}(z)S\Big(\mathcal{T}_{p,q}\Big)(z) - \Big(w\theta_0 f_{p,q}\Big)(z) \\ &+ \Big(\mathcal{T}\theta_0((x-c)g_{p,q}(z))\Big)(z) - (z-c)\Big(\Big(\mathcal{T}\theta_0 g_{p,q}\Big) - \Big(\mathcal{T}_{p,q}\theta_0 f_{p,q}\Big)\Big)(z). \end{aligned}$$

Therefore, using (62), the last equation becomes

$$f_{p,q}(z)S(w)(z) = (z-c)f_{p,q}(z)S\Big(\mathcal{T}_{p,q}\Big)(z) + f_{p,q}(z),$$

which readily gives

$$S(w)(z) = (z-c)S\Big(\mathcal{T}_{p,q}\Big)(z) + 1.$$

Then, using (4) and by Remark 2.1, we get

$$w = (x-c)\mathcal{T}_{p,q},$$

so,

$$(w)_n = \left(\mathcal{T}_{p,q}\right)_{n+1} - c\left(\mathcal{T}_{p,q}\right)_n, \ \ n \geq 0.$$

Hence, using (25) we have the desired relation.
$(d) \Rightarrow (e)$ By hypothesis we have

$$\begin{aligned}(w)_n &= \left(\mathcal{T}_{p,q}\right)_{n+1} - c\left(\mathcal{T}_{p,q}\right)_n \\ &= \left((x-c)\mathcal{T}_{p,q}\right)_n, \ \ n \geq 0.\end{aligned}$$

Then,

$$w = (x-c)\mathcal{T}_{p,q}. \tag{63}$$

Using Lemma 3.3 we conclude that w is a second degree form. As a consequence, w is a semiclassical form. Based on the property (14), the relation (63) becomes

$$S(w)(z) = (z-c)S\left(\mathcal{T}_{p,q}\right)(z) + 1. \tag{64}$$

Taking formal derivatives in the last equation we get

$$S'(w)(z) = (z-c)S'(\mathcal{T}_{p,q})(z) + S(\mathcal{T}_{p,q})(z).$$

After combining the latter two expressions we obtain

$$S'(\mathcal{T}_{p,q})(z) = \frac{(z-c)S'(w)(z) - S(w)(z) + 1}{(z-c)^2}. \tag{65}$$

Replacing (64) and (65) in (23), and multiplying both sides of the resulting equation by $(z-c)^2$, one obtains

$$\phi_w(z)S'(w)(z) = C_w(z)S(w)(z) + D_w(z), \tag{66}$$

where the polynomials ϕ_w, C_w and D_w are

$$\begin{aligned}\phi_w(z) &= (z-c)\Phi(z), \\ C_w(z) &= \Phi(z) + (z-c)C_0^{p,q}(z), \\ D_w(z) &= -\Phi(z) - (z-c)C_0^{p,q}(z) + (z-c)^2 D_0^{p,q}(z).\end{aligned}$$

Therefore, it follows from (24) that $S(w)(z)$ fulfills (66) with

$$\begin{aligned}\phi_w(z) &= (z-c)(z^2-1), \\ C_w(z) &= (p+q)z^2 + \big(q-p-c(p+q-1)\big)z - c(q-p) - 1, \\ D_w(z) &= -\big(q-p+c(p+q+1)\big)z + c^2(p+q) + c(q-p) + 1.\end{aligned}$$

Now, we see that conditions $\Phi(c) \neq 0$, $C_0^{p,q}(1) \neq 0$ and $C_0^{p,q}(-1) \neq 0$ hold. Then ϕ_w, C_w, and D_w are coprime. Since $\deg D_w \leq 1$ and $\deg C_w = 2$ the class of w is one.
$(e) \Rightarrow (a)$ It is easy to verify that the form $(x-c)\mathcal{T}_{p,q}$ satisfies the same functional equation as w with $\deg \phi_w = \deg \psi_w + 1 = 3$, as well as $\left((x-c)\mathcal{T}_{p,q}\right)_0 = \frac{p-q}{p+q+1} - c = (w)_0$ and $\left((x-c)\mathcal{T}_{p,q}\right)_1 = \frac{2(p+\frac{1}{2})(p+\frac{3}{2})+(q+\frac{1}{2})(q+\frac{3}{2})}{(p+q+2)(p+q+1)} - 1 - c\frac{p-q}{p+q+1} = (w)_1$. From Lemma 6.2 we conclude that $w = (x-c)\mathcal{T}_{p,q}$. Thus, the proof is finished. □

7. Open problems

- In this work we have analyzed several examples of second degree linear forms which are also semiclassical of class $s = 1$. An interesting question is to describe all second degree semiclassical forms of class $s = 1$. We conjecture that the examples studied in our contribution constitute the only available cases.

- Very few examples of second degree forms which are semiclassical of class $s \geq 2$ are known in the literature. In particular, in [32] by means of the quadratic decomposition, an example of symmetric semiclassical forms of class $s = 2$ that is of second degree is determined. The description of all symmetric linear forms which are of second degree and semiclassical of class $s = 2$ seems to be an open problem despite the fact that some examples of symmetric semiclassical linear forms are considered in [33] and [20]. To check among them what are second degree forms seems to be an interesting exercise.

- In [35] all quasi-symmetric semiclassical orthogonal polynomial sequences of class two are studied. The coefficients of the corresponding three term recurrence relation are obtained as well as the integral representations of the linear forms. In [18] the author deduces the second degree forms among those analyzed in [35].

- Rational spectral transformations of second degree forms preserve such a family. Notice that according to [37] they are generated by Christoffel, Geronimus, association and anti-association transformations. An interesting problem is to describe the set of the transformations of linear forms such that the second degree character is preserved.

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