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Some Essential Spectra of Unbounded Operator Matrices Pencils with Non-Diagonal Domain and Application

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Abstract. In this paper, we investigate the stability of some essential spectra of a 2×2 block operator matrices pencil with unbounded entries and with non-diagonal domain (i.e a domain consisting of vectors which satisfy certain relations between their components) by using the resolvent of this kind of matrix operator in terms of the union of the essential spectra of the restriction of its diagonal operators entries. Furthermore, an example of two-group transport operators pencils is presented to illustrate the validity of the main results.

1. Introduction

Numerous mathematical and physical problems lead to operator pencils, $T - \lambda S$ (see for example [11, 17, 21]). Recently, the spectral theory of operator matrices attracts the attention of many mathematicians for study and characterize the essential spectra with different methods. The obtained results are used in many physical problems, for example, transport operator. (see, for example [8, 10, 12, 13, 20]).
In this paper, we are mainly concerned with the study of the spectral theory of operator matrices pencils of the form

$$\mathcal{A} - \lambda M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} - \lambda \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \tag{1}$$

defined on $E \times F$ product of Banach spaces, where $\mathcal{A}$ is an unbounded matrix operator with non-diagonal domain and M is a bounded and invertible matrix operator. Many authors suggested a spectral theory for operators in the form $\mathcal{A} - \lambda M$, where $M = I$, (see [10, 13, 20]).
The characterization and the investigation of some essential spectra of block operator matrices pencils with diagonal domain case, have drawn the attention of several authors involving the corresponding Schur complements, see [12]. Later, in paper [25], the author improves the previous results by considering the case of matrix operator pencils with domain consisting of vectors satisfying one relation between their components expressed as:
$\Gamma_X f = \Gamma_Y g$ for $\begin{pmatrix} f \\ g \end{pmatrix} \in (\mathrm{dom}\,(A) \cap \mathrm{dom}\,(C)) \times (\mathrm{dom}\,(B) \cap \mathrm{dom}\,(D))$ where Γ_X and Γ_Y are two linear operators.
In [22], R. Nagel has paid attention to the research of the problem related to spectral properties of 2×2 operator matrices $\mathcal{A}$ with non-diagonal domain defined by two relations between their components, that is:

$$\mathrm{dom}\,(\mathcal{A}) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathrm{dom}\,(A_m) \times \mathrm{dom}\,(D_m) \text{ such that } \begin{matrix} \phi_1(f) = \psi_2(g) \\ \text{and} \\ \phi_2(g) = \psi_1(f) \end{matrix} \right\}, \tag{2}$$

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for continuous linear operators ϕ_i and ψ_i, $i = 1, 2$, A_m and D_m are two maximal operators. The purpose of this work consists principally in studying some essential spectra of operator matrices with non-diagonal domain, given in [2] in the case of matrix operator pencil. For this, to achieve this goal, first, we determine the expression of the resolvent $(\mathcal{A} - \lambda M)^{-1}$ for some convenable λ. More precisely, the idea is to associate to the pair $(\mathcal{A}, M)$ a pair $(\mathcal{A}_0, M)$, which is more easier to deal with and we prove that

$$\sigma_{e,k}(\mathcal{A}, M) = \sigma_{e,k}(A_0, M_1) \cup \sigma_{e,k}(D_0, M_4)$$

where $A_0 = A_m \mid_{\ker \phi_1}$, $D_0 = D_m \mid_{\ker \phi_2}$ and $k \in \{1, 2, 3, 4, r, l\}$. (For more details see Theorems 3.10 and 3.11). In the last section, we will apply the results obtained in Section 3 to exploit several results from perturbation theory and spectral theory to obtain information about the M-essential spectra of the following two-group transport operator acts in the space $X_p \times X_p$, where $X_p := L_p([-a, a] \times [-1, 1]; dxdv)$, $a > 0$ and $p \geq 1$. Let

$$\mathcal{A}_H = T + K,$$

where T, K and M are defined by

$$T := \begin{pmatrix} T_{m_1} & 0 \\ 0 & T_{m_4} \end{pmatrix}, \; K := \begin{pmatrix} 0 & K_{12} \\ K_{21} & 0 \end{pmatrix} \text{ and } M := \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}.$$

The operators T_{m_i}, $i = 1, 4$ are the so-called streaming operators, defined by:

$$\begin{cases} T_{m_i} : \mathrm{dom}\,(T_{m_i}) \subseteq X_p \longrightarrow X_p \\ \qquad\qquad f \longmapsto T_{m_i} f = -v\frac{\partial f}{\partial x} - \sigma_i(v) f, \\ \mathrm{dom}\,(T_{m_i}) = \{f \in X_p \text{ such that } v\frac{\partial f}{\partial x} \in X_p\} := \mathcal{W}_p. \end{cases}$$

The linear bounded collision operators $K_{jj'}$, for $(j, j') \in \{(1, 2), (2, 1)\}$, are defined on X_p by:

$$\begin{cases} K_{jj'} : X_p \longrightarrow X_p \\ \qquad u \longmapsto \displaystyle\int_{-1}^{1} k_{jj'}(x, v, v') u(x, v') dv', \end{cases}$$

(see Section 4 for more details).

Our paper is organized as follow: In the next section, we give some preliminary results and notations used in the sequel of the paper. Section 3 focuses on the characterization of the essential spectra of unbounded operator matrices pencils with non-diagonal domain. We end this paper by applying our results to determine the M-essential spectra of two-group transport equations.

2. Preliminary results

In this section, we gather some auxiliary notations and definitions that we will need in the rest of the paper. let X and Y be two Banach spaces. We denote by $\mathcal{L}(X, Y)$ (resp. $\mathcal{C}(X, Y)$) the set of all bounded (resp. closed, densely defined) linear operators from X into Y. We denote by $\mathcal{K}(X, Y)$ the subspace of all compact operators. If $X = Y$, then $\mathcal{L}(X, Y)$, $\mathcal{C}(X, Y)$ and $\mathcal{K}(X, Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$ and $\mathcal{K}(X)$ respectively.
We will consider the set of polynomially compact operators which are denoted by $\mathcal{PK}(X)$ and defined as:

$$\mathcal{PK}(X) := \left\{ \begin{array}{l} T \in \mathcal{L}(X); \text{ there exists a non zero complex polynomial} \\ P(z) = \sum_{r=0}^{p} a_r z^r \text{ with } P(1) \neq 0 \text{ such that } P(T) \in \mathcal{K}(X) \end{array} \right\}.$$

For $T \in \mathcal{C}(X, Y)$, we write $\mathrm{dom}\,(T) \subset X$ for the domain, $\ker(T)$ for the null space and $\mathrm{ran}\,(T) \subset Y$ for the range of T. The nullity, $\alpha(T)$, of T is defined as the dimension of $\ker(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of $\mathrm{ran}\,(T)$ in Y.
For $T \in \mathcal{C}(X, Y)$ and S be a bounded operator from X into Y, we define the S-resolvent set $\rho(T, S)$ of the pair (T, S) by:

$$\rho(T, S) = \{\lambda \in \mathbb{C} \text{ such that } \mathcal{R}_\lambda(T, S) = (T - \lambda S)^{-1} \text{ exists and is bounded}\}.$$

For $\lambda \in \rho(T,S)$, $\mathcal{R}_\lambda(T,S)$ is the S-resolvent of T. The S-spectrum $\sigma(T,S)$ is the complement of $\rho(T,S)$ in the complex plane, see([8]).

In what follows, we need to introduce some important classes of operators. The set of upper semi-Fredholm operators from X into Y is defined by:

$$\Phi_+(X,Y) = \{T \in C(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } \operatorname{ran}(T) \text{ is closed in } Y\}.$$

and the set of lower semi-Fredholm operators from X into Y is defined by:

$$\Phi_-(X,Y) = \{T \in C(X,Y) \text{ such that } \beta(T) < \infty \text{ and } \operatorname{ran}(T) \text{ is closed in } Y\}.$$

$\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$ (resp. $\Phi_\pm(X,Y) := \Phi_+(X,Y) \cup \Phi_-(X,Y)$) denotes the set of Fredholm (resp. semi-Fredholm) operators from X into Y . For such an operator, we define the index $\operatorname{ind}(T)$ by: $\operatorname{ind}(T) = \alpha(T) - \beta(T)$.
If $X = Y$, then $\Phi(X,Y)$, $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ are replaced by $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$ respectively.

Definition 2.1. [12, Definition 2.4, p. 4] *Let X and Y be two Banach spaces.*
An operator $T \in \mathcal{L}(X,Y)$ is said to have a left (resp. a right) Fredholm inverse if there exists an operator $T_l \in \mathcal{L}(Y,X)$ (resp. $T_r \in \mathcal{L}(Y,X)$) such that $T_lT - Id \in \mathcal{K}(X)$ (resp. $TT_r - Id \in \mathcal{K}(Y)$). The operator T_l(resp. T_r) is called left (resp. right) Fredholm inverse of T. ◇

In [12],the authors proved that if $T \in C(X,Y)$, then they have same above definition with an hypothesis that is $\operatorname{dom}(T)$ endow with the graph norm.

Definition 2.2. *Let X and Y be two Banach spaces. An operator $T \in \mathcal{L}(X,Y)$ is said to be weakly compact if $T(B)$ is relatively weakly compact in Y for every bounded $B \in X$.*

◇

The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X,Y)$. If $X = Y$ the family of weakly compact operators on X, $\mathcal{W}(X) := \mathcal{W}(X,X)$ is a two-sided closed ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$(see [5]).

Definition 2.3. *Let X and Y be two Banach spaces. An operator $T \in \mathcal{L}(X,Y)$ is said to be strictly singular if the restriction of T to any infinite-dimensional subspace of X is not a homeomorphism.* ◇

Let $\mathcal{S}(X,Y)$ denote the set of strictly singular operators from X to Y. The concept of strictly singular operators was introduced in the pioneering paper by T. Kato [14]) as a generalization of the notion of compact operators. For a detailed study of the properties of strictly singular operators, we refer to [14]. Note that $\mathcal{S}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$. If $X = Y$, $\mathcal{S}(X) := \mathcal{S}(X,X)$ is a two-sided closed ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. If X is a Hilbert space, then $\mathcal{S}(X) = \mathcal{K}(X)$. The class of weakly compact operators in L_1-spaces (resp. $C(\Omega)$- spaces with Ω is a compact Hausdorff space) is nothing else than the family of strictly singular operators on L_1-space (resp. $C(\Omega)$-space) see [23].

In order to state our main results, let us introduce some definitions on Fredholm perturbations and then continue with some lemmas and propositions:

Definition 2.4. *Let X and Y be two Banach spaces and $F \in \mathcal{L}(X,Y)$.*
(i) F is called a Fredholm perturbation if $T + F \in \Phi(X,Y)$ whenever $T \in \Phi(X,Y)$.
(ii) F is called an upper (resp. lower) semi-Fredholm perturbation if $T + F \in \Phi_+(X,Y)$ (resp. $T + F \in \Phi_-(X,Y)$) whenever $T \in \Phi_+(X,Y)$. (resp. $T \in \Phi_-(X,Y)$).
(iii) F is called a left (resp. right) Fredholm perturbation if $T + F \in \Phi_l(X,Y)$ (resp. $T + F \in \Phi_r(X,Y)$) whenever $T \in \Phi_l(X,Y)$. (resp. $T \in \Phi_r(X,Y)$). ◇

We denote by $\mathcal{F}(X,Y)$ the set of Fredholm perturbations, by $\mathcal{F}_+(X,Y)$ (resp. $\mathcal{F}_-(X,Y)$) the set of upper semi-Fredholm (resp. lower semi-Fredholm) perturbations and by $\mathcal{F}_l(X,Y)$ (resp. $\mathcal{F}_r(X,Y)$) the set of left (resp. right) Fredholm perturbations.
If $X = Y$ we write $\mathcal{F}(X)$, $\mathcal{F}_+(X)$, $\mathcal{F}_-(X)$, $\mathcal{F}_l(X)$ and $\mathcal{F}_r(X)$ for $\mathcal{F}(X,X)$, $\mathcal{F}_+(X,X)$, $\mathcal{F}_-(X,X)$, $\mathcal{F}_l(X,X)$ and $\mathcal{F}_r(X,X)$, respectively.

Remark 2.5. *Let $\Phi^b(X,Y)$, $\Phi^b_l(X,Y)$ and $\Phi^b_r(X,Y)$ denote respectively the sets $\Phi(X,Y)\cap\mathcal{L}(X,Y)$, $\Phi_l(X,Y)\cap\mathcal{L}(X,Y)$ and $\Phi_r(X,Y)\cap\mathcal{L}(X,Y)$. In Definition* 2.4*, if we replace $\Phi(X,Y)$, $\Phi_l(X,Y)$ and $\Phi_r(X,Y)$ by $\Phi^b(X,Y)$, $\Phi^b_l(X,Y)$ and $\Phi^b_r(X,Y)$, we obtain the sets $\mathcal{F}^b(X,Y)$, $\mathcal{F}^b_l(X,Y)$ and $\mathcal{F}^b_r(X,Y)$ respectively.*

The sets of Fredholm and semi-Fredholm perturbations were introduced and investigated in [4]. In particular, it is shown that $\mathcal{F}^b(X,Y)$, $\mathcal{F}^b_+(X,Y)$ and $\mathcal{F}^b_-(X,Y)$ are closed subsets of $\mathcal{L}(X,Y)$ and if $X=Y$, then $\mathcal{F}^b(X):=\mathcal{F}^b(X,X)$, $\mathcal{F}^b_+(X)=\mathcal{F}^b_+(X,X)$ and $\mathcal{F}^b_-(X)=\mathcal{F}^b_-(X,X)$ are closed two-sided ideals of $\mathcal{L}(X)$.
In [12], it was proved that if $X=Y$, then $\mathcal{F}^b_l(X):=\mathcal{F}^b_l(X,X)$ and $\mathcal{F}^b_r(X):=\mathcal{F}^b_r(X,X)$ are two-sided ideals of $\mathcal{L}(X)$ and we have:

$$\mathcal{K}(X,Y)\subseteq\mathcal{F}^b_l(X,Y)\subseteq\mathcal{F}^b(X,Y) \tag{3}$$

and

$$\mathcal{K}(X,Y)\subseteq\mathcal{F}^b_r(X,Y)\subseteq\mathcal{F}^b(X,Y). \tag{4}$$

Let us recall the following useful results on Fredholm perturbations theory of 2×2 block operator matrices established in [12].

Theorem 2.6. [12, Theorems 3.1-3.2] *Let X_1 and X_2 be two Banach spaces and $F:=\begin{pmatrix} F_{11} & F_{12}\\ F_{21} & F_{22}\end{pmatrix}$ where $F_{ij}\in\mathcal{L}(X_i,X_j)\ \forall\ i,j=1,2$. Then :*
(i) $F\in\mathcal{F}^b(X_1\times X_2)\Leftrightarrow F_{ij}\in\mathcal{F}^b(X_i,X_j)\ \forall\ i,j=1,2.$
(ii) $F\in\mathcal{F}^b_r(X_1\times X_2)\Leftrightarrow F_{ij}\in\mathcal{F}^b_r(X_i,X_j)\ \forall\ i,j=1,2.$
(iii) $F\in\mathcal{F}^b_l(X_1\times X_2)\Leftrightarrow F_{ij}\in\mathcal{F}^b_l(X_i,X_j)\ \forall\ i,j=1,2.$

$\diamond$

Our concern in this paper is about the following definitions of some essential spectra of operators pencils. For $T\in\mathcal{C}(X,Y)$ and $S\in\mathcal{L}(X,Y)$, we consider

$$\begin{aligned}
\sigma_{e,l}(T,S) &:= \{\lambda\in\mathbb{C}\ :\ T-\lambda S\notin\Phi_l(X,Y)\},\\
\sigma_{e,r}(T,S) &:= \{\lambda\in\mathbb{C}\ :\ T-\lambda S\notin\Phi_r(X,Y)\},\\
\sigma_{e,1}(T,S) &:= \{\lambda\in\mathbb{C}\ :\ T-\lambda S\notin\Phi_+(X,Y)\},\\
\sigma_{e,2}(T,S) &:= \{\lambda\in\mathbb{C}\ :\ T-\lambda S\notin\Phi_-(X,Y)\},\\
\sigma_{e,3}(T,S) &:= \{\lambda\in\mathbb{C}\ :\ T-\lambda S\notin\Phi_\pm(X,Y)\},\\
\sigma_{e,4}(T,S) &:= \{\lambda\in\mathbb{C}\ :\ T-\lambda S\notin\Phi(X,Y)\}.
\end{aligned}$$

These sets can be ordered as

$$\sigma_{e,1}(T,S)\cap\sigma_{e,2}(T,S)=\sigma_{e,3}(T,S)\subset\sigma_{e,4}(T,S)=\sigma_{e,l}(T,S)\cup\sigma_{e,r}(T,S). \tag{5}$$

and

$$\sigma_{e,1}(T,S)\subset\sigma_{e,l}(T,S)\subset\sigma_{e,4}(T,S), \tag{6}$$

$$\sigma_{e,2}(T,S)\subset\sigma_{e,r}(T,S)\subset\sigma_{e,4}(T,S). \tag{7}$$

Remark 2.7. *We mention that if $S=Id$, we recover the usual definition of the essential spectra of a closed densely defined linear operator T, see* [7, 16, 24].

Remark 2.8. (*i*) *It follows from* [16, pp. 779], *that if $p>1$*

$$\mathcal{K}(X_p)=\mathcal{F}_+(X_p)=\mathcal{F}_-(X_p)=\mathcal{F}(X_p)=\mathcal{S}(X_p).$$

and if $p=1$, according to Theorem 1 *in* [23, pp. 779], *we have $\mathcal{S}(X_1)=\mathcal{W}(X_1)$.*
(*ii*) *For two Banach spaces X and Y, the last assertion with Eqs.* (6) *and* (7) *reveals*

$$\mathcal{F}_r(X_p)=\mathcal{F}_l(X_p)=\mathcal{F}(X_p).$$

We recall some stability results of the essential spectra of unbounded operator subjected to Fredholm perturbations which is essential to provide the main purpose of this paper.

Theorem 2.9. [1, 11] *Let T_1 and T_2 two closed densely defined linear operators on X and S an invertible operator on X.*

(*i*) *If for some $\lambda \in \rho(T_1, S) \cap \rho(T_2, S)$, the operator $(T_1 - \lambda S)^{-1} - (T_2 - \lambda S)^{-1} \in \mathcal{F}_r^b(X)$, then*

$$\sigma_{e,r}(T_1, S) = \sigma_{e,r}(T_2, S).$$

(*ii*) *If for some $\lambda \in \rho(T_1, S) \cap \rho(T_2, S)$, the operator $(T_1 - \lambda S)^{-1} - (T_2 - \lambda S)^{-1} \in \mathcal{F}_l^b(X)$, then*

$$\sigma_{e,l}(T_1, S) = \sigma_{e,l}(T_2, S).$$

(*iii*) *If for some $\lambda \in \rho(T_1, S) \cap \rho(T_2, S)$, the operator $(T_1 - \lambda S)^{-1} - (T_2 - \lambda S)^{-1} \in \mathcal{F}_+^b(X)$, then*

$$\sigma_{e,1}(T_1, S) = \sigma_{e,1}(T_2, S).$$

(*v*) *If for some $\lambda \in \rho(T_1, S) \cap \rho(T_2, S)$, the operator $(T_1 - \lambda S)^{-1} - (T_2 - \lambda S)^{-1} \in \mathcal{F}_-^b(X)$, then*

$$\sigma_{e,2}(T_1, S) = \sigma_{e,2}(T_2, S).$$

♢

3. Stability of the essential spectra of an operator matrices pencils with non-diagonal domain

This section deals with the spectral theory of 2×2 operator matrix pencil $\mathcal{A} - \lambda M$ on the product of two Banach space E and F. First, we start by giving a simplified description of the resolvent of $\mathcal{A} - \lambda M$ which are needed to provide a new characterization of some essential spectra of this kind matrix in terms of the union of the essential spectra of the restriction of its diagonal operators entries.

3.1. Expression of the resolvent of an operator matrices pencils with non-diagonal domain

Let us consider the unbounded block operator matrix pencil defined as a product of simpler operators on the Banach space $E \times F$ having the form

$$\mathcal{A} - \lambda M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} - \lambda \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

To treat this problem in a functional analytic setting, we consider the following assumptions introduced in Nagel [22].
(H_1) A_m and D_m two closed, densely defined linear operators with maximal domains $\mathrm{dom}\,(A_m)$ in E and $\mathrm{dom}\,(D_m)$ in F.
(H_2) Let X and Y be two Banach spaces (called "spaces of boundary conditions"), endow $\mathrm{dom}\,(A_m)$ and $\mathrm{dom}\,(D_m)$ with the graph norm and define continuous linear operators ϕ_i and ψ_i for $i = 1, 2$ as in the following diagram:

$$\begin{array}{ccc} E \supset \mathrm{dom}\,(A_m) & \xrightarrow{\phi_1} & X \\ & \searrow^{\psi_2} & \\ & & \\ F \supset \mathrm{dom}\,(D_m) & \xrightarrow[\phi_2]{} & Y \end{array} \qquad \begin{array}{c} \\ \\ \psi_1 \nearrow \\ \\ \end{array}$$

(H_3) ϕ_1 and ϕ_2 are surjective.

Definition 3.1. *On the Banach space $E \times F$ we consider the non-diagonal domain*

$$\mathrm{dom}\,(\mathcal{A} - \lambda M) = \mathrm{dom}\,(\mathcal{A}) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathrm{dom}\,(A_m) \times \mathrm{dom}\,(D_m) \text{ such that } \begin{array}{c} \phi_1(f) = \psi_2(g) \\ \text{and} \\ \phi_2(g) = \psi_1(f) \end{array} \right\}, \tag{8}$$

to define the pencil matrix $\mathcal{A} - \lambda M$ by

$$(\mathcal{A} - \lambda M)\begin{pmatrix} f \\ g \end{pmatrix} = (\mathcal{A}_m - \lambda M)\begin{pmatrix} f \\ g \end{pmatrix}, \text{ for } \begin{pmatrix} f \\ g \end{pmatrix} \in \text{dom}\,(\mathcal{A} - \lambda M).$$

Here, $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$ is a bounded and invertible operator and $\mathcal{A}$ is of the form

$$\mathcal{A}_m = \begin{pmatrix} A_m & B \\ C & D_m \end{pmatrix}$$

where B, C are bounded linear operators such that $B \in \mathcal{L}(\text{dom}\,(D_m), E)$ and $C \in \mathcal{L}(\text{dom}\,(A_m), F)$.

◊

Remark 3.2. *In view of the continuity assumption on the operators ϕ_1, ϕ_2, ψ_1 and ψ_2, the domain $\text{dom}\,(\mathcal{A})$ is closed in $\text{dom}\,(A_m) \times \text{dom}\,(D_m)$ with respect to the graph norm. Hence $(\mathcal{A}, \text{dom}\,(\mathcal{A}))$ is a closed operator on $E \times F$*

As a first towards, the description of the resolvent of unbounded operator matrix pencil $\mathcal{A} - \lambda M$ with non-diagonal domain will be investigated by associating $\mathcal{A} - \lambda M$ with an operator matrix pencil $\mathcal{A}_0 - \lambda M_d$ where $M_d = \begin{pmatrix} M_1 & 0 \\ 0 & M_4 \end{pmatrix}$ is bounded and invertible operator acting in $E \times F$ and $\mathcal{A}_0$ is of the form $\begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix}$ which is easier to deal with since it has diagonal domain, where $A_0 = A_m|_{\ker \phi_1}$ and $D_0 = D_m|_{\ker \phi_2}$ represent the restriction of the operator A_m to $\ker \phi_1$ and D_m to $\ker \phi_2$ respectively.

Remark 3.3. *From the definition of the operator A_0 (resp. D_0) one can easily check that $\phi_1(\text{dom}\,(A_0)) = \{0\}$ (resp. $\phi_1(\text{dom}\,(D_0)) = \{0\}$) and thus the operator A_0 (resp. D_0) is closed. So $\mathcal{A}_0$ is closed.*

Now, let us recall the following lemma explaining the relation between the pencil operators $\mathcal{A} - \lambda M$ and $\mathcal{A}_0 - \lambda M_d$ using the lemma 2.4 in [22] that can be rephrased in the case of linear operator pencil as follows:

Lemma 3.4. (*i*) *For $\lambda \in \rho(A_0, M_1)$(resp. $\lambda \in \rho(D_0, M_4)$), the following decomposition holds:*

$$\text{dom}\,(A_m) = \text{dom}\,(A_0) \oplus \ker(A_m - \lambda M_1)$$

$$(\textit{resp.}\ \text{dom}\,(D_m) = \text{dom}\,(D_0) \oplus \ker(D_m - \lambda M_4)).$$

(*ii*) *For $\lambda \in \rho(A_0, M_1)$ (resp. $\lambda \in \rho(D_0, M_4)$), the following operator*

$$\phi_{\lambda, M_1} := \phi_1|_{\ker(A_m - \lambda M_1)} \text{ and } \phi_{\lambda, M_4} := \phi_2|_{\ker(D_m - \lambda M_4)}.$$

is a continuous bijection from $\ker(A_m - \lambda M_1)$ onto X (resp. from $\ker(D_m - \lambda M_4)$ onto Y). ◊

Proof. (*i*) Let $\lambda \in \rho(A_0, M_1)$. Because $(A_0 - \lambda M_1)$ is invertible, then we can deduce that

$$\text{dom}\,(A_0) \cap \ker(A_m - \lambda M_1) = \{0\}.$$

Now, we show that

$$\text{dom}\,(A_m) = \text{dom}\,(A_0) + \ker(A_m - \lambda M_1).$$

To do this, it is clear that

$$\text{dom}\,(A_0) + \ker(A_m - \lambda M_1) \subset \text{dom}\,(A_m).$$

For any $f \in \text{dom}\,(A_m)$, $\exists\, g \in \text{dom}\,(A_0)$ such that

$$g = (A_0 - \lambda M_1)^{-1}(A_m - \lambda M_1)f$$

Then, $f - g \in \ker(A_m - \lambda M_1)$.
A same reasoning as helps us to reach the result of $\text{dom}\,(D_m) = \text{dom}\,(D_0) \oplus \ker(D_m - \lambda M_4)$, which ends the proof of this assertion.

(*ii*) We know that ϕ_1 et ϕ_2 are bounded then ϕ_{λ,M_1} and ϕ_{λ,M_4} there are.
The injectivity of the operator ϕ_{λ,M_1} follows from the following fact:

$$\ker(\phi_{\lambda,M_1}) = \ker\phi_1 \cap \ker(A_m - \lambda M_1) = \ker(A_0 - \lambda M_1) = \{0\}.$$

Now Using the item (*i*) and the linearity of ϕ_{λ,M_1}, we get

$$Im(\phi_{\lambda,M_1}) = \phi_1(\ker(A_m - \lambda M_1)\phi_1(\mathrm{dom}\,(A_m)) = Im(\phi_1)$$

As long as ϕ_1 is surjective, then $Im(\phi_{\lambda,M_1}) = X$. Hence ϕ_{λ,M_1} is surjective.
A same reasoning to reach the result of ϕ_{λ,M_2} □

As a direct consequence of the above lemma, for $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$, we can define the following operators

$$\mathrm{K}_{\lambda,M_1} := \phi^{-1}_{\lambda,M_1} \circ \psi_2 \ \text{ and } \ \mathrm{L}_{\lambda,M_4} := \phi^{-1}_{\lambda,M_4} \circ \psi_1.$$

Note that $\mathrm{K}_{\lambda,M_1} \in \mathcal{L}(\mathrm{dom}\,(D_m), \mathrm{dom}\,(A_m))$ and $\mathrm{L}_{\lambda,M_4} \in \mathcal{L}(\mathrm{dom}\,(A_m), \mathrm{dom}\,(D_m))$,

$$\phi_{\lambda,M_1}(\mathrm{K}_{\lambda,M_1} g) = \psi_2(g) \ \text{ for } \ g \in \mathrm{dom}\,(D_m) \ \text{ and } \ \phi_{\lambda,M_4}(\mathrm{L}_{\lambda,M_4} f) = \psi_1(f) \ \text{ for } \ f \in \mathrm{dom}\,(A_m).$$

The following factorizations may be used to formulate the key tool for our investigations.

Theorem 3.5. *Let $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$, Then,*

$$(\mathcal{A} - \lambda M) = (\mathcal{A}_0 - \lambda M_d)\mathcal{Q}_{\lambda,M} \quad \text{on } \ \mathrm{dom}\,(\mathcal{A}) \tag{9}$$

where

$$\mathcal{Q}_{\lambda,M} = \begin{pmatrix} Id & -\mathrm{K}_{\lambda,M_1} + (A_0 - \lambda M_1)^{-1}(B - \lambda M_2) \\ -\mathrm{L}_{\lambda,M_4} + (D_0 - \lambda M_4)^{-1}(C - \lambda M_3) & Id \end{pmatrix} \tag{10}$$

◊

Proof. We decompose $\mathcal{Q}_{\lambda,M}$ in the form

$$\mathcal{Q}_{\lambda,M} \ = \ \mathcal{B}_{\lambda,M} + \mathcal{C}_{\lambda,M}$$

where

$$\mathcal{B}_{\lambda,M} := \begin{pmatrix} Id & -\mathrm{K}_{\lambda,M_1} \\ -\mathrm{L}_{\lambda,M_4} & Id \end{pmatrix} \ , \ \mathcal{C}_{\lambda,M} = \begin{pmatrix} 0 & (A_0 - \lambda M_1)^{-1}(B - \lambda M_2) \\ (D_0 - \lambda M_4)^{-1}(C - \lambda M_3) & 0 \end{pmatrix}.$$

are bounded operators matrices defined on $\mathrm{dom}\,(A_m) \times \mathrm{dom}\,(D_m)$.
The proof can show in two steps:
Step 1: Let $\mathcal{H}_\lambda = (\mathcal{A}_0 - \lambda M_d)\mathcal{Q}_{\lambda,M}$, then

$$\mathrm{dom}\,(\mathcal{H}_\lambda) \ = \ \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathrm{dom}\,(A_m) \times \mathrm{dom}\,(D_m) \text{ tel que } \mathcal{Q}_{\lambda,M} \begin{pmatrix} f \\ g \end{pmatrix} \in \mathrm{dom}\,(\mathcal{A}_0) \right\}$$

Using the definitions and the linearity of K_{λ,M_1} and L_{λ,M_4}, we obtain

$$\begin{aligned} \mathrm{dom}\,(\mathcal{H}_\lambda) \ &= \ \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathrm{dom}\,(A_m) \times \mathrm{dom}\,(D_m) \text{ tel que } \begin{matrix} \phi_1(f) = \psi_2(g), \\ \phi_2(g) = \psi_1(f). \end{matrix} \right\} \\ &= \ \mathrm{dom}\,(\mathcal{A}). \end{aligned}$$

Step 2: On $\mathrm{dom}\,(\mathcal{A})$, we prove the decomposition (9).
We consider the unbounded operator matrices $\mathcal{A}_d = \begin{pmatrix} A_m & 0 \\ 0 & D_m \end{pmatrix}$ with non-diagonal domain $\mathrm{dom}\,(\mathcal{A})$ expressed in (8), and for $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$, the operator matrices pencil $(\mathcal{A}_d - \lambda M_d)$ can be decomposed as follows:

$$(\mathcal{A}_d - \lambda M_d) = (\mathcal{A}_0 - \lambda M_d)\mathcal{B}_{\lambda,M} \ \text{ on } \ \mathrm{dom}\,(\mathcal{A}_d),$$

then we get

$$\begin{aligned}(\mathcal{A}_0 - \lambda M_d)\mathcal{Q}_{\lambda,M}\begin{pmatrix} f \\ g \end{pmatrix} &= (\mathcal{A}_d - \lambda M_d)\begin{pmatrix} f \\ g \end{pmatrix} + (\mathcal{A}_0 - \lambda M_d)\mathcal{C}_{\lambda,M}\begin{pmatrix} f \\ g \end{pmatrix} \\ &= (\mathcal{A} - \lambda M)\begin{pmatrix} f \\ g \end{pmatrix}, \text{ for all } \begin{pmatrix} f \\ g \end{pmatrix} \in \text{dom}\,(\mathcal{A}).\end{aligned}$$

□

In view of the above decomposition of $\mathcal{A} - \lambda M$, we shall describe its resolvent. For this reason, we impose some conditions on the components of the matrix operator $\mathcal{Q}_{\lambda,M}$ and the matrix operator $\mathcal{A}_0 - \lambda M_d$ to show the invertibility of the operator matrices pencil $\mathcal{A} - \lambda M$. First, we will provide the invertibility of $\mathcal{Q}_{\lambda,M}$. For this, we consider the following notation

$$\begin{cases} G_\lambda = -\mathrm{K}_{\lambda,M_1} + (A_0 - \lambda M_1)^{-1}(B - \lambda M_2) \in \mathcal{L}(\text{dom}\,(D_m), \text{dom}\,(A_m)), \\ F_\lambda = -\mathrm{L}_{\lambda,M_4} + (D_0 - \lambda M_4)^{-1}(C - \lambda M_3) \in \mathcal{L}(\text{dom}\,(A_m), \text{dom}\,(D_m)). \end{cases} \tag{11}$$

Lemma 3.6. *For $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$*

$$\mathcal{Q}_{\lambda,M} \text{ is invertible in } \mathcal{L}(\text{dom}\,(A_m) \times \text{dom}\,(D_m)) \quad \Leftrightarrow \quad Id - F_\lambda G_\lambda \text{ is invertible in } \mathcal{L}(\text{dom}\,(D_m)).$$

◊

Proof. For $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$, according to the Frobenuis-Schur factorization, the operator $\mathcal{Q}_{\lambda,M}$ can be written in the following form

$$\mathcal{Q}_{\lambda,M} = \underbrace{\begin{pmatrix} Id & 0 \\ F_\lambda & Id \end{pmatrix}}_{U}\underbrace{\begin{pmatrix} Id & 0 \\ 0 & Id - F_\lambda G_\lambda \end{pmatrix}}_{V}\underbrace{\begin{pmatrix} Id & G_\lambda \\ 0 & Id \end{pmatrix}}_{W} \tag{12}$$

Since U and W are invertible and have a bounded inverse in $\text{dom}\,(A_m) \times \text{dom}\,(D_m)$. Hence, $\mathcal{Q}_{\lambda,M}$ is invertible if and only if V is invertible in $\text{dom}\,(A_m) \times \text{dom}\,(D_m)$ if and only if $Id - F_\lambda G_\lambda$ is invertible in $\text{dom}\,(D_m)$. □

In order to describe the resolvent of the operator matrices pencil $\mathcal{R}_\lambda(\mathcal{A}, M) = (\mathcal{A} - \lambda M)^{-1}$, we start our investigations with the following result.

Theorem 3.7. *For $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$.*

(i) If $Id - F_\lambda G_\lambda \in \mathcal{L}(\text{dom}\,(D_m))$ is invertible, then $\mathcal{A} - \lambda M$ is invertible in $\mathcal{L}(E \times F)$.

(ii)Furthermore, if $F_\lambda G_\lambda \in \mathcal{PK}(\text{dom}\,(D_m))$ then,

$$\mathcal{A} - \lambda M \text{ is invertible in } \mathcal{L}(E \times F) \quad \Leftrightarrow \quad Id - F_\lambda G_\lambda \text{ is invertible in } \mathcal{L}(\text{dom}\,(D_m)).$$

◊

Proof. (*i*) Let $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$, then $\lambda \in \rho(\mathcal{A}_0, M_d)$ and according to hypothesis $1 \in \rho(\mathbb{FG})$ and by the lemma 3.6 and Theorem 3.5, we get $\lambda \in \rho(\mathcal{A}, M)$.
(*ii*) We suppose that $\lambda \in \rho(\mathcal{A}, M)$, then $\mathcal{A} - \lambda M$ is injective. Consequently, the equation (9) of Theorem 3.5, gives that $Id - \mathbb{FG}$ is injective.
Under the hypothesis $\mathbb{FG} \in \mathcal{PK}(\mathcal{D}(D_m))$, and using Theorem 2.2 in [7], give $(Id - \mathbb{FG})^{-1}$ invertible and have a bounded inverse. Then, $Id - \mathbb{FG}$ is invertible.

□

By the decomposition $(\mathcal{A} - \lambda M) = (\mathcal{A}_0 - \lambda M_d)\mathcal{Q}_{\lambda,M}$ and Theorem 3.7, we deduce the expression of the resolvent $\mathcal{R}_\lambda(\mathcal{A}, M) := (\mathcal{A} - \lambda M)^{-1}$.

Theorem 3.8. *Let $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$ such that $1 \in \rho(F_\lambda G_\lambda)$, we obtain*

$$\mathcal{R}_\lambda(\mathcal{A}, M) = \mathcal{R}_\lambda(\mathcal{A}_0, M_d) + \mathcal{M}_f \tag{13}$$

Here,

$$\mathcal{M}_f = \begin{pmatrix} G_\lambda(Id - F_\lambda G_\lambda)^{-1} F_\lambda \mathcal{R}_\lambda(A_0, M_1) & G_\lambda[Id + (Id - F_\lambda G_\lambda)^{-1} F_\lambda G_\lambda]\mathcal{R}_\lambda(D_0, M_4) \\ (Id - F_\lambda G_\lambda)^{-1} F_\lambda \mathcal{R}_\lambda(A_0, M_1) & (Id - F_\lambda G_\lambda)^{-1} F_\lambda G_\lambda \mathcal{R}_\lambda(D_0, M_4) \end{pmatrix}. \tag{14}$$

◇

Proof. For $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$ such that $1 \in \rho(F_\lambda G_\lambda)$, then $\lambda \in \rho(\mathcal{A}, M)$ and by a simple calculus, we have

$$\mathcal{R}_\lambda(\mathcal{A}, M) = \begin{pmatrix} [Id + G_\lambda(Id - F_\lambda G_\lambda)^{-1} F_\lambda]\mathcal{R}_\lambda(A_0, M_1) & -G_\lambda(Id - F_\lambda G_\lambda)^{-1}\mathcal{R}_\lambda(D_0, M_4) \\ -(Id - F_\lambda G_\lambda)^{-1} F_\lambda \mathcal{R}_\lambda(A_0, M_1) & (Id - F_\lambda G_\lambda)^{-1}\mathcal{R}_\lambda(D_0, M_4) \end{pmatrix}, \tag{15}$$

and we can write

$$(Id - F_\lambda G_\lambda)^{-1} = Id + (Id - F_\lambda G_\lambda)^{-1} F_\lambda G_\lambda \text{ on } \operatorname{dom}(D_m).$$

Consequently,

$$(Id - F_\lambda G_\lambda)^{-1}\mathcal{R}_\lambda(D_0, M_4) = \mathcal{R}_\lambda(D_0, M_4) + (Id - F_\lambda G_\lambda)^{-1} F_\lambda G_\lambda \mathcal{R}_\lambda(D_0, M_4).$$

$$G_\lambda(Id - F_\lambda G_\lambda)^{-1}\mathcal{R}_\lambda(D_0, M_4) = G_\lambda \mathcal{R}_\lambda(D_0, M_4) + G_\lambda(Id - F_\lambda G_\lambda)^{-1} F_\lambda G_\lambda \mathcal{R}_\lambda(D_0, M_4).$$

Then, thanks to its above expressions, we can rewrite the new entries of the resolvent (15) to obtain our result. □

3.2. Main results

Our aim in this subsection is to characterize some essential spectra of unbounded operator matrices pencils. To do this, we need to determine the resolvent $\mathcal{R}_\lambda(\mathcal{A}_0, M)$ of the operator pencil $\mathcal{A}_0 - \lambda M$. We consider the following notation

$$G_{\lambda,d} = -\lambda(A_0 - \lambda M_1)^{-1} M_2 \in \mathcal{L}(F, \operatorname{dom}(A_0)),$$
$$F_{\lambda,d} = -\lambda(D_0 - \lambda M_4)^{-1} M_3 \in \mathcal{L}(E, \operatorname{dom}(D_0)).$$

Lemma 3.9. *For $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$ such that $1 \in \rho(F_{\lambda,d} G_{\lambda,d})$, then $\lambda \in \rho(\mathcal{A}_0, M)$ and we have*

$$\mathcal{R}_\lambda(\mathcal{A}_0, M) = \mathcal{R}_\lambda(\mathcal{A}_0, M_d) + \mathcal{M}'_f \tag{16}$$

with

$$\mathcal{M}'_f = \begin{pmatrix} G_{\lambda,d}(Id - F_{\lambda,d} G_{\lambda,d})^{-1} F_{\lambda,d} \mathcal{R}_\lambda(A_0, M_1) & G_{\lambda,d}[Id + (Id - F_{\lambda,d} G_{\lambda,d})^{-1} F_{\lambda,d} G_{\lambda,d}]\mathcal{R}_\lambda(D_0, M_4) \\ (Id - F_{\lambda,d} G_{\lambda,d})^{-1} F_{\lambda,d} \mathcal{R}_\lambda(A_0, M_1) & (Id - F_{\lambda,d} G_{\lambda,d})^{-1} F_{\lambda,d} G_{\lambda,d} \mathcal{R}_\lambda(D_0, M_4) \end{pmatrix}. \tag{17}$$

◇

Proof. In theorem 3.5, for $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$, replacing the operator matrix $\mathcal{A}$ by $\mathcal{A}_0$ then we get the following decomposition of $\mathcal{A}_0 - \lambda M$:

$$(\mathcal{A}_0 - \lambda M) = (\mathcal{A}_0 - \lambda M_d)\mathcal{Q}_{\lambda,M} \quad \text{on } \operatorname{dom}(\mathcal{A}_0). \tag{18}$$

Indeed, if $A_m = A_0$ and $D_m = D_0$ then K_{λ,M_1} and L_{λ,M_4} are null operators respectively, and with $B = C = 0$ we obtain $F_\lambda = F_{\lambda,d}$ and $G_\lambda = G_{\lambda,d}$. Then, the invertibility of the operator matrices $\mathcal{A}_0 - \lambda M$ is satisfied if $1 \in \rho(F_{\lambda,d} G_{\lambda,d})$.
Hence, we obtain the expression of the resolvent $\mathcal{R}_\lambda(\mathcal{A}_0, M)$ which is indicated in the lemma below by using the same method as in theorem 3.8. □

Now, we are in the position to express the first main results of this subsection.

Theorem 3.10. *Let M be a bounded and invertible operator matrices on $E \times F$ and $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$ such that $1 \in \rho(F_\lambda G_\lambda)$ and $1 \in \rho(F_{\lambda,d} G_{\lambda,d})$. Then, we have:*

(*i*) *If M_2 and M_3 are right Fredholm perturbation and if $F_\lambda \mathcal{R}_\lambda(A_0, M_1) \in \mathcal{F}_r^b(E, \mathrm{dom}\,(D_m))$ and $G_\lambda \mathcal{R}_\lambda(D_0, M_4) \in \mathcal{F}_r^b(F, \mathrm{dom}\,(A_m))$, then*

$$\sigma_{e,r}(\mathcal{A}, M) = \sigma_{e,r}(A_0, M_1) \cup \sigma_{e,r}(D_0, M_4).$$

(*ii*) *If M_2 and M_3 are left Fredholm perturbation and if $F_\lambda \mathcal{R}_\lambda(A_0, M_1) \in \mathcal{F}_l^b(E, \mathrm{dom}\,(D_m))$ and $G_\lambda \mathcal{R}_\lambda(D_0, M_4) \in \mathcal{F}_l^b(F, \mathrm{dom}\,(A_m))$, then*

$$\sigma_{e,l}(\mathcal{A}, M) = \sigma_{e,l}(A_0, M_1) \cup \sigma_{e,l}(D_0, M_4).$$

(*iii*) *If M_2 and M_3 are Fredholm perturbation and if $F_\lambda \mathcal{R}_\lambda(A_0, M_1) \in \mathcal{F}^b(E, \mathrm{dom}\,(D_m))$ and $G_\lambda \mathcal{R}_\lambda(D_0, M_4) \in \mathcal{F}^b(F, \mathrm{dom}\,(A_m))$, then*

$$\sigma_{e,4}(\mathcal{A}, M) = \sigma_{e,4}(A_0, M_1) \cup \sigma_{e,4}(D_0, M_4),$$

♢

Proof. Based on Theorem 2.9, we will prove the Fredholmness perturbation of the operator $\mathcal{R}_\lambda(\mathcal{A}, M) - \mathcal{R}_\lambda(\mathcal{A}_0, M)$, Then, it follows from the two equations 13 and 16, it is sufficient to show that all the entries of the block operators matrices $\mathcal{M}_f$ and $\mathcal{M}'_f$ are Fredholm perturbations.

(*i*) From the assumptions $F_\lambda \mathcal{R}_\lambda(A_0, M_1) \in \mathcal{F}_r^b(E, \mathrm{dom}\,(D_m))$ and $G_\lambda \mathcal{R}_\lambda(D_0, M_4) \in \mathcal{F}_r^b(F, \mathrm{dom}\,(A_m))$ and Theorem 3.2 in [12], we deduce that the all entries of the matrix $\mathcal{M}_f$ are right Fredholm perturbations operators.

On the other hand, the right Fredholmness perturbations of M_2 and M_3 result that the operators $G_{\lambda,d}$ and $F_{\lambda,d}$ are right fredholm perturbations. Then all entries of the matrix $\mathcal{M}'_f$ are right Fredholm perturbations operators by using Theorem 3.2 in [12].

Hence according to Theorem 2.6 (*ii*), we can deduce that $\mathcal{M}_f \in \mathcal{F}_r^b(E \times F)$ and $\mathcal{M}'_f \in \mathcal{F}_r^b(E \times F)$. Consequently, for $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$, we obtained

$$\mathcal{R}_\lambda(\mathcal{A}, M) - \mathcal{R}_\lambda(\mathcal{A}_0, M) \in \mathcal{F}_r^b(E \times F).$$

According to Theorem 2.9 (*i*), one gets

$$\sigma_{e,r}(\mathcal{A}, M) = \sigma_{e,r}(\mathcal{A}_0, M). \tag{19}$$

As $\begin{pmatrix} 0 & M_2 \\ M_3 & 0 \end{pmatrix} \in \mathcal{F}_r^b(E \times F)$, and using Proposition 3.1 in [1], we find that

$$\begin{aligned} \sigma_{e,r}(\mathcal{A}_0, M) &= \sigma_{e,r}\left(\begin{pmatrix} A_0 - \lambda M_1 & 0 \\ 0 & D_0 - \lambda M_4 \end{pmatrix} - \lambda \begin{pmatrix} 0 & M_2 \\ M_3 & 0 \end{pmatrix}\right) \\ &= \sigma_{e,r}\begin{pmatrix} A_0 - \lambda M_1 & 0 \\ 0 & D_0 - \lambda M_4 \end{pmatrix} \end{aligned}$$

Furthermore

$$\sigma_{e,r}(\mathcal{A}_0, M) = \sigma_{e,r}(A_0, M_1) \cup \sigma_{e,r}(D_0, M_4). \tag{20}$$

Then by Eq (19) and (20) we obtain

$$\sigma_{e,r}(\mathcal{A}, M) = \sigma_{e,r}(A_0, M_1) \cup \sigma_{e,r}(D_0, M_4).$$

The use of Theorems 2.6 (*iii*), 2.9 (*ii*) and according to [12, Theorem 3.2] allows us to reach the result of assertion (*ii*) in a similar way as in the item (*i*).

(*iii*) From Eq (5), the first result of this item is an immediate consequence of the items (*i*) and (*ii*). □

We can translate the results of the above Theorem in terms of some essential spectra of type $\sigma_{e,k}(.)$ for $k \in \{1, 2, 3\}$.

Theorem 3.11. *Let M be an invertible operator matrices on $E \times F$ and consider $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$ such that $1 \in \rho(F_\lambda G_\lambda)$ and $1 \in \rho(F_{\lambda,d} G_{\lambda,d})$. Then*
(i) If $M_2 \in \mathcal{F}_+^b(E \times F)$, $M_3 \in \mathcal{F}_+^b(F \times E)$ and $\mathcal{R}_\lambda(\mathcal{A}, M) - \mathcal{R}_\lambda(\mathcal{A}_0, M) \in \mathcal{F}_+^b(E \times F)$, then

$$\sigma_{e,1}(\mathcal{A}, M) = \sigma_{e,1}(A_0, M_1) \cup \sigma_{e,1}(D_0, M_4).$$

(ii) If $M_2 \in \mathcal{F}_-^b(E \times F)$, $M_3 \in \mathcal{F}_-^b(F \times E)$ and $\mathcal{R}_\lambda(\mathcal{A}, M) - \mathcal{R}_\lambda(\mathcal{A}_0, M) \in \mathcal{F}_-^b(E \times F)$ then

$$\sigma_{e,2}(\mathcal{A}, M) = \sigma_{e,2}(A_0, M_1) \cup \sigma_{e,2}(D_0, M_4).$$

(iii) If $M_2 \in \mathcal{F}_+^b(E\times F) \cap \mathcal{F}_-^b(E\times F)$, $M_3 \in \mathcal{F}_+^b(F\times E) \cap \mathcal{F}_-^b(E\times F)$ and $\mathcal{R}_\lambda(\mathcal{A}, M) - \mathcal{R}_\lambda(\mathcal{A}_0, M) \in \mathcal{F}_+^b(E\times F) \cap \mathcal{F}_-^b(E\times F)$ then

$$\sigma_{e,3}(\mathcal{A}, M) = \sigma_{e,3}(A_0, M_1) \cup \sigma_{e,3}(D_0, M_4).$$

◊

Proof.

(*i*) For $\lambda \in \rho(A_0, M_1) \cap \rho(D_0, M_4)$ such that $1 \in \rho(F_\lambda G_\lambda)$, we infer by Theorem 3.7 that $\lambda \in \rho(\mathcal{A}, M) \cap \rho(\mathcal{A}_0, M_d)$ and if adding the condition $1 \in \rho(F_{\lambda,d} G_{\lambda,d})$ we get $\lambda \in \rho(\mathcal{A}, M) \cap \rho(\mathcal{A}_0, M)$. Together with the fact that $\mathcal{R}_\lambda(\mathcal{A}, M) - \mathcal{R}_\lambda(\mathcal{A}_0, M) \in \mathcal{F}_+^b(E \times F)$ and based on Theorem 2.9 (*iii*), we obtain

$$\sigma_{e,1}(\mathcal{A}, M) = \sigma_{e,1}(\mathcal{A}_0, M). \tag{21}$$

Since $M_2, M_3 \in \mathcal{F}_+^b(E \times F)$, we deduce from Theorem (2.1) in [11]

$$\sigma_{e,1}(\mathcal{A}_0, M) = \sigma_{e,1}\left(\begin{pmatrix} A_0 - \lambda M_1 & 0 \\ 0 & D_0 - \lambda M_4 \end{pmatrix} - \lambda \begin{pmatrix} 0 & M_2 \\ M_3 & 0 \end{pmatrix}\right) = \sigma_{e,1}(\mathcal{A}_0, M_d). \tag{22}$$

As $\mathcal{A}_0 - \lambda M_d$ is a diagonal operators matrix, this shows that

$$\sigma_{e,1}(\mathcal{A}_0, M_d) = \sigma_{e,1}(A_0, M_1) \cup \sigma_{e,1}(D_0, M_4). \tag{23}$$

So by Eqs (21), (22) and (23) we infer that

$$\sigma_{e,1}(\mathcal{A}, M) = \sigma_{e,1}(A_0, M_1) \cup \sigma_{e,1}(D_0, M_4).$$

(*ii*) A same reasoning as helps us to reach the result of item (*ii*).

(*iii*) According to Eq (5) we see that this assertion is a consequence of the items (*i*) and (*ii*).

□

Remark 3.12. *It is noted that in the book* [8]*, Jeribi supposed that the operator matrix*

$$M(\lambda) =: \begin{pmatrix} 0 & M_1(A - \lambda M_1)^{-1} - M_2 \\ (C - \lambda M_3)^{-1} M_1 - M_3 & (C - \lambda M_3)^{-1} M_1 (A - \lambda M_1)^{-1} \end{pmatrix} \in \mathcal{I}(E \times F),$$

(where $\mathcal{I}(E \times F)$ is an arbitrary nonzero two-sided ideal of $\mathcal{L}(E \times F)$ satisfying that $\mathcal{I}(E \times F) \subset \mathcal{F}(E \times F)$) to characterize the essential spectra of unbounded operator matrix pencil with diagonal domain in terms of its Schur complement. But in our case we suppose only that M_2, M_3, $F_\lambda \mathcal{R}_\lambda(A_0, M_1)$ and $G_\lambda \mathcal{R}_\lambda(D_0, M_4)$ are Fredholm perturbations and by using the difference between the resolvent of two block operator matrices pencils, (see Theorems 3.10 *and* 3.11*) we investigate the essential spectra of the matrix pencil $\mathcal{A} - \lambda M$ in terms of the essential spectra pencil of the restriction of its diagonal operators entries.*

4. Application to two-group transport operator matrices pencils

In this section, we study the essential spectra of two-group transport operator matrices pencils with non-diagonal domain on L_p-space.
Let

$$X_p := L_p([-a,a]\times[-1,1];dxdv), a>0 \text{ and } p\geq 1.$$

We consider the following two-group transport operator matrices pencil:

$$\mathcal{A}_H - \lambda M := \begin{pmatrix} T_{m_1}-\lambda M_1 & K_{12}-\lambda M_2 \\ K_{21}-\lambda M_3 & T_{m_4}-\lambda M_4 \end{pmatrix},$$

where

- The closed linear operator $T_{m_i}, i=1,4$ is defined on its maximal domain dom (T_{m_i}) as:

$$\begin{cases} T_{m_i} : \text{dom}\,(T_{m_i}) \subseteq X_p \longrightarrow X_p \\ \qquad f \longmapsto T_{m_i} f = -v\frac{\partial f}{\partial x} - \sigma_i(v) f, \\ \text{dom}\,(T_{m_i}) := \mathcal{W}_p := \{f \in X_p \text{ such that } v\frac{\partial f}{\partial x} \in X_p\}, \end{cases}$$

  where the collision frequency $\sigma_i(.) \in \mathcal{L}^\infty(-1,1)$ for $i=1,4$.

- $K_{jj'}$: are bounded linear collision operator, for $(j,j') \in \{(1,2),(2,1)\}$ on X_p by:

$$\begin{cases} K_{jj'} : X_p \longrightarrow X_p \\ \qquad u \longmapsto \displaystyle\int_{-1}^{1} k_{jj'}(x,v,v')u(x,v')dv', \end{cases} \tag{24}$$

  with the scattering kernel $K_{jj'} : x \in [-a,a] \longrightarrow K(x) \in \mathcal{L}(L_p([-1,1],dv))$, assumed to be measurable.

- The coefficients $M_i,\ \ i=1,4$ is defined by:

$$\begin{cases} M_i : X_p \longrightarrow X_p \\ \qquad u \longmapsto M_i u(x,v) = \eta_i(v)u(x,v), \quad i = 1.4 \end{cases}$$

  where $\eta_i(.) \in \mathcal{L}^\infty(-1.1)$ and M_2, M_3 are bounded operators on X_p.
  Let

$$\lambda_j^* := \inf_{v\in[-1,1]} \sigma_j(v) \text{ and } \mu_j^* := \inf_{v\in[-1,1]} \eta_j(v), \text{ for } j=1,4$$

  and assume that $\mu_j^* > 0,\ \ j=1,4.$

To verify the hypotheses of Theorem 3.10, we shall define the operator $\mathcal{A}_H - \lambda M$ on the domain:

$$\text{dom}\,(\mathcal{A}_H) = \left\{ \mathcal{V} = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{W}_p \times \mathcal{W}_p \text{ such that } \mathcal{V}^i = H\mathcal{V}^o \right\},$$

where $\mathcal{V}^o$ and $\mathcal{V}^i$ represent respectively the outgoing and the incoming fluxes related by a boundary operator H.
We consider the boundary spaces:

$$X_p^o := L_p(\{-a\}\times[-1,0];|v|dv) \times L_p(\{a\}\times[0,1];|v|dv) = X_{1,p}^o \times X_{2,p}^o$$

and

$$X_p^i := L_p(\{-a\}\times[0,1];|v|dv) \times L_p(\{a\}\times[-1,0];|v|dv) = X_{1,p}^i \times X_{2,p}^i$$

(see [3] for more details). We will assume that the operator H is giving by

$$\begin{array}{rcl} H : X_p^o \times X_p^o & \to & X_p^i \times X_p^i \\ \begin{pmatrix} f \\ g \end{pmatrix} & \mapsto & H\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & H_{12} \\ H_{21} & 0 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} \end{array}$$

Clearly, the boundaries condition, $\mathcal{V}^i = H\mathcal{V}^o$, of the unbounded operator matrix $\mathcal{A}_H$ can be related to a coupled conditions

$$\begin{cases}\phi_1(f) = \psi_2(g)\\ \phi_2(g) = \psi_1(f)\end{cases}$$

modeled by the bounded linear operators ϕ_k and ψ_k, $k = 1, 2$ acting like in the diagram

$$\begin{array}{ccc} X_p \supset \mathrm{dom}\,(T_{m_1}) = \mathcal{W}_p & \xrightarrow{\phi_1} & X_p^i \\ & \searrow \quad \nearrow_{\psi_2} & \\ & \nearrow_{\phi_2} \quad \searrow^{\psi_1} & \\ X_p \supset \mathrm{dom}\,(T_{m_4}) = \mathcal{W}_p & \xrightarrow{\phi_2} & X_p^i \end{array}$$

and defined as follows

$$\begin{cases}\phi_k : W_p \longrightarrow X_p^i\\ \quad u \longmapsto u^i\end{cases}, \quad \begin{cases}\psi_1 : W_p \longrightarrow X_p^i\\ \quad f \longmapsto H_{21}f^o\end{cases} \quad \text{and} \quad \begin{cases}\psi_2 : W_p \longrightarrow X_p^i\\ \quad g \longmapsto H_{12}g^o\end{cases}.$$

Let T_i, $i = 1, 4$ be the closed, densely defined operator with non empty resolvent set, defined respectively by:

$$\begin{cases}T_1 := T_{m_1}|_{\ker\phi_1},\\ \mathrm{dom}\,(T_1) := \left\{f \in \mathcal{W}_p \text{ such that } f^i = 0\right\}\end{cases}$$

$$\begin{cases}T_4 := T_{m_4}|_{\ker\phi_2},\\ \mathrm{dom}\,(T_4) := \left\{g \in \mathcal{W}_p \text{ such that } g^i = 0\right\}.\end{cases}$$

Then, we obtain the operator matrices $\mathcal{A}_0 := \begin{pmatrix} T_1 & 0\\ 0 & T_4\end{pmatrix}$ with diagonal domain, $\mathrm{dom}\,(\mathcal{A}_0) = \mathrm{dom}\,(T_1) \times \mathrm{dom}\,(T_4)$.

Remark 4.1. 1. *It is well known from Remark* 4.1 *in* [20] *that the operators* T_{m_i}, $i = 1.4$ *are closed, densely defined linear operators with nonempty resolvent set. Hence, the assumption* (H_1) *is satisfied.*

2. *the trace mapping* ϕ_i, $i = 1, 2$ *is surjective, which was establish by Dautray and Lion in*[3]. *(see Theorem* 1, *p* 252 *for more details).*
Then for $\lambda \in \varrho(T_1, M_1) \cap \varrho(T_4, M_4)$, *the restrictions*

$$\phi_{\lambda,M_1} := \phi_1|_{\ker(T_{m_1} - \lambda M_1)} \text{ and } \phi_{\lambda,M_4} := \phi_2|_{\ker(T_{m_4} - \lambda M_4)}$$

of ϕ_1 *and* ϕ_2 *are invertible with bounded inverses operators*

$$\phi^{-1}_{\lambda,M_1} \in \mathcal{L}(X_p^i, \ker(T_{m_1} - \lambda M_1)) \subseteq \mathcal{L}(X_p^i, \mathcal{W}_p), \text{ for } p \geq 1.$$

and

$$\phi^{-1}_{\lambda,M_4} \in \mathcal{L}(X_p^i, \ker(T_{m_4} - \lambda M_4)) \subseteq \mathcal{L}(X_p^i, \mathcal{W}_p), \text{ for } p \geq 1.$$

♢

The following Lemma is crucial for the expression of the operators K_{λ,M_1} and L_{λ,M_4}.

Lemma 4.2. *Let* $\lambda \in \rho(T_1, M_1) \cap \rho(T_4, M_4)$, *the bounded operators* K_{λ,M_1} *and* L_{λ,M_4} *are defined by*

$$\begin{cases}K_{\lambda,M_1} : \mathcal{W}_p \longrightarrow \ker(T_{m_1} - \lambda M_1) \subset \mathcal{W}_p\\ \quad g \longmapsto K_{\lambda,M_1}g(x,v) = \chi_{(-1,0)}(v)H_{12}g(-a,v)e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))}{|v|}|a-x|}\\ \qquad\qquad +\chi_{(0,1)}(v)H_{12}g(a,v)e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))}{|v|}|x+a|},\end{cases}$$

and

$$\begin{cases}L_{\lambda,M_4} : \mathcal{W}_p \longrightarrow \ker(T_{m_4} - \lambda M_4) \subset \mathcal{W}_p\\ \quad f \longmapsto L_{\lambda,M_4}f(x,v) = \chi_{(-1,0)}(v)H_{21}f(-a,v)e^{-\frac{(\lambda\eta_4(v)+\sigma_2(v))}{|v|}|a-x|}\\ \qquad\qquad +\chi_{(0,1)}(v)H_{21}f(a,v)e^{-\frac{(\lambda\eta_4(v)+\sigma_2(v))}{|v|}|x+a|}.\end{cases}$$

♢

Proof. Note that the expression of the operators K_{λ,M_1} and L_{λ,M_4} may be checked using the following steps:
Step 1. Determine the expression of $\ker(T_{m_1} - \lambda M_1)$ and $\ker(T_{m_4} - \lambda M_4)$, for $\lambda \in \rho(T_1, M_1) \cap \rho(T_4, M_4)$. For this, we consider $\Gamma \in \mathrm{dom}\,(T_{m_1})$ and $\Upsilon \in \mathrm{dom}\,(T_{m_4})$.
An easy computation reveals that the solution of the equation $(T_{m_1} - \lambda M_1)\Gamma(x,v) = 0$ and $(T_{m_4} - \lambda M_4)\Upsilon(x,v) = 0$ are formally given by

$$\Gamma(x,v) = \begin{cases} \Gamma(a,v)e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))}{|v|}|a-x|}, & -1 < v < 0 \\ \Gamma(-a,v)e^{-\frac{(\lambda\eta_1(v)+\sigma_1(v))}{|v|}|a+x|}, & 0 < v < 1 \end{cases}$$

and

$$\Upsilon(x,v) = \begin{cases} \Upsilon(a,v)e^{-\frac{(\lambda\eta_4(v)+\sigma_4(v))}{|v|}|a-x|}, & -1 < v < 0 \\ \Upsilon(-a,v)e^{-\frac{(\lambda\eta_4(v)+\sigma_4(v))}{|v|}|a+x|}, & 0 < v < 1 \end{cases}$$

Step 2. For $\lambda \in \rho(T_1, M_1) \cap \rho(T_4, M_4)$, construct the operators K_{λ,M_1} and L_{λ,M_4} satisfying

$$\phi_{\lambda,M_1}(\mathrm{K}_{\lambda,M_1}g) = \psi_2(g) \ \text{ for } \ g \in \mathrm{dom}\,(T_{m_1}) \ \text{ and } \ \phi_{\lambda,M_4}(\mathrm{L}_{\lambda,M_4}f) = \psi_1(f) \ \text{ for } \ f \in \mathrm{dom}\,(T_{m_4}).$$

Indeed, according to Step 1, it suffices to establish the expressions of $\Gamma(a,v)$, $\Gamma(-a,v)$, $\Upsilon(a,v)$ and $\Upsilon(-a,v)$ which have to satisfy

$$\begin{cases} \mathrm{K}_{\lambda,M_1}g(x,v) = \gamma\Gamma(a,v) + \epsilon\Gamma(-a,v) \\ \gamma\Gamma^i(a,v) + \epsilon\Gamma^i(-a,v) = H_{12}g^o(x,v) \end{cases}$$

and

$$\begin{cases} \mathrm{L}_{\lambda,M_4}f(x,v) = \mu\Upsilon(a,v) + \nu\Upsilon(-a,v) \\ \mu\Upsilon^i(a,v) + \mu\Upsilon^i(-a,v) = H_{21}f^o(x,v) \end{cases}$$

Then,

$$\begin{cases} \Gamma(a,v) = H_{12}g(-a,v), & -1 < v < 0, \\ \Gamma(-a,v) = H_{12}g(a,v), & 0 < v < 1, \end{cases}$$

and

$$\begin{cases} \Upsilon(a,v) = H_{21}f(-a,v), & -1 < v < 0, \\ \Upsilon(-a,v) = H_{21}f(a,v), & 0 < v < 1, \end{cases}$$

which finds an explicit from of the bounded operators K_{λ,M_1} and L_{λ,M_4}. □

To compute the essential spectra of $(\mathcal{A}_H - \lambda M)$, we shall prove the Fredholmness perturbations of the operators $F_\lambda(T_1 - \lambda M_1)^{-1}$ and $G_\lambda(T_4 - \lambda M_4)^{-1}$. To do this, the following definition introduced by M. Kharroubi in [18] is required.

Definition 4.3. [18] *A collision operator $K_{jj'}$ in the form* (24)*, is said to be regular if it satisfies the following conditions:*

$$\begin{cases} -\textit{the function } K_{jj'}(.) \textit{ is mesurable}, \\ -\textit{there exists a compact subset } C \subset \mathcal{L}(L_p([-1,1],dv)) \textit{ such that } : \\ \ K_{jj'}(x) \in C \textit{ a.e on } [-a,a], \\ -K_{jj'}(x) \in \mathcal{K}(L_p([-1,1],dv)) \textit{ a.e on } [-a,a] \end{cases}$$

where $\mathcal{K}(L_p([-1,1],dv))$ is the set of compact operators on $L_p([-1,1],dv)$.

In order to verify the hypotheses of Theorem 3.10, We use the next lemma established in [11] .

Lemma 4.4. *Let $\lambda \in \rho(T_1, M_1) \cap \rho(T_4, M_4)$.*
If $K_{jj'}(x,v,v'),(j,j') \in \{(1,2),(2,1)\}$, defines a regular operator, then the operator $(T_i - \lambda M_i)^{-1}K_{jj'}$, $i = 1,4$, is compact on X_p, for $1 < p < \infty$ and weakly compact on X_1.

♢

By identification, we have

$$\begin{cases} G_\lambda = -\mathrm{K}_{\lambda,M_1} + (T_1 - \lambda M_1)^{-1}(K_{12} - \lambda M_2) \\ F_\lambda = -\mathrm{L}_{\lambda,M_4} + (T_4 - \lambda M_4)^{-1}(K_{21} - \lambda M_3) \end{cases}$$

and as a consequence for the previous lemma, the following result holds

Lemma 4.5. *Let $\lambda \in \rho(T_1, M_1) \cap \rho(T_4, M_4)$.*
(i) If M_2 and M_3 are a Fredholm perturbation on X_1 and $K_{jj'}$, $(j, j') \in \{(1,2),(2,1)\}$, is a regular operator, and if $H_{jj'} \in \mathcal{W}(X_1)$, then the operators $F_\lambda(T_1 - \lambda M_1)^{-1}$ and $G_\lambda(T_4 - \lambda M_4)^{-1})$ are weakly compact on X_1.
(ii) If M_2 and M_3 are a Fredholm perturbation on X_p and $K_{jj'}$, $(j, j') \in \{(1,2),(2,1)\}$, is a regular operator, and if $H_{jj'} \in \mathcal{K}(X_p)$, then the operators $F_\lambda(T_1 - \lambda M_1)^{-1}$ and $G_\lambda(T_4 - \lambda M_4)^{-1})$ are compact on X_p for $p > 1$. ♢

Proof. Let us write, for $\lambda \in \rho(T_1, M_1) \cap \rho(T_4, M_4)$, the operator $G_\lambda(T_4 - \lambda M_4)^{-1}$ (resp. $F_\lambda(T_1 - \lambda M_1)^{-1}$) as:

$$G_\lambda(T_4 - \lambda M_4)^{-1} = -\mathrm{K}_{\lambda,M_1}(T_4 - \lambda M_4)^{-1} + (T_1 - \lambda M_1)^{-1}(K_{12} - \lambda M_2)(T_4 - \lambda M_4)^{-1}$$

and

$$F_\lambda(T_1 - \lambda M_1)^{-1} = -\mathrm{L}_{\lambda,M_4}(T_1 - \lambda M_1)^{-1} + (T_4 - \lambda M_4)^{-1}(K_{21} - \lambda M_3)(T_1 - \lambda M_1)^{-1}$$

(*i*) We deduce from $H_{ij} \in \mathcal{W}(X_1)$ for $i, j = 1, 2$ that the operator K_{λ,M_1} (resp. L_{λ,M_4}) is weakly compact on X_1. So, the fact that the set $\mathcal{W}(X_1)$ is a closed two sided ideal of $\mathcal{L}(X_1)$, we have $\mathrm{K}_{\lambda,M_1}(T_4 - \lambda M_4)^{-1} \in \mathcal{W}(X_1)$ and $\mathrm{L}_{\lambda,M_4}(T_1 - \lambda M_1)^{-1} \in \mathcal{W}(X_1)$. Using Lemma 4.4 and the assumptions for the fredholmness perturbation of the operators M_2 and M_3, we deduce the result.
(*ii*) The use of Lemma 4.4 and the fact that the set $\mathcal{K}(X_p)$, for $p > 1$ is a closed two sided ideal of $\mathcal{L}(X_p)$ allows us to reach the result of assertion (*ii*) in a similar ways as in the item (*i*). □

Remark 4.6.
1. *For the remainder, we observe that if H_{12} is compact on $X_p, p > 1$ (resp. weakly compact on X_1), K_{12} defines a regular operator, then $F_\lambda G_\lambda \in \mathcal{K}(X_p)$ (resp. $F_\lambda G_\lambda \in \mathcal{W}(X_1)$). Hence, one has $[F_\lambda G_\lambda]^2 \in \mathcal{K}(X_p)$, $\forall p \geq 1$, we deduce that $F_\lambda G_\lambda \in \mathcal{PK}(X)$, $\forall p \geq 1$*
2. *Taking account from the last item and Theorem* 2.2 *in* [7] *we infer that the following properties are equivalent:*
   - $1 \in \rho(F_\lambda G_\lambda)$.
   - *$I - F_\lambda G_\lambda$ is invertible.*
   - *$I - F_\lambda G_\lambda$ is injective.*

The following proposition makes precise the injectivity properties.

Proposition 4.7. *Let $\lambda \in \rho(T_1, M_1) \cap \rho(T_4, M_4)$, then the operator $I - F_\lambda G_\lambda$ is injective.*

♢

Proof. Let $\lambda \in \rho(T_1, M_1) \cap \rho(T_4, M_4)$ and $h \in \ker(I - F_\lambda G_\lambda)$. Then we will solve the following equation:

$$(I - F_\lambda G_\lambda)h = 0.$$

The explicit expression of F_λ and G_λ and their properties yield that to solve the equation

$$\Big(T_{m_4} - \lambda M_4 - (k_{21} - \lambda M_3)(T_1 - \lambda M_1)^{-1}(k_{12} - \lambda M_2)\Big)h = 0.$$

Since $\lambda \in \rho(T_1, M_1) \cap \rho(T_4, M_4)$ and the use of Remark 3.1 in [10] assert that $\lambda \in \rho(T_1, M_1) \cap \rho(T_4, M_4) \cap \rho\Big(\Big[T_4 - (k_{21} - \lambda M_3)(T_1 - \lambda M_1)^{-1}(k_{12} - \lambda M_2)\Big], M_4\Big)$. That is $T_{m_4} - \lambda M_4 - (k_{21} - \lambda M_3)(T_1 - \lambda M_1)^{-1}(k_{12} - \lambda M_2)$ is injective and so $h = 0$. Hence, this argument yields the injectivity of the desert operator. □

Remark 4.8.
1. *By Remark* 4.6 *and Theorem* 3.7, *allows us to deduce that the matrix operator pencil $(\mathcal{A}_H - \lambda M)$ is invertible with bounded inverse.*
2. *By replacing the operator matrix $\mathcal{A}_H$ by $\mathcal{A}_0$ i.e (T_1 and T_2 are nothing else the streaming operators with vacuum boundaries conditions), we obtain $K_{\lambda,M_1} = L_{\lambda,M_4} = 0$ and $B = C = 0$. Then, $F_\lambda = F_{\lambda,d}$ and $G_\lambda = G_{\lambda,d}$. Hence, by Remark* 4.6 *and Proposition* 4.7, *we get $1 \in \rho(F_{\lambda,d}G_{\lambda,d})$. Furthermore we deduce that the matrix operator pencil $(\mathcal{A}_0 - \lambda M)$ is invertible with bounded inverse.*

We close this section with the precise picture of the essential spectra of two-group transport operators pencils with non-diagonal domain.

Theorem 4.9. *For $\lambda \in \rho(T_1, M_1) \cap \rho(T_4, M_4)$, if the operators M_2 and M_3 are Fredholm perturbations, K_{12} and K_{21} are regular operators and we assume that H_{12} and H_{21} are strictly singular operators on X_p for $p > 1$. Then,*

$$\begin{aligned}\sigma_{e,k}(\mathcal{A}_H, M) &= \sigma_{e,k}(T_1, M_1) \cup \sigma_{e,k}(T_4, M_4) \\ &= \{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -min(\frac{\lambda_1^*}{\mu_1^*}, \frac{\lambda_4^*}{\mu_4^*})\}, \ \ k = 4, 5, l, r.\end{aligned}$$

♢

Proof. Let $\lambda \in \rho(T_1, M_1) \cap \rho(T_4, M_4)$. Lemma 4.2, Remark 4.6, Theorem 2.6, Proposition 4.7 and Remark 4.8 assert that $\lambda \in \rho(T_1, M_1) \cap \rho(T_4, M_4) \cap \rho(\mathcal{A}_H, M) \cap \rho(\mathcal{A}_0, M)$
and

$$(\mathcal{A}_H - \lambda M)^{-1} - (\mathcal{A}_0 - \lambda M)^{-1} \in \mathcal{S}(X_p), \ \ p > 1$$

Consequently, Theorem 3.10 reveals that

$$\sigma_{e,k}(\mathcal{A}_H - \lambda M) = \sigma_{e,k}(\mathcal{A}_0 - \lambda M) = \sigma_{e,k}(T_1, M_1) \cup \sigma_{e,k}(T_4, M_4), \ \ \forall\, k \in \{4, r, l\}.$$

And Theorem 3.2 in [11] shows that

$$\sigma_{e,4}(T_1, M_1) = \{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -\frac{\lambda_1^*}{\mu_1^*})\},$$

and

$$\sigma_{e,4}(T_4, M_4) = \{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -\frac{\lambda_4^*}{\mu_4^*})\}.$$

Consequently, Eqs. (6) and (7) amounts that

$$\sigma_{e,r}(T_1, M_1) = \sigma_{e,l}(T_1, M_1) = \sigma_{e,4}(T_1, M_1),$$

and

$$\sigma_{e,r}(T_4, M_4) = \sigma_{e,l}(T_4, M_4) = \sigma_{e,4}(T_4, M_4).$$

Therefore, we get

$$\sigma_{e,k}(\mathcal{A}_H, M) = \{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -min(\frac{\lambda_1^*}{\mu_1^*}, \frac{\lambda_4^*}{\mu_4^*})\}, \ \ \forall\, k \in \{4, l, r\}.$$

□

Conclusion: Sufficient conditions are reduced to the study of invertibility conditions of unbounded operator matrix pencil and this study is applied to the new investigation of spectral properties of matrix operator pencil with non-diagonal domain(see Theorem 3.10 and 3.11) in a fast manner of computation. Such a result exploit the resolvent expression involving an elegant use of the notion of Fredholm type properties of 2×2 operator matrix (see Theorem 2.6)and pursue our alternative approach of studying the essential spectra of non maximal operator matrix pencil given in [8, 11, 12, 25]. Finally, under less hypotheses, sufficient conditions in terms of collision operators assuring the stability of some essential spectra of unbounded operator matrices pencils with non-diagonal domain obtained in [11, 25].

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