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On Generalized W_2-Curvature Tensor of Para-Kenmotsu Manifolds

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Abstract. The object of the present paper is to generalize W_2-curvature tensor of para-Kenmotsu manifold with the help of a new generalized (0,2) symmetric tensor $\mathcal{Z}$ introduced by Mantica and Suh [11]. Various geometric properties of generalized W_2-curvature tensor of para-Kenmotsu manifold have been studied. It is shown that a generalized W_2 ϕ-symmetric para-Kenmotsu manifold is an Einstein manifold.

1. Introduction

The W_2 and E-tensor fields were introduced by G.P. Pokhariyal and R.S. Mishra [15] in 1970. They studied these tensor fields and their relativistic significance in a Riemannian manifold. Further, in 1980, G.P. Pokhariyal [14] carried out the study of these tensor fields in a Sasakian manifolds. Later on, in 1986, properties of W_2 and E-tensor fields were further explored by K. Matsumoto, S. Ianus and I. Mihai [12] on P-Sasakian manifolds . The W_2-curvature tensor has been studied by many other authors such as U.C. De and A. Sarkar [7], A. Yildiz and U.C. De [21] and many others. The W_2-curvature tensor is defined by [15]

$$W_2(X,Y,U) = R(X,Y,U) + \frac{1}{n-1}[g(X,U)QY - g(Y,U)QX], \tag{1}$$

where Q is a Ricci tensor of type (1,1), i.e., $S(X,Y) = g(QX,Y)$; S being the type (0,2) Ricci tensor. Afterwards several researchers have carried out the study of W_2-curvature tensor in a variety of directions such as [13, 18, 19].

Several years ago, the notion of paracontact metric structures were introduced in [8]. Since the publication of [3–5, 22], paracontact metric manifolds have been studied by many authors in recent years. The importance of para-Kenmotsu geometry, have been pointed out especially in the last years by several papers highlighting the exchanges with the theory of para-Kähler manifolds and its role in semi-Riemannian geometry and mathematical physics [6, 9, 10, 17].

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In this paper, we consider the generalized W_2-curvature tensor of para-Kenmotsu manifolds and study some properties of generalized W_2-curvature tensor. The organisation of the paper is as follows: After preliminaries on para-Kenmotsu manifold in section 2, we briefly describe the generalized W_2-curvature tensor on a para-Kenmotsu manifold in section 3 and also study some properties of the generalized W_2-curvature tensor in a para-Kenmotsu manifold. In section 4, we prove that a generalized W_2 semi-symmetric para-Kenmotsu manifold is an η-Einstein manifold. Further in the section 5, we show that a generalized W_2 Ricci semi-symmetric para-Kenmotsu manifold is either an Einstein manifold or $\psi = 0$ on it. In the last section, we prove that a generalized W_2 ϕ-symmetric para-Kenmotsu manifold is an Einstein manifold.

2. Preliminaries

The notion of an almost para-contact manifold was introduced by I. Sato [16]. An n-dimensional differentiable manifold M^n is said to have almost para-contact structure (ϕ, ξ, η), where ϕ is a tensor field of type $(1, 1)$, ξ is a vector field known as characteristic vector field and η is a 1-form satisfying the following relations

$$\phi^2(X) = X - \eta(X)\xi, \tag{2}$$

$$\eta(\phi X) = 0, \tag{3}$$

$$\phi(\xi) = 0 \tag{4}$$

and

$$\eta(\xi) = 1. \tag{5}$$

A differentiable manifold with an almost para-contact structure (ϕ, ξ, η) is called an almost para-contact manifold. Further, if the manifold M^n has a semi-Riemannian metric g satisfying

$$\eta(X) = g(X, \xi) \tag{6}$$

and

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{7}$$

then the structure (ϕ, ξ, η, g) satisfying conditions (2) to (7) is called an almost para-contact Riemannian structure and the manifold M^n with such a structure is called an almost para-contact Riemannian manifold [1, 16].

On a para-Kenmotsu manifold [2, 17], the following relations hold:

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{8}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{9}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{10}$$

$$\eta(R(X, Y, Z)) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{11}$$

$$R(X, Y, \xi) = \eta(X)Y - \eta(Y)X, \tag{12}$$

$$R(X, \xi, Y) = -R(\xi, X, Y) = g(X, Y)\xi - \eta(Y)X, \tag{13}$$

$$S(\phi X, \phi Y) = -(n-1)g(\phi X, \phi Y), \tag{14}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{15}$$

$$Q\xi = -(n-1)\xi, \tag{16}$$

$$r = -n(n-1), \tag{17}$$

for any vector fields X, Y, Z, where Q is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$, S is the Ricci tensor and r is the scalar curvature.

In [2], Blaga has given an example of para-Kenmotsu manifold:

Example 2.1. *[2] We consider the three dimensional manifold $M^3 = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are the standard co-ordinates in $\mathbb{R}^3$. The vector fields*

$$e_1 := \frac{\partial}{\partial x}, \quad e_2 := \frac{\partial}{\partial y}, \quad e_3 := -\frac{\partial}{\partial z}$$

are linearly independent at each point of the manifold.
Define

$$\phi := \frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial x} \otimes dy, \quad \xi := -\frac{\partial}{\partial z}, \quad \eta := -dz,$$

$$g := dx \otimes dx - dy \otimes dy + dz \otimes dz.$$

Then it follows that

$$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0,$$

$$\eta(e_1) = 0, \quad \eta(e_2) = 0, \quad \eta(e_3) = 1.$$

Let ∇ be the Levi-Civita connetion with respect to metric g. Then, we have

$$[e_1, e_2] = 0, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0.$$

The Riemannian connection ∇ of the metric g is deduced from Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned}$$

Then Koszul's formula yields

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_2} e_3 = e_2,$$

$$\nabla_{e_3} e_1 = e_1, \quad \nabla_{e_3} e_2 = e_2, \quad \nabla_{e_3} e_3 = 0.$$

These results show that the manifold satisfies

$$\nabla_X \xi = X - \eta(X)\xi,$$

for $\xi = e_3$. Hence, the manifold under consideration is para-Kenmotsu manifold of dimension three.

A para-Kenmotsu is said to be an η-Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{18}$$

for any vector fields X, Y, where a and b are functions on M^n.

3. Generalized W_2-curvature tensor of a para-Kenmotsu manifold

In this section, we give a brief account of generalized W_2-curvature tensor of a para-Kenmotsu manifold and study various geometric properties of it.

Now we consider W_2-curvature tensor field for the para-Kenmotsu manifold which is given by the following relation

$$W_2(X,Y,U) = R(X,Y,U) + \frac{1}{(n-1)}[g(X,U)QY - g(Y,U)QX]. \tag{19}$$

Also, the $(0,4)$ type tensor field $'W_2$ is given by

$$'W_2(X,Y,U,V) = {}'R(X,Y,U,V) + \frac{1}{(n-1)}[g(X,U)S(Y,V) - g(Y,U)S(X,V)] \tag{20}$$

where

$$'W_2(X,Y,U,V) = g(W_2(X,Y,U),V)$$

and

$$'R(X,Y,U,V) = g(R(X,Y,U),V)$$

for arbitrary vector fields X,Y,U,V.

Differentiating covariantly equation (19) with respect to V, we get

$$(\nabla_V W_2)(X,Y)U = (\nabla_V R)(X,Y)U + \frac{1}{(n-1)}[g(X,U)(\nabla_V Q)Y - g(Y,U)(\nabla_V Q)X]. \tag{21}$$

Divergence of W_2-curvature tensor given by equation (19), is

$$(divW_2)(X,Y)U = (divR)(X,Y)U + \frac{1}{(n-1)}[g(X,U)(div(Q)Y) - g(Y,U)(div(Q)X)]. \tag{22}$$

But

$$(divR)(X,Y)U = (\nabla_X S)(Y,U) - (\nabla_Y S)(X,U), \tag{23}$$

By equations (22) and (23), gives

$$(divW_2)(X,Y)U = [(\nabla_X S)(Y,U) - (\nabla_Y S)(X,U)] + \frac{1}{(n-1)}[g(X,U)(div(Q)Y) - g(Y,U)(div(Q)X)]. \tag{24}$$

A new generalized $(0,2)$ symmetric tensor $\mathcal{Z}$ is defined by Mantica and Suh [11]

$$\mathcal{Z}(X,Y) = S(X,Y) + \psi g(X,Y), \tag{25}$$

where ψ is an arbitrary scalar function.
From equation (25), we have

$$\mathcal{Z}(\phi X,\phi Y) = S(\phi X,\phi Y) + \psi g(\phi X,\phi Y), \tag{26}$$

which, on using equations (7) and (14), gives

$$\mathcal{Z}(\phi X,\phi Y) = [\psi - (n-1)][-g(X,Y) + \eta(X)\eta(Y)]. \tag{27}$$

From equation (20), we have

$$'W_2(X,Y,U,V) = 'R(X,Y,U,V) + \frac{1}{(n-1)}[g(X,U)S(Y,V) - g(Y,U)S(X,V)]. \tag{28}$$

In view of equation (25), the above equation reduces to

$$\begin{aligned}'W_2(X,Y,U,V) = {}& 'R(X,Y,U,V) + \frac{1}{(n-1)}[\mathcal{Z}(Y,V)g(X,U) - \mathcal{Z}(X,V)g(Y,U)] \\ &+ \frac{\psi}{(n-1)}[g(Y,U)g(X,V) - g(Y,V)g(X,U)].\end{aligned} \tag{29}$$

We now put

$$'W_2^*(X,Y,U,V) = 'R(X,Y,U,V) + \frac{1}{(n-1)}[g(X,U)\mathcal{Z}(Y,V) - g(Y,U)\mathcal{Z}(X,V)]. \tag{30}$$

Then from the equation (29), we get

$$'W_2^*(X,Y,U,V) = 'W_2(X,Y,U,V) - \frac{\psi}{(n-1)}[g(X,V)g(Y,U) - g(Y,V)g(X,U)]. \tag{31}$$

The tensor field $'W_2^*$ defined by equation (30) is called the generalized W_2-curvature tensor of para-Kenmotsu manifold.

Obviously if ψ=0, then from equation (31), we have

$$'W_2^*(X,Y,U,V) = 'W_2(X,Y,U,V). \tag{32}$$

Thus, we may write the following theorem.

Theorem 3.1. *If the scalar function ψ vanishes on the para-Kenmotsu manifold, then the W_2-curvature tensor and generalized W_2-curvature tensor coincide.*

Theorem 3.2. *Generalized W_2-curvature tensor $'W_2^*$ of a para-Kenmotsu manifold is*

(a) skew symmetric in the first two slots,
(b) skew symmetric in the last two slots,
(c) symmetric in the pair of slots.

Proof: (a) From equation (31), we have

$$'W_2^*(Y,X,U,V) = 'W_2(Y,X,U,V) - \frac{\psi}{(n-1)}[g(X,U)g(Y,V) - g(Y,U)g(X,V)]. \tag{33}$$

Now adding equations (31) and (33) and using the fact that

$$'W_2(X,Y,U,V) + 'W_2(Y,X,U,V) = 0,$$

we get

$$'W_2^*(X,Y,U,V) = -'W_2^*(Y,X,U,V),$$

which shows that the generalized W_2-curvature tensor $'W_2^*$ is skew symmetric in the first two slots.

(b) Again, from equation (31), we have

$$'W_2^*(X,Y,V,U) = \, 'W_2(X,Y,V,U) - \frac{\psi}{(n-1)}[g(Y,V)g(X,U) - g(X,V)g(Y,U)]. \tag{34}$$

Now adding equations (31) and (34) and using the fact that

$$'W_2(X,Y,U,V) + \, 'W_2(X,Y,V,U) = 0,$$

we obtain

$$'W_2^*(X,Y,U,V) = -\, 'W_2^*(X,Y,V,U),$$

which shows that the generalized W_2-curvature tensor $'W_2^*$ is skew symmetric in the last two slots.

(c) From equation (31), interchanging pair of slots, we have

$$'W_2^*(U,V,X,Y) = \, 'W_2(U,V,X,Y) - \frac{\psi}{(n-1)}[g(U,Y)g(V,X) - g(U,X)g(V,Y)]. \tag{35}$$

In view of the fact that

$$'W_2(X,Y,U,V) = \, 'W_2(U,V,X,Y),$$

we get from equations (31) and (35)

$$'W_2^*(X,Y,U,V) = \, 'W_2^*(U,V,X,Y),$$

which shows that the generalized W_2-curvature tensor $'W_2^*$ is symmetric in pair of slots.

Theorem 3.3. *Generalized W_2-curvature tensor of a para-Kenmotsu manifold satisfies Bianchi's first identity.*

Proof: From equation (31), we have

$$W_2^*(X,Y,U) = W_2(X,Y,U) - \frac{\psi}{(n-1)}[g(Y,U)X - g(X,U)Y]. \tag{36}$$

Writing two more equations by cyclic permutations of X, Y and U in the above equation, we get

$$W_2^*(Y,U,X) = W_2(Y,U,X) - \frac{\psi}{(n-1)}[g(U,X)Y - g(Y,X)U)] \tag{37}$$

and

$$W_2^*(U,X,Y) = W_2(U,X,Y) - \frac{\psi}{(n-1)}[g(X,Y)U - g(U,Y)X)]. \tag{38}$$

Adding equations (36), (37) and (38) and using the fact that

$$W_2(X,Y,U) + W_2(Y,U,X) + W_2(U,X,Y) = 0,$$

we get

$$W_2^*(X,Y,U) + W_2^*(Y,U,X) + W_2^*(U,X,Y) = 0,$$

which shows that the generalized W_2-curvature tensor of a para-Kenmotsu manifold satisfies Bianchi's first identity.

Theorem 3.4. *Generalized W_2-curvature tensor of a para-Kenmotsu manifold satisfies the following identities:*

$$\text{(a) } W_2^*(\xi, Y, U) = -W_2^*(Y, \xi, U) = \left[\frac{n-1+\psi}{n-1}\right]\eta(U)Y + \frac{1}{(n-1)}[\eta(U)QY - \psi g(Y,U)\xi] \tag{39}$$

$$\text{(b) } W_2^*(X, Y, \xi) = \left[\frac{n-1+\psi}{n-1}\right][\eta(X)Y - \eta(Y)X] + \frac{1}{(n-1)}[\eta(X)QY - \eta(Y)QX] \tag{40}$$

$$\text{(c) } \eta(W_2^*(U, V, Y)) = \frac{\psi}{(n-1)}[g(U,Y)\eta(V) - g(V,Y)\eta(U)]. \tag{41}$$

Proof: (a) Putting $X = \xi$ in the equation (36), we have

$$W_2^*(\xi, Y, U) = W_2(\xi, Y, U) - \frac{\psi}{(n-1)}[g(Y,U)\xi - g(\xi,U)Y],$$

which, on using equations (6), (13), (16), (19), yields the desired result.

(b) Again, putting $U = \xi$ in the equation (36), we have

$$W_2^*(X, Y, \xi) = W_2(X, Y, \xi) - \frac{\psi}{(n-1)}[g(Y,\xi)X - g(X,\xi)Y].$$

Now, using equations (6), (12), (19) in the above equation, we obtain the required result.

(c) Taking the inner product with ξ in equation (36), we have

$$\eta(W_2^*(U, V, Y)) = \eta(W_2(U, V, Y)) - \frac{\psi}{(n-1)}[g(V,Y)\eta(U) - g(U,Y)\eta(V)],$$

which, on using equations (11), (16), (19), gives the desired result.

4. Generalized W_2 semi-symmetric para-Kenmotsu manifold

Definition 4.1. *A para-Kenmotsu manifold is said to be semi-symmetric if it satisfies the condition*

$$R(X,Y) \cdot R = 0, \tag{42}$$

where $R(X,Y)$ is considered as the derivation of the tensor algebra at each point of the manifold.

Definition 4.2. *A para-Kenmotsu manifold is said to be generalized W_2 semi-symmetric if it satisfies the condition*

$$R(X,Y) \cdot W_2^* = 0, \tag{43}$$

where W_2^* is the generalized W_2-curvature tensor and $R(X,Y)$ is considered as the derivation of the tensor algebra at each point of the manifold.

Theorem 4.3. *A generalized W_2 semi-symmetric para-Kenmotsu manifold is an η-Einstein manifold.*

Proof: Consider $(R(\xi, X) \cdot W_2^*)(U, V, Y) = 0,$

for any vector fields X, Y, U, V, where W_2^* is generalized W_2-curvature tensor.
Then we have

$$0 = R(\xi, X, W_2^*(U, V, Y)) - W_2^*(R(\xi, X, U), V, Y) - W_2^*(U, R(\xi, X, V), Y) - W_2^*(U, V, R(\xi, X, Y)). \tag{44}$$

In view of the equation (13), the above equation takes the form

$$\begin{aligned} 0 &= \eta(W_2^*(U, V, Y))X -' W_2^*(U, V, Y, X)\xi - \eta(U)W_2^*(X, V, Y) + g(X, U)W_2^*(\xi, V, Y) \\ &- \eta(V)W_2^*(U, X, Y) + g(X, V)W_2^*(U, \xi, Y) - \eta(Y)W_2^*(U, V, X) + g(X, Y)W_2^*(U, V, \xi). \end{aligned}$$

Taking the inner product of above equation with ξ and using equations (5), (16), (31), (39), (40) and (41), we get

$$\begin{aligned} 'W_2(U, V, Y, X) &= -\frac{\psi}{(n-1)}[g(X, U)\eta(V)\eta(Y) - g(X, V)\eta(U)\eta(Y)] \\ &+ \left[\frac{n-1+\psi}{(n-1)}\right] g(X, U)\eta(V)\eta(Y) - g(X, U)\eta(Y)\eta(V) \\ &- \left[\frac{n-1+\psi}{(n-1)}\right] g(X, V)\eta(U)\eta(Y) + g(X, V)\eta(Y)\eta(U). \end{aligned}$$

By virtue of equation (20), the above equation reduces to

$$'R(U, V, Y, X) = -\frac{1}{(n-1)}[g(Y, U)S(X, V) - g(Y, V)S(X, U)].$$

Let $\{e_i : i = 1, 2, ..., n\}$ be an orthonormal basis with $\nabla_{e_i} e_j = 0$. Putting $X = U = e_i$ in the above equation and taking summation over i, we get

$$S(Y, V) = -ng(Y, V) + \eta(Y)\eta(V).$$

This shows that the generalized W_2 semi-symmetric para-Kenmotsu manifold is an η-Einstein manifold.

5. Generalized W_2 Ricci semi-symmetric para-Kenmotsu manifold

Definition 5.1. *A para-Kenmotsu manifold M is said to be Ricci semi-symmetric if the condition*

$$R(X, Y) \cdot S = 0, \tag{45}$$

holds for all vector fields X, Y.

Definition 5.2. *A para-Kenmotsu manifold is said to be generalized W_2 Ricci semi-symmetric if the condition*

$$W_2^*(X, Y) \cdot S = 0, \tag{46}$$

holds for all vector fields X, Y, where W_2^* is generalized W_2-curvature tensor of a para-Kenmotsu manifold.

Theorem 5.3. *A generalized W_2 Ricci semi-symmetric para-Kenmotsu manifold is either an Einstein manifold or $\psi = 0$ on it.*

Proof: Consider

$$(W_2^*(\xi, X) \cdot S)(U, V) = 0,$$

which gives

$$S(W_2^*(\xi, X, U), V) + S(U, W_2^*(\xi, X, V)) = 0.$$

Using equations (15), (16) and (39) in the above equation, we get

$$\frac{\psi}{(n-1)}[S(X, V)\eta(U) + S(X, U)\eta(V)] + \psi[g(X, U)\eta(V) + g(X, V)\eta(U)] = 0.$$

Putting $U = \xi$ in the above equation and using (5), (6) and (15), we get

$$\psi[S(X, V) + g(X, V)(n-1)] = 0,$$

which gives either $\psi = 0$ or

$$S(X, V) = -(n-1)g(X, V).$$

This shows that the generalized W_2 Ricci semi-symmetric para-Kenmotsu manifold is an Einstein manifold.

6. Generalized W_2 ϕ-symmetric para-Kenmotsu manifold

Definition 6.1. *A para-Kenmotsu manifold M^n is said to be locally ϕ-symmetric if*

$$\phi^2((\nabla_V R)(X, Y, U)) = 0, \tag{47}$$

for all vector fields X, Y, U, V orthogonal to ξ.

Definition 6.2. *A para-Kenmotsu manifold is said to be ϕ-symmetric if*

$$\phi^2((\nabla_V R)(X, Y, U)) = 0, \tag{48}$$

for all vector fields X, Y, U, V.

These notions were introduced by Takahashi for Sasakian manifold [20]. Analogous to these definitons, we consider

Definition 6.3. *A para-Kenmotsu manifold M^n is said to be a generalized W_2 locally ϕ-symmetric para-Kenmotsu manifold if*

$$\phi^2((\nabla_V W_2^*)(X, Y, U)) = 0, \tag{49}$$

for all vector fields X, Y, U, V orthogonal to ξ.

Definition 6.4. *A para-Kenmotsu manifold M^n is said to be a generalized W_2 ϕ-symmetric para-Kenmotsu manifold if*

$$\phi^2((\nabla_V W_2^*)(X, Y, U)) = 0, \tag{50}$$

for all vector fields X, Y, U, V.

Theorem 6.5. *A generalized W_2 ϕ-symmetric para Kenmotsu manifold is an Einstein manifold.*

Proof: Taking the covariant derivative of equation (36) with respect to the vector field V, we obtain

$$(\nabla_V W_2^*)(X,Y,U) = (\nabla_V W_2)(X,Y,U) - \frac{dr(\psi)}{(n-1)}[g(Y,U)X - g(X,U)Y]. \tag{51}$$

Using equation (21) in the above equation, it yields

$$\begin{aligned}(\nabla_V W_2^*)(X,Y,U) = (\nabla_V R)(X,Y,U) - \frac{dr(\psi)}{(n-1)}[g(Y,U)X - g(X,U)Y] \\ +\frac{1}{(n-1)}[g(X,U)(\nabla_V Q)Y - g(Y,U)(\nabla_V Q)X],\end{aligned} \tag{52}$$

Assume that the para-Kenmotsu manifold is generalized W_2 ϕ-symmetric, i.e., satisfies

$$\phi^2((\nabla_V W_2^*)(X,Y,U)) = 0,$$

for all vector fields, which on using equation (2), gives

$$(\nabla_V W_2^*)(X,Y,U) = \eta((\nabla_V W_2^*)(X,Y,U))\xi.$$

Using equation (52) in the above equation, we get

$$\begin{aligned}(\nabla_V R)(X,Y,U) - \frac{dr(\psi)}{(n-1)}[g(Y,U)X - g(X,U)Y)] + \frac{1}{(n-1)}[g(X,U)(\nabla_V Q)Y - g(Y,U)(\nabla_V Q)X] \\ = \eta((\nabla_V R)(X,Y,U))\xi - \frac{dr(\psi)}{(n-1)}[g(Y,U)\eta(X) - g(X,U)\eta(Y)]\xi \\ +\frac{1}{(n-1)}[g(X,U)\eta((\nabla_V Q)Y) - g(Y,U)\eta((\nabla_V Q)X)]\xi,\end{aligned}$$

Taking the inner product of the above equation with W, we get

$$\begin{aligned}g((\nabla_V R)(X,Y,U),W) - \frac{dr(\psi)}{(n-1)}[g(Y,U)g(X,W) - g(X,U)g(Y,W)] \\ +\frac{1}{(n-1)}[g(X,U)g((\nabla_V Q)Y,W) - g(Y,U)g((\nabla_V Q)X,W)] \\ = \eta((\nabla_V R)(X,Y,U))\eta(W) - \frac{dr(\psi)}{(n-1)}[g(Y,U)\eta(X)\eta(W) \\ -g(X,U)\eta(Y)\eta(W)] + \frac{1}{(n-1)}[g(X,U)\eta((\nabla_V Q)Y)\eta(W) \\ -g(Y,U)\eta((\nabla_V Q)X)\eta(W)],\end{aligned}$$

Putting $X = W = e_i$ in the above equation and taking the summation over i, we obtain

$$\begin{aligned}(\nabla_V S)(Y,U) + \frac{1}{(n-1)}[g((\nabla_V Q)Y,U) - g(Y,U)g((\nabla_V Q)e_i,e_i)] \\ -dr(\psi)g(Y,U) - \eta((\nabla_V R)(e_i,Y,U))\eta(e_i) - \frac{1}{(n-1)}[\eta((\nabla_V Q)Y)\eta(U) \\ -g(Y,U)\eta((\nabla_V Q)e_i)\eta(e_i)] + \frac{dr(\psi)}{(n-1)}[g(Y,U) - \eta(Y)\eta(U)] = 0.\end{aligned}$$

Taking $U = \xi$ in the above equation, we have

$$\begin{aligned}(\nabla_V S)(Y,\xi) - \eta((\nabla_V R)(e_i,Y,\xi))\eta(e_i) - dr(\psi)\eta(Y) \\ -\frac{1}{(n-1)}[dr(V)\eta(Y) - \eta((\nabla_V Q)e_i)\eta(e_i)\eta(Y)] = 0.\end{aligned} \tag{53}$$

The second term on L.H.S. of equation (53) takes the form and denoting it by E which is of the form

$$E = \eta((\nabla_V R)(e_i, Y, \xi))\eta(e_i) = g((\nabla_V R)(e_i, Y, \xi), \xi)g(e_i, \xi).$$

In this case E vanishes. Namely we have

$$\begin{aligned} g((\nabla_V R)(e_i, Y, \xi), \xi) = g(\nabla_V R(e_i, Y, \xi), \xi) - g(R(\nabla_V e_i, Y, \xi), \xi) \\ - g(R(e_i, \nabla_V Y, \xi), \xi) - g(R(e_i, Y, \nabla_V \xi), \xi). \end{aligned} \tag{54}$$

Since $\nabla_X e_i = 0$ and using equation (12) in (54), we get

$$g(R(e_i, \nabla_V Y, \xi), \xi) = 0.$$

In view of $g(R(e_i, Y, \xi), \xi) + g(R(\xi, \xi, Y), e_i) = 0$, we have

$$g(\nabla_V R(e_i, Y, \xi), \xi) + g(R(e_i, Y, \xi), \nabla_V \xi) = 0.$$

Using this fact in equation (54), we get

$$g((\nabla_V R)(e_i, Y, \xi), \xi) = 0. \tag{55}$$

Also

$$\eta((\nabla_V Q)e_i)\eta(e_i) = g((\nabla_V Q)e_i, \xi)g(e_i, \xi) = g((\nabla_V Q)\xi, \xi).$$

Using equations (9) and (15), we get

$$\eta((\nabla_V Q)e_i)\eta(e_i) = 0. \tag{56}$$

Using equations (55) and (56) in (53), we have

$$(\nabla_V S)(Y, \xi) = dr(\psi)\eta(Y) + \frac{1}{(n-1)}dr(V)\eta(Y). \tag{57}$$

Taking $Y = \xi$ in the above equation and using equations (5) and (15), we get

$$dr(\psi) = -\frac{dr(V)}{(n-1)}, \tag{58}$$

which shows that r is constant. Now, we have

$$(\nabla_V S)(Y, \xi) = \nabla_V S(Y, \xi) - S(\nabla_V Y, \xi) - S(Y, \nabla_V \xi).$$

Then by using (9), (10), (15) in the above equation, it follows that

$$(\nabla_V S)(Y, \xi) = -S(Y, V) - (n-1)g(Y, V). \tag{59}$$

So from equations (57), (58) and (59), we get

$$S(Y, V) = -(n-1)g(Y, V),$$

which shows that M^n is an Einstein manifold.

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