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Products and Inverses of Multidiagonal Matrices with Equally Spaced Diagonals

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Abstract. Let n, k be fixed natural numbers with $1 \le k \le n$ and let $A_{n+1,k,2k,\ldots,sk}$ denote an $(n+1)\times(n+1)$ complex multidiagonal matrix having $s = [n/k]$ sub- and superdiagonals at distances $k, 2k, \ldots, sk$ from the main diagonal. We prove that the set $\mathcal{MD}_{n,k}$ of all such multidiagonal matrices is closed under multiplication and powers with positive exponents. Moreover the subset of $\mathcal{MD}_{n,k}$ consisting of all nonsingular matrices is closed under taking inverses and powers with negative exponents. In particular we obtain that the inverse of a nonsingular matrix $A_{n+1,k}$ (called k-tridigonal) is in $\mathcal{MD}_{n,k}$, moreover if $n+1 \le 2k$ then $A^{-1}_{n+1,k}$ is also k-tridigonal. Using this fact we give an explicit formula for this inverse.

1. Introduction

Multidiagonal matrices have a wide range of applications in various fields of mathematics and engineering. Among them matrices with equally spaced diagonals have much nicer properties than those with arbitrarily spaced diagonals (see [1, 9, 10] and their references). Here we study how multidiagonal matrices with equally spaced diagonals behave under multiplication, taking inverse and powers.

Let n, k be fixed natural numbers with $1 \le k \le n$ and let $\mathcal{M}_n$ denote the set of $n \times n$ complex matrices. A matrix $A = (a_{ij})_{i,j=0}^{n}, \in \mathcal{M}_{n+1}$ is called $(k, 2k, \ldots, sk)$- multidiagonal if $a_{ij} = 0$ if $|i-j| \ne lk$, for $l = 0, 1, \ldots, s$ where $sk \le n$. Such matrices will be denoted by $A_{n+1,k,2k,\ldots,sk}$ (supressing for the moment their dependence from the diagonals). Such matrices are called k-tridiagonal if $s = 1$ and $(k, 2k)$-pentadiagonal if $s = 2$. Clearly the maximal number s of sub- and superdiagonals in $A_{n+1,k,2k,\ldots,sk}$ is $[n/k]$.

Let $\mathcal{MD}_{n,k}$ be the set of all $A_{n+1,k,2k,\ldots,sk}$ matrices with $s = [n/k]$. We prove that the set $\mathcal{MD}_{n,k}$ is closed under multiplication, and taking positive and (for nonsingular matrices also) negative powers. Since matrices $A_{n+1,k,2k,\ldots,s'k}$ with $1 \le s' \le [n/k]$ also belong to $\mathcal{MD}_{n,k}$ (by taking the diagonals $(s'+1)k, (s'+2)k, \ldots, sk$ to be zero) we obtain that the inverse of a k-tridiagonal matrix belongs to $\mathcal{MD}_{n,k}$. Moreover, if $n+1 \le 2k$ then the inverse of a k-tridiagonal matrix is also k-tridiagonal. Using this we find the explicit inverse of such k-tridiagonal matrices.

The articles [2, 15] are related to the structure of the product of tridiagonal matrices. Their investigations are based on the result that the product of two different 1-tridiagonal Toeplitz matrices is a $(1, 2)$-pentadiagonal imperfect Toeplitz matrix (where the first and last elements of the main diagonal are

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different from the other ones). In [17] the authors use Toeplitz $(1,2)$-pentadiagonal matrices to study orthogonal polynomials on the unit circle.

In the paper [18] a representation is given for the powers of k-tridiagonal ℓ-Toeplitz matrices (where the sub- and superdiagonals are ℓ-periodic) and a formula is derived for their eigenpairs. [16] studies matrices with one sub- and one superdiagonal only (not necessarily symmetrically located) and proves that its eigenvalues are constant multiples of roots of unity.

2. Multidiagonal matrices as the sum of their diagonals

In the sequel (unless otherwise said) all matrices will be in $\mathcal{M}_{n+1}$. Let now $A_{n+1,k,2k,\dots,sk} = (a_{ij})_{i,j=0}^{n}$ where $s = [n/k]$. Denote its sub-, main, superdiagonal vectors extended to $n+1$ dimensional vectors by adding the necessary number of zeros after their last coordinates by

$$\begin{aligned}
\mathbf{v}_{-s} &= (v_{-s,0}, \dots, v_{-s,n-sk}, 0, \dots, 0),\\
&\vdots\\
\mathbf{v}_{-1} &= (v_{-1,0}, \dots, v_{-1,n-k}, 0, \dots, 0),\\
\mathbf{v}_{0} &= (v_{0,0}, \dots, v_{0,n}),\\
\mathbf{v}_{1} &= (v_{1,0}, \dots, v_{1,n-k}, 0, \dots, 0),\\
&\vdots\\
\mathbf{v}_{s} &= (v_{s,0}, \dots, v_{s,n-sk}, 0, \dots, 0).
\end{aligned} \tag{1}$$

This means that for $i, j = 0, \dots, n$

$$a_{ij} = \begin{cases} v_{p,j} & \text{if } j - i = pk, p = -s, \dots, 0, \dots, s, \\ 0 & \text{otherwise.} \end{cases}$$

For this matrix we also use the notations

$$A = A_{n+1,k,2k,\dots,sk} = A(\mathbf{v}_{-s}, \dots, \mathbf{v}_0, \dots, \mathbf{v}_s) = A_{n+1,k,2k,\dots,sk}(\mathbf{v}_{-s}, \dots, \mathbf{v}_0, \dots, \mathbf{v}_s) \tag{2}$$

always trying to choose the most convenient one. Here we have to remark that only the nonzero coordinates of the diagonal vectors take part in building our matrix. Clearly all matrices of $\mathcal{MD}_{n,k}$ can be written in the form (2).

Introduce the elementary nilpotent matrix $N = (n_{ij})$ with

$$n_{ij} = \begin{cases} 1 & \text{if } i - j = -1, \\ 0 & \text{otherwise.} \end{cases}$$

N contains one single unit superdiagonal right above the main diagonal, and its transpose N^T contains one single unit subdiagonal immediately below the main diagonal.

It is easy to check that raising N to power $k > 0$ moves its single unit superdiagonal to distance k above the main diagonal.

The Moore-Penrose inverse N^+ of N is its transpose, i.e $N^+ = N^T$. Let $N^0 := E$ (the unit matrix in $\mathcal{M}_{n+1}$) and define the negative powers of N by

$$N^{-k} := (N^+)^k = (N^T)^k \quad (k \in \mathbb{N}).$$

Then N^{-k} has a single unit subdiagonal at distance k below the main diagonal. For $|k| \geq n+1$ the matrices N^k become zero matrices.

Let

$$D(\mathbf{v}) := \mathrm{Diag}(v_0, v_1, \dots, v_n)$$

be the diagonal matrix with main diagonal $\mathbf{v} = (v_0, v_1, \dots, v_n)$.

The new form of our multidiagonal matrix is

$$\begin{aligned} &A(\mathbf{v}_{-s},\dots,\mathbf{v}_0,\dots,\mathbf{v}_s) = \\ &\quad N^{-sk}D(\mathbf{v}_{-s}) + \cdots + N^{-k}D(\mathbf{v}_{-1}) + D(\mathbf{v}_0) + D(\mathbf{v}_1)N^k + \cdots + D(\mathbf{v}_s)N^{sk}. \end{aligned} \tag{3}$$

Now, our matrix is the sum of matrices with a single diagonal and it is easy to identify these matrices with their corresponding diagonals. We are grateful to Prof. Cs. Hegedűs for proposing us the use of the nilpotent matrix N to describe multidiagonal matrices.

Define the operator τ and its inverse by

$$\tau\mathbf{v} := (v_1,\dots,v_n,v_{n+1}), \quad \tau^{-1}\mathbf{v} := (v_{-1},v_0,\dots,v_{n-1}) \tag{4}$$

where for any vector $\mathbf{v} = (v_0,\dots,v_n) \in \mathbb{C}^{n+1}$

$$v_k = 0 \text{ if } k > n \text{ or if } k < 0. \tag{5}$$

This means that the effect of τ^k on any vector $\mathbf{v}$ is the increase of the subscripts of its coordinates by k. Clearly $\tau^j\mathbf{v}$ is zero vector for $|j| > n$.

The $*$ product of two vectors $\mathbf{v}$ and $\mathbf{w} = (w_0,\dots,w_n)$ is defined coordinate-wise by

$$\mathbf{v} * \mathbf{w} := (v_0w_0,\dots,v_nw_n).$$

The operation $*$ is commutative, associative, $D(\mathbf{v})D(\mathbf{w}) = D(\mathbf{v} * \mathbf{w})$ and

$$\tau^i(\mathbf{v} * \mathbf{w}) = (\tau^i\mathbf{v}) * (\tau^i\mathbf{w}), \ \tau^i(\tau^j\mathbf{v}) = \tau^{i+j}\mathbf{v} \tag{6}$$

for any $\mathbf{v},\mathbf{w} \in \mathbb{C}^{n+1}$ and for integers i, j.

For the multiplication of powers of N we shall use the identities

$$N^iN^j = N^{i+j} \ \text{ if } i,j \ge 0 \text{ or if } i,j \le 0, \tag{7}$$

and for nonnegative i, j the identities

$$\begin{aligned} N^iN^{-j} &= D(\tau^i\mathbf{1})N^{i-j} && \text{if } i-j \ge 0, \\ N^{-j}N^i &= D(\tau^{-j}\mathbf{1})N^{i-j} && \text{if } i-j \ge 0, \\ N^iN^{-j} &= N^{-(j-i)}D(\tau^j\mathbf{1}) && \text{if } i-j \le 0, \\ N^{-j}N^i &= N^{-(j-i)}D(\tau^{-i}\mathbf{1}) && \text{if } i-j \le 0, \end{aligned} \tag{8}$$

where $\mathbf{1} = (1,\dots,1) \in \mathbb{C}^{n+1}$ is the unit vector. Clearly $\mathbf{1} * \mathbf{v} = \mathbf{v}$ for any $\mathbf{v} \in \mathbb{C}^{n+1}$.

The order of factors in the products $D(\mathbf{v})N^{\pm j}$ can be changed by help of the identities

$$\begin{aligned} D(\mathbf{v})N^{-j} &= N^{-j}D(\tau^j\mathbf{v}), \quad & N^jD(\mathbf{v}) &= D(\tau^j\mathbf{v})N^j, \\ N^{-j}D(\mathbf{v}) &= D(\tau^{-j}\mathbf{v})N^{-j}, \quad & D(\mathbf{v})N^j &= N^jD(\tau^{-j}\mathbf{v}), \end{aligned} \tag{9}$$

valid for any $\mathbf{v} \in \mathbb{C}^{n+1}$ and for nonnegative integer values of j.

3. Products, inverses and powers of some multidiagonal matrices

Theorem 3.1. *(i) The set $\mathcal{MD}_{n,k}$ is closed under multiplication and taking powers with positive exponents.*
(ii) The subset of $\mathcal{MD}_{n,k}$ consisting of all nonsingular matrices is closed under taking inverses and powers with negative (and also nonnegative) exponents.

Proof. Let

$$\begin{aligned} V &= \sum_{i=1}^{s} N^{-ik} D(\mathbf{v}_{-i}) + \sum_{i=0}^{s} D(\mathbf{v}_i) N^{ik} \\ W &= \sum_{j=1}^{s} N^{-jk} D(\mathbf{w}_{-j}) + \sum_{j=0}^{s} D(\mathbf{w}_j) N^{jk} \end{aligned} \tag{10}$$

be two matrices in $\mathcal{MD}_{n,k}$ where the vectors $\mathbf{v}_i, (i = -s, \dots, 0, \dots, s)$ are defined by (1), and

$$\mathbf{w}_j = (w_{j,0}, \dots, w_{j,n-jk}, 0, \dots, 0) \quad (j = -s, \dots, 0, \dots, s).$$

The product VW can be decomposed into four sums

$$\begin{aligned} VW &= \sum_{i=1}^{s} \sum_{j=1}^{s} N^{-ik} D(\mathbf{v}_{-i}) N^{-jk} D(\mathbf{w}_{-j}) + \sum_{i=1}^{s} \sum_{j=0}^{s} N^{-ik} D(\mathbf{v}_{-i}) D(\mathbf{w}_j) N^{jk} \\ &+ \sum_{i=0}^{s} \sum_{j=1}^{s} D(\mathbf{v}_i) N^{ik} N^{-jk} D(\mathbf{w}_{-j}) + \sum_{i=0}^{s} \sum_{j=0}^{s} D(\mathbf{v}_i) N^{ik} D(\mathbf{w}_j) N^{jk}. \end{aligned} \tag{11}$$

We transform the summands by help of the relations (8) and (9) as follows:

$$\begin{aligned} &N^{-ik} D(\mathbf{v}_{-i}) N^{-jk} D(\mathbf{w}_{-j}) = N^{-ik} N^{-jk} D(\tau^{ik} \mathbf{v}_{-i}) D(\mathbf{w}_{-j}) = N^{-(i+j)k} D(\tau^{ik} \mathbf{v}_{-i} * \mathbf{w}_{-j}), \\ &N^{-ik} D(\mathbf{v}_{-i}) D(\mathbf{w}_j) N^{jk} = N^{-ik} D(\mathbf{v}_{-i} * \mathbf{w}_j) N^{jk} = D(\tau^{-ik}(\mathbf{v}_{-i} * \mathbf{w}_j)) N^{-ik} N^{jk}, \\ &D(\mathbf{v}_i) N^{ik} N^{-jk} D(\mathbf{w}_{-j}) = N^{ik} D(\tau^{-ik} \mathbf{v}_i) N^{-jk} D(\mathbf{w}_{-j}) = N^{ik} N^{-jk} D(\tau^{(j-i)k} \mathbf{v}_i * \mathbf{w}_{-j}), \\ &D(\mathbf{v}_i) N^{ik} D(\mathbf{w}_j) N^{jk} = D(\mathbf{v}_i * \tau^{ik} \mathbf{w}_j) N^{(i+j)k}. \end{aligned}$$

The expressions in the second and third line require further transformations using again (8),(9) and the properties of the $*$. The last expression in the second line is transformed as follows.

If $i - j \le 0$ then we get

$$\begin{aligned} D(\tau^{-ik}(\mathbf{v}_{-i} * \mathbf{w}_j) N^{-ik} N^{jk} &= D(\tau^{-ik}(\mathbf{v}_{-i} * \mathbf{w}_j)) D(\tau^{-ik} \mathbf{1}) N^{(j-i)k} \\ &= D(\tau^{-ik}(\mathbf{v}_{-i} * \mathbf{w}_j)) N^{(j-i)k}, \end{aligned}$$

since

$$\tau^{-ik}(\mathbf{v}_{-i} * \mathbf{w}_j) * \tau^{-ik} \mathbf{1} = \tau^{-ik}(\mathbf{v}_{-i} * \mathbf{w}_j * \mathbf{1}) = \tau^{-ik}(\mathbf{v}_{-i} * \mathbf{w}_j).$$

If $i - j > 0$ then we obtain

$$\begin{aligned} D(\tau^{-ik}(\mathbf{v}_{-i} * \mathbf{w}_j)) N^{-ik} N^{jk} &= D(\tau^{-ik}(\mathbf{v}_{-i} * \mathbf{w}_j)) D(\tau^{-jk} \mathbf{1}) N^{-(i-j)k} \\ &= D\left(\tau^{-ik}(\mathbf{v}_{-i} * \mathbf{w}_j) * \tau^{-jk} \mathbf{1}\right) N^{-(i-j)k} \\ &= N^{-(i-j)k} D\left(\tau^{(i-j)k}(\tau^{-ik}(\mathbf{v}_{-i} * \mathbf{w}_j) * \tau^{-jk} \mathbf{1})\right) = N^{-(i-j)k} D(\tau^{-jk}(\mathbf{v}_{-i} * \mathbf{w}_j)), \end{aligned}$$

since

$$\begin{aligned} \tau^{(i-j)k}\left(\tau^{-ik}(\mathbf{v}_{-i} * \mathbf{w}_j) * \tau^{-jk} \mathbf{1}\right) &= \tau^{(i-j)k}(\tau^{-ik} \mathbf{v}_{-i} * \tau^{-ik} \mathbf{w}_j * \tau^{-jk} \mathbf{1}) \\ &= \tau^{(i-j)k}\left(\tau^{-ik} \mathbf{v}_{-i} * \tau^{-jk}(\tau^{(j-i)k} \mathbf{w}_j * \mathbf{1})\right) = \tau^{(i-j)k}(\tau^{-ik} \mathbf{v}_{-i} * \tau^{-ik} \mathbf{w}_j) \\ &= \tau^{-jk}(\mathbf{v}_{-i} * \mathbf{w}_j). \end{aligned}$$

We transform the last expression in the third line similarly.

If $i - j \le 0$ then we get

$$\begin{aligned} N^{ik} N^{-jk} D(\tau^{(j-i)k} \mathbf{v}_i * \mathbf{w}_{-j}) &= N^{-(j-i)k} D(\tau^{jk} \mathbf{1}) D(\tau^{(j-i)k} \mathbf{v}_i * \mathbf{w}_{-j}) \\ &= N^{-(j-i)k} D(\tau^{jk} \mathbf{1} * \tau^{(j-i)k} \mathbf{v}_i * \mathbf{w}_{-j}) = N^{-(j-i)k} D(\tau^{jk}(\mathbf{1} * \tau^{-ik} \mathbf{v}_i) * \mathbf{w}_{-j}) \\ &= N^{-(j-i)k} D(\tau^{(j-i)k} \mathbf{v}_i * \mathbf{w}_{-j}). \end{aligned}$$

For $i - j > 0$ we have

$$\begin{aligned} N^{ik} N^{-jk} D(\tau^{(j-i)k} \mathbf{v}_i * \mathbf{w}_{-j}) &= D(\tau^{ik} \mathbf{1}) N^{(i-j)k} D(\tau^{(j-i)k} \mathbf{v}_i * \mathbf{w}_{-j}) \\ &= D(\tau^{ik} \mathbf{1}) D\left(\tau^{(i-j)k}(\tau^{(j-i)k} \mathbf{v}_i * \mathbf{w}_{-j})\right) N^{(i-j)k} = D(\mathbf{v}_i * \tau^{(i-j)k} \mathbf{w}_{-j}) N^{(i-j)k} \end{aligned}$$

since

$$\tau^{ik}\mathbf{1} * \tau^{(i-j)k}(\tau^{(j-i)k}\mathbf{v}_i * \mathbf{w}_{-j}) = \tau^{ik}\mathbf{1} * \mathbf{v}_i * \tau^{(i-j)k}\mathbf{w}_{-j}$$
$$= \tau^{ik}(\mathbf{1} * \tau^{-ik}\mathbf{v}_i) * \tau^{(i-j)k}\mathbf{w}_{-j} = \mathbf{v}_i * \tau^{(i-j)k}\mathbf{w}_{-j}.$$

Using these new forms of the summands and splitting the second and third sums into two we can rewrite (11) as

$$\begin{aligned} VW &= \sum_{i=1}^{s}\sum_{j=1}^{s} N^{-(i+j)k} D(\tau^{ik}\mathbf{v}_{-i} * \mathbf{w}_{-j}) + \sum_{i=1}^{s}\sum_{j=0, i\le j}^{s} D(\tau^{-ik}(\mathbf{v}_{-i} * \mathbf{w}_j))N^{(j-i)k} \\ &+ \sum_{i=1}^{s}\sum_{j=0, i>j}^{s} N^{-(i-j)k} D(\tau^{-jk}(\mathbf{v}_{-i} * \mathbf{w}_j)) + \sum_{i=0}^{s}\sum_{j=1, i\le j}^{s} N^{-(j-i)k} D(\tau^{(j-i)k}\mathbf{v}_i * \mathbf{w}_{-j}) \\ &+ \sum_{i=0}^{s}\sum_{j=1, i>j}^{s} D(\mathbf{v}_i * \tau^{(i-j)k}\mathbf{w}_{-j})N^{(i-j)k} + \sum_{i=0}^{s}\sum_{j=0}^{s} D(\mathbf{v}_i * \tau^{ik}\mathbf{w}_j)N^{(i+j)k}. \end{aligned} \tag{12}$$

Using the rules $D(\mathbf{v})N^p + D(\mathbf{w})N^p = D(\mathbf{v}+\mathbf{w})N^p$, $N^{-p}D(\mathbf{v}) + N^{-p}D(\mathbf{w}) = N^{-p}D(\mathbf{v}+\mathbf{w})$ for $p \ge 0, \mathbf{v}, \mathbf{w} \in \mathbb{C}^{n+1}$ we add those terms of (12) for which the exponents of N are the same nonnegative or negative numbers and omit those terms where the absolute value of the exponents of N is greater than n.

The result is

$$VW = \sum_{p=1}^{s} N^{-pk} D(\mathbf{z}_{-p}) + D(\mathbf{z}_0) + \sum_{p=1}^{s} D(\mathbf{z}_p)N^{pk} \tag{13}$$

with suitable vectors $\mathbf{z}_p$ $(p = -s, \dots, 0, \dots, s)$ proving that the set $\mathcal{MD}_{n,k}$ is closed under taking products. This clearly implies that it is also closed under taking powers with positive exponents, completing the proof of (i).

To prove (ii) take a nonsingular matrix $V \in \mathcal{MD}_{n,k}$ and let

$$\det(V - \lambda E) = \sum_{j=0}^{n+1} \nu_j \lambda^j$$

be the characteristic polynomial of V, where $\nu_j \in \mathbb{C}$, in particular $\nu_{n+1} = (-1)^{n+1}$ and $\nu_0 = \det(V) \neq 0$. By the Cayley-Hamilton theorem we have $\sum_{j=0}^{n+1} \nu_j V^j = O$ (where O is the zero matrix) therefore

$$E = V\left(-\sum_{j=1}^{n+1} \frac{\nu_j}{\nu_0} V^{j-1}\right) = \left(-\sum_{j=1}^{n+1} \frac{\nu_j}{\nu_0} V^{j-1}\right)V$$

showing that

$$V^{-1} = -\sum_{j=1}^{n+1} \frac{\nu_j}{\nu_0} V^{j-1} \in \mathcal{MD}_{n,k}$$

and completing the proof. □

4. Inverse of the k-tridiagonal matrix $A_{n+1,k}$ if $n + 1 \le 2k$

On k-tridiagonal matrices see the recent survey [8]. The evaluation of the inverse of k-tridiagonal matrices is a very common subject. In [3–7] numerical methods, formulae, algorithms were given for such inverses. Most complicated inverses of k-tridiagonal matrices are for $k = 1$, if k is larger then the inverses are getting simpler as several sub- superdiagonals will be zero vectors.

Fonseca and Petronilho [11] (using results of [13, 19]) found explicit formula for the inverse of a general 1-tridiagonal matrix, while Fonseca and Yılmaz [12] proved that any k-tridiagonal matrix is congruent to the direct sum of 1-tridiagonal matrices.

To be more precise let $n+1=kq+p, 0\le p<k$ then any k-tridiagonal matrix is permutational equivalent to the direct sum of p pieces of $(q+1)\times(q+1)$ and $k-p$ pieces of $q\times q$ type 1-tridiagonal matrices (see [12] also [14]) and the same is true for the inverses. Using these and [11] one could obtain formulae for the entries of the inverse of a k-tridiagonal matrix. But these inverse-entries were given by help of two recursive sequences built up from the entries of the original matrix, thus to express the inverse-entries in terms of the entries of $A_{n+1,k}$ would be simply too complicated, hence useless.

The inverse of $A_{n+1,k}$ is the simplest if $n+1\le 2k$ since in this case the inverse is also k-tridiagonal. In the next theorem the entries of $A^{-1}_{n+1,k}$ are expressed by simple formulae *in terms of the entries* of $A_{n+1,k}$ provided that $n+1\le 2k$. If $2k<n+1\le 3k$ the inverse of $A_{n+1,k}$ is $(k,2k)$-pentadiagonal, if $3k<n+1\le 4k$ the inverse is $(k,2k,3k)$-heptadiagonal, calculation of their entries in terms of the entries of $A_{n+1,k}$ is possible, but would require much more efforts. These inverses (if found) could help to find simpler formulae for the inverses of k-tridiagonal matrices as the existing ones (see [6] Theorem 2.2,[7]).

Theorem 4.1. *(j) If $n+1\le 2k$ then the k-tridiagonal matrix*

$$A=N^{-k}D(\mathbf{a})+D(\mathbf{b})+D(\mathbf{c})N^{k} \tag{14}$$

where

$$\mathbf{a}=(a_0,\dots,a_{n-k},0,\dots,0),\mathbf{b}=(b_0,\dots,b_n),\mathbf{c}=(c_0,\dots,c_{n-k},0,\dots,0)$$

is nonsingular if and only if

$$\begin{aligned} b_j &\ne 0\,(j=n+1-k,\dots,k-1)\\ b_jb_{j+k}-a_jc_j &\ne 0,\ (j=0,\dots,n-k)). \end{aligned} \tag{15}$$

(jj) If (15) *holds then A^{-1} is also k-tridiagonal and is of the form*

$$A^{-1}=N^{-k}D(\mathbf{x})+D(\mathbf{y})+D(\mathbf{z})N^{k}$$

where

$$\begin{aligned}
\mathbf{x}&=\left(\frac{-a_0}{b_0b_k-a_0c_0},\dots,\frac{-a_{n-k}}{b_{n-k}b_n-a_{n-k}c_{n-k}},0,\dots,0\right),\\
\mathbf{y}&=\left(\frac{b_k}{b_0b_k-a_0c_0},\dots,\frac{b_n}{b_{n-k}b_n-a_{n-k}c_{n-k}},\frac{1}{b_{n+1-k}},\dots,\frac{1}{b_{k-1}},\frac{b_0}{b_kb_0-a_0c_0},\dots,\frac{b_{n-k}}{b_nb_{n-k}-a_{n-k}c_{n-k}}\right),\\
\mathbf{z}&=\left(\frac{-c_0}{b_0b_k-a_0c_0},\dots,\frac{-c_{n-k}}{b_{n-k}b_n-a_{n-k}c_{n-k}},0,\dots,0\right).
\end{aligned}$$

Proof. The determinant of A is by the known formula (see e.g. [6, 7])

$$\det A=\prod_{j=0}^{n}f_j \tag{16}$$

where

$$f_j=\begin{cases} b_j & \text{if } j=0,\dots,k-1,\\ b_j-a_{j-k}c_{j-k}/f_{j-k} & \text{if } j=k,\dots,n.\end{cases}$$

To define f_j for $j=k,\dots,n$ we have to assume $f_j\ne 0$ for $j=0,\dots,n-k$. However formula (16) is valid without this assumption as after simplifications the fractions disappear (see [9]). In our case $n-k\le k-1$ and the product in (16) can be simplified to

$$\begin{aligned}
\det A &=\left(\prod_{j=0}^{k-1}b_j\right)\left(\prod_{j=k}^{n}\left(b_j-a_{j-k}c_{j-k}/f_{j-k}\right)\right)=\left(\prod_{j=0}^{k-1}b_j\right)\left(\prod_{j=0}^{n-k}\left(b_{j+k}-a_jb_j/b_j\right)\right)\\
&=\left(\prod_{j=n+1-k}^{k-1}b_j\right)\left(\prod_{j=0}^{n-k}\left(b_jb_{j+k}-a_jc_j\right)\right).
\end{aligned}$$

This shows that A is nonsingular if and only if (15) holds, proving (j).

If $n+1 \le 2k$ and (15) holds then we have seen that A^{-1} is also k-tridiagonal thus we may write it as

$$A^{-1} = X = N^{-k}D(\mathbf{x}) + D(\mathbf{y}) + D(\mathbf{z})N^k$$

where

$$\mathbf{x} = (x_0, \ldots, x_{n-k}, 0, \ldots, 0),\ \mathbf{y} = (y_0, \ldots, y_n),\ \mathbf{z} = (z_0, \ldots, z_{n-k}, 0, \ldots, 0).$$

Expanding the product AX we get

$$\begin{aligned} AX \ &= N^{-k}D(\mathbf{a})N^{-k}D(\mathbf{x}) + N^{-k}D(\mathbf{a})D(\mathbf{y}) + N^{-k}D(\mathbf{a})D(\mathbf{z})N^k \\ &+D(\mathbf{b})N^{-k}D(\mathbf{x}) + D(\mathbf{b})D(\mathbf{y}) + D(\mathbf{b})D(\mathbf{z})N^k \\ &+D(\mathbf{c})N^kN^{-k}D(\mathbf{x}) + D(\mathbf{c})N^kD(\mathbf{y}) + D(\mathbf{c})N^kD(\mathbf{z})N^k. \end{aligned} \tag{17}$$

Using suitable relations of (9) we rewrite the first term of (17) as

$$N^{-k}D(\mathbf{a})N^{-k}D(\mathbf{x}) = N^{-k}N^{-k}D(\tau^k\mathbf{a})D(\mathbf{x}) = N^{-2k}D(\tau^k\mathbf{a} * \mathbf{x}),$$

the second term as $N^{-k}D(\mathbf{a} * \mathbf{y})$.

The third term can be written as

$$\begin{aligned} N^{-k}D(\mathbf{a})D(\mathbf{z})N^k \ &= D(\tau^{-k}\mathbf{a})N^{-k}D(\mathbf{z})N^k = D(\tau^{-k}\mathbf{a})D(\tau^{-k}\mathbf{z})N^{-k}N^k \\ &= D(\tau^{-k}(\mathbf{a} * \mathbf{z})D(\tau^{-k}\mathbf{1})N^0 = D(\tau^{-k}(\mathbf{a} * \mathbf{z})). \end{aligned}$$

Rewriting the other terms in a similar way we finally get that

$$\begin{aligned} AX \ &= N^{-2k}D(\tau^k\mathbf{a} * \mathbf{x}) + N^{-k}D(\mathbf{a} * \mathbf{y} + \tau^k\mathbf{b} * \mathbf{x}) \\ &+D(\tau^{-k}(\mathbf{a} * \mathbf{z}) + \mathbf{b} * \mathbf{y} + \mathbf{c} * \mathbf{x}) \\ &+D(\mathbf{b} * \mathbf{z} + \mathbf{c} * \tau^k\mathbf{y})N^k + D(\mathbf{c} * \tau^k\mathbf{z})N^{2k}. \end{aligned} \tag{18}$$

In our case $N^{\pm 2k}$= zero matrix, hence the equations of the linear inhomogeneous system $AX = E$ can be written as

$$\begin{aligned} &\mathbf{a} * \mathbf{y} + \tau^k\mathbf{b} * \mathbf{x} = \mathbf{0},\ \mathbf{b} * \mathbf{z} + \mathbf{c} * \tau^k\mathbf{y} = \mathbf{0}, \\ &\tau^{-k}(\mathbf{a} * \mathbf{z}) + \mathbf{b} * \mathbf{y} + \mathbf{c} * \mathbf{x} = \mathbf{1} \end{aligned} \tag{19}$$

where $\mathbf{0}$ is the $n+1$ dimensional zero vector. The unknowns are the nonzero coordinates of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ numbering to $n+1+2(n+1-k) = 3(n+1)-2k$. System (19) is in detailed form

$$\begin{aligned} \mathbf{0} &= \mathbf{a} * \mathbf{y} + \tau^k\mathbf{b} * \mathbf{x} = (a_0, \ldots, a_{n-k}, 0, \ldots, 0) * (y_0, \ldots, y_n) \\ &+(b_k, \ldots, b_n, 0, \ldots, 0) * (x_0, \ldots, x_{n-k}, 0, \ldots, 0) \\ &= (\underbrace{b_kx_0 + a_0y_0, \ldots, b_nx_{n-k} + a_{n-k}y_{n-k}}_{n+1-k}, \underbrace{0, \ldots, 0}_{k})) \end{aligned}$$

$$\begin{aligned} \mathbf{0} &= \mathbf{b} * \mathbf{z} + \mathbf{c} * \tau^k\mathbf{y} = (b_0, \ldots, b_n) * (z_0, \ldots, z_{n-k}, 0, \ldots, 0 \\ &+(c_0, \ldots, c_{n-k}, 0, \ldots, 0) * (y_k, \ldots, y_n, 0, \ldots, 0) \\ &= (\underbrace{c_0y_k + b_0z_0, \ldots, c_{n-k}y_n + b_{n-k}z_{n-k}}_{n+1-k}, \underbrace{0, \ldots, 0}_{k}) \end{aligned}$$

$$\begin{aligned} \mathbf{1} &= \tau^{-k}(\mathbf{a} * \mathbf{z}) + \mathbf{b} * \mathbf{y} + \mathbf{c} * \mathbf{x} = (0, \ldots, 0, a_0z_0, \ldots, a_{n-k}z_{n-k}) \\ &+(b_0, \ldots, b_n) * (y_0, \ldots, y_n) + (c_0, \ldots, c_{n-k}, 0, \ldots, 0) * (x_0, \ldots, x_{n-k}, 0, \ldots, 0) \\ &= (\underbrace{c_0x_0 + b_0y_0, \ldots, c_{n-k}x_{n-k} + b_{n-k}y_{n-k}}_{n+1-k}, \underbrace{0, \ldots, 0}_{k}) \\ &+(\underbrace{0, \ldots, 0}_{n+1-k}, \underbrace{b_{n+1-k}y_{n+1-k}, \ldots, b_{k-1}y_{k-1}}_{2k-(n+1)}, \underbrace{0, \ldots, 0}_{n+1-k}) \\ &+(\underbrace{0, \ldots, 0}_{k}, \underbrace{b_ky_k + a_0z_0, \ldots, b_ny_n + a_{n-k}z_{n-k}}_{n+1-k}). \end{aligned}$$

In the first and second group the last k equations are trivial ($0 = 0$) thus these are omitted. The remaining number of our (non trivial) equations is $2(n+1-k) + n + 1 = 3(n+1) - 2k$, the same as the number of unknowns.

Next we solve this system. The unknowns $y_{n+1-k}, \ldots, y_{k-1}$ obtained easily as

$$y_j = \frac{1}{b_j} \ (j = n+1-k, \ldots, k-1).$$

Collect the remaining unknowns into one column vector and the corresponding free terms also into one vector

$$\begin{aligned} \mathbf{x}^* &= (x_0, \ldots, x_{n-k}, y_0, \ldots, y_{n-k}, y_k, \ldots, y_n, z_0, \ldots, z_{n-k})^T \\ \mathbf{b}^* &= (\underbrace{0, \ldots, 0,}_{2(n+1-k)} \underbrace{1, \ldots, 1}_{2(n+1-k)})^T. \end{aligned}$$

Denoting by U the matrix of the reduced system it can be written as $U\mathbf{x}^* = \mathbf{b}^*$.

This reduced system has $4(n+1-k)$ equations and unknowns. In detailed form

$$\left(\begin{array}{ccc|ccc|ccc|ccc}
b_k x_0 & & & a_0 y_0 & & & & & & & & \\
 & \ddots & & & \ddots & & & & & & & \\
 & & b_n x_{n-k} & & & a_{n-k} y_{n-k} & & & & & & \\
\hline
 & & & & & & c_0 y_k & & & b_0 z_0 & & \\
 & & & & & & & \ddots & & & \ddots & \\
 & & & & & & & & c_{n-k} y_n & & & b_{n-k} z_{n-k} \\
\hline
c_0 x_0 & & & b_0 y_0 & & & & & & & & \\
 & \ddots & & & \ddots & & & & & & & \\
 & & c_{n-k} x_{n-k} & & & b_{n-k} y_{n-k} & & & & & & \\
\hline
 & & & & & & b_k y_k & & & a_0 z_0 & & \\
 & & & & & & & \ddots & & & \ddots & \\
 & & & & & & & & b_n y_n & & & a_{n-k} z_{n-k}
\end{array}\right) = \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ \hline 1 \\ \vdots \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{array}\right)$$

which shows that our system consists of four groups of equations, each of them with $n + 1 - k$ equations of similar structures. Number the equations starting by zero. Multiply the jth equations of the first system by $-c_j$ and add these to the jth equations of the third system multiplied by b_{k+j} for $j = 0, \ldots, n-k$. Our system goes over into

$$\left(\begin{array}{ccc|ccc|ccc|ccc}
b_k x_0 & & & a_0 y_0 & & & & & & & & \\
 & \ddots & & & \ddots & & & & & & & \\
 & & b_n x_{n-k} & & & a_{n-k} y_{n-k} & & & & & & \\
\hline
 & & & & & & c_0 y_k & & & b_0 z_0 & & \\
 & & & & & & & \ddots & & & \ddots & \\
 & & & & & & & & c_{n-k} y_n & & & b_{n-k} z_{n-k} \\
\hline
 & & & (b_0 b_k - a_0 c_0) y_0 & & & & & & & & \\
 & & & & \ddots & & & & & & & \\
 & & & & & (b_{n-k} b_n - a_{n-k} c_{n-k}) y_{n-k} & & & & & & \\
\hline
 & & & & & & b_k y_k & & & a_0 z_0 & & \\
 & & & & & & & \ddots & & & \ddots & \\
 & & & & & & & & b_n y_n & & & a_{n-k} z_{n-k}
\end{array}\right) = \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ \hline b_k \\ \vdots \\ b_n \\ \hline 1 \\ \vdots \\ 1 \end{array}\right)$$

From the third group of equations we get immediately that

$$y_j = \frac{b_{k+j}}{b_j b_{k+j} - a_j c_j} \ (j = 0, \ldots, n-k).$$

To continue our calculations we temporally assume that $b_{k+j} \neq 0$, $(j = 0, \dots, n-k)$. Then from the first group of equations we obtain that

$$x_j = \frac{-a_j y_j}{b_{k+j}} = \frac{-a_j}{b_j b_{k+j} - a_j c_j} \ (j = 0, \dots, n-k). \tag{20}$$

Multiply the jth equations of the second group by $-a_j$ and add them to the jth equations of the fourth group multiplied by b_j for $j = 0, \dots, n-k$. Then the fourth group of equations go over into

$$(b_{k+j} b_j - a_j c_j) y_{k+j} = b_j,$$

hence

$$y_{k+j} = \frac{b_j}{b_{k+j} b_j - a_j c_j}, \ (j = 0, \dots, n-k).$$

Finally multiply the jth equations of the second group by $-b_{j+k}$ and add them to the jth equations of the fourth group multiplied by c_j for $j = 0, \dots, n-k$. Then the fourth group of equations become

$$(-b_{k+j} b_j + a_j c_j) z_j = c_j,$$

thus

$$z_j = \frac{-c_j}{b_{k+j} b_j - a_j c_j}, \ (j = 0, \dots, n-k).$$

Now we justify (20) without our temporally assumption. Namely if $b_{k+j} = 0$ for some $j = 0, \dots, n-k$ then change it a little to $b'_{k+j} \neq 0$ such that the factor $b_j b'_{j+k} - a_j c_j \neq 0$. Then we obtain

$$x'_j = \frac{-a_j}{b_j b'_{k+j} - a_j c_j}$$

taking the limit $b'_{k+j} \to 0 = b_{k+j}$ justifies the validity of the final formula for x_j. $\square$

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