Filomat 36:2 (2022), 703–713
https://doi.org/10.2298/FIL2202703B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Separation Axioms, Urysohn's Lemma and Tietze Extention Theorem for Extended Pseudo-Quasi-Semi Metric Spaces

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Abstract. In this paper, we characterize each of various forms of T_0, T_1, T_2, and pre-Hausdorff extended pseudo-quasi-semi metric spaces as well as examine how these generalizations are related. Moreover, we give some invariance properties of these T_0, T_1, and T_2 extended pseudo-quasi-semi metric spaces and investigate the relationship among each of irreducible T_i, $i = 1, 2$ extended pseudo-quasi-semi metric spaces. Finally, we present Urysohn's Lemma and Tietze Extention Theorem for extended pseudo-quasi-semi metric spaces.

1. Introduction

The extended pseudo-quasi-semi metric spaces were defined in 1988 by E. Lowen and R. Lowen [11] with their corresponding non-expansive mappings. They are the most general category of metric spaces which is cartesian closed and hereditary topological [11].

There are several ways to generalize the usual T_0-axiom of topology to topological categories [2, 13, 16] which are used to define various forms of Hausdorff objects [2] in arbitrary topological categories. Also, in 1991, Baran [2] gave a generalization of the usual T_1 and pre-Hausdorff axioms of topology to topological categories that are used to define each of T_3, T_4, and completely regular objects of an arbitrary topological category [7].

In General Topology, one of the most important uses of separation properties is theorems such as the Urysohn's Lemma and the Tietze Extension Theorem. In view of this, it is useful to be able to extend these various notions to arbitrary topological categories.

Note that if (X, d) is an extended pseudo-quasi-semi metric space, then d does not induce a topology on X since d does not fulfil the triangle inequality. The main goal of this paper is to characterize each of various forms of T_0, T_1, T_2, and pre-Hausdorff extended pseudo-quasi-semi metric spaces and give some invariance properties of these subcategories $\mathbf{T_i pqsMet}$ of T_i-extended pseudo-quasi-semi metric spaces, $i = 0, 1, 2$ as well as to present Urysohn's Lemma and Tietze Extention Theorem for extended pseudo-quasi-semi metric spaces and to investigate the relationship among each of irreducible T_i, $i = 1, 2$ extended pseudo-quasi-semi metric spaces.

2020 *Mathematics Subject Classification*. Primary 54B30; Secondary 18B99, 18D15, 54A05, 54A20, 54D10
Keywords. Topological category, pre-Hausdorff objects, Hausdorff objects, extended pseudo-quasi-semi metric spaces
Received: 21 February 2021; Revised: 11 July 2021; Accepted: 16 July 2021
Communicated by Ljubiša D.R. Kočinac
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2. Preliminaries

Recall, in [11], that an extended pseudo-quasi-semi metric space is a pair (X, d), where X is a set and $d : X \times X \to [0, \infty]$ is a function which fulfills $d(x, x) = 0$ for all $x \in X$. Moreover, if $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$, then (X, d) is called an extended pseudo-quasi metric space.

In addition, if for all $x, y \in X$, $d(x, y) = d(y, x)$, then (X, d) is called an extended pseudo-metric space [12]. A map $f : (X, d) \to (Y, e)$ between extended pseudo-quasi-semi metric spaces is said to be a non-expansive if it fulfills the property $e(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$. The construct of extended pseudo-quasi-semi metric spaces and non-expansive maps is denoted by **pqsMet** which is a cartesian closed and hereditary topological category [11, 14] in the sense of [1, 15].

Proposition 2.1. *(1) Let I be an index set, $\{(X_i, d_i), i \in I\}$ be a class of extended pseudo-quasi-semi metric spaces, X be a nonempty set, and $\{f_i : X \to X_i, i \in I\}$ be a source in the category* **Set***, the category of sets and functions. A source $\{f_i : (X, d) \to (X_i, d_i), i \in I\}$ in* **pqsMet** *is an initial lift if and only if for all $x, y \in X, d(x, y) = \sup_{i \in I}(d_i(f_i(x), f_i(y)))$ [11, 14].*

(2) Let $\{(X_i, d_i), i \in I\}$ be a class of extended pseudo-quasi-semi metric spaces and X be a nonempty set. A sink $\{f_i : (X_i, d_i) \to (X, d), i \in I\}$ is final in **pqsMet** *if and only if for all $x, y \in X, d(x, y) = \inf\{(d_i(x_i, y_i)) :$ there exist $x_i, y_i \in X_i$ such that $f_i(x_i) = x$ and $f_i(y_i) = y, i \in I\}$ [11, 14]. In particular, let $X = \coprod_{i \in I} X_i$ and define*

$$d((k, x), (j, y)) = \begin{cases} d_k(x, y) & \text{if } k = j \\ \infty & \text{if } k \neq j \end{cases}$$

for all $(k, x), (j, y) \in X$. (X, d) is the coproduct of $\{(X_i, d_i), i \in I\}$ extended pseudo-quasi-semi metric spaces, i.e., $\{i_k : (X_k, d_k) \to (X, d), k \in I\}$ is a final lift of $\{i_k : X_k \to X, k \in I\}$, where i_k are the canonical injection maps.

(3) The discrete extended pseudo-quasi-semi metric structure d on X is given by

$$d_{dis}(a, b) = \begin{cases} 0 & \text{if } a = b \\ \infty & \text{if } a \neq b \end{cases}$$

for all $a, b \in X$.

3. Separation Axioms In pqsMet

Let B be a nonempty set. Two distinct copies of B^2 identified along the diagonal Δ is called the wedge at Δ denoted by $B^2 \bigvee_\Delta B^2$ [2]. A point (x, y) in $B^2 \vee_\Delta B^2$ will be denoted by $((x, y)_1, (x, y)_2)$ if $(x, y)_1$ is in the first (resp. $(x, y)_2$ is in the second) component of $B^2 \vee_\Delta B^2$. Clearly $(x, y)_1 = (x, y)_2$ if and only if $x = y$ [2]. Let $i_k : B^2 \to B^2 \coprod B^2, k = 1, 2$ be the canonical injection maps and $q : B^2 \coprod B^2 \to B^2 \vee_\Delta B^2$ be the quotient map.

The principal axis map $A : B^2 \vee_\Delta B^2 \to B^3$ is given by $A(x, y)_1 = (x, y, x)$ and $A(x, y)_2 = (x, x, y)$. The skewed axis map $S : B^2 \vee_\Delta B^2 \to B^3$ is given by $S(x, y)_1 = (x, y, y)$ and $S(x, y)_2 = (x, x, y)$ and the fold map, $\nabla : B^2 \vee_\Delta B^2 \to B^2$ is given by $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$ [2].

Definition 3.1. ([2, 13]) Let $U : \mathcal{E} \to$ **Set** be topological, X an object in $\mathcal{E}$ with $U(X) = B$.

(1) If the initial lift of the U-source $\{A : B^2 \vee_\Delta B^2 \to U(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2)) = B^2\}$ is discrete, then X is called a $\overline{T_0}$ object, where D is the discrete functor which is a left adjoint to U.

(2) If the initial lift of the U-source $\{id : B^2 \vee_\Delta B^2 \to U(B^2 \vee_\Delta B^2)' = B^2 \vee_\Delta B^2$ and $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2)) = B^2\}$ is discrete, then X is called T'_0 object, where $(B^2 \vee_\Delta B^2)'$ is the final lift of the U-sink $\{q \circ i_1, q \circ i_2 : U(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$, the maps i_1, i_2 and q are defined above.

(3) If X does not contain an indiscrete subspace with (at least) two points, then X is called a T_0 object.

(4) If the initial lift of the U-source $\{S : B^2 \vee_\Delta B^2 \to U(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2)) = B^2\}$ is discrete, then X is called a T_1 object.

Remark 3.2. Note that for the category **Top** of topological spaces and continuous functions, all of $\overline{T_0}$, T'_0 and T_0 are equivalent and they reduce to the usual T_0 separation axiom [2, 13, 16] and T_1 reduces to the usual T_1 separation axiom [2].

Theorem 3.3. *(1) An extended pseudo-quasi-semi metric space (X, d) is T_0 if and only if for each pair x and y in X, $d(x, y) = 0 = d(y, x)$ implies $x = y$.*

(2) An extended pseudo-quasi-semi metric space (X, d) is T_1 if and only if for every distinct pair x and y in X $d(x, y) = \infty = d(y, x)$.

Proof. The proof of (1) is well known and the proof of (2) is given in [9]. □

Theorem 3.4. *An extended pseudo-quasi-semi metric space (X, d) is $\overline{T}_0$ if and only if for every distinct pair x and y in X, $d(x, y) = \infty$ or $d(y, x) = \infty$.*

Proof. The proof is similar to the proof of Theorem 3.3(2) by using the principal axis map instead of the skewed axis map. □

Theorem 3.5. *Every extended pseudo-quasi-semi metric space (X, d) is $T_0^{'}$.*

Proof. Let (X, d) be an extended pseudo-quasi-semi metric space, d_2 be the final extended pseudo-quasi-semi metric structure on $X^2 \coprod X^2$ induced by the canonical injection maps $i_1, i_2 : (X^2, d^2) \to X^2 \coprod X^2$, d_1 be the quotient extended pseudo-quasi-semi metric structure on $X^2 \vee_\Delta X^2$ induced by the quotient map $q : (X^2 \coprod X^2, d_2) \to X^2 \vee_\Delta X^2$, and $\bar{d}$ be the initial structure on $X^2 \vee_\Delta X^2$ induced by $id : X^2 \vee_\Delta X^2 \to (X^2 \vee_\Delta X^2, d_1)$ and $\triangledown : X^2 \vee_\Delta X^2 \to (X^2, d_{dis})$, where d_{dis} is discrete structure on X^2.

Let u and v be any points in $X^2 \vee_\Delta X^2$.

If $u = v$, then $\bar{d}(u, v) = 0$.

Suppose that $\triangledown(u) = (x, x) = \triangledown(v)$ for some $x \in X$. It follows that $u = (x, x)_k = v, k = 1, 2$ and $q^{-1}(u) = \{(x, x)_1, (x, x)_2\} = q^{-1}(v)$. Note that $d_{dis}(\triangledown(u), \triangledown(v)) = 0$ and by Proposition 2.1, $d_1(u, v) = d_2(q^{-1}(u), q^{-1}(v)) = d_2(\{(x, x)_1, (x, x)_2\}, \{(x, x)_1, (x, x)_2\}) = 0$. Hence, by Proposition 2.1, $\bar{d}(u, v) = 0$.

Suppose that $u \neq v$ and $\triangledown(u) = (x, y) = \triangledown(v)$, for some $(x, y) \in X^2$ with $x \neq y$. Since $u \neq v$, we must have $u = (x, y)_1, v = (x, y)_2$ or $u = (x, y)_2, v = (x, y)_1$.

If $u = (x, y)_1$ and $v = (x, y)_2$, then $d_{dis}(\triangledown(u), \triangledown(v)) = 0$ and by Proposition 2.1, $d_1(u, v) = \infty$ since $u = (x, y)_1$ and $v = (x, y)_2$ are in different component of the wedge and $x \neq y$ and consequently, $\bar{d}(u, v) = \infty$. Similarly, if $u = (x, y)_2$ and $v = (x, y)_1$, then $\bar{d}(u, v) = \infty$.

If $u \neq v$ and $\triangledown(u) \neq \triangledown(v)$, then, by Proposition 2.1, $d_{dis}(\triangledown(u), \triangledown(v)) = \infty$, and consequently, $\bar{d}(u, v) = \infty$.

Hence, by Proposition 2.1, $\bar{d}$ is discrete structure on $X^2 \vee_\Delta X^2$ and by Definition 3.1, (X, d) is $T_0^{'}$. □

Definition 3.6. ([2]) Let $U : \mathcal{E} \to$ **Set** be topological and X an object in $\mathcal{E}$ with $U(X) = B$.

(1) If the initial structure on $B^2 \bigvee_\Delta B^2$ induced from the U-source $S : B^2 \bigvee_\Delta B^2 \to U(X^3) = B^3$ and the final structure on $B^2 \bigvee_\Delta B^2$ induced from the U-sink $\{q \circ i_1, q \circ i_2 : U(X^2) = B^2 \to B^2 \bigvee_\Delta B^2\}$ coincide, then X is called a $PreT_2'$ object.

(2) If the initial lift of the U-source $S : B^2 \bigvee_\Delta B^2 \to U(X^3) = B^3$ and the initial lift of the U-source $A : B^2 \bigvee_\Delta B^2 \to U(X^3) = B^3$ coincide, then X is called a $Pre\overline{T}_2$ object.

Remark 3.7. (1) For the category **Top** of topological spaces both $Pre\overline{T_2}$ and $PreT_2'$ are equivalent and they reduce to for any two distinct points, if there is a neighborhood of one missing the other, then the two points have disjoint neighborhoods [2].

(2) In any topological category, by Theorem 3.1 of [6], it is shown that $PreT_2'$ implies $Pre\overline{T}_2$, but the reverse implication is not true, in general.

Lemma 3.8. *Suppose $a, b, c \in [0, \infty]$. Then,* $\sup\{a, b\} = \sup\{a, c\} = \sup\{b, c\} = \sup\{a, b, c\}$ *if and only if we have either $a = b \geq c$ or $a = c \geq b$ or $b = c \geq a$.*

Proof. If $b = \sup\{a, b\} = \sup\{a, c\}$, then $b = c \geq a$ and if $a = \sup\{a, b\} = \sup\{b, c\}$, then $a = c \geq b$. If $a = \sup\{a, c\} = \sup\{b, c\}$, then $a = b \geq c$.

The converse is clear. □

Lemma 3.9. *Suppose $a, b, c, d \in [0, \infty]$. $\sup\{a, b, c\} = \sup\{a, b, d\} = \sup\{a, c, d\} = \sup\{b, c, d\}$ if and only if either $a = b \geq c, d$ or $a = c \geq b, d$ or $a = d \geq b, c$ or $b = c \geq a, d$ or $b = d \geq a, c$ or $c = d \geq a, b$.*

Proof. Suppose $a = \sup\{a, b, c\} = \sup\{b, c, d\}$. Then $a = b \geq c, d$ or $a = c \geq b, d$ or $a = d \geq b, c$. $b = \sup\{a, b, c\} = \sup\{a, c, d\}$ implies $b = a \geq c, d$ or $b = c \geq a, d$ or $b = d \geq a, c$. If $c = \sup\{a, b, c\} = \sup\{a, b, d\}$, then, $c = a \geq b, d$ or $c = b \geq a, d$ or $c = d \geq a, b$. If $d = \sup\{a, b, d\} = \sup\{a, b, c\}$, then, $d = a \geq b, c$ or $d = b \geq a, c$ or $d = c \geq a, b$.

The converse is clear. □

Theorem 3.10. *An extended-pseudo-quasi-semi metric space (X, d) is $Pre\overline{T}_2$ if and only if the following conditions are satisfied.*

(1) d is symmetric.

(2) For any three distinct points x, y, z of X, we have either $d(x, y) = d(y, z) \geq d(x, z)$ or $d(x, y) = d(x, z) \geq d(y, z)$ or $d(x, z) = d(z, y) \geq d(x, y)$.

(3) For any four distinct points x, y, z, w of X, we have either $d(x, z) = d(y, z) \geq d(x, w), d(y, w)$ or $d(x, z) = d(x, w) \geq d(y, z), d(y, w)$ or $d(x, z) = d(y, w) \geq d(y, z), d(x, w)$ or $d(y, z) = d(x, w) \geq d(x, z), d(y, w)$ or $d(y, z) = d(y, w) \geq d(x, z), d(x, w)$ or $d(x, w) = d(y, w) \geq d(x, z), d(y, z)$.

Proof. Suppose (X, d) is $Pre\overline{T}_2$ and $x, y \in X$ with $x \neq y$. Let $u = (x, y)_1$ and $v = (x, y)_2$. Note that $u, v \in X^2 \vee_\Delta X^2$ and since (X, d) is $Pre\overline{T}_2$, it follows from Proposition 2.1 and Definition 3.6 that

$$\begin{aligned} d(y, x) &= \sup\{d(\pi_i S(u), \pi_i S(v)), i = 1, 2, 3\} \\ &= \sup\{d(\pi_i A(u), \pi_i A(v)), i = 1, 2, 3\} \\ &= \sup\{d(x, y), d(y, x)\} \\ &= \sup\{d(\pi_i A(v), \pi_i A(u)), i = 1, 2, 3\} \\ &= \sup\{d(\pi_i S(v), \pi_i S(u)), i = 1, 2, 3\} \\ &= d(x, y) \end{aligned}$$

where $\pi_i : X^3 \to X$, $i = 1, 2, 3$ are the projection maps and consequently, $d(x, y) = d(y, x)$, i.e., d is symmetric.

Let x, y, z be any three distinct points of X. Since (X, d) is $Pre\overline{T}_2$, it follows from Proposition 2.1 and Definition 3.6 that

$$\begin{aligned} \sup\{d(y, x), d(z, x)\} &= \sup\{d(\pi_i S((y, z)_1), \pi_i S((x, z)_2)), i = 1, 2, 3\} \\ &= \sup\{d(\pi_i A((y, z)_1), \pi_i A((x, z)_2)), i = 1, 2, 3\} \\ &= \sup\{d(y, x), d(z, x), d(y, z)\}, \sup\{d(x, y), d(z, y)\} \\ &= \sup\{d(\pi_i S((x, z)_1), \pi_i S((y, z)_2)), i = 1, 2, 3\} \\ &= \sup\{d(\pi_i A((x, z)_1), \pi_i A((y, z)_2)), i = 1, 2, 3\} \\ &= \sup\{d(x, y), d(z, y), d(x, z)\}, \end{aligned}$$

and

$$\begin{aligned} \sup\{d(x, z), d(y, z)\} &= \sup\{d(\pi_i S((x, y)_1), \pi_i S((z, y)_2)), i = 1, 2, 3\} \\ &= \sup\{d(\pi_i A((x, y)_1), \pi_i A((z, y)_2)), i = 1, 2, 3\} \\ &= \sup\{d(x, z), d(y, z), d(x, y)\}. \end{aligned}$$

Since d is symmetric,

$$\begin{aligned} \sup\{d(y, x), d(z, x)\} &= \sup\{d(y, x), d(z, x), d(y, z)\} \\ &= \sup\{d(x, y), d(z, y)\} \\ &= \sup\{d(z, x), d(y, z)\} \end{aligned}$$

and in Lemma 3.8, taking $a = d(x, y), b = d(x, z), c = d(y, z)$, we get either $d(x, y) = d(y, z) \geq d(x, z)\}$ or $d(x, y) = d(x, z) \geq d(y, z)$ or $d(x, z) = d(z, y) \geq d(x, y)$.

Let x, y, z, w be any four distinct points of X. Since (X, d) is $Pre\overline{T}_2$, by Proposition 2.1 and Definition 3.6,

$$\begin{aligned}\sup\{d(x,z), d(y,z), d(y,w)\} &= \sup\{d(\pi_i S((x,y)_1), \pi_i S((z,w)_2)), i = 1,2,3\}\\ &= \sup\{d(\pi_i A((x,y)_1), \pi_i A((z,w)_2)), i = 1,2,3\}\\ &= \sup\{d(x,z), d(y,z), d(x,w)\},\end{aligned}$$

$$\begin{aligned}\sup\{d(x,w), d(y,z), d(y,w)\} &= \sup\{d(\pi_i S((x,y)_1), \pi_i S((w,z)_2)), i = 1,2,3\}\\ &= \sup\{d(\pi_i A((x,y)_1), \pi_i A((w,z)_2)), i = 1,2,3\}\\ &= \sup\{d(x,z), d(y,w), d(x,w)\},\end{aligned}$$

and

$$\begin{aligned}\sup\{d(w,y), d(z,y), d(z,x)\} &= \sup\{d(\pi_i S((w,z)_1), \pi_i S((y,x)_2)), i = 1,2,3\}\\ &= \sup\{d(\pi_i A((w,z)_1), \pi_i A((y,x)_2)), i = 1,2,3\}\\ &= \sup\{d(w,y), d(z,y), d(w,x)\}.\end{aligned}$$

Since d is symmetric, we have

$$\begin{aligned}\sup\{d(x,z), d(y,z), d(y,w)\} &= \sup\{d(x,z), d(y,z), d(x,w)\}\\ &= \sup\{d(x,w), d(y,z), d(y,w)\}\\ &= \sup\{d(x,z), d(y,w), d(x,w)\}.\end{aligned}$$

In Lemma 3.9, take $a = d(x,z), b = d(y,z), c = d(x,w)$, and $d = d(y,w)$. Then we get either $d(x,z) = d(y,z) \geq d(x,w), d(y,w)$ or $d(x,z) = d(x,w) \geq d(y,w), d(y,z)$ or $d(x,z) = d(y,w) \geq d(x,w), d(y,z)$ or $d(y,z) = d(x,w) \geq d(x,z), d(y,w)$ or $d(y,w) = d(y,z) \geq d(x,z), d(x,w)$ or $d(y,w) = d(x,w) \geq d(x,z), d(y,z)$.

Conversely, suppose that the conditions hold. Then, we will show that (X, d) is $Pre\overline{T}_2$, i.e., by Definition 3.6, for any points u and v of the wedge $X^2 \bigvee_\Delta X^2$, $d_A(u,v) = d_S(u,v)$, where d_A and d_S are the initial structures on $X^2 \vee_\Delta X^2$ induced by $A : X^2 \vee_\Delta X^2 \to (X^3, d^3)$ and $S : X^2 \vee_\Delta X^2 \to (X^3, d^3)$, respectively ($d^3$ is the product extended pseudo-quasi-semi metric structure on X^3).

First, note that d_A and d_S are symmetric since d is symmetric by the assumption (1).

If $u = v$, then $d_A(u,v) = 0 = d_S(u,v)$.

Suppose the distinct points u and v are in the same component of the wedge.

If $u = (x,y)_k$ and $v = (z,w)_k$ for $x, y, z, w \in X$ and $k = 1,2$, then, by Proposition 2.1,

$$d_A(u,v) = \sup\{d(\pi_i A(u), \pi_i A(v)), i = 1,2,3\} = \sup\{d(y,w), d(x,z)\}, \text{ and}$$
$$d_S(u,v) = \sup\{d(\pi_i S(u), \pi_i S(v)), i = 1,2,3\} = \sup\{d(y,w), d(x,z)\}\ .$$

and consequently, $d_A(u,v) = d_S(u,v)$

Suppose the distinct points u and v are in the different component of the wedge. We have the following cases for u and v :

Case 1. $u = (x,y)_k$ or $(y,x)_k$ and $v = (x,y)_j$ or $(y,x)_j$ for $x \neq y$, $k \neq j$ and $k, j = 1,2$.

If $u = (x,y)_1$ and $v = (x,y)_2$ (resp. $v = (y,x)_2$), then $d_A(u,v) = \sup\{d(y,x), d(x,y)\}$ and $d_S(u,v) = d(y,x)$ (resp. $d_S(u,v) = \sup\{d(y,x), d(x,y)\}$ and $d_A(u,v) = d(y,x)$), and consequently, $d_A(u,v) = d_S(u,v)$ since d is symmetric.

Similarly if $u = (y,x)_1$ and $v = (x,y)_2$ ($v = (y,x)_2$), then $d_A(u,v) = d_S(u,v)$.

Case 2. $u = (x,y)_k, (x,z)_k, (y,z)_k, (y,x)_k, (z,x)_k$ or $(z,y)_k$ and $v = (x,y)_j, (x,z)_j, (y,z)_j, (y,x)_j, (z,x)_j$ or $(z,y)_j$ for three distinct points x, y, z of X, $k \neq j$ and $k, j = 1,2$. If $u = (x,z)_1$ and $v = (x,y)_2$ or $(z,y)_2$,(resp. $u = (x,y)_1$ and $v = (x,z)_2$ or $(y,z)_2$, $u = (y,z)_1$ and $v = (x,y)_2$, $u = (z,y)_1$ and $v = (x,z)_2$), then

$$d_A(u,v) = \sup\{d(x,y), d(x,z)\} \text{ and } d_S(u,v) = \sup\{d(y,z), d(x,z)\},$$
$$(\text{resp. } d_S(u,v) = \sup\{d(x,y), d(y,z)\}, d_S(u,v) = \sup\{d(x,z), d(y,z), d(x,y)\}).$$

If $d(x,y) = d(y,z) \geq d(x,z)$, then $d_A(u,v) = \sup\{d(x,y), d(x,z)\} = d(x,y) = d(y,z) = d_S(u,v)$.

If $d(x,y) = d(x,z) \geq d(y,z)$, then $d_A(u,v) = \sup\{d(x,y), d(x,z)\} = d(x,y) = d(x,z) = d_S(u,v)$.

If $d(x,z) = d(z,y) \geq d(x,y)$, then $d_A(u,v) = \sup\{d(x,y), d(x,z)\} = d(x,z) = d(y,z) = d_S(u,v)$.

If $u = (x,z)_1$ and $v = (y,x)_2$ or $(y,z)_2$ (resp. $u = (y,x)_1$ and $v = (x,z)_2$ or $(y,z)_2$, $u = (y,z)_1$ and $v = (y,x)_2$ or $(z,x)_2$), then $d_A(u,v) = \sup\{d(x,y), d(y,z)\}$ and $d_S(u,v) = \sup\{d(x,y), d(x,z), d(y,z)\}$, $d_S(u,v) = \sup\{d(x,y), d(x,z)\}$ (resp. $d_S(u,v) = \sup\{d(x,z), d(y,z)\}$).

If $d(x,y) = d(y,z) \geq d(x,z)$, then $d_A(u,v) = \sup\{d(x,y), d(x,z)\} = d(x,y) = d(y,z) = d_S(u,v)$.

If $d(x,y) = d(x,z) \geq d(y,z)$, then $d_A(u,v) = \sup\{d(x,y), d(x,z)\} = d(x,y) = d(x,z) = d_S(u,v)$.

If $d(x,z) = d(z,y) \geq d(x,y)$, then $d_A(u,v) = \sup\{d(x,y), d(x,z)\} = d(x,z) = d(y,z) = d_S(u,v)$.

If $u = (z,x)_1$ and $v = (x,y)_2$ or $(z,y)_2$ (resp. $u = (z,y)_1$ and $v = (y,x)_2$) or $(z,x)_2$), $u = (y,x)_1$ and $v = (z,y)_2$, $u = (y,x)_1$ and $v = (z,y)_2$), then $d_A(u,v) = \sup\{d(y,z), d(x,z)\}$ and $d_S(u,v) = \sup\{d(x,z), d(x,y)\}$ (resp. $d_S(u,v) = \sup\{d(x,y), d(y,z)\}$, $d_S(u,v) = \sup\{d(x,y), d(x,z)d(y,z)\}$).

If $u = (x,z)_1$ and $v = (y,z)_2$ or $u = (z,x)_1$ and $v = (y,x)_2$ (resp. $u = (y,z)_1$ and $v = (x,z)_2$), $u = (z,y)_1$ and $v = (x,y)_2$), $u = (x,y)_1$ and $v = (z,y)_2$, $u = (y,x)_1$ and $v = (z,x)_2$), then $d_A(u,v) = \sup\{d(x,y), d(y,z), d(x,z)\}$ and $d_S(u,v) = \sup\{d(y,z), d(x,y)\}$ (resp. $d_S(u,v) = \sup\{d(x,y), d(x,z)\}$, $d_S(u,v)$).

If $d(x,y) = d(y,z) \geq d(x,z)$ or $d(x,y) = d(x,z) \geq d(y,z)$ or $d(x,z) = d(z,y) \geq d(x,y)$, then it follows easily that $d_A(u,v) = d_S(u,v)$.

If $u = (a,b)_k$ or $u = (b,a)_k$ and $v = (a,b)_j$ or $v = (b,a)_j$ for $a \neq b, a, b = x, y, z, k \neq j$ and $k, j = 1, 2$, then by the case 1, we have $d_A(u,v) = d_S(u,v)$.

Case 3. Let x, y, z, w be four distinct points of X. If $u = (x,y)_1$ and $v = (z,w)_2$ or $u = (z,w)_1$ and $v = (x,y)_2$, then $d_A(u,v) = \sup\{d(y,z), d(x,w), d(x,z)\}$ and $d_S(u,v) = \sup\{d(y,w), d(x,z), d(y,z)\}$ (resp. $d_S(u,v) = \sup\{d(y,w), d(x,z), d(x,w)\}$).

If $u = (x,y)_1$ and $v = (w,z)_2$ or $u = (w,z)_1$ and $v = (x,y)_2$, then $d_A(u,v) = \sup\{d(y,w), d(x,w), d(x,z)\}$ and $d_S(u,v) = \sup\{d(x,w), d(y,w), d(y,z)\}$ (resp. $d_S(u,v) = \sup\{d(y,z), d(x,z), d(x,w)\}$).

If $u = (y,x)_1$ and $v = (z,w)_2$ or $u = (z,w)_1$ and $v = (y,x)_2$, then $d_A(u,v) = \sup\{d(y,z), d(y,w), d(x,z)\}$ and $d_S(u,v) = \sup\{d(x,w), d(x,z), d(y,z)\}$ (resp. $d_S(u,v) = \sup\{d(y,z), d(y,w), d(x,w)\}$).

If $u = (y,x)_1$ and $v = (w,z)_2$ or $u = (w,z)_1$ and $v = (y,x)_2$, then $d_A(u,v) = \sup\{d(y,w), d(x,w), d(y,z)\}$ and $d_S(u,v) = \sup\{d(x,w), d(y,w), d(x,z)\}$ (resp. $d_S(u,v) = \sup\{d(y,z), d(y,w), d(x,z)\}$).

If $d(x,z) = d(y,z) \geq d(x,w), d(y,w)$, then $d_A(u,v) = d(x,z) = d(x,w) = d_S(u,v)$.

If $d(x,z) = d(x,w) \geq d(y,z), d(y,w)$, then $d_A(u,v) = d(x,z) = d(y,w) = d_S(u,v)$.

If $d(x,z) = d(y,w) \geq d(y,z), d(x,w)$, then $d_A(u,v) = d(x,z) = d(y,w) = d_S(u,v)$.

If $d(y,z) = d(x,w) \geq d(x,z), d(y,w)$, then $d_A(u,v) = d(y,z) = d(x,w) = d_S(u,v)$.

If $d(y,z) = d(y,w) \geq d(x,z), d(x,w)$, then $d_A(u,v) = d(y,z) = d(y,w) = d_S(u,v)$.

If $d(x,w) = d(y,w) \geq d(x,z), d(y,z)$, then $d_A(u,v) = d(x,w) = d(y,w) = d_S(u,v)$.

Similarly, if $u = (x,z)_1$ and $v = (y,w)_2$ (resp. $u = (y,w)_1$ and $v = (x,z)_2$, $u = (x,z)_1$ and $v = (w,y)_2$, $u = (w,y)_1$ and $v = (x,z)_2$, $u = (w,y)_1$ and $v = (z,x)_2$, $u = (z,x)_1$ and $v = (w,y)_2$, $u = (z,x)_1$ and $v = (y,w)_2$, $u = (y,w)_1$ and $v = (z,x)_2$) and if $u = (x,w)_1$ and $v = (y,z)_2$ (resp. $u = (y,z)_1$ and $v = (x,w)_2$, $u = (x,w)_1$ and $v = (z,y)_2$, $u = (z,y)_1$ and $v = (x,w)_2$, $u = (w,x)_1$ and $v = (y,z)_2$, $u = (y,z)_1$ and $v = (w,x)_2$, $u = (w,x)_1$ and $v = (z,y)_2$, $u = (z,y)_1$ and $v = (w,x)_2$), by the assumption (3), $d_A(u,v) = d_S(u,v)$.

Hence, for all points u and v in the wedge $X^2 \bigvee_\Delta X^2$, $d_A(u,v) = d_S(u,v)$ and by Definition 3.6, (X,d) is $Pre\overline{T}_2$. □

Theorem 3.11. *An extended pseudo-quasi-semi metric space (X,d) is $PreT'_2$ if and only if for all $x, y \in X$ with $x \neq y$, $d(x,y) = \infty$ and $d(y,x) = \infty$.*

Proof. Suppose that (X,d) is $PreT'_2$ and $x, y \in X$ with $x \neq y$. Let d_2 be the final extended pseudo-quasi-semi metric structure on $X^2 \coprod X^2$ induced by the canonical injection maps $i_1, i_2 : (X^2, d^2) \to X^2 \coprod X^2$ and d_1 be the quotient extended pseudo-quasi-semi metric structure on $X^2 \vee_\Delta X^2$ induced by the quotient map $q : (X^2 \coprod X^2, d_2) \to X^2 \vee_\Delta X^2$. Suppose that for $u = (x,y)_1$ and $v = (x,y)_2$ with $x \neq y$. Then $\sup\{d(\pi_i S(u), \pi_i S(v)), i = 1,2,3\} = d(y,x)$, where $\pi_i : X^3 \to X$, $i = 1,2,3$ are the projection maps and by Proposition 2.1,

$d_1(u,v) = d_2(q^{-1}(u), q^{-1}(v)) = d_2(\{(x,y)_1\}, \{(x,y)_2\}) = \infty.$

Since (X,d) is $PreT_2'$, by Definition 3.6,
$d(y,x) = \sup\{d(\pi_i S(u), \pi_i S(v)), i = 1,2,3\} = d_1(u,v) = \infty$ which shows $d(y,x) = \infty$.
Similarly, if $u = (x,y)_2$ and $v = (x,y)_1$ with $x \neq y$, then
$d(x,y) = \sup\{d(\pi_i S(u), \pi_i S(v)), i = 1,2,3\} = d_1(u,v) = \infty$, i.e., $d(x,y) = \infty$.

Conversely, suppose that $d(x,y) = \infty$ and $d(y,x) = \infty$ for all $x,y \in X$ with $x \neq y$. Let d_S be the initial structure on $X^2 \vee_\Delta X^2$ induced from the principle axis map $S : X^2 \vee_\Delta X^2 \to (X^3, d^3)$, where d^3 is the product extended pseudo-quasi-semi metric structure on X^3, d_1 and d_2 be extended pseudo-quasi-semi metric structures defined as above. Note that the composition of final lifts is final.

We need to show that (X,d) is $PreT_2'$, i.e., by Definition 3.6, the initial structure d_S and the final structure d_1 are equal, i.e., $d_1 = d_S$.

Let u and v be any points in $X^2 \vee_\Delta X^2$.

If $u = v$, then $d_1(u,v) = 0 = d_S(u,v)$. Suppose that $u \neq v$ and they are in the same component of the wedge with $u \neq v$, i.e., $u = (x,y)_k$ and $v = (z,w)_k$ for $k = 1,2$ and $x,y,z,w \in X$.

If $x \neq y$ and $z = w$, then by Proposition 2.1,

$$d_S(u,v) = \sup\{d(\pi_i S(u), \pi_i S(v)), i = 1,2,3\} = \sup\{d(x,z)d(y,z)\} and$$

$$\begin{aligned}
d_1(u,v) &= d_2(q^{-1}(u), q^{-1}(v)) \\
&= d_2(\{(x,y)_k\}, \{(z,z)_1, (z,z)_2\}) \\
&= \inf\{d_2((x,y)_k, (z,z)_1), d_2((x,y)_k, (z,z)_2), k = 1,2\} \\
&= d_2((x,y)_k, (z,z)_k) = d^2((x,y)_k, (z,z)_k) \\
&= \sup\{d(x,z), d(y,z)\}
\end{aligned}$$

and consequently, $d_1(u,v) = d_S(u,v)$.

If $x = y$ and $z \neq w$, then by Proposition 2.1,

$$d_S(u,v) = \sup\{d(\pi_i S(u), \pi_i S(v)), i = 1,2,3\} = \sup\{d(x,z)d(x,w)\}$$

and

$$\begin{aligned}
d_1(u,v) &= d_2(q^{-1}(u), q^{-1}(v)) \\
&= d_2(\{(x,x)_1, (x,x)_2\}, \{(z,w)_k\}) \\
&= d_2((x,y)_k, (z,z)_k) \\
&= d^2((x,x)_k, (z,w)_k) \\
&= \sup\{d(x,z), d(x,w)\}
\end{aligned}$$

and consequently, $d_1(u,v) = d_S(u,v)$.

If $x = y \neq z = w$, then by Proposition 2.1,

$$d_S(u,v) = \sup\{d(\pi_i S(u), \pi_i S(v)), i = 1,2,3\} = d(x,z)$$

and

$$\begin{aligned}
d_1(u,v) &= d_2(q^{-1}(u), q^{-1}(v)) \\
&= d_2(\{(x,x)_1, (x,x)_2\}, \{(z,z)_1, (z,z)_2\}) \\
&= d_2((x,x)_k, (z,z)_k) \\
&= d^2((x,x)_k, (z,z)_k) \\
&= d(x,z)
\end{aligned}$$

for $k = 1,2$ and consequently, $d_1(u,v) = d_S(u,v)$.

If $x \neq y$ and $z \neq w$, then by Proposition 2.1,

$$d_S(u,v) = \sup\{d(\pi_i S(u), \pi_i S(v)), i = 1,2,3\} = \sup\{d(x,z)d(y,w)\} \text{ and}$$
$$d_1(u,v) = d_2(q^{-1}(u), q^{-1}(v)) = d_2(\{(x,y)_k\}, \{(z,w)_k\}) = d^2((x,y)_k, (z,w)_k) = \sup\{d(x,z), d(y,w)\}$$

and consequently, $d_1(u, v) = d_S(u, v)$.

Suppose that $u \neq v$ and they are in the different component of the wedge $X^2 \vee_\Delta X^2$, i.e., $u = (x, y)_i$ and $v = (z, w)_j$ for $i, j = 1, 2$, with $i \neq j$ and $x, y, z, w \in X$. If $x = z \neq y = w$, $u = (x, y)_1$ and $v = (x, y)_2$ (resp. $u = (x, y)_2$ and $v = (x, y)_1$), then by Proposition 2.1, $d_S(u, v) = d(y, x)$ (resp. $d(x, y)$) and $d_1(u, v) = d_2(q^{-1}(u), q^{-1}(v) = d_2(u, v) = \infty$ (resp. $d_1(u, v) = \infty$). Since $x \neq y$, by the assumption $d(y, x) = \infty = d(x, y) =$, and consequently, $d_1(u, v) = d_S(u, v)$.

Similarly, if $z = y \neq w = x$, then, by Proposition 2.1, $d_1(u, v) = d_S(u, v)$.

Suppose that $(x, y) \neq (z, w)$.

Assume that $x \neq z \neq y = w$. If $u = (x, y)_1$ and $v = (z, y)_2$, then

$$d_S(u, v) = \sup\{d(\pi_i S(u), \pi_i S(v)), i = 1, 2, 3\} = \sup\{d(y, z)d(x, z)\} = \infty$$

by the assumption $d(x, z) = \infty$ since $x \neq z$ and $d_1(u, v) = \infty$ and consequently, $d_1(u, v) = d_S(u, v)$.

Assume that $x = z \neq y \neq w$. If $u = (x, y)_1$ and $v = (x, w)_2$, then by assumption and Proposition 2.1, $d_S(u, v) = \sup\{d(\pi_i S(u), \pi_i S(v)), i = 1, 2, 3\} = \sup\{d(y, x)d(y, w)\} = \infty$ and
$d_1(u, v) = d_2(q^{-1}(u), q^{-1}(v) = d_2(u, v) = \infty$.

Suppose that $x \neq z, y \neq w$. If $u = (x, y)_1$ and $v = (z, w)_2$, then, by Proposition 2.1 and assumption, $d_S(u, v) = \sup\{d(\pi_i S(u), \pi_i S(v)), i = 1, 2, 3\} = \sup\{d(x, z), d(y, z), d(y, w)\} = \infty$ and $d_1(u, v) = d_2(u, v) = \infty$.

Hence, for all points u and v on the wedge $X^2 \bigvee_\Delta X^2$, we have $d_1(u, v) = d_S(u, v)$ and by Definition 3.6, (X, d) is $PreT'_2$. □

Definition 3.12. ([5]) Let $\mathcal{E}$ be a topological category and X an object in $\mathcal{E}$.

(1) If X is T'_0 and $PreT'_2$, then X is called T'_2,

(2) If X is $\overline{T}_0$ and $Pre\overline{T}_2$, then X is called $\overline{T}_2$,

(3) If X is T'_0 and $Pre\overline{T}_2$, then X is called KT_2,

(4) If X is T_0 and $Pre\overline{T}_2$, then X is called NT_2.

Theorem 3.13. *An extended-pseudo-quasi-semi metric space (X, d) is KT_2 if and only if (X, d) is $Pre\overline{T}_2$.*

Proof. It follows from Theorems 3.4, 3.10, and Definition 3.12. □

Theorem 3.14. *An extended-pseudo-quasi-semi metric space (X, d) is NT_2 if and only if the following conditions are satisfied.*

(1) (X, d) an extended-semi metric space.

(2) For any three distinct points x, y, z of X, we have either $d(x, y) = d(y, z) \geq d(x, z)$ or $d(x, y) = d(x, z) \geq d(y, z)$ or $d(x, z) = d(z, y) \geq d(x, y)$.

(3) For any four distinct points x, y, z, w of X, we have either $d(x, z) = d(y, z) \geq d(x, w), d(y, w)$ or $d(x, z) = d(x, w) \geq d(y, z), d(y, w)$ or $d(x, z) = d(y, w) \geq d(y, z), d(x, w)$ or $d(y, z) = d(x, w) \geq d(x, z), d(y, w)$ or $d(y, z) = d(y, w) \geq d(x, z), d(x, w)$ or $d(x, w) = d(y, w) \geq d(x, z), d(y, z)$.

Proof. It follows from Theorems 3.3, 3.10 and Definition 3.12. □

Theorem 3.15. *An extended-pseudo-quasi-semi metric space (X, d) is $\overline{T}_2$ (resp. T'_2) if and only if (X, d) is discrete.*

Proof. It follows from Theorems 3.3-3.5, 3.10, 3.11, and Definition 3.12. □

Theorem 3.16. *Let $\mathcal{E}$ be a topological category and X an object in $\mathcal{E}$. If X is T'_2, then X is KT_2.*

Proof. Suppose X is T'_2. By Definition 3.12, X is T'_0 and $PreT'_2$. By Theorem 3.1 of [6], X is $Pre\overline{T}_2$ and consequently, X is T'_0 and $Pre\overline{T}_2$, i.e., by Definition 3.12, X is KT_2. □

Let **TpqsMet** be the full subcategory of a topological category **pqsMet** consisting of all T extended pseudo-quasi-semi metric spaces, where $T = Pre\overline{T}_2, T'_0, NT_2, \overline{T}_2, T'_2, KT_2$.

Theorem 3.17. *(1)* **KT_2pqsMet** *is a topological category and the full subcategories* **Pre$\overline{T}_2$pqsMet** *and* **KT_2pqsMet** *are isomorphic.*

(2) **T_0'pqsMet** *is a cartesian closed and hereditary topological category.*

(3) The full subcategories **$\overline{T}_2$pqsMet** *and* **T_2'pqsMet** *are isomorphic.*

Proof. (1) By Theorem 3.4 of [8], **Pre$\overline{T}_2$pqsMet** is a topological category. By Theorem 3.10 and Theorem 3.13, both **Pre$\overline{T}_2$pqsMet** and **KT_2pqsMet** are isomorphic categories and consequently, **KT_2pqsMet** is a topological category.

(2) By Theorem 3.5, both **T_0'pqsMet** and **pqsMet** are isomorphic categories and consequently, **T_0'pqsMet** is a cartesian closed and hereditary topological category [11].

(3) By Theorem 3.15, the full subcategories **$\overline{T}_2$pqsMet** and **T_2'pqsMet** of **pqsMet** are isomorphic. □

Remark 3.18. (1) Note that for the category **Top** of topological spaces, by Definition 3.12, Remark 3.2, and Remark 3.7, all of $\overline{T_2}$, T_2', NT_2, and KT_2 are equivalent and they reduce to the usual T_2 separation axiom.

(2) In **pqsMet**, by Theorems 3.13-3.15, $\overline{T}_2 \Leftrightarrow T_2' \Rightarrow NT_2 \Rightarrow KT_2$ but the reverse implication is not true, in general. As an example, if (R, d) is the indiscrete extended pseudo-quasi-semi metric space, i.e., $d(a, b) = 0$ for all $a, b \in R$, where R is the set of reel numbers, then by Theorem 3.13, (R, d) is KT_2 but by Theorems 3.13-3.15, (R, d) is not $\overline{T}_2, T_2'$ and NT_2. Also, if $X = \{x, y, z\}$ and d is defined as $d(x, x) = 0 = d(y, y) = d(z, z), d(x, y) = 1 = d(y, x), d(x, z) = 3 = d(z, x) = d(y, z) = d(z, y)$. By Theorem 3.14, (X, d) is NT_2 but by Theorem 3.15, (X, d) is not $\overline{T}_2$ (resp. T_2').

(3) By Theorem 3.17, the categories **Pre$\overline{T}_2$pqsMet**, **T_0'pqsMet**, and **KT_2pqsMet** have all limits and colimits. By Theorems 3.3, 3.4, and 3.15, the categories **$\overline{T}_2$pqsMet**, **T_1pqsMet**, **T_0pqsMet**, **$\overline{T}_0$pqsMet**, and **T_2'pqsMet** are hereditary and productive.

(4) In any topological category, by Theorem 3.16, Theorem 2.7 of [5] and Remark 3.2 of [6], both T_2' and $\overline{T}_2$ implies KT_2. The relationships among $\overline{T_2}$, T_2', NT_2, and KT_2 objects in some well known topological categories are investigated in Remark 2.8(1-7) of [5]. Moreover, for an arbitrary topological category, we have $\overline{T}_0$ implies T_0' ([4], Theorem 3.2) but the reverse implication is generally not true by Theorem 3.4, Theorem 3.5 and Remark 3.18. Also, there are no implications between T_0 and each of $\overline{T}_0$ and T_0' ([4], Remark 3.6).

Let $U : \mathcal{E} \to$ **Set** be topological functor and X be an object in $\mathcal{E}$ with $U(X) = B$. Let M be a nonempty subset of B. We denote by X/M the final lift of the epi U-sink $q : U(X) = B \to B/M = (B\backslash M) \cup \{*\}$, where q is the epi map that is the identity on $B\backslash M$ and identifying M with a point * [3].

Recall, in [3], that $M \subset X$ is strongly closed if and only if X/M is T_1 at * and $M \subset X$ is closed if and only if $\{*\}$, the image of M, is closed in X/M.

Theorem 3.19. ([10]) *Let (X, d) be an extended pseudo-quasi-semi metric space and $\emptyset \neq M \subset X$.*

(1) M is strongly closed if and only if $d(M, x) = \infty = d(x, M)$ for all $x \in X$ with $x \notin M$.

(2) M is closed if and only if $d(M, x) = \infty$ or $d(x, M) = \infty$ for all $x \in X$ with $x \notin M$.

Theorem 3.20. *Let (X, d) be an extended pseudo-quasi-semi metric space.*

(1) If (X, d) is $\overline{T}_2$ (resp. T_2' or T_1), then each subset of X is (strongly) closed.

(2) If $N \subset X$ is (strongly) closed and $M \subset N$ is (strongly) closed, then $M \subset X$ is (strongly) closed.

(3) If (X, d) is KT_2 (resp. NT_2), then a subset M of X is strongly closed if and only if it is closed.

Proof. (1) follows from Theorems 3.3, 3.15, and 3.18.

(2) Suppose $N \subset X$ and $M \subset N$ are strongly closed. Let d_N be the initial extended pseudo-quasi-semi metric structure on N induced by the inclusion map $i : N \to (X, d)$ and d_M be the initial extended pseudo-quasi-semi metric structure on M induced by the inclusion map $i : M \to (N, d_N)$. Let $x \in X$, $x \notin M$ and $x \notin N$. By Proposition 2.1, $d_M(x, M) = d_N(x, M) = d(x, M)$ and $d_M(M, x) = d_N(M, x) = d(M, x)$ and by Theorem 3.19, $d(M, x) = \infty = d(x, M)$ since $N \subset X$ is strongly closed.

Suppose $x \in N$. Since $x \notin M$ and $M \subset N$ is strongly closed by Theorem 3.19, $d_N(x, M) = \infty = d_N(M, x)$ and by Proposition 2.1, $d(x, M) = \infty = d(M, x)$. Hence, by Theorem 3.19, $M \subset X$ is strongly closed.

The proof for closedness is similar.

(3) Suppose $\emptyset \neq M \subset X$ and $x \in X$ with $x \notin M$. By Theorems 3.13 and 3.14, $d(x, y) = d(y, x)$ for all $y \in M$. By Theorem 3.19, M is strongly closed if and only if it is closed. □

Theorem 3.21. (Urysohn's Lemma) *Let (X, d) be a KT_2 (resp. NT_2) extended pseudo-quasi-semi metric space and $M, N \subset X$ be nonempty disjoint closed subset of X. Then, there exists a non-expansive mapping $f : (X, d) \to ([0, 1], e)$, where e is any extended pseudo-quasi-semi metric structure on $[0, 1]$, such that $f(M) = \{0\}$ and $f(N) = \{1\}$.*

Proof. Define $f : (X, d) \to ([0, 1], e)$, where e is any extended pseudo-quasi-semi metric structure on $[0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in M \\ 1 & \text{if } x \notin M \end{cases}$$

for $x \in X$.

Note that $f(M) = \{0\}$ and $f(N) = \{1\}$. We show that f is a non-expansive mapping.

Let $x, y \in X$. If $x, y \in M$ or $x, y \in M^C$, then $e(f(x), f(y)) = 0 = e(f(y), f(x)) \leq d(x, y)$. If $x \in M$ and $y \in M^C$ (resp. $y \in M$ and $x \in M^C$), then by Theorem 3.19 , $e(f(x), f(y)) \leq d(x, y) = \infty$.

Hence, f is a non-expansive mapping such that $f(M) = \{0\}$ and $f(N) = \{1\}$. □

Theorem 3.22. *Let (X, d) be a $\overline{T}_2$ (resp. T_2' or T_1) extended pseudo-quasi-semi metric space and $M, N \subset X$ be nonempty disjoint subset of X. Then, there exists a non-expansive mapping $f : (X, d) \to ([0, 1], e)$, where e is any extended pseudo-quasi-semi metric structure on $[0, 1]$, such that $f(M) = \{0\}$ and $f(N) = \{1\}$.*

Proof. The proof is similar to the proof of Theorem 3.21 by using Theorems 3.3(2), 3.15, and 3.20. □

Theorem 3.23. (Tietze Extention Theorem) *Let (X, d) be a KT_2 (resp. NT_2) extended pseudo-quasi-semi metric space and $A \subset X$ be nonempty closed subspace of X. Then, every non-expansive mapping $f : (A, d) \to ([0, 1], e)$, where e is any extended pseudo-quasi-semi metric structure on $[0, 1]$ has a non-expansive extention mapping $g : (X, d) \to ([0, 1], e)$.*

Proof. Suppose (X, d) is a KT_2 (resp. NT_2) extended pseudo-quasi-semi metric space, A is nonempty closed subspace of X, and $f : (A, d_A) \to ([0, 1], e)$ is a non-expansive mapping, where d_A is the initial extended pseudo-quasi-semi metric structure on A induced by the inclusion map $i : A \to (X, d)$ and where e is any extended pseudo-quasi-semi metric structure on $[0, 1]$.

Define $g : (X, d) \to ([0, 1], e)$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

for $x \in X$.

Note that $g(x) = f(x)$ for all $x \in A$. We show that g is a non-expansive mapping.

Let $x, y \in X$. If $x, y \in A$, then $e(g(x), g(y)) = e(f(x), f(y)) \leq d(x, y)$ since f is a non-expansive mapping. If $x \in A$ and $y \in A^C$ (resp. $y \in A$ and $x \in A^C$), then by Theorems 3.13 and 3.14, $e(g(x), g(y)) \leq d(x, y) = \infty$. If $x, y \in A^C$, then $e(f(x), f(y)) = 0 \leq d(x, y)$.

Hence, g is a non-expansive extention mapping of f. □

Theorem 3.24. *Let (X, d) be a $\overline{T}_2$ (resp. T_2' or T_1) extended pseudo-quasi-semi metric space and A be any nonempty subspace of X. Then, every non-expansive mapping $f : (A, d) \to ([0, 1], e)$, where e is any extended pseudo-quasi-semi metric structure on $[0, 1]$ has a non-expansive extention mapping $g : (X, d) \to ([0, 1], e)$.*

Proof. The proof is similar to the proof of Theorem 3.23 by using Theorems 3.3(2), 3.15, and 3.20. □

Let $U : \mathcal{E} \to \mathbf{Set}$ be a topological functor and X be an object in $\mathcal{E}$. Recall, in [10], that X is said to be (strongly) irreducible if M, N are (strongly) closed subobjects of X and $X = M \cup N$, then $M = X$ or $N = X$.

Note that if a topological space (X, τ) is T_1, then the notions of irreducible spaces and strongly irreducible spaces coincide and if (X, τ) is nonempty irreducible and T_2, then (X, τ) must be a one-point space [10].

Theorem 3.25. *Let (X, d) be an extended pseudo-quasi-semi metric space.*

(1) If (X, d) is a nonempty (strongly) irreducible and $\overline{T}_2$ (resp. T_2' or T_1), then (X, d) must be a one-point space.

(2) If (X, d) is (strongly) irreducible and KT_2 (resp. NT_2), then (X, d) may not be a one-point space.

(3) If (X, d) is KT_2 (resp. NT_2), then (X, d) is strongly irreducible if and only if (X, d) is irreducible.

Proof. (1) Suppose that (X, d) is nonempty (strongly) irreducible $\overline{T}_2$ (resp. T_2' or T_1) and X has at least two points, x and y. Let $M = \{x\}$. By Theorem 3.20, M and M^C are proper (strongly) closed and $X = M \cup M^C$, a contradiction. Hence, (X, d) must be a one-point space.

(2) Let $X = \{x, y\}$ and d is defined as $d(x, x) = 0 = d(y, y), d(x, y) = 1 = d(y, x)$. By Theorems 3.13 and 3.14, (X, d) is KT_2 and NT_2. By Theorem 3.19, (X, d) is (strongly) irreducible but (X, d) is not a one-point space.

(3) By Theorem 5.4 of [10], if (X, d) is irreducible, then (X, d) is strongly irreducible. Suppose that (X, d) is strongly irreducible KT_2 (resp. NT_2) and $X = M \cup N$, where M, N are closed subsets of X. By Theorem 3.20, M and N are strongly closed subsets of X. Since (X, d) is strongly irreducible, then $M = X$ or $N = X$ and consequently, (X, d) is irreducible. □

Acknowledgement

We would like to thank the referees for their kind suggestions.

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