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Certain Dynamical Aspects of a Family $f_\lambda(z) = \lambda \frac{e^z}{z+1}$ for $z \in \mathbb{C}$ when $\lambda < 0$

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Abstract. We study the change of dynamics of transcendental meromorphic functions $f_\lambda = \lambda \frac{e^z}{z+1}$ for $z \in \mathbb{C}$ when λ varies on the negative real axis. It is shown that there is a $\widehat{\lambda}$ such that the Fatou set of f_λ is empty for $\lambda < \widehat{\lambda}$ whereas the Fatou set is an invariant parabolic basin corresponding to a real rationally indifferent fixed point $\widehat{x}$ if $\lambda = \widehat{\lambda}$. In fact, the Fatou set is an invariant attracting basin of a real negative fixed point $\widehat{a}_\lambda$ if $\widehat{\lambda} < \lambda < 0$. Also the dynamics of f_λ^n for $n \geq 2$ at the fixed points is investigated for different values of λ. As a generalization of f_λ, we observed some dynamical issues for the class of entire maps $F_{\lambda,a,m}(z) = \lambda(z+a)^m \exp(z)$ where $\lambda, a \in \mathbb{C}$ and $m \in \mathbb{N}$.

1. Introduction

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic function. Then the Fatou set, denoted by $\mathcal{F}(f)$ is defined by

$\mathcal{F}(f) = \{z \in \widehat{\mathbb{C}} : \{f^n : n \in \mathbb{N}\}$ is defined and normal in some neighbourhood of $z\}$

and the Julia set, denoted by $\mathcal{J}(f)$, is the complement of $\mathcal{F}(f)$ in $\widehat{\mathbb{C}}$. The Fatou set is open and the Julia set is perfect. Roughly speaking, the Fatou set is the set where iterative behaviour is relatively tame. i.e., points close to each other behave similarly while the Julia set is the set of points where the nearby points behave in a drastically different way under the iteration of the given function. The orbits here are extremely sensitive to initial conditions. Another interesting property is that if the interior is nonempty, then the Julia set coincides with $\widehat{\mathbb{C}}$. Periodic points play very crucial role in the iteration theory because their orbits are finite and they sometimes control the dynamics locally. A point z_0 is called a periodic point of f if $f^n(z_0) = z_0$ for some $n \geq 1$. The smallest n with this property is called the period of z_0. For a periodic point z_0 of period n, the orbit $O^+(z_0) = \{z_0, f(z_0), ..., f^{n-1}(z_0)\}$ is called the cycle of z_0. The number $\lambda = (f^n)'(z_0)$ is called multiplier of z_0.

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A periodic point is called attracting, indifferent or repelling as in the case with $|\lambda| < 1$, $|\lambda| = 1$ or $|\lambda| > 1$. Moreover, an attracting periodic point is called superattracting if $\lambda = 0$. An indifferent periodic point is called rationally indifferent or irrationally indifferent according to $\lambda^m = 1$ for some $m \in \mathbb{N}$ or $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{Q}^c$. A periodic point of order one is called a fixed point. The attracting periodic points are always in the Fatou set while, repelling and rationally indifferent periodic points are in the Julia set. For irrationally indifferent periodic points it is difficult to decide whether it is in the Fatou set or in the Julia set.

A point $a \in \widehat{\mathbb{C}}$ is said to be a non-singular value (of the inverse function f^{-1}) if it has a neighbourhood V such that $f : f^{-1}(V) \to V$ is an unbranched cover. We call a point $a \in \widehat{\mathbb{C}}$ a singular value of f if for every open neighborhood U of a, there exists a component V of $f^{-1}(U)$ such that $f : V \to U$ is not bijective. Denote the set of singular values of f by $sing(f^{-1})$. This is the set of closure of critical values and asymptotic values of f. A critical value is the image of a critical point, that is, $f(z_0)$ where $f'(z_0) = 0$. A point $a \in \widehat{\mathbb{C}}$ is an asymptotic value of f if there exists a curve $\gamma : [0, \infty) \to \mathbb{C}$ with $\lim_{t\to\infty} |\gamma(t)| = \infty$ such that $a = \lim_{t\to\infty} f(\gamma(t))$. For a comprehensive definition of singular values, one can see [6].

For any transcendental function f, we have $sing(f^{-1}) \neq \phi$. If f is transcendental entire function then $\infty \in sing(f^{-1})$. If f is transcendental meromorphic function then it has at least one singular value. One of the main reason for which singular values are important is the fact that any attracting fixed point or cycle must have a singular value in its immediate basin of attraction. It follows that if f has n singular values, then it has at most n attracting cycles. Same is the case for rationally indifferent cycles.

A maximal connected domain U of normality of the iterates of f is called a component of the Fatou set. Then $f^n(U)$ is contained in a component of $\mathcal{F}(f)$ which we denote by U_n. A component U is preperiodic if there exists $n > m \geq 0$ such that $U_n = U_m$. If this happens for $m = 0$ and with smallest $n \geq 1$, then U is called periodic with period n, and $\{U, U_1, ..., U_{n-1}\}$ is called a cycle of components. If $n = 1$ that is, $U_1 = U$ then U is called invariant.

If U is a periodic component of period p then we have one of the following possibilities: Attracting domain, Parabolic domain, Siegel disk, Herman ring and Baker domain. A component that is not preperiodic is called a wandering component. A Baker wandering domain is a particular type of wandering Fatou component U of a function f such that for large n, $f^n(U)$ is contained in a bounded multiply connected Fatou component U_n that surrounds the origin and $U_n \to \infty$ as $n \to \infty$.

Study of change in dynamics in a one parameter family is a well pursued theme. For example the exponential family (λe^z, $\lambda \in \mathbb{C}$) has been investigated by many researchers [8, 15]. It was shown by Devaney et al.[8] that for the exponential family λe^z with $\lambda > 0$, there are two different dynamical behaviors, depending on whether $0 < \lambda < 1/e$ or $\lambda > 1/e$. The Julia sets of these maps have the interesting property that they explode as the parameter λ crosses the value $1/e$. Precisely, they have shown that when $\lambda < 1/e$ the Julia set is a connected nowhere dense subset of the right half plane, but when $\lambda > 1/e$, the Julia set is the whole plane. This is well known as chaotic burst in the Julia sets and it has been observed in [12] for the one parameter family of Joukowski-exponential maps $\{g_\lambda = \lambda(e^z + 1 + \frac{1}{e^z+1}) : \lambda > 0\}$ at the parameter value $\lambda^* \approx 0.266$. In spite of this similarity, it is observed that the Julia set of g_λ is disconnected. In fact it is a disjoint union of two completely invariant subsets one of which is totally disconnected. Joukowski-exponential maps have two asymptotic values, ∞ and 2λ (finite) like exponential maps and have an additional singular (critical) value -2λ. Additionally, each function in these two families are periodic. This paper is an attempt to study a similar family of meromorphic maps which are not periodic.

Let $f_\lambda(z) = \lambda \frac{e^z}{z+1}, \lambda < 0$. It has a single pole at -1 which is not an omitted value. The set of singular values of f_λ, denoted by $sing(f_\lambda^{-1})$, is $\{\lambda, 0, \infty\}$. The asymptotic values are 0 and ∞. The finite singular values have a single forward orbit. We see that the asymptotic value 0 is also the omitted value for the function. Since all the singular values are on the real line, it is important to know how the function behaves on $\mathbb{R}$. The dynamics of the one parameter family of transcendental meromorphic functions $f_\lambda(z) = \lambda \frac{e^z}{z+1}$ for $z \in \mathbb{C}$, $\lambda > 0$ is already studied in [7].

The dynamics of the family of transcendental meromorphic functions $\mathcal{K} = \{f_\lambda(z) = \lambda f(z) : f(z) = \frac{e^z}{z+1}$ for $z \in \mathbb{C}$ and $\lambda \in \mathbb{R} \setminus \{0\}\}$ is studied in this paper. The function f_λ and λe^z have some properties in common. For example, each of them has exactly two transcendental singularities, one over 0 and another over ∞.

The singular values of a function influence the dynamics of the function in a number of ways. For a survey on these one can refer to [10], [13] & [4]. Functions with finitely many singular values has been studied in [1]. In this line, the study of exponential family gives much richness into the literature. Some of the key contributors in this direction are L. Rempe [9, 15], Schleicher & Zimmer [16] & Devaney [8] to name a few. One of the main motivation behind considering this family of functions is that, it is in some sense a simpler function than functions where the set of essential singularities form a compact set. The exponential family λe^z, $\lambda \in \mathbb{C}$ has only one finite singular value which is the asymptotic value. But for the family $f_\lambda(z) = \lambda \frac{e^z}{z+1}$ for $z \in \mathbb{C}$, $\lambda < 0$ has two finite singular values one of which is the asymptotic value. The dynamics here is less complicated in comparison to functions with more singular values and a clear understanding of exponential dynamics can give some intuition for further study of meromorphic functions. In Section 3 of this paper we have investigated some dynamical issues of the class of functions $\mathcal{M} = \{F_{\lambda,a,m}(z) = \lambda(z+a)^m e^z : \lambda, a \in \mathbb{C},\ m \in \mathbb{N}\}$.

Let $f(z) = \frac{e^z}{z+1}$. The following result regarding the nature of singularities of the inverse function from [7] is an important observation and will be used later.

Proposition 1.1. *The number of singularities of f^{-1} lying each over 0 and ∞ are exactly one which are direct. Moreover, both the singularities are of logarithmic type.*

The escaping set of transcendental meromorphic functions is defined as, $I(f) = \{z : f^n(z)$ is defined for $n \in \mathbb{N}, f^n(z) \to \infty$ as $n \to \infty\}$. The following remark follows from [2].

Remark 1.2. *Since f is a meromorphic function with a direct singularity over infinity, $I(f) \cap \mathcal{J}(f)$ contains a continua and $I(f)$ has an unbounded component.*

So for $f(z) = \frac{e^z}{z+1}$, $I(f)$ has an unbounded component and $I(f) \cap J(f)$ contains a continua.

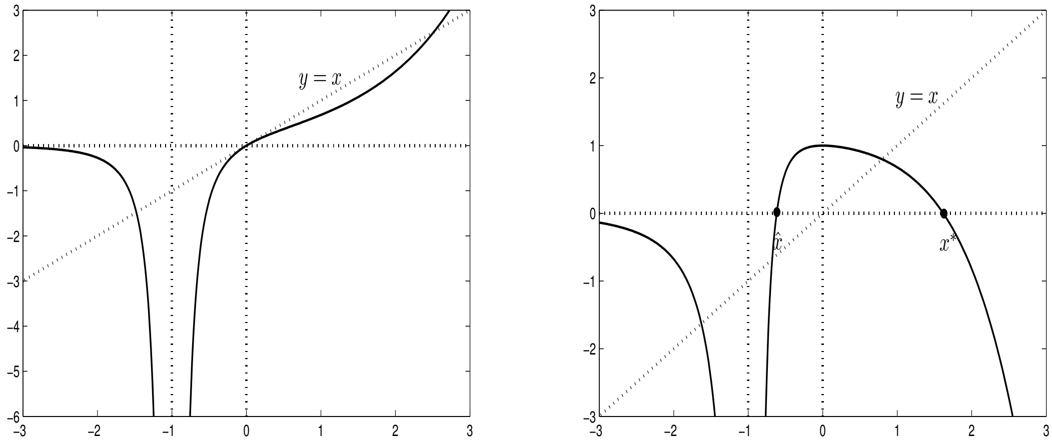
2. Dynamics of $f_\lambda(z)$ when $\lambda < 0$

Let us consider the function $f(x) = \frac{e^x}{1+x}$ when $x \in \mathbb{R}$. It is clear that $f(x) > 0$ when $x > -1$, $f(x) < 0$ when $x < -1$ and $f(x)$ is continuous everywhere except at the point $x = -1$. Since $f'(x) = \frac{xe^x}{(1+x)^2}$, the function $f(x)$ is strictly increasing in $(0,\infty)$ and is strictly decreasing in $(-\infty,-1)$ and $(-1,0)$. As $f''(x) = \frac{(1+x^2)e^x}{(1+x)^3} < 0$ for $x < -1$, $f'(x)$ is decreasing in $(-\infty,-1)$ and is increasing in $(-1,\infty)$. Note that $\lim_{x\to-1} f'(x) = -\infty$, $\lim_{x\to\infty} f'(x) = \infty$ and $\lim_{x\to-\infty} f'(x) = 0$. As f is decreasing in $(-\infty,0)$ and increasing $(0,\infty)$, it attains its local minima at $x = 0$ and the minimum value is $f(0) = 1$. Moreover, $f(x) \to 0$ when $x \to -\infty$ and $f(x) \to +\infty$ when $x \to +\infty$.

The function $\phi(x) = f(x) - xf'(x)$ is continuous except at $x = -1$. We have $\phi(0) = 1$, $\phi(x) \to -\infty$ as $x \to \infty$ and $\phi(x) \to -\infty$ as $x \to -1$. Again $\phi'(x) = -xf''(x)$ is positive when $-1 < x < 0$ and negative when $x > 0$. $\phi(x)$ is increasing when $-1 < x < 0$ and decreasing when $x > 0$ (see Figure 1). So by Intermediate value theorem, $\phi(x)$ has a zero namely, $\widehat{x}$ in the interval $(-1,0)$ and has another zero namely x^* in the interval $(0,\infty)$ such that,

$$\phi(x) \begin{cases} < 0 & \text{for } x^* < x < \infty, \\ = 0 & \text{for } x = x^*, \\ > 0 & \text{for } \widehat{x} < x < x^*, \\ = 0 & \text{for } x = \widehat{x}\,, \\ < 0 & \text{for } -1 < x < \widehat{x} < 0, \\ < 0 & \text{for } -\infty < x < -1\,. \end{cases} \tag{2.1}$$

Clearly $\widehat{x} \in (-1,0)$ is the solution of $\phi(x)$. Then we have $f(\widehat{x}) = \widehat{x}f'(\widehat{x})$. This implies $f'(\widehat{x}) = \frac{f(\widehat{x})}{\widehat{x}}$. Now $f_\lambda(x) = \frac{\lambda e^x}{1+x} = \lambda f(x)$. If we put $x = \widehat{x}$ then $f_\lambda(\widehat{x}) = \lambda f(\widehat{x}) = \lambda\widehat{x}f'(\widehat{x})$. If $\lambda f'(\widehat{x}) = 1$ then $f_\lambda(\widehat{x}) = \widehat{x}$ i.e., $\widehat{x}$ is the fixed point of f_λ. Thus for making $\widehat{x}$ as fixed point of f_λ, we choose $\lambda = \frac{1}{f'(\widehat{x})} = \widehat{\lambda}$ (say). Numerical computation gives us, $\widehat{x} = (\frac{1}{2} - \frac{\sqrt{5}}{2}) \cong -0.618033988749895$ which is incidentally the golden ratio and $\widehat{\lambda} \cong -0.437971479322040$.



(a) (b)

Figure 1: Graphs of (*a*) $f'(x)$ and (*b*) $\phi(x)$

Now let us define $g_\lambda(x) = f_\lambda(x) - x$ for $x \in \mathbb{R}$. Then $g'_\lambda(x) = \lambda \frac{xe^x}{(1+x)^2} - 1$ and $g''_\lambda(x) = \lambda \frac{(1+x^2)e^x}{(1+x)^3} < 0$ for $\lambda < 0$ and $x \in (-1, 0)$. Therefore, $g'_\lambda(x)$ is decreasing in $(-1, 0)$ and $\lim\limits_{x\to -1^+} g'_\lambda(x) = \infty$. Since $f'(0) = 0$, $g'_\lambda(0) = -1$ and $g'_\lambda(x)$ is continuous and strictly decreasing in $(-1, 0)$, there exists a point $x_\lambda \in (-1, 0)$ such that, for $\lambda < 0$

$$g'_\lambda(x) \begin{cases} > 0 & \text{for } x \in (-1, x_\lambda), \\ = 0 & \text{for } x = x_\lambda, \\ < 0 & \text{for } x \in (x_\lambda, 0). \end{cases} \tag{2.2}$$

Thus $g_\lambda(x)$ for $\lambda < 0$, increases strictly in $(-1, x_\lambda)$ and decreases strictly in $(x_\lambda, 0)$. Thus g_λ attains its maximum at x_λ. Thus $g'_\lambda(x_\lambda) = 0$. This implies $f'_\lambda(x_\lambda) - 1 = 0$. Then by some elementary calculations we get that $\lambda = \frac{1}{f'(x_\lambda)}$. Also we can see that $\lim\limits_{x\to -1^+} g_\lambda(x) = -\infty$.

The Julia set of f_λ undergoes a change as λ passes through the value $\widehat{\lambda}$. When $\lambda = \widehat{\lambda}$ the graph of f_λ is tangent to the diagonal line at $x = \widehat{x}$, so that the two fixed points $\widehat{a_\lambda}$ and $\widehat{r_\lambda}$ coincide to become one neutral fixed point (see Figure 2). For $\widehat{\lambda} < \lambda < 0$ then f_λ has two negative real fixed points. For $\lambda < \widehat{\lambda}$ the fixed points disappear from the real line. This phenomenon is famously known as saddle-node bifurcation. The following theorem describes the dynamics of f_λ as λ passes through $\widehat{\lambda}$.

Theorem 2.1. *Let $f_\lambda \in \mathcal{K}$, and $\lambda < 0$. Then, the following are true. (see Figure 2)*

1. *For $\widehat{\lambda} < \lambda < 0$, f_λ has two negative real fixed points $\widehat{a}_\lambda$ and $\widehat{r}_\lambda$ with $\widehat{a}_\lambda > \widehat{r}_\lambda$, where $\widehat{a}_\lambda$ is attracting and $\widehat{r}_\lambda$ is repelling. Further, $\lim\limits_{n\to\infty} f^n_\lambda(x) = \widehat{a}_\lambda$ for $\widehat{r}_\lambda < x < 0$.*

2. *For $\lambda = \widehat{\lambda}$, f_λ has only one negative real fixed point $x = \widehat{x}$ and $x = \widehat{x}$ is rationally indifferent. Further, $\lim\limits_{n\to\infty} f^n_\lambda(x) = \widehat{x}$ for $\widehat{x} \le x < 0$.*

3. *For $\lambda < \widehat{\lambda}$, f_λ has no real fixed point.*

Proof. 1. *If $\widehat{\lambda} < \lambda < 0$, then $\frac{1}{f'(\widehat{x})} < \frac{1}{f'(x_\lambda)} < 0$. Since f' is increasing in $(-1, \infty)$, we have $x_\lambda < \widehat{x}$. So $\phi(x_\lambda) < 0$ by Equation 2.1. Now $\phi(x_\lambda) = f(x_\lambda) - x_\lambda f'(x_\lambda) < 0$. That is, $\frac{f(x_\lambda)}{f'(x_\lambda)} - x_\lambda > 0$ as $f'(x_\lambda) < 0$. This implies that*

$g_\lambda(x_\lambda) = f_\lambda(x_\lambda) - x_\lambda = \frac{\lambda e^{x_\lambda}}{1+x_\lambda} - x_\lambda = \lambda f(x_\lambda) - x_\lambda = \frac{f(x_\lambda)}{f'(x_\lambda)} - x_\lambda > 0$. But $g_\lambda(0) = \lambda < 0$ and by Equation 2.2, there exists two real numbers $-1 < \widehat{r}_\lambda < x_\lambda < \widehat{a}_\lambda < 0$ such that $g_\lambda(\widehat{a}_\lambda) = g_\lambda(\widehat{r}_\lambda) = 0$. Thus f_λ has exactly two fixed points $\widehat{a}_\lambda$ and $\widehat{r}_\lambda$ in the interval $(-1, 0)$. Again as f' is increasing in $(-1, 0)$ and $\lambda < 0$, we get $f'_\lambda(\widehat{a}_\lambda) < f'_\lambda(x_\lambda) = 1 < f'_\lambda(\widehat{r}_\lambda)$. So $\widehat{a}_\lambda$ is the attracting fixed point and $\widehat{r}_\lambda$ is the repelling fixed point of f_λ for $\lambda < 0$. Again $f_\lambda(x) < x$ for $\widehat{a}_\lambda \leq x < 0$ and $f_\lambda(x) > x$ for $\widehat{r}_\lambda < x < \widehat{a}_\lambda$. Since $f_\lambda(x)$ is increasing in $(-1, 0)$, the sequence $\{f^n_\lambda(x)\}_{n\geq 0}$ is decreasing and bounded below by $\widehat{a}_\lambda$ for $\widehat{a}_\lambda < x \leq 0$. Also $\{f^n_\lambda(x)\}_{n\geq 0}$ is increasing and bounded above by $\widehat{a}_\lambda$ for $\widehat{r}_\lambda < x < \widehat{a}_\lambda$. Hence by the monotone convergence theorem $\lim_{n\to\infty} f^n_\lambda(x) = \widehat{a}_\lambda$ for $\widehat{r}_\lambda < x \leq 0$.

2. *If $\lambda = \widehat{\lambda}$, then $f'(x_\lambda) = f'(\widehat{x})$ and injectivity of f' implies that $x_\lambda = \widehat{x}$. We can see that $g_{\widehat{\lambda}}(\widehat{x}) = f_{\widehat{\lambda}}(\widehat{x}) - \widehat{x} = \frac{\widehat{\lambda} e^{\widehat{x}}}{1+\widehat{x}} - \widehat{x} = \widehat{\lambda} f(\widehat{x}) - \widehat{x} = \frac{f(\widehat{x})}{f'(\widehat{x})} - \widehat{x}$. Now $\phi(\widehat{x}) = f(\widehat{x}) - \widehat{x} f'(\widehat{x}) = 0$. Thus $f(\widehat{x}) = \widehat{x} f'(\widehat{x})$. Therefore, $g_{\widehat{\lambda}}(\widehat{x}) = 0$. Hence $f_{\widehat{\lambda}}(x)$ has only one fixed point in the negative real axis and it is rationally indifferent. The sequence $\{f^n_{\widehat{\lambda}}(x)\}_{n\geq 0}$ is decreasing and bounded below by $\widehat{x}$ for $\widehat{x} < x \leq 0$. By monotone convergence theorem, we have $\lim_{n\to\infty} f^n_{\widehat{\lambda}}(x) = \widehat{x}$ for $\widehat{x} \leq x < 0$.*

3. *If $\lambda < \widehat{\lambda} < 0$, then $\frac{1}{f'(x_\lambda)} < \frac{1}{f'(\widehat{x})}$ and it follows that $\widehat{x} < x_\lambda$ since f' is increasing in $(-1, \infty)$. So, by Equation 2.1, $\phi(x_\lambda) = f(x_\lambda) - x_\lambda f'(x_\lambda) > 0$. Since $f'(x_\lambda) < 0$ thus $\frac{f(x_\lambda)}{f'(x_\lambda)} - x_\lambda < 0$. Now $g_\lambda(x_\lambda) = f_\lambda(x_\lambda) - x_\lambda = \lambda f(x_\lambda) - x_\lambda = \frac{f(x_\lambda)}{f'(x_\lambda)} - x_\lambda < 0$. Since the maximum value of g_λ in $(-1, 0)$ is less than zero thus $g_\lambda(x)$ is negative for all $x \in (-1, 0)$. This proves that f_λ has no real fixed point.*

□

2.1. The Fatou set of f_λ when $\lambda < 0$

It is known that functions which have finite number of singular values do not have wandering domain nor Baker domain [3]. So f_λ when $\lambda < 0$, has no wandering domain nor Baker domain. In particular, f_λ has no Baker wandering domain. Each function in the class $\mathcal{K}$ is with only one pole and that is why it can not have a Herman ring as well [11].

Theorem 2.2. *Let $f_\lambda \in \mathcal{K}$ and $\lambda < 0$. Then, the dynamics of f_λ is as follows.*

1. *If $\widehat{\lambda} < \lambda < 0$, then the Fatou set $\mathcal{F}(f_\lambda)$ is an invariant attracting basin of a real negative fixed point $\widehat{a}_\lambda$.*
2. *If $\lambda = \widehat{\lambda}$, then the Fatou set $\mathcal{F}(f_\lambda)$ is an invariant parabolic basin corresponding to a real rationally indifferent fixed point $\widehat{x}$.*
3. *If $\lambda < \widehat{\lambda}$, then the Fatou set $\mathcal{F}(f_\lambda)$ does not contain any invariant attracting or parabolic basin and hence show that $\mathcal{F}(f_\lambda) = \phi$.*

Proof. Here, $sing(f_\lambda^{-1}) = \{\lambda, 0, \infty\} \subseteq \mathbb{R} \cup \{\infty\}$. Thus $\overline{O^+(sing f_\lambda^{-1})} \subseteq \mathbb{R} \cup \{\infty\}$. We know that if f_λ has Herman ring or Siegel disc then $U_j \subseteq \overline{O^+(sing f_\lambda^{-1})}$ where U_j is the boundary of Herman ring or Siegel disc. But $U_j \subseteq \overline{O^+(sing f_\lambda^{-1})} \subseteq \mathbb{R} \cup \{\infty\}$ is not possible by [14]. Therefore f_λ has no Herman ring or Siegel disc. Since f_λ has finitely many singular values so the function f_λ has neither wandering domain nor a Baker domain. So any periodic Fatou components corresponds to a real non-repelling periodic point.

If $\widehat{\lambda} < \lambda < 0$, then f_λ has only one real negative fixed point $\widehat{a}_\lambda$. Then the Fatou set $\mathcal{F}(f_\lambda)$ is an invariant attracting basin of the fixed point $\widehat{a}_\lambda$.

If $\lambda = \widehat{\lambda}$, then f_λ has a real rationally indifferent fixed point $\widehat{x}$ and there corresponds the Fatou set $\mathcal{F}(f_\lambda)$ which is an invariant parabolic basin.

If $\lambda < \widehat{\lambda}$, then f_λ has no real fixed point. In this case $\mathcal{F}(f_\lambda)$ does not contain any invariant attracting or parabolic basin. Here $\mathcal{F}(f_\lambda) = \phi$.

□

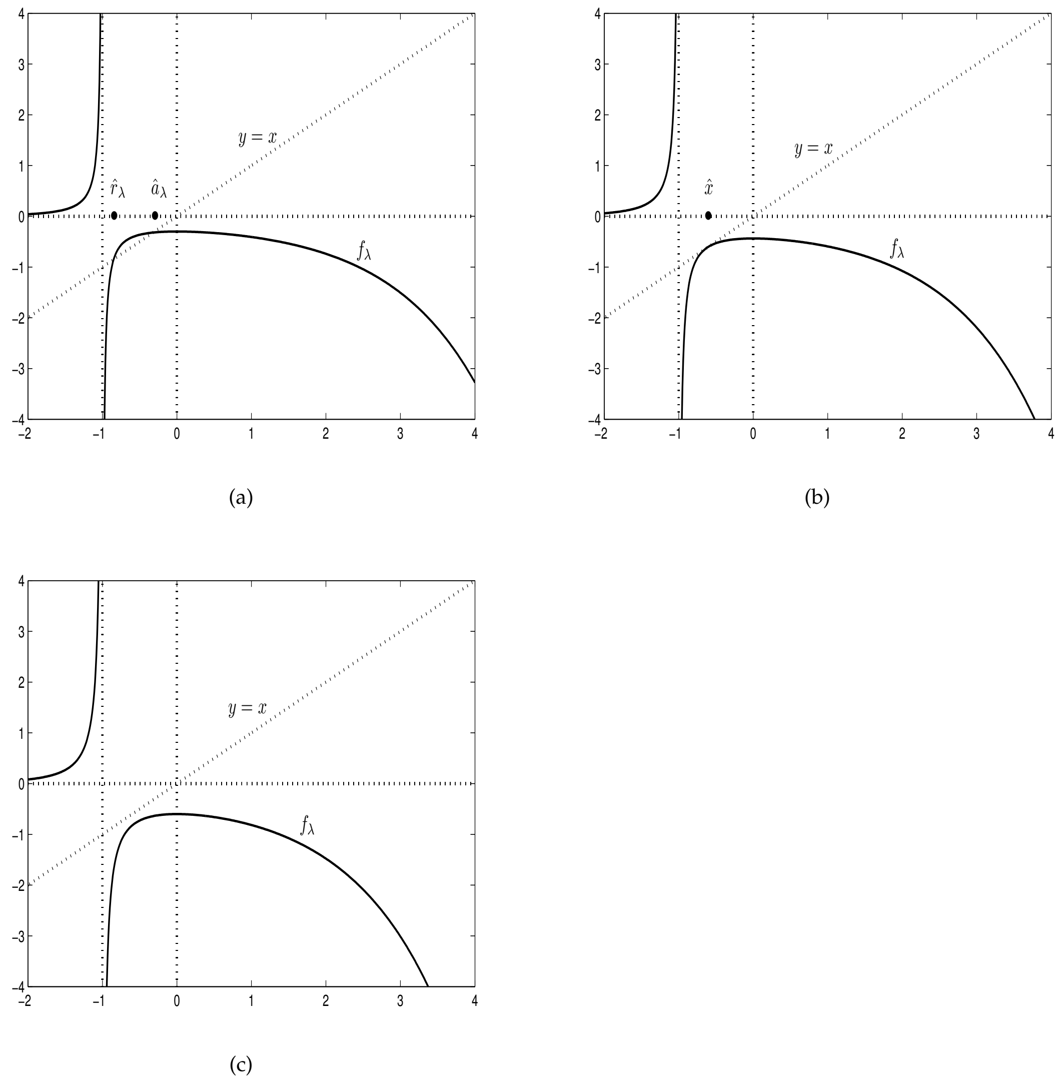
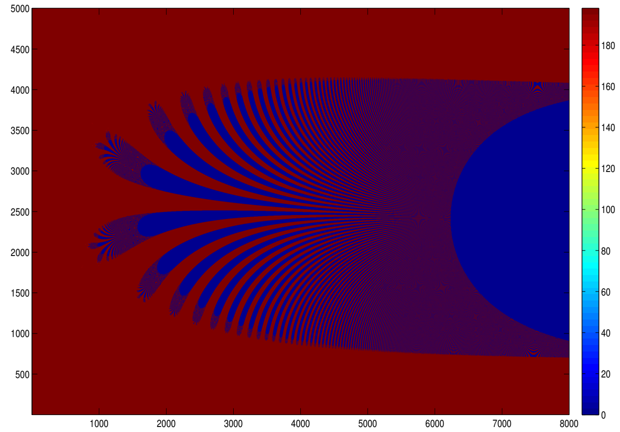
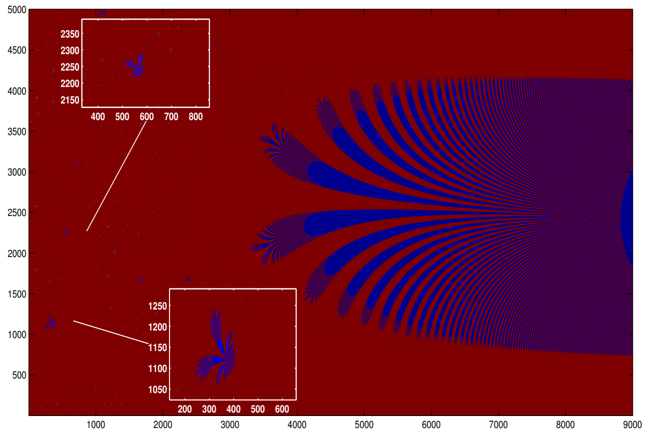


Figure 2: Graphs of $f_\lambda(x) = \lambda \frac{e^x}{1+x}$, $\lambda < 0$ for (a) $\lambda > \widehat{\lambda}$ (b) $\lambda = \widehat{\lambda}$ (c) $\lambda < \widehat{\lambda}$



(a)



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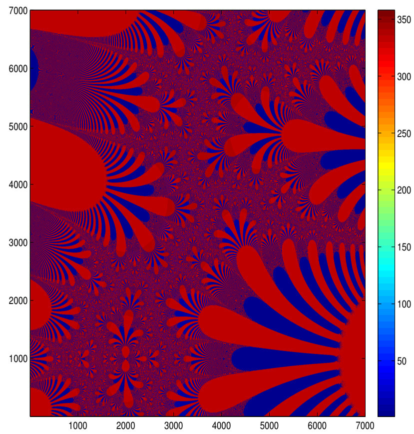

(c)

Figure 3: Dynamical Plane of $f_\lambda(z)$, $\lambda < 0$ for (a) $\lambda = -0.3$ (b) $\lambda = \widehat{\lambda} = -0.437$ (c) $\lambda = -1$

2.2. *The Julia set of f_λ when $\lambda < 0$*

Since the Julia set is the complement of the Fatou set, thus one can easily find out the Julia set of f_λ when $\lambda < 0$ by using Theorem 2.2. When $\widehat{\lambda} < \lambda < 0$, then the Julia set of f_λ will be the complement of invariant attracting basin of the negative attracting real fixed point $\widehat{a}_\lambda$. When $\lambda = \widehat{\lambda}$, then the Julia set of f_λ will be the complement of the parabolic basin corresponding to the rationally indifferent real fixed point $\widehat{x}$. When $\lambda < \widehat{\lambda}$, then $\mathcal{J}(f_\lambda) = \widehat{\mathbb{C}}$.

The Julia sets of f_λ for different negative values of λ are generated in the rectangular domain $R(z) = \{z \in \mathbb{C} : -3 \leq \Re(z) \leq 8, -5 \leq \Im(z) \leq 5\}$, where 500 iterations of the functions are considered. The red region and blue region in the Figure 3 are approximations to the Fatou set and the Julia set. We use Matlab as a tool to draw pictures here.

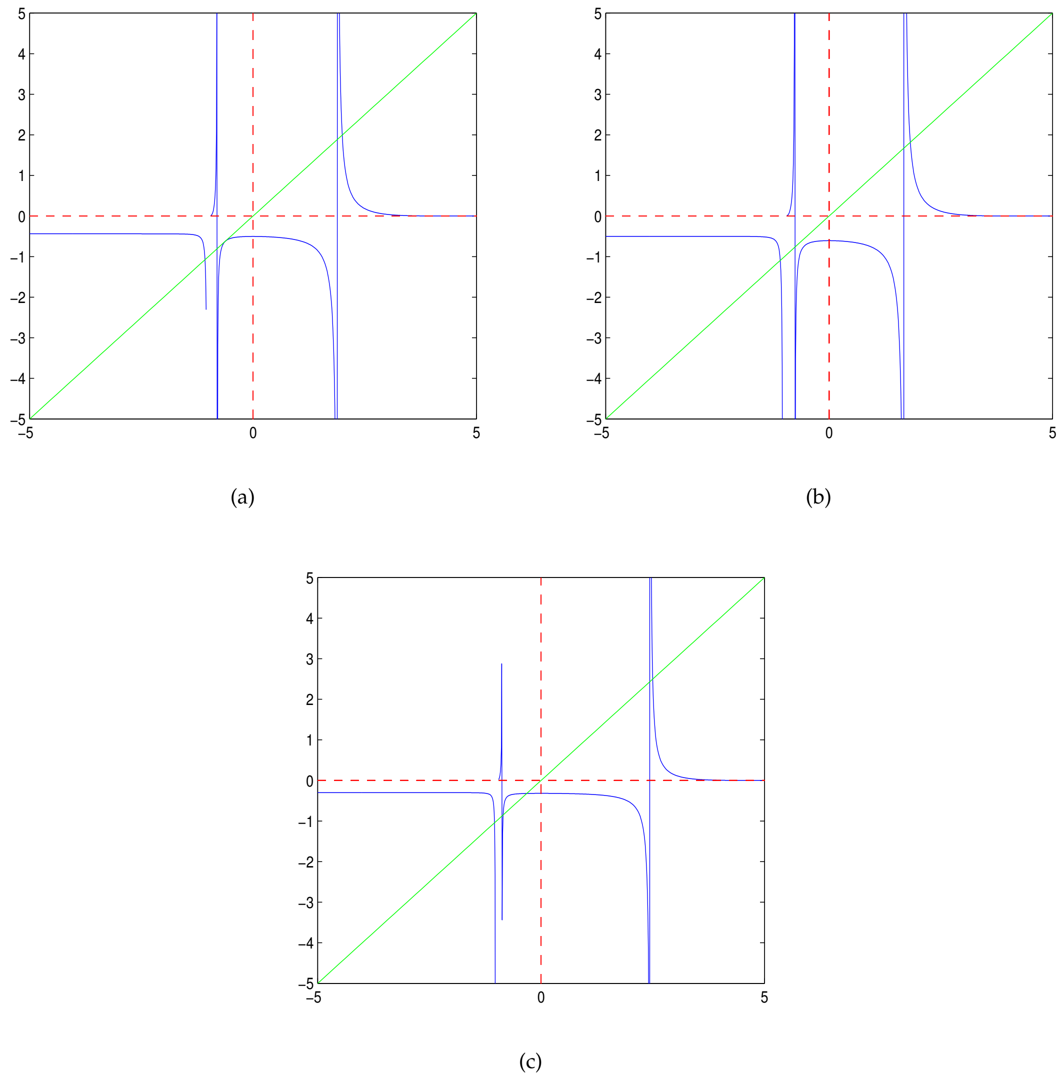

Figure 4: Graphs of $f_\lambda^2(x)$, for $(a)\lambda = \widehat{\lambda}(= -0.4379)$ (b) $\lambda(= -0.5) < \widehat{\lambda}$ (c) $\lambda(= -0.3) > \widehat{\lambda}$

2.3. *Dynamics of the map* ${f_\lambda}^n$ *for* $n \geq 2$

We know that $\mathcal{F}(f_\lambda) = \mathcal{F}({f_\lambda}^n)$ and $\mathcal{J}(f_\lambda) = \mathcal{J}({f_\lambda}^n)$ for all $n \geq 2$. Since the Julia and the Fatou set of f_λ is already known to us thus we can find out the Julia and the Fatou set of ${f_\lambda}^n$ for any $n \geq 2$. Since 0 is a finite singular value of f_λ and $f_\lambda(0) = \lambda$ thus f_λ have a single forward orbit and the orbit tends either to the attracting fixed point $\widehat{a}_\lambda$ or to the parabolic fixed point $\widehat{r}_\lambda$. No singular orbits accumulate the real periodic points of period greater than or equal to two. When $\widehat{\lambda} < \lambda < 0$, then ${f_\lambda}^n$ $(n \geq 2)$ has two real fixed points namely $\widehat{a}_\lambda$ and $\widehat{r}_\lambda$ where $\widehat{a}_\lambda$ is attracting and $\widehat{r}_\lambda$ is repelling fixed point. Since f_λ and hence ${f_\lambda}^n$ has only one singular orbit, thus ${f_\lambda}^n$ $(n \geq 2)$ does not possess any other attracting periodic point. In this case, all the other perodic points are repelling. When $\lambda = \widehat{\lambda}$, then ${f_\lambda}^n$ $(n \geq 2)$ has a real rationally indifferent fixed point $\widehat{x}$. Here, ${f_\lambda}^n$ $(n \geq 2)$ does not possess any other rationally indifferent fixed point due to the existence of single singular orbit. Here also other periodic points are repelling. When $\lambda < \widehat{\lambda}$, then for $n \geq 2$, $\mathcal{J}({f_\lambda}^n) = \widehat{\mathbb{C}}$. For $n = 2$, one can see the Figure 4.

Now we are giving Table 1 of comparison of dynamics between four classes of functions.

Comparison of dynamics			
Dynamics of $\lambda\frac{e^z}{z+1}, \lambda < 0$	**Dynamics of** $\lambda\frac{e^z}{z+1}, \lambda > 0$	**Dynamics of** $\lambda e^z, \lambda > 0$	**Dynamics of** $\lambda(e^z + 1 + \frac{1}{e^z+1}), \lambda > 0$
1. Meromorphic with one pole at -1	1. Meromorphic with one pole at -1	1. Entire	1. Meromorphic with poles at $z_k = i\pi(2k+1)$, $k \in \mathbb{Z}$
2. Critical value is λ, $\lambda < 0$	2. Critical value is λ, $\lambda > 0$	2. No critical value	2. Critical value is -2λ
3. Not periodic	3. Not periodic	3. Periodic	3. Periodic
4. The asymptotic values are 0 and ∞	4. The asymptotic values are 0 and ∞	4. The asymptotic values are 0 and ∞	4. The asymptotic values are 2λ and ∞
5. The number of singularities each over 0 and ∞ is one and those are logarithmic	5. The number of singularities each over 0 and ∞ is one and those are logarithmic	5. The number of singularities each over 0 and ∞ is one and those are logarithmic	5. At least one direct singularity over ∞ and one logarithmic singularity over 2λ
6. The Julia set is $\widehat{\mathbb{C}}$ for $\lambda < \widehat{\lambda} \approx -0.44$	6. The Julia set is $\widehat{\mathbb{C}}$ for $\lambda > \lambda^* \approx 0.84$	6. The Julia set is $\widehat{\mathbb{C}}$ for $\lambda > \frac{1}{e}$	6. The Julia set is $\widehat{\mathbb{C}}$ for $\lambda > \lambda^* \approx 0.26$
7. The Fatou set is an invariant attracting basin when $\widehat{\lambda} < \lambda < 0$	7. The Fatou set is an invariant attracting basin when $0 < \lambda < \lambda^*$	7. The Fatou set is the complement of nowhere dense subset of the right half plane when $0 < \lambda < \frac{1}{e}$	7. The Fatou set is non empty and the Julia set is disconnected when $0 < \lambda < \lambda^*$
8. The Fatou set is a parabolic basin when $\lambda = \widehat{\lambda}$	8. The Fatou set is a parabolic basin when $\lambda = \lambda^*$	8. The Fatou set is the complement of nowhere dense subset of the right half plane when $0 < \lambda < \frac{1}{e}$	8. The Fatou set is non empty and the Julia set is disconnected when $0 < \lambda < \lambda^*$
9. Observed saddle-node bifurcation at $\lambda = \widehat{\lambda}$	9. Observed saddle-node bifurcation at $\lambda = \widehat{\lambda}$	9. Observed chaotic burst at $\lambda = \frac{1}{e}$	9. Observed chaotic burst at $\lambda^* \approx 0.26$

Table 1: Comparison table of dynamics between $\lambda\frac{e^z}{z+1}$ $(\lambda < 0)$, $\lambda\frac{e^z}{z+1}$ $(\lambda > 0)$, $\lambda e^z(\lambda > 0)$ and $\lambda(e^z + 1 + \frac{1}{e^z+1})$ $(\lambda > 0)$

3. Dynamics of the map $F_{\lambda,a,m}(z) = \lambda(z + a)^m e^z$ where $\lambda, a \in \mathbb{C}$ and $m \in \mathbb{N}$

The map $F_{\lambda,a,m}(z) = \lambda(z + a)^m e^z$, where $\lambda, a \in \mathbb{C}$ and $m \in \mathbb{N}$ is an entire function with a zero of order m at $z = -a$. The case when $a = m = 0$, the dynamics of the map is studied by Devaney and Durkin in [8]. The critical values of the map are $\{w : w = F_{\lambda,a,m}(z)$ such that $F'_{\lambda,a,m}(z) = 0\}$. The critical points of $F_{\lambda,a,m}$ are $\{z \in \mathbb{C} : z = -a$ or $z = -(a + m),\ m \geq 1\}$. When $a = 0$, the function has a supper attracting fixed point at $z = 0$. Let $\mathcal{M} = \{F_{\lambda,a,m}(z) = \lambda(z + a)^m e^z : \lambda, a \in \mathbb{C},\ m \in \mathbb{N}\}$.

Lemma 3.1. *For each $m \geq 1$, the order of the entire function $F_{\lambda,a,m}(z) = \lambda(z + a)^m e^z$ is one.*

Proof. We know that the order $\mu = \limsup_{x\to\infty} \frac{\log\log M(r)}{\log r}$, where $M(r) = \max_{|z|=r} |F_{\lambda,a,m}(z)|$. We can assume $a = 0$ and $\lambda = 1$. So, $M(r) = \max_{|z|=r} |z^m e^z|$. That is, $M(r) = \max_{\theta} |r^m e^{im\theta} e^{re^{i\theta}}| = \max_{\theta}\{r^m |e^{re^{i\theta}}|\} = \max_{\theta}\{r^m e^{r\cos\theta}\} = r^m e^r$. Now, $\mu = \limsup_{r\to\infty} \frac{\log\log\{r^m e^r\}}{\log r} = \limsup_{r\to\infty} \frac{\log(m\log r + r)}{\log r} \approx \limsup_{r\to\infty} \frac{\log(m\log r)}{\log r}$. It follows that the order of $F_{\lambda,a,m}$ is one. □

Lemma 3.2. *Each function in the class $\mathcal{M}$ has a finite number of asymptotic values. Thus functions in $\mathcal{M}$ can have at most finitely many singular values.*

Proof. Any function in the class has a finite number of critical values. By Denjoy-Carleman-Ahlfors Theorem [5] the inverse function of a meromorphic function of finite order ρ can have at most 2ρ direct singularities. Further, each direct singularity corresponds to an asymptotic value. From Lemma 3.1 and using a result of Bergweiler & Eremenko [6] (See Corollary 3) the number of asymptotic values is at most 2. It follows that the number of singular values of each function of the class $\mathcal{M}$ is finite. □

Let S denote the set of singular values of f. The map $f : \mathbb{C} \setminus f^{-1}(S) \to \widehat{\mathbb{C}} \setminus S$ is an unbranched covering. Let $a \in \mathbb{R}$, then all the functions have a real zero of order $m \geq 1$ at $-a$. If a is purely imaginary, then any sufficiently small neighborhood U of 0, $F^{-1}_{\lambda,a,m}(U)$ has two components H_l and B. The component H_l contains $\{z \in \mathbb{C} : \Re z < -l$ for some real number $l > 0\}$ and B is a small enough neighborhood of $-a$. A point $a \in \mathbb{C}$ is locally omitted by f if $\exists\, r > 0$ and a component G of the set $f^{-1}(B_r(a))$ such that $f(z) \neq a$ in G. It follows that 0 is a locally omitted value of the function for the component H_l.

The following result is proved in [5].

Proposition 3.1. *Let f be an entire function of finite order, and let $a \in \mathbb{C}$ be either a critical value or a locally omitted value. If D is a simply connected region that does not contain a, then $f^{-1}(D)$ is disconnected.*

The following result immediately follows from the above proposition. Notice that the point 0 is a locally omitted value as well as a critical value whenever $m > 1$.

Theorem 3.2. *Let $m > 1$ and $D \subset \mathbb{C}$ be any simply connected region and D does not intersect any critical value of $F_{\lambda,a,m}$. Then $F^{-1}_{\lambda,a,m}(D)$ is disconnected.*

We get the following result from [5].

Theorem 3.3. *Let f be an entire function of finite order, and $a \in \mathbb{C}$ be a locally omitted value. Then a is the projection of a logarithmic singularity of f^{-1}.*

Since the map $F_{\lambda,a,m}(z) = \lambda(z + a)^m e^z$, $m \in \mathbb{N}$ has a direct singularity over infinity, the escaping set $I(F_{\lambda,a,m}(z))$ has an unbounded component by Remark 1.2. Moreover, $I(F_{\lambda,a,m}(z)) \cap J(F_{\lambda,a,m}(z))$ contains a continua. Since the functions have only finitely many singular values, there is no wandering domain.

4. Future prospects

In the line of the works as carried out in this paper, one may think to construct different classes of families of functions and try to investigate their dynamics. This may be an active area of research to the future workers of this branch.

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References

[1] I. N. Baker, J. Kotus, L. Yinian, *Iterates of meromorphic functions. III: Preperiodic domains*, Ergodic Theory Dynam. Systems 11 (1991) no. 4 603–618.
[2] W. Bergweiler, *Dynamics of meromorphic functions with direct and logarithmic singularities*, Proc. London Math. Soc. (3) 97 (2008) 368–400.
[3] W. Bergweiler, *Iteration of meromorphic functions*, Bull. of Amer. Math. Soc. 29 (1993) no. 2 151–188.
[4] W. Bergweiler, *Singularities in Baker domains*, Computational Methods and Function Theory 1 (1) (2001) 41–49.
[5] W. Bergweiler, A. Eremenko, *Direct singularities and completely invariant domains of entire functions*, Illinois J. Math. 52 (2008) no. 1 243–259.
[6] W. Bergweiler, A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoamericana 11 (1995) no. 2 355–373.
[7] T. Chakra, T. Nayak, K. Senapati, *Iteration of certain Exponential-like Meromorphic Functions*, Proc. Indian Acad. Sci. (Math. Sci.) (2018).
[8] R. L. Devaney, M. B. Durkin, *The exploding exponential and other chaotic bursts in complex dynamics*, Amer. Math. Monthly 98 (1991) 217–233.
[9] M. Förster, L. Rempe, D. Schleicher, *Classification of escaping exponential maps*, Proc. Amer. Math. Soc. 136 (2008) 651–663.
[10] F. Mingliang, *Baker domains and singularities for certain meromorphic functions*, Indian J. Pure Appl. Math. 32 (1)(2001) 91–98.
[11] T. Nayak, *Herman rings of meromorphic maps with an omitted value*, Proceedings of the Amer. Math. Soc. 144 (2) (2016) 587–597.
[12] T. Nayak, M. G. P. Prasad, *Joukowski-exponential maps*, Complex Anal. Oper. Theory 8 (2014) no-5 1061–1076.
[13] T. Nayak, J. H. Zheng, *Omitted values and dynamics of transcendental meromorphic functions*, J. London Math. Soc. 83 (1) (2011) 121–136.
[14] M. G. P. Prasad, T. Nayak, *Dynamics of* $\lambda \tanh e^z$, Discrete and Continuous Dynam. Systems- Series A 19 (2007) (1) 121–138.
[15] L. Rempe, *The escaping set of the exponential*, Ergodic Theory and Dynam. Systems 30 (2010) 595–599.
[16] D. Schleicher, J. Zimmer, *Escaping points of exponential maps*, J. London Math. Soc. 2 (67) (2003) 380–400.