Neutrosophic Soft Semi-Regularization and Neutrosophic Soft Sub-Maximality

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Abstract. In this study, our target point is to focus on neutrosophic soft semi-regularization spaces connected with neutrosophic soft topological spaces and examine their properties. First, we define the neutrosophic soft sub-maximal space and present the evidences for the existence of a neutrosophic soft sub-maximal space for every neutrosophic soft topological space. In this document, we focused on the relationship of neutrosophic soft sub-maximal spaces and neutrosophic soft semi-regular spaces with these spaces. Also, we find that this relationship is very close and it is minimal or maximal depending on some definite properties which are called neutrosophic soft semi-regular properties. This led us to examine the semi-regularity of different properties. After all, we introduced some types of functions in neutrosophic soft topological spaces that correspond with some types of functions previously defined in many topological spaces of different types and revealed the behaviours of these functions according to the cases where their domain or codomain spaces are replaced by their semi-regularization spaces.

1. Introduction

The fact that a topological space had a distinctive semi-regular space that was coarser than itself attracted the attention of many scientists and was the focus of their studies as in [14, 16, 22]. In [14], the concept of sub-maximal space was presented to the world of mathematics for the first time as a new study topic for scientists. In [16], focusing on this new concept, Cameron studied in detail on its properties. As the needs of people in daily life changed and technology advanced, some studies on general topology remained inadequate. For this reason, it has become inevitable for scientists to re-examine some of the issues that are the cornerstones of General Mathematics, as in [15, 19], in accordance with the new theories put forward. Again, new types of theory were needed to keep up with the technology that continued to evolve. Thereupon, in 1999, Molodstov [21] presented the concept of soft set to the scientific world as a new tool to overcome this problem. Immediately afterwards, in 2005, Smarandache [23] appeared in the scientific world with the concept of neutrosophic set. Many scientists from around the world have evaluated these new theories to create new fields of study and have used them extensively in their studies as in [1, 5–7, 9, 13, 18, 24]. Maji [20], who was among the scientists who evaluated these new theories well, came up with the concept of neutrosophic soft set. This new type of set inspired Bera to describe the neutrosophic soft topological space in [12]. In [8], Aras, Ozturk and Bayramov reinterpreted this set

concept in their own way and presented their own definition of neutrosophic soft topological space. This new type of topological space has provided an opportunity for mathematicians to come up with new ideas. In particular , Acikgoz and Esenbel made good use of this new opportunity in their studies as in [3, 4]. The main purpose of this study is to interpret these new ideas in neutrosophic soft topological spaces. In Section 2, the concept of semi-regularization space, one of the cornerstones of our study, is introduced and its characterizations are examined in detail. The concepts of neutrosophic soft ro-equivalent and neutrosophic soft sub-maximal space, whose definitions are given in this section, are also among the cornerstones of our study. Ultimately, the existence of a neutrosophic soft sub-maximal space as an expansion of any given neutrosophic soft topological space is established. It is seen that some certain properties of a neutrosophic soft topological space are also shared by its semi-regular space and vice-versa. In this paper, such a property is called neutrosophic soft semi-regular property. Ultimately, it is concluded that every neutrosophic soft maximal P (minimal P) topological space is neutrosophic soft sub-maximal (resp. semi-regular) if P has a neutrosophic soft semi-regular property. This makes possible that certain neutrosophic soft topological spaces are isolated and recognised to be neutrosophic soft semi-regular or sub-maximal. Inherently, the above results require a wide research to determine which properties are neutrosophic soft semi-regular. In Section 3, such a study is taken up and certain properties are examined to include them into the class of neutrosophic soft semi-regular properties. Obviously, there exist many different properties which we can test along this line. Eventually, in Section 4, we adapt some certain known mappings on neutrosophic soft topological spaces and effort to figure out the behaviour of some types of mappings between two neutrosophic soft topological spaces in cases that their domain and/or co-domain are replaced with their neutrosophic soft semi-regularization spaces. Throughout the paper, without any explanation, we use the symbols and definitions introduced in [8, 10, 12, 17, 20, 21, 23].

2. Preliminaries

In this section, we present the basic definitions and theorems related to neutrosophic soft set theory.

Definition 2.1. ([23]) *A neutrosophic set A on the universe set X is defined as:*

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\},$$

where $T, I, F : X \to]^-0, 1^+[$ and $^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$.

Scientifically, membership functions, indeterminacy functions and non-membership functions of a neutrosophic set take value from real standart or nonstandart subsets of $]^-0, 1^+[$. However, these subsets are sometimes inconvenient to be used in real life applications such as economical and engineering problems. On account of this fact, we consider the neutrosophic sets, whose membership function, indeterminacy functions and non-membership functions take values from subsets of $[0, 1]$.

Definition 2.2. ([21]) *Let X be an initial universe, E be a set of all parameters and $P(X)$ denote the power set of X. A pair (F, E) is called a soft set over X, where F is a mapping given by $F : E \longrightarrow P(X)$. In other words, the soft set is a parameterized family of subsets of the set X. For $e \in E$, $F(e)$ may be considered as the set of e-elements of the soft set (F, E) or as the set of e-approximate elements of the soft set, i.e. $(F, E) = \{(e, F(e)) : e \in E, F : E \longrightarrow P(X)\}$.*

After the neutrosophic soft set was defined by Maji [20], this concept was modified by Deli and Broumi [17] as given below:

Definition 2.3. ([17]) *Let X be an initial universe set and E be a set of parameters. Let $NS(X)$ denote the set of all neutrosophic sets of X. Then, a neutrosophic soft set $\left(\widetilde{F}, E\right)$ over X is a set defined by a set valued function $\widetilde{F}$ representing a mapping $\widetilde{F} : E \to NS(X)$, where $\widetilde{F}$ is called the approximate function of the*

neutrosophic soft set $\left(\widetilde{F}, E\right)$. In other words, the neutrosophic soft set is a parametrized family of some elements of the set $NS(X)$ and therefore it can be written as a set of ordered pairs:

$$\left(\widetilde{F}, E\right) = \left\{\left(e, \left\langle x, T_{\widetilde{F}(e)}(x), I_{\widetilde{F}(e)}(x), F_{\widetilde{F}(e)}(x)\right\rangle : x \in X\right) : e \in E\right\}$$

where $T_{\widetilde{F}(e)}(x), I_{\widetilde{F}(e)}(x), F_{\widetilde{F}(e)}(x) \in [0, 1]$ are respectively called the truth-membership, indeterminacy-membership and falsity-membership function of $\widetilde{F}(e)$. Since the supremum of each T, I, F is 1, the inequality $0 \leq T_{\widetilde{F}(e)}(x) + I_{\widetilde{F}(e)}(x) + F_{\widetilde{F}(e)}(x) \leq 3$ is obvious.

Definition 2.4. ([12]) Let $\left(\widetilde{F}, E\right)$ be a neutrosophic soft set over the universe set X. The complement of $\left(\widetilde{F}, E\right)$ is denoted by $\left(\widetilde{F}, E\right)^c$ and is defined by

$$\left(\widetilde{F}, E\right)^c = \left\{\left(e, \left\langle x, F_{\widetilde{F}(e)}(x), 1 - I_{\widetilde{F}(e)}(x), T_{\widetilde{F}(e)}(x)\right\rangle : x \in X\right) : e \in E\right\}.$$

It is obvious that $\left[\left(\widetilde{F}, E\right)^c\right]^c = \left(\widetilde{F}, E\right)$.

Definition 2.5. ([20]) Let $\left(\widetilde{F}, E\right)$ and $\left(\widetilde{G}, E\right)$ be two neutrosophic soft sets over the universe set X. $\left(\widetilde{F}, E\right)$ is said to be a neutrosophic soft subset of $\left(\widetilde{G}, E\right)$ if

$$T_{\widetilde{F}(e)}(x) \leq T_{\widetilde{G}(e)}(x), I_{\widetilde{F}(e)}(x) \leq I_{\widetilde{G}(e)}(x), F_{\widetilde{F}(e)}(x) \geq F_{\widetilde{G}(e)}(x), \forall e \in E, \forall x \in X.$$

It is denoted by $\left(\widetilde{F}, E\right) \subseteq \left(\widetilde{G}, E\right)$. $\left(\widetilde{F}, E\right)$ is said to be neutrosophic soft equal to $\left(\widetilde{G}, E\right)$ if $\left(\widetilde{F}, E\right) \subseteq \left(\widetilde{G}, E\right)$ and $\left(\widetilde{G}, E\right) \subseteq \left(\widetilde{F}, E\right)$. It is denoted by $\left(\widetilde{F}, E\right) = \left(\widetilde{G}, E\right)$.

Definition 2.6. ([8]) Let $\left(\widetilde{F}_1, E\right)$ and $\left(\widetilde{F}_2, E\right)$ be two neutrosophic soft sets over the universe set X. Then, their union is denoted by $\left(\widetilde{F}_1, E\right) \cup \left(\widetilde{F}_2, E\right) = \left(\widetilde{F}_3, E\right)$ and is defined by $\left(\widetilde{F}_3, E\right) = \left\{\left(e, \left\langle x, T_{\widetilde{F}_3(e)}(x), I_{\widetilde{F}_3(e)}(x), F_{\widetilde{F}_3(e)}(x)\right\rangle : x \in X\right) e \in E\right\}$, where

$$T_{\widetilde{F}_3(e)}(x) = \max\left\{T_{\widetilde{F}_1(e)}(x), T_{\widetilde{F}_2(e)}(x)\right\},$$

$$I_{\widetilde{F}_3(e)}(x) = \max\left\{I_{\widetilde{F}_1(e)}(x), I_{\widetilde{F}_2(e)}(x)\right\},$$

$$F_{\widetilde{F}_3(e)}(x) = \min\left\{F_{\widetilde{F}_1(e)}(x), F_{\widetilde{F}_2(e)}(x)\right\}.$$

Definition 2.7. ([8]) Let $\left(\widetilde{F}_1, E\right)$ and $\left(\widetilde{F}_2, E\right)$ be two neutrosophic soft sets over the universe set X. Then, their intersection is denoted by $\left(\widetilde{F}_1, E\right) \cap \left(\widetilde{F}_2, E\right) = \left(\widetilde{F}_4, E\right)$ and is defined by

$$\left(\widetilde{F}_4, E\right) = \left\{\left(e, \left\langle x, T_{\widetilde{F}_4(e)}(x), I_{\widetilde{F}_4(e)}(x), F_{\widetilde{F}_4(e)}(x)\right\rangle : x \in X\right) e \in E\right\},$$

where

$$T_{\widetilde{F}_4(e)}(x) = \min\left\{T_{\widetilde{F}_1(e)}(x), T_{\widetilde{F}_2(e)}(x)\right\},$$

$$I_{\widetilde{F}_4(e)}(x) = \min\left\{I_{\widetilde{F}_1(e)}(x), I_{\widetilde{F}_2(e)}(x)\right\},$$

$$F_{\widetilde{F}_4(e)}(x) = \max\left\{F_{\widetilde{F}_1(e)}(x), F_{\widetilde{F}_2(e)}(x)\right\}.$$

Definition 2.8. ([8]) A neutrosophic soft set $\left(\widetilde{F}, E\right)$ over the universe set X is said to be a null neutrosophic soft set if $T_{\widetilde{F}(e)}(x) = 0, I_{\widetilde{F}(e)}(x) = 0, F_{\widetilde{F}(e)}(x) = 1; \forall e \in E, \forall x \in X$. It is denoted by $0_{(X,E)}$.

Definition 2.9. ([8]) A neutrosophic soft set $\left(\widetilde{F}, E\right)$ over the universe set X is said to be an absolute neutrosophic soft set if $T_{\widetilde{F}(e)}(x) = 1$, $I_{\widetilde{F}(e)}(x) = 1$, $F_{\widetilde{F}(e)}(x) = 0; \forall e \in E$, $\forall x \in X$. It is denoted by $1_{(X,E)}$. Clearly $0^c_{(X,E)} = 1_{(X,E)}$ and $1^c_{(X,E)} = 0_{(X,E)}$.

Definition 2.10. ([8]) Let $NSS(X, E)$ be the family of all neutrosophic soft sets over the universe set X and $\tau \subset NSS(X, E)$. Then, τ is said to be a neutrosophic soft topology on X, if
  1. $0_{(X,E)}$ and $1_{(X,E)}$ belong to τ,
  2. the union of any number of neutrosophic soft sets in τ belongs to τ,
  3. the intersection of a finite number of neutrosophic soft sets in τ belongs to τ.
  Then, (X, τ, E) is said to be a neutrosophic soft topological space over X. Each member of τ is said to be a neutrosophic soft open set [3].

Definition 2.11. ([8]) Let (X, τ, E) be a neutrosophic soft topological space over X and $\left(\widetilde{F}, E\right)$ be a neutrosophic soft set over X. Then $\left(\widetilde{F}, E\right)$ is said to be a neutrosophic soft closed set iff its complement is a neutrosophic soft open set.

Definition 2.12. ([8]) Let $NSS(X, E)$ be the family of all neutrosophic soft sets over the universe set X. Then, neutrosophic soft set $x^e_{(\alpha,\beta,\gamma)}$ is called a neutrosophic soft point for every $x \in X$, $0 < \alpha, \beta, \gamma \le 1, e \in E$ and is defined as

$$x^e_{(\alpha,\beta,\gamma)}(e')(y) = \begin{cases} (\alpha, \beta, \gamma), & \text{if } e' = e \text{ and } y = x \\ (0, 0, 1), & \text{if } e' \ne e \text{ or } y \ne x \end{cases}$$

It is clear that every neutrosophic soft set is the union of its neutrosophic soft points.

Definition 2.13. ([8]) Let $\left(\widetilde{F}, E\right)$ be a neutrosophic soft set over the universe set X. We say that $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F}, E\right)$ is read as belonging to the neutrosophic soft set $\left(\widetilde{F}, E\right)$, whenever

$$\alpha \le T_{\widetilde{F}(e)}(x), \beta \le I_{\widetilde{F}(e)}(x) \text{ and } \gamma \ge F_{\widetilde{F}(e)}(x).$$

Definition 2.14. ([8]) Let $x^e_{(\alpha,\beta,\gamma)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ be two neutrosophic soft points. For the neutrosophic soft points $x^e_{(\alpha,\beta,\gamma)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ over a common universe X, we say that the neutrosophic soft points are distinct points if $x^e_{(\alpha,\beta,\gamma)} \cap y^{e'}_{(\alpha',\beta',\gamma')} = 0_{(X,E)}$. It is clear that $x^e_{(\alpha,\beta,\gamma)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ are distinct neutrosophic soft points if and only if $x \ne y$ or $e \ne e'$.

Definition 2.15. ([10]) Let $\left(\widetilde{F}, E_1\right)$, $\left(\widetilde{G}, E_2\right)$ be two neutrosophic soft sets over the universal set X. Then, their cartesian product is another neutrosophic soft set $\left(\widetilde{K}, E_3\right) = \left(\widetilde{F}, E_1\right) \times \left(\widetilde{G}, E_2\right)$, where $E_3 = E_1 \times E_2$ and $\widetilde{K}(e_1, e_2) = \widetilde{F}(e_1) \times \widetilde{G}(e_2)$. The truth, indeterminacy and falsity membership of $\left(\widetilde{K}, E_3\right)$ are given by $\forall e_1 \in E_1, \forall e_2 \in E_2, \forall x \in X$,

$$\begin{aligned} T_{\widetilde{K}(e_1,e_2)}(x) &= \min\left\{T_{\widetilde{F}(e_1)}(x), T_{\widetilde{G}(e_2)}(x)\right\}, \\ I_{\widetilde{K}(e_1,e_2)}(x) &= I_{\widetilde{F}(e_1)}(x) \cdot I_{\widetilde{G}(e_2)}(x), \\ F_{\widetilde{K}(e_1,e_2)}(x) &= \max\left\{F_{\widetilde{F}(e_1)}(x), F_{\widetilde{G}(e_2)}(x)\right\} \end{aligned}$$

This definition can be extended for more than two neutrosophic soft sets.

Definition 2.16. ([10]) A neutrosophic soft relation $\widetilde{R}$ between two neutrosophic soft sets $\left(\widetilde{F}, E_1\right)$ and $\left(\widetilde{G}, E_2\right)$ over the common universe X is the neutrosophic soft subset of $\left(\widetilde{F}, E_1\right) \times \left(\widetilde{G}, E_2\right)$. Clearly, it is another neutrosophic soft set $\left(\widetilde{R}, E_3\right)$, where $E_3 \subseteq E_1 \times E_2$ and $\widetilde{R}(e_1, e_2) = \widetilde{F}(e_1) \times \widetilde{G}(e_2)$ for $(e_1, e_2) \in E_3$.

Definition 2.17. ([10]) Let $\left(\widetilde{F}, E_1\right)$, $\left(\widetilde{G}, E_2\right)$ be two neutrosophic soft sets over the universal set X and f be a neutrosophic soft relation defined on $\left(\widetilde{F}, E_1\right) \times \left(\widetilde{G}, E_2\right)$. Then, f is called neutrosophic soft function, if f associates each element of $\left(\widetilde{F}, E_1\right)$ with the unique element of $\left(\widetilde{G}, E_2\right)$. We write $f : \left(\widetilde{F}, E_1\right) \to \left(\widetilde{G}, E_2\right)$ as a neutrosophic soft function or a mapping. For $x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F}, E_1\right)$ and $y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{G}, E_2\right)$, when $x^e_{(\alpha,\beta,\gamma)} \times y^{e'}_{(\alpha',\beta',\gamma')} \in f$, we denote it by $f\left(x^e_{(\alpha,\beta,\gamma)}\right) = y^{e'}_{(\alpha',\beta',\gamma')}$. Here, $\left(\widetilde{F}, E_1\right)$ and $\left(\widetilde{G}, E_2\right)$ are called domain and codomain respectively and $y^{e'}_{(\alpha',\beta',\gamma')}$ is the image of $x^e_{(\alpha,\beta,\gamma)}$ under f.

Definition 2.18. ([10]) Let $f : (F, A) \to (G, B)$ be a neutrosophic soft function over the universal set U. If there exists another neutrosophic soft function $g : (G, B) \to (F, A)$ with $g \circ f : (F, A) \to (F, A)$ and $f \circ g : (G, B) \to (G, B)$ such that $g \circ f = I_{(F,A)}$ and $f \circ g = I_{(G,B)}$ then g is called the inverse neutrosophic soft function of f. It is denoted by f^{-1} and is defined as $F(a) \times G(b) \in f^{-1}$ if and only if $G(b) \times F(a) \in f$.

3. Neutrosophic Soft Semi-regularization and Neutrosophic Soft Sub-maximal Spaces

Definition 3.1. A neutrosophic soft set $\left(\widetilde{F}, E\right)$ is said to be neutrosophic soft quasi-coincident (neutrosophic soft q-coincident, for short) with $\left(\widetilde{G}, E\right)$, denoted by $\left(\widetilde{F}, E\right) q \left(\widetilde{G}, E\right)$, if and only if $\left(\widetilde{F}, E\right) \nsubseteq \left(\widetilde{G}, E\right)^c$. If $\left(\widetilde{F}, E\right)$ is not neutrosophic soft quasi-coincident with $\left(\widetilde{G}, E\right)$, we denote by $\left(\widetilde{F}, E\right) \overline{q} \left(\widetilde{G}, E\right)$.

Definition 3.2. A neutrosophic soft set $\left(\widetilde{F}, E\right)$ in a neutrosophic soft topological space (X, τ, E) is said to be a neutrosophic soft q-neighbourhood of a neutrosophic soft point $x^e_{(\alpha,\beta,\gamma)}$ if and only if there exists a neutrosophic soft open set $\left(\widetilde{G}, E\right)$ such that $x^e_{(\alpha,\beta,\gamma)} q \left(\widetilde{G}, E\right) \subset \left(\widetilde{F}, E\right)$.

Definition 3.3. A neutrosophic soft point $x^e_{(\alpha,\beta,\gamma)}$ is said to be a neutrosophic soft cluster point of a neutrosophic soft set $\left(\widetilde{F}, E\right)$ if and only if every neutrosophic soft open q-neighbourhood $\left(\widetilde{G}, E\right)$ of $x^e_{(\alpha,\beta,\gamma)}$ is neutrosophic soft q-coincident with $\left(\widetilde{F}, E\right)$. The collection of all neutrosophic soft cluster points of $\left(\widetilde{F}, E\right)$ is called the neutrosophic soft closure of $\left(\widetilde{F}, E\right)$ and denoted by $\overline{\left(\widetilde{F}, E\right)}$.

Definition 3.4. A neutrosophic soft point $x^e_{(\alpha,\beta,\gamma)}$ is said to be a neutrosophic soft interior point of a neutrosophic soft set $\left(\widetilde{F}, E\right)$ if and only if there exists a neutrosophic soft open q-neighborhood $\left(\widetilde{G}, E\right)$ of $x^e_{(\alpha,\beta,\gamma)}$ is such that $\left(\widetilde{G}, E\right) \subset \left(\widetilde{F}, E\right)$. The collection of all neutrosophic soft interior points of $\left(\widetilde{F}, E\right)$ is called the neutrosophic soft interior of $\left(\widetilde{F}, E\right)$ and denoted by $\left(\widetilde{F}, E\right)^\circ$.

For a neutrosophic soft set $\left(\widetilde{A}, E\right)$ in a neutrosophic soft topological space (X, τ, E), the notations $\tau - NScl\left(\widetilde{A}, E\right)$ and $\tau - NSint\left(\widetilde{A}, E\right)$ will respectively stand for the neutrosophic soft closure and neutrosophic soft interior of A.

Definition 3.5. A neutrosophic soft set $\left(\widetilde{F}, E\right)$ in a neutrosophic soft topological space (X, τ, E) is called a neutrosophic soft regular open set if and only if $\left(\widetilde{F}, E\right) = \left[\overline{\left(\widetilde{F}, E\right)}\right]^\circ$. The complement of a neutrosophic soft regular open set is called a neutrosophic soft regular closed set.

Equivalently, a neutrosophic soft set $\left(\widetilde{U}, E\right)$ in a neutrosophic soft topological space (X, τ, E) is called a neutrosophic soft regular closed set if and only if $\left(\widetilde{U}, E\right) = \overline{\left[\left(\widetilde{U}, E\right)^\circ\right]}$. The complement of a neutrosophic soft regular closed set is called a neutrosophic soft regular open.

Definition 3.6. Let (X, τ, E) be a neutrosophic soft topological space. A subfamily β of τ is a neutrosophic soft base for τ if and only if every member of τ can be expressed as the union of some members of β.

Definition 3.7. Consider that (X, τ, E) is a neutrosophic soft topological space. The set of all neutrosophic soft regular open sets in (X, τ, E) forms a base for some neutrosophic soft topology on X. This topology is called the neutrosophic soft semi-regularization topology of τ, to be denoted by τ_s. The inclusion $\tau_S \subseteq \tau$ holds. (X, τ_S, E) is called the neutrosophic soft semi-regularization space or simply the neutrosophic soft semi-regularization of (X, τ, E).

We can define a neutrosophic soft topology (X, τ, E) to be neutrosophic soft semi-regular if and only if the neutrosophic soft regular open sets in (X, τ, E) form a base for the neutrosophic soft topology τ on X. Thus, according to the above definition, (X, τ, E) is neutrosophic soft semi-regular if and only if $\tau = \tau_S$.

Example 3.8. Consider that $X = \{x, y\}$ is a universe, $E = \{a, b\}$ is a parametric set, the neutrosophic soft sets $\left(\widetilde{F}, E\right)$ and $\left(\widetilde{G}, E\right)$ are defined as $\widetilde{F}(a) = \{\langle x, 0.2, 0.2, 0.8 \rangle, \langle y, 0.2, 0.2, 0.8 \rangle\}$, $\widetilde{F}(b) = \{\langle x, 0.2, 0.2, 0.8 \rangle, \langle y, 0.2, 0.2, 0.8 \rangle\}$, $\widetilde{G}(a) = \{\langle x, 0.7, 0.7, 0.3 \rangle, \langle y, 0.7, 0.7, 0.3 \rangle\}$ and $\widetilde{G}(b) = \{\langle x, 0.7, 0.7, 0.3 \rangle, \langle y, 0.7, 0.7, 0.3 \rangle\}$. The family

$$\tau = \left\{ 0_{(X,E)}, 1_{(X,E)}, \left(\widetilde{F}, E\right), \left(\widetilde{G}, E\right) \right\}$$

is a neutrosophic soft topology over X. Then, (X, τ, E) is a neutrosophic soft semi-regular topological space.

The characterization of a neutrosophic soft semi-regular space is as follows.

Theorem 3.9. *Consider that (X, τ, E) is a neutrosophic soft topological space. The following statements are equivalent:*
(a) (X, τ, E) is neutrosophic soft semi-regular;
(b) for every neutrosophic soft open set $\left(\widetilde{U}, E\right)$ and every neutrosophic soft point $x^e_{(\alpha,\beta,\gamma)}$ with $x^e_{(\alpha,\beta,\gamma)} q \left(\widetilde{U}, E\right)$, there exists a neutrosophic soft open set $\left(\widetilde{V}, E\right)$ such that

$$x^e_{(\alpha,\beta,\gamma)} q \left(\widetilde{V}, E\right) \subseteq \left[\overline{\left(\widetilde{V}, E\right)}\right]^{\circ} \subseteq \left(\widetilde{U}, E\right);$$

(c) for every neutrosophic soft closed set $\left(\widetilde{A}, E\right)$ and every neutrosophic soft point $x^e_{(\alpha,\beta,\gamma)} \notin \left(\widetilde{A}, E\right)$, there exists a neutrosophic soft regular closed set $\left(\widetilde{B}, E\right)$ such that $\left(\widetilde{A}, E\right) \subseteq \left(\widetilde{B}, E\right)$ and $x^e_{(\alpha,\beta,\gamma)} \notin \left(\widetilde{B}, E\right)$;

(d) for every neutrosophic soft set $\left(\widetilde{A}, E\right)$ in (X, τ, E) and every neutrosophic soft open set $\left(\widetilde{B}, E\right)$ with $\left(\widetilde{A}, E\right) q \left(\widetilde{B}, E\right)$, there exists a neutrosophic soft regular open set $\left(\widetilde{U}, E\right)$ such that $\left(\widetilde{A}, E\right) q \left(\widetilde{U}, E\right) \subseteq \left(\widetilde{B}, E\right)$.

Proof. $(a) \Rightarrow (b)$ Assume that (X, τ, E) is neutrosophic soft semi-regular. Consider a neutrosophic soft point $x^e_{(\alpha,\beta,\gamma)}$ and a neutrosophic soft open set $\left(\widetilde{U}, E\right)$ such that $x^e_{(\alpha,\beta,\gamma)} q \left(\widetilde{U}, E\right)$. Since (X, τ, E) is neutrosophic soft semi-regular, $\left(\widetilde{U}, E\right)$ is the union of some neutrosophic soft regular open sets in (X, τ, E). Then, there exists a neutrosophic soft regular open set $\left(\widetilde{V}, E\right)$ such that $x^e_{(\alpha,\beta,\gamma)} q \left(\widetilde{V}, E\right) \subset \left(\widetilde{U}, E\right)$. Since $\left(\widetilde{V}, E\right)$ is neutrosophic soft regular open, $x^e_{(\alpha,\beta,\gamma)} q \left(\widetilde{V}, E\right) \subseteq \left[\overline{\left(\widetilde{V}, E\right)}\right]^{\circ} \subseteq \left(\widetilde{U}, E\right)$.

$(b) \Rightarrow (c)$ Consider a neutrosophic soft point $x^e_{(\alpha,\beta,\gamma)}$ and a neutrosophic soft closed set $\left(\widetilde{A}, E\right)$ such that $x^e_{(\alpha,\beta,\gamma)} \notin \left(\widetilde{A}, E\right)$. Then, $x^e_{(\alpha,\beta,\gamma)} q \left(\widetilde{A}, E\right)^c$ and $\left(\widetilde{A}, E\right)^c$ is neutrosophic soft open. From our assumption, there exists a neutrosophic soft open set $\left(\widetilde{V}, E\right)$ such that $x^e_{(\alpha,\beta,\gamma)} q \left(\widetilde{V}, E\right) \subseteq \left[\overline{\left(\widetilde{V}, E\right)}\right]^{\circ} \subseteq \left(\widetilde{A}, E\right)^c$. Obviously, $\left[\overline{\left(\widetilde{V}, E\right)}\right]^{\circ} \subseteq \overline{\left[\overline{\left(\widetilde{V}, E\right)}\right]^{\circ}}$. Since $\overline{\left(\widetilde{V}, E\right)}$ is neutrosophic soft closed, $\overline{\left[\overline{\left(\widetilde{V}, E\right)}\right]^{\circ}} \subseteq \overline{\left(\widetilde{V}, E\right)}$. So, $\left[\overline{\left[\overline{\left(\widetilde{V}, E\right)}\right]^{\circ}}\right]^{\circ} \subseteq \left[\overline{\left(\widetilde{V}, E\right)}\right]^{\circ}$. This

means that $\left[\overline{\left(\widetilde{V},E\right)}\right]^{\circ} = \left[\left[\overline{\left(\widetilde{V},E\right)}\right]^{\circ}\right]^{\circ}$. Then, $\left[\overline{\left(\widetilde{V},E\right)}\right]^{\circ}$ neutrosophic soft regular open. Say $\left(\widetilde{B},E\right) = \left[\left[\overline{\left(\widetilde{V},E\right)}\right]^{\circ}\right]^{c}$.

So, $\left(\widetilde{A},E\right) \subseteq \left(\widetilde{B},E\right)$ and $x^{e}_{(\alpha,\beta,\gamma)} \notin \left(\widetilde{B},E\right)$, where $\left(\widetilde{B},E\right)$ is neutrosophic soft regular closed.

(*c*) $\Rightarrow$ (*d*) Consider a neutrosophic soft set $\left(\widetilde{A},E\right)$ and a neutrosophic soft open set $\left(\widetilde{B},E\right)$ such that $\left(\widetilde{A},E\right) q \left(\widetilde{B},E\right)$. Then, $\left(\widetilde{A},E\right) \subset \left(\widetilde{B},E\right)^{c}$ and $\left(\widetilde{B},E\right)^{c}$ is neutrosophic soft closed. So, there exists a neutrosophic soft point $x^{e}_{(\alpha,\beta,\gamma)} \in \left(\widetilde{A},E\right)$ such that $x^{e}_{(\alpha,\beta,\gamma)} \notin \left(\widetilde{B},E\right)^{c}$. From our assumption, there exists a neutrosophic soft regular closed set $\left(\widetilde{V},E\right)$ such that $\left(\widetilde{B},E\right)^{c} \subset \left(\widetilde{V},E\right)$ and $x^{e}_{(\alpha,\beta,\gamma)} \notin \left(\widetilde{V},E\right)$. Say $\left(\widetilde{U},E\right) = \left(\widetilde{V},E\right)^{c}$. Then, $\left(\widetilde{A},E\right) q \left(\widetilde{U},E\right)$ and $\left(\widetilde{U},E\right) \subset \left(\widetilde{B},E\right)$, where $\left(\widetilde{U},E\right)$ is neutrosophic soft regular open.

(*d*) $\Rightarrow$ (*a*) Consider that (X,τ,E) is a neutrosophic soft topological space and $\left(\widetilde{A},E\right)$ is a neutrosophic soft open set in (X,τ,E). For every neutrosophic point $x^{e}_{(\alpha,\beta,\gamma)} \in \left(\widetilde{A},E\right)$, there exists a neutrosophic open q-neighbourhood $\left(\widetilde{B},E\right)$ of $x^{e}_{(\alpha,\beta,\gamma)}$ such that $\left(\widetilde{B},E\right) \subset \left(\widetilde{A},E\right)$. Then, there exists a neutrosophic soft open set $\left(\widetilde{V},E\right)$ such that $x^{e}_{(\alpha,\beta,\gamma)} q \left(\widetilde{V},E\right) \subset \left(\widetilde{B},E\right)$. From our assumption, there exists a neutrosophic regular open set $\left(\widetilde{U},E\right)$ such that $x^{e}_{(\alpha,\beta,\gamma)} q \left(\widetilde{U},E\right) \subset \left(\widetilde{V},E\right)$. So, $x^{e}_{(\alpha,\beta,\gamma)} q \left(\widetilde{U},E\right) \subset \left(\widetilde{A},E\right)$. This implies that every neutrosophic soft open set in (X,τ,E) is the union of neutrosophic soft regular open sets. Therefore, (X,τ,E) is neutrosophic soft semi-regular. $\square$

Theorem 3.10. *Consider that* (X,τ_{s},E) *denote the neutrosophic soft semi-regularization of a neutrosophic soft topological space* (X,τ,E). *Then, for every neutrosophic soft open set* $\left(\widetilde{U},E\right)$ *in* (X,τ,E):

(*a*) $\tau - NScl\left(\widetilde{U},E\right) = \tau_{S} - NScl\left(\widetilde{U},E\right)$ *and*

(*b*) $\tau - NSint\left(\tau - NScl\left(\widetilde{U},E\right)\right) = \tau_{S} - NSint\left(\tau_{S} - NScl\left(\widetilde{U},E\right)\right)$.

Proof. (a) Consider that $\left(\widetilde{U},E\right)$ is a neutrosophic soft open set in (X,τ,E). Clearly, $\tau - NSint\left(\tau - NScl\left(\widetilde{U},E\right)\right) \subset \tau - NScl\left(\widetilde{U},E\right)$. It is easily seen that $\tau - NScl\left(\tau - NSint\left(\tau - NScl\left(\widetilde{U},E\right)\right)\right) \subset \tau - NScl\left(\widetilde{U},E\right)$. Since $\left(\widetilde{U},E\right)$ is neutrosophic soft open in (X,τ,E), $\left(\widetilde{U},E\right) \subset \tau - NSint\left(\tau - NScl\left(\widetilde{U},E\right)\right)$. Clearly $\tau - NScl\left(\widetilde{U},E\right) \subset \tau - NScl\left(\tau - NSint\left(\tau - NScl\left(\widetilde{U},E\right)\right)\right)$. This means that $\tau - NScl\left(\widetilde{U},E\right) = \tau - NScl\left(\tau - NSint\left(\tau - NScl\left(\widetilde{U},E\right)\right)\right)$. So $\tau - NScl\left(\widetilde{U},E\right)$ is neutrosophic soft regular closed and $\left[\tau - NScl\left(\widetilde{U},E\right)\right]^{c}$ is neutrosophic soft regular open in (X,τ,E). This implies that $\tau - NScl\left(\widetilde{U},E\right)$ is neutrosophic soft closed in (X,τ_{S},E). Hence $\tau_{S} - NScl\left(\widetilde{U},E\right) \subset \tau - NScl\left(\widetilde{U},E\right)$. Also, τ being finer than τ_{S}, $\tau - NScl\left(\widetilde{U},E\right) \subset \tau_{S} - NScl\left(\widetilde{U},E\right)$. Thus $\tau - NScl\left(\widetilde{U},E\right) = \tau_{S} - NScl\left(\widetilde{U},E\right)$, for all $\left(\widetilde{U},E\right)$ in τ.

(b) For any $\left(\widetilde{U},E\right) \in \tau$, using (a) we have $\tau - NSint\left(\tau - NScl\left(\widetilde{U},E\right)\right) \subseteq \tau_{S} - NScl\left(\widetilde{U},E\right)$. Since

$$\tau - NSint\left(\tau - NScl\left(\widetilde{U},E\right)\right) \in \tau_{S},$$

we have

$$\tau - NSint\left(\tau - NScl\left(\widetilde{U},E\right)\right) \subseteq \tau_{S} - NSint\left(\tau_{S} - NScl\left(\widetilde{U},E\right)\right).$$

Again, τ being finer than τ_{S},

$$\tau_{S} - NSint\left(\tau_{S} - NScl\left(\widetilde{U},E\right)\right) \subseteq \tau - NSint\left(\tau - NScl\left(\widetilde{U},E\right)\right).$$

Thus, $\tau - NSint\left(\tau - NScl\left(\widetilde{U},E\right)\right) = \tau_{S} - NSint\left(\tau_{S} - NScl\left(\widetilde{U},E\right)\right)$. $\square$

Corollary 3.11. *The set of all neutrosophic soft regular open sets of (X, τ, E) is that of all neutrosophic soft regular open sets in (X, τ_s, E). Thus neutrosophic soft semi-regularizations in (X, τ, E) and (X, τ_s, E) are the identical.*

Corollary 3.12. *For any neutrosophic soft topological space (X, τ, E), (X, τ_s, E) is neutrosophic soft semi-regular.*

Theorem 3.13. *Consider that (X, τ, E) is a neutrosophic soft topological space (X, τ, E) and (X, τ_0, E) be any neutrosophic soft semi-regular space such that $\tau_s \subset \tau_0 \subset \tau$. Then $\tau_0 = \tau_s$.*

Proof. Consider that $\left(\widetilde{U}, E\right)$ be any neutrosophic soft regular open set in (X, τ_0, E). Since

$$\tau_s \subset \tau_0 \subset \tau \text{ and since } \tau_S - NScl\left(\widetilde{U}, E\right) = \tau_S - NScl\left(\widetilde{U}, E\right), \text{ we have}$$
$$\tau_S - NScl\left(\widetilde{U}, E\right) = \tau_0 - NScl\left(\widetilde{U}, E\right). \text{ Again,}$$
$$\tau_S - NSint\left(\tau_S - NScl\left(\widetilde{U}, E\right)\right) = \tau - NSint\left(\tau - NScl\left(\widetilde{U}, E\right)\right). \text{ Then,}$$
$$\tau - NSint\left(\tau - NScl\left(\widetilde{U}, E\right)\right) = \tau_0 - NSint\left(\tau_0 - NScl\left(\widetilde{U}, E\right)\right) = \left(\widetilde{U}, E\right).$$

Hence $\left(\widetilde{U}, E\right) \in \tau_S$ and consequently, $\tau_0 = \tau_S$. $\square$

Corollary 3.14. *For a neutrosophic soft topological space (X, τ, E), among all the neutrosophic soft semi-regular spaces which are weaker than (X, τ, E), (X, τ_s, E) is the finest neutrosophic soft semi-regular space.*

Definition 3.15. Two neutrosophic soft topological spaces (X, τ, E) and (X, ϑ, E) are said to be neutrosophic soft ro-equivalent if $\tau_S = \vartheta_S$.

Definition 3.16. A neutrosophic soft topological space (X, τ, E) is said to be an expansion of a neutrosophic soft topological space (X, τ, E) if ϑ is coarser than τ (i.e., $\vartheta \subset \tau$).

Definition 3.17. A property P of a neutrosophic soft topological space is called expansive iff whenever a neutrosophic soft topological space (X, τ, E) has the property P, so does any expansion of the space.

Theorem 3.18. *An expansion (X, ϑ, E) of a neutrosophic soft topological space (X, τ, E) is neutrosophic soft ro-equivalent to (X, τ, E) if and only if $\tau - NScl\left(\widetilde{U}, E\right) = \vartheta - NScl\left(\widetilde{U}, E\right)$, for all $\left(\widetilde{U}, E\right) \in \vartheta$.*

Proof. Consider that (X, τ, E) be neutrosophic soft ro-equivalent to (X, ϑ, E) so that $\tau_S = \vartheta_S$. Consider that $\left(\widetilde{U}, E\right) \in \vartheta$. Then, $\vartheta - NScl\left(\widetilde{U}, E\right) \subseteq \tau - NScl\left(\widetilde{U}, E\right)$. If $\left(\widetilde{V}, E\right) = \left[\vartheta - NScl\left(\widetilde{U}, E\right)\right]^c$, then $\left(\widetilde{V}, E\right) = \vartheta - NSint\left(\vartheta - NScl\left(\widetilde{V}, E\right)\right)$ and hence, $\left(\widetilde{V}, E\right) \in \vartheta_S = \tau_S$. Thus, $\left(\widetilde{V}, E\right) \in \tau$ so that $\vartheta - NScl\left(\widetilde{U}, E\right)$ is neutrosophic soft closed in (X, τ, E). Consequently, $\tau - NScl\left(\widetilde{U}, E\right) \subseteq \vartheta - NScl\left(\widetilde{U}, E\right)$. Thus, we have $\tau - NScl\left(\widetilde{U}, E\right) = \vartheta - NScl\left(\widetilde{U}, E\right)$.

Conversely, let $\left(\widetilde{U}, E\right)$ be neutrosophic soft regular open in (X, ϑ, E). Then,

$$\left(\widetilde{U}, E\right) = \vartheta - NSint\left(\vartheta - NScl\left(\widetilde{U}, E\right)\right) = \left[\vartheta - NScl\left(\widetilde{V}, E\right)\right]^c,$$

where $\left(\widetilde{V}, E\right) = \left[\vartheta - NScl\left(\widetilde{U}, E\right)\right]^c$. Since $\left(\widetilde{V}, E\right) \in \vartheta$, $\tau - NScl\left(\widetilde{V}, E\right) = \vartheta - NScl\left(\widetilde{V}, E\right)$ and hence, $\left(\widetilde{U}, E\right) = \left[\tau - NScl\left(\widetilde{V}, E\right)\right]^c \in \tau$. Thus, $\vartheta_S \subset \tau$ which in view of Corollary 3.14. yields $\vartheta_S \subset \tau_S$. Again, $\tau \subset \vartheta$. Then, $\tau_S \subset \vartheta$. This implies that $\tau_S \subset \vartheta_S$ (by Corollary 3.14.). Hence, $\tau_S = \vartheta_S$. $\square$

Definition 3.19. Consider that (X, τ, E) is a neutrosophic soft semi-regular space, φ is the set of all neutrosophic soft topologies τ_α's on X such that (X, τ_α, E) is neutrosophic soft ro-equivalent to (X, τ, E). Then φ is partially ordered by the set inclusion relation. Then (X, τ^*, E), where τ^* is a maximal element of φ, which is defined as a neutrosophic soft sub-maximal space.

Lemma 3.20. *Consider that $\left(\widetilde{A}, E\right)$ and $\left(\widetilde{B}, E\right)$ any two neutrosophic soft open sets in a neutrosophic soft topological space (X, τ, E). If $\left(\widetilde{A}, E\right) \overline{q} \left(\widetilde{B}, E\right)$ then $\overline{\left(\widetilde{A}, E\right)} \overline{q} \left(\widetilde{B}, E\right)$ and $\left(\widetilde{A}, E\right) \overline{q} \left(\widetilde{B}, E\right)$.*

Theorem 3.21. *For every neutrosophic soft topological space, there exists a neutrosophic soft sub-maximal space which is an expansion of the given neutrosophic soft topological space.*

Proof. Consider that (X, τ, E) is a neutrosophic soft topological space. Let φ denote the set of all neutrosophic soft topologies τ_α's on X such that (X, τ_α, E) is neutrosophic soft ro-equivalent to (X, τ, E). Then φ is a poset under the set-inclusion relation. Consider that φ_1 be the sub-collection of φ such that $\tau_\alpha \in \varphi_1$ iff $\tau \subset \tau_\alpha$. Then φ_1 is also a poset under the identical relation as in φ. Consider that φ_0 is a chain in φ_1 and let $\psi_0 = \bigcup \{\tau_\alpha : \tau_\alpha \in \psi_0\}$. It is obvious that ψ_0 is a base for some neutrosophic soft topology ϑ (say) on X such that $\tau \subset \vartheta$. We claim that $\tau_S = \vartheta_S$. In fact, let $\left(\widetilde{U}, E\right) \in \vartheta$. In view of Theorem 3.18, it suffices to show that $\tau - NScl\left(\widetilde{U}, E\right) = \vartheta - NScl\left(\widetilde{U}, E\right)$. Obviously,
$\vartheta - NScl\left(\widetilde{U}, E\right) \subseteq \tau - NScl\left(\widetilde{U}, E\right)$. Next, suppose $x^e_{(\alpha,\beta,\gamma)}$ is a neutrosophic soft point such that $x^e_{(\alpha,\beta,\gamma)} \in \tau - NScl\left(\widetilde{U}, E\right)$, and let $\left(\widetilde{W}, E\right)$ be any neutrosophic soft open q-neighborhood of $x^e_{(\alpha,\beta,\gamma)}$ in (X, ϑ, E). Then there exists $\left(\widetilde{B}, E\right) \in \Psi_0$ such that $x^e_{(\alpha,\beta,\gamma)} q \left(\widetilde{B}, E\right) \subseteq \left(\widetilde{W}, E\right)$. Now, $\left(\widetilde{B}, E\right) \in \tau_\alpha$, for some α for which $\tau_\alpha \in \varphi_0$. Consider that $\left(\widetilde{W}, E\right)^* = \tau_\alpha - NSint\left(\tau_\alpha - NScl\left(\widetilde{B}, E\right)\right)$. Then, $\left(\widetilde{W}, E\right)^* \in (\tau_\alpha)_s = \tau_S \subset \tau$, i.e. $\left(\widetilde{W}, E\right)^* \in \tau$. Thus, $\left(\widetilde{W}, E\right)^*$ is neutrosophic soft open q-neighborhood of $x^e_{(\alpha,\beta,\gamma)}$ in (X, τ, E) and hence, $\left(\widetilde{W}, E\right)^* q \left(\widetilde{U}, E\right)$. So there exists a neutrosophic soft point $y^{e'}_{(\alpha',\beta',\gamma')}$ in (X, τ, E) such that $y^{e'}_{(\alpha',\beta',\gamma')} q \left(\widetilde{U}, E\right)$ and hence there exists $\left(\widetilde{U}_Y, E\right) \in \psi_0$ such that $y^{e'}_{(\alpha',\beta',\gamma')} q \left(\widetilde{U}_Y, E\right) \subseteq \left(\widetilde{U}, E\right)$. Now, $\left(\widetilde{U}_Y, E\right) \in \tau_\beta$, for some $\tau_\beta \in \varphi_0$. If possible, let $\left(\widetilde{W}, E\right) \overline{q} \left(\widetilde{U}, E\right)$. Then, $\left(\widetilde{B}, E\right) \overline{q} \left(\widetilde{U}_Y, E\right)$. Since (X, τ_α, E) and $\left(X, \tau_\beta, E\right)$ are comparable and every is neutrosophic soft ro-equivalent to (X, τ_s, E), by *Theorem 3.18.*, $\tau_\alpha - NScl\left(\widetilde{B}, E\right) = \tau_\beta - NScl\left(\widetilde{B}, E\right)$. Therefore by Lemma 3.20, $\left[\tau_\alpha - NScl\left(\widetilde{B}, E\right)\right] q \left(\widetilde{U}_Y, E\right)$. Since $\left(\widetilde{W}, E\right)^* \subseteq \tau_\alpha - NScl\left(\widetilde{B}, E\right)$, we have $\left(\widetilde{W}, E\right)^* \overline{q} \left(\widetilde{U}, E\right)$ and hence $y^{e'}_{(\alpha',\beta',\gamma')} \overline{q} \left(\widetilde{U}_Y, E\right)$ which is a contradiction. Thus $x^e_{(\alpha,\beta,\gamma)} \in \vartheta - NScl\left(\widetilde{U}, E\right)$ and we finally conclude that $\tau - NScl\left(\widetilde{U}, E\right) = \vartheta - NScl\left(\widetilde{U}, E\right)$. Hence $\tau_S = \vartheta_S$ so that $\vartheta \in \psi_0$. Openly ϑ is an upper bound of ψ_0. By Zorn's lemma, ψ_1 has a maximal element τ^* (say). This maximal element is also a maximal element of φ. Hence (X, τ^*, E) is a neutrosophic soft sub-maximal space which is an expansion of (X, τ, E). $\square$

Definition 3.22. A property P is said to is a neutrosophic soft semi-regular property provided that a neutrosophic soft topological space (X, τ, E) possesses the property P if its neutrosophic soft semi-regularization space (X, τ_S, E) possesses the property.

Definition 3.23. A neutrosophic soft topological space (X, τ, E) is said to be maximal (minimal) with respect to a property P if whenever a neutrosophic soft topological space (X, ϑ, E) has the property P, one possesses $\vartheta \subset \tau$ (resp. $\tau \subset \vartheta$). On the upshot, we shall say such a space maximal P (resp. minimal P).

Theorem 3.24. *Consider that P is a neutrosophic soft semi-regular property. Then each maximal P (minimal P) neutrosophic soft topological space is neutrosophic soft sub-maximal (resp. neutrosophic soft semi-regular).*

Proof. Consider that (X, τ, E) has the maximal property P (minimal property P). If it is not neutrosophic soft sub-maximal (semi-regular), there exists a neutrosophic soft sub-maximal space (X, τ^*, E) which is strictly finer (resp. neutrosophic soft semi-regular space (X, τ_S, E) which is strictly weaker) than (X, τ, E) which has the property P. This contradicts the maximality (resp. minimality) of the space (X, τ, E) with respect to the property P. $\square$

4. Neutrosophic Soft Semi-regular Properties

In the last section, it has just seen that sub-maximality and semi-regularity of neutrosophic soft topological spaces are respectively the necessary conditions for them to be maximal and minimal with respect to neutrosophic soft semi-regular properties. This motivates us to investigate the different properties of neutrosophic soft topological spaces in order to ascertain whether these are neutrosophic soft semi-regular properties. We start with a new approach to the concept of neutrosophic soft separation axioms in the following manner.

Definition 4.1. A neutrosophic soft topological space (X, τ, E) is said to be a neutrosophic soft T_0-space if for every pair of distinct neutrosophic soft points $x^e_{(\alpha,\beta,\gamma)}$, $y^{e'}_{(\alpha',\beta',\gamma')}$ there exist neutrosophic open soft sets $(\widetilde{F}, E), (\widetilde{G}, E)$ such that $x^e_{(\alpha,\beta,\gamma)} \in (\widetilde{F}, E)$, $y^{e'}_{(\alpha',\beta',\gamma')} \in (\widetilde{F}, E)^c$ or $x^e_{(\alpha,\beta,\gamma)} \in (\widetilde{G}, E)^c$, $y^{e'}_{(\alpha',\beta',\gamma')} \in (\widetilde{G}, E)$.

Definition 4.2. A neutrosophic soft topological space (X, τ, E) is said to be a neutrosophic soft T_1-space if for every pair of distinct neutrosophic soft points $x^e_{(\alpha,\beta,\gamma)}$, $y^{e'}_{(\alpha',\beta',\gamma')}$ there exists neutrosophic open soft sets $(\widetilde{F}, E)$ and $(\widetilde{G}, E)$ such that $x^e_{(\alpha,\beta,\gamma)} \in (\widetilde{F}, E)$, $y^{e'}_{(\alpha',\beta',\gamma')} \in (\widetilde{F}, E)^c$ and $x^e_{(\alpha,\beta,\gamma)} \in (\widetilde{G}, E)^c$, $y^{e'}_{(\alpha',\beta',\gamma')} \in (\widetilde{G}, E)$.

Definition 4.3. A neutrosophic soft topological space (X, τ, E) is said to be a neutrosophic soft T_2-space if for every pair of distinct neutrosophic soft points $x^e_{(\alpha,\beta,\gamma)}$, $y^{e'}_{(\alpha',\beta',\gamma')}$ there exists neutrosophic open soft sets $(\widetilde{F}, E)$ and $(\widetilde{G}, E)$ such that

$$x^e_{(\alpha,\beta,\gamma)} \in (\widetilde{F}, E), \ y^{e'}_{(\alpha',\beta',\gamma')} \in (\widetilde{F}, E)^c, \ y^{e'}_{(\alpha',\beta',\gamma')} \in (\widetilde{G}, E), x^e_{(\alpha,\beta,\gamma)} \in (\widetilde{G}, E)^c \text{ and } (\widetilde{F}, E) \subset (\widetilde{G}, E)^c.$$

It is obvious that the property of a space to be neutrosophic soft T_0 or neutrosophic soft T_1 is an expansive property. Thus we have the following theorem:

Theorem 4.4. *A neutrosophic soft topological space (X, τ, E) is neutrosophic soft T_0 (neutrosophic soft T_1) if (X, τ, E) is neutrosophic soft T_1 (resp. neutrosophic soft T_2).*

Proof. Straightforward. $\square$

Theorem 4.5. *Neutrosophic soft T_2- property is a neutrosophic soft semi-regular property.*

Proof. Consider that (X, τ, E) is neutrosophic soft T_2 and $x^e_{(\alpha,\beta,\gamma)}$, $y^{e'}_{(\alpha',\beta',\gamma')}$ are two distinct neutrosophic soft points in (X, τ, E). Then, $x^e_{(\alpha,\beta,\gamma)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ have neutrosophic soft open neighbourhoods $(\widetilde{U}, E)$ and $(\widetilde{V}, E)$ respectively in (X, τ, E) such that

$$x^e_{(\alpha,\beta,\gamma)} \in (\widetilde{U}, E), y^{e'}_{(\alpha',\beta',\gamma')} \in (\widetilde{U}, E)^c, y^{e'}_{(\alpha',\beta',\gamma')} \in (\widetilde{V}, E), x^e_{(\alpha,\beta,\gamma)} \in (\widetilde{V}, E)^c \text{ and } (\widetilde{U}, E) \subset (\widetilde{V}, E)^c.$$

This implies that $(\widetilde{U}, E) \overline{q} (\widetilde{V}, E)$. Then by Lemma 3.20 we have

$$\left[\tau - NScl(\widetilde{U}, E)\right] \overline{q} (\widetilde{V}, E) \text{ and } \tau - NSint(\tau - NScl(\widetilde{U}, E)) \overline{q} (\widetilde{V}, E).$$

Again by the same lemma we have $\tau - NSint(\tau - NScl(\widetilde{U}, E)) \overline{q} \left[\tau - NScl(\widetilde{V}, E)\right]$. Consider that us put

$$\tau - NSint(\tau - NScl(\widetilde{U}, E)) = (\widetilde{U}_S, E) \text{ and } \tau - NSint(\tau - NScl(\widetilde{V}, E)) = (\widetilde{V}_S, E).$$

Then $(\widetilde{U}_S, E)$ and $(\widetilde{V}_S, E)$ are neutrosophic soft open neighbourhoods of $x^e_{(\alpha,\beta,\gamma)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ in (X, τ_S, E) respectively such that $(\widetilde{U}, E) \overline{q} (\widetilde{V}, E)$. $\square$

Definition 4.6. A neutrosophic soft topological space (X, τ, E) is said to be neutrosophic soft regular if every neutrosophic soft open set $\left(\widetilde{U}, E\right)$ in (X, τ, E) is a union of neutrosophic soft open sets $\left(\widetilde{V}_\alpha, E\right)$'s in (X, τ, E) such that $\overline{\left(\widetilde{V}_\alpha, E\right)} \subseteq \left(\widetilde{U}, E\right)$, for every α.

Theorem 4.7. *A neutrosophic soft topological space (X, τ, E) is neutrosophic soft regular if for every neutrosophic soft point $x^e_{(\alpha,\beta,\gamma)}$ in (X, τ, E) and for every neutrosophic soft open q-neighbourhood $\left(\widetilde{U}, E\right)$ of $x^e_{(\alpha,\beta,\gamma)}$, there exists a neutrosophic soft open set $\left(\widetilde{V}, E\right)$ such that $x^e_{(\alpha,\beta,\gamma)} q \left(\widetilde{V}, E\right) \subseteq \overline{\left(\widetilde{V}, E\right)} \subseteq \left(\widetilde{U}, E\right)$.*

Proof. It is omitted. $\square$

Definition 4.8. A neutrosophic soft topological space (X, τ, E) is said to be neutrosophic soft almost regular if for every neutrosophic soft point $x^e_{(\alpha,\beta,\gamma)}$ in (X, τ, E) and for every neutrosophic soft regular open q-neighbourhood $\left(\widetilde{U}, E\right)$ of $x^e_{(\alpha,\beta,\gamma)}$, there exists a neutrosophic soft regular open q- neighbourhood $\left(\widetilde{V}, E\right)$ of $x^e_{(\alpha,\beta,\gamma)}$ such that $\overline{\left(\widetilde{V}, E\right)} \subseteq \left(\widetilde{U}, E\right)$.

By virtue of Theorem 3.9. and Theorem 4.7. it is obvious that a neutrosophic soft regular space is neutrosophic soft almost regular as well as neutrosophic soft semi-regular.

Theorem 4.9. *A neutrosophic soft topological space (X, τ, E) is neutrosophic soft almost regular if and only if (X, τ_S, E) is neutrosophic soft regular.*

Proof. Necessity: Consider that $x^e_{(\alpha,\beta,\gamma)}$ is any neutrosophic soft point in (X, τ, E) and $\left(\widetilde{U}, E\right)$ be any neutrosophic soft open q-neighbourhood of $x^e_{(\alpha,\beta,\gamma)}$ in (X, τ_S, E). Then there exists a neutrosophic soft regular open set $\left(\widetilde{V}, E\right)$ in (X, τ, E) such that $x^e_{(\alpha,\beta,\gamma)} q \left(\widetilde{V}, E\right) \subseteq \left(\widetilde{U}, E\right)$. By neutrosophic soft almost regularity of (X, τ, E), there exists a neutrosophic soft regular open q-neighbourhood $\left(\widetilde{W}, E\right)$ of $x^e_{(\alpha,\beta,\gamma)}$ in (X, τ, E), i.e. a neutrosophic soft open q-neighbourhood $\left(\widetilde{W}, E\right)$ of $x^e_{(\alpha,\beta,\gamma)}$ in (X, τ_S, E) such that

$$\tau_S - NScl\left(\widetilde{W}, E\right) = \tau - NScl\left(\widetilde{W}, E\right) \subseteq \left(\widetilde{V}, E\right) \subseteq \left(\widetilde{U}, E\right).$$

Thus, (X, τ_S, E) is neutrosophic soft regular, by Theorem 4.7.

Sufficiency: Consider that $x^e_{(\alpha,\beta,\gamma)}$ is any neutrosophic soft point in (X, τ, E) and $\left(\widetilde{U}, E\right)$ be any neutrosophic soft regular open q- neighbourhood of $x^e_{(\alpha,\beta,\gamma)}$ in (X, τ, E). Then $\left(\widetilde{U}, E\right)$ is a neutrosophic soft open q-neighbourhood of $x^e_{(\alpha,\beta,\gamma)}$ in (X, τ_S, E). So by neutrosophic soft regularity of (X, τ_S, E) there exists a neutrosophic soft open q-neighbourhood $\left(\widetilde{V}, E\right)$ of $x^e_{(\alpha,\beta,\gamma)}$ in (X, τ_S, E) such that $\tau_S - NScl\left(\widetilde{V}, E\right) \subseteq \left(\widetilde{U}, E\right)$. Again, there exists a neutrosophic soft regular open q-neighbourhood $\left(\widetilde{W}, E\right)$ of $x^e_{(\alpha,\beta,\gamma)}$ in (X, τ, E) such that $\left(\widetilde{W}, E\right) \subseteq \left(\widetilde{V}, E\right)$ and hence $\overline{\left(\widetilde{W}, E\right)} \subseteq \tau_S - NScl\left(\widetilde{W}, E\right) \subseteq \tau_S - NScl\left(\widetilde{V}, E\right) \subseteq \left(\widetilde{U}, E\right)$. Hence (X, τ, E) is neutrosophic soft almost regular. $\square$

Corollary 4.10. *A neutrosophic soft topological space is neutrosophic soft semi-regular and neutrosophic soft almost regular iff it is neutrosophic soft regular.*

Corollary 4.11. *For any neutrosophic soft topological space (X, τ, E), its neutrosophic soft semi-regularization (X, τ_S, E) is neutrosophic soft almost regular iff it is neutrosophic soft regular.*

Corollary 4.12. *Neutrosophic soft almost regularity is a neutrosophic soft semi-regular property.*

Definition 4.13. A neutrosophic soft topological space (X, τ, E) is said to be a neutrosophic soft Urysohn space if for every pair of distinct neutrosophic soft points $x^e_{(\alpha,\beta,\gamma)}$, $y^{e'}_{(\alpha',\beta',\gamma')}$ there exist neutrosophic open soft sets $\left(\widetilde{F}, E\right)$ and $\left(\widetilde{G}, E\right)$ such that

$$x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{F}, E\right),\ y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{F}, E\right)^c,\ y^{e'}_{(\alpha',\beta',\gamma')} \in \left(\widetilde{G}, E\right),\ x^e_{(\alpha,\beta,\gamma)} \in \left(\widetilde{G}, E\right)^c \text{ and } \overline{\left(\widetilde{F}, E\right)} \subset \left[\overline{\left(\widetilde{G}, E\right)}\right]^c.$$

Theorem 4.14. *The neutrosophic soft Urysohn property is a neutrosophic soft semi-regular property.*

Proof. Consider that (X, τ, E) is neutrosophic soft Urysohn and $x^e_{(\alpha,\beta,\gamma)}$, $y^{e'}_{(\alpha',\beta',\gamma')}$ are two distinct neutrosophic soft points in (X, τ, E). Then, $x^e_{(\alpha,\beta,\gamma)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ have neutrosophic soft open neighbourhoods $\left(\widetilde{U}, E\right)$ and $\left(\widetilde{V}, E\right)$ respectively in (X, τ, E) such that $\left[\tau - NScl\left(\widetilde{U}, E\right)\right] \subset \left[\tau - NScl\left(\widetilde{V}, E\right)\right]^c$. Put $\tau - NSint\left(\tau - NScl\left(\widetilde{U}, E\right)\right) = \left(\widetilde{U}_S, E\right)$ and $\tau - NSint\left(\tau - NScl\left(\widetilde{V}, E\right)\right) = \left(\widetilde{V}_S, E\right)$. Then, $\left(\widetilde{V}, E\right) \subset \left(\widetilde{V}_S, E\right)^c$ and consequently, $\tau_S - NScl\left(\widetilde{U}_S, E\right) = \tau - NScl\left(\widetilde{U}, E\right)$, $\tau_S - NScl\left(\widetilde{V}_S, E\right) = \tau - NScl\left(\widetilde{V}, E\right)$ and $\left[\tau - NScl\left(\widetilde{U}_S, E\right)\right] \subset \left[\tau - NScl\left(\widetilde{V}_S, E\right)\right]^c$, where $\left(\widetilde{U}_S, E\right)$ and $\left(\widetilde{V}_S, E\right)$ are neutrosophic soft open neighbourhoods of $x^e_{(\alpha,\beta,\gamma)}$ and $y^{e'}_{(\alpha',\beta',\gamma')}$ respectively in (X, τ_S, E). Hence (X, τ_S, E) is neutrosophic soft Urysohn. $\square$

It is obvious that the neutrosophic soft Urysohn property is an expansive property and thus the converse part follows.

Definition 4.15. A neutrosophic soft topological space (X, τ, E) is said to be
(a) *neutrosophic soft compact* [11] if every neutrosophic soft open cover in (X, τ, E) has a finite subcover,
(b) *neutrosophic soft nearly compact* if every neutrosophic soft open cover in (X, τ, E) by neutrosophic soft regular open sets has a finite sub-cover,
(c) *neutrosophic soft almost compact* if every neutrosophic soft cover in (X, τ, E) has a finite neutrosophic soft proximate sub-cover (i.e., there exists a finite sub-collection Γ_0 of the given neutrosophic soft cover Γ (say) such that $\left\{\overline{\left(\widetilde{U}, E\right)} : \left(\widetilde{U}, E\right) \in \Gamma_0\right\}$ is a neutrosophic soft cover in (X, τ, E)).
Openly, we obtain the diagram below:

<div align="center">

Neutrosophic soft compactness
↓
Neutrosophic soft near compactness
↓
Neutrosophic soft almost compactness

</div>
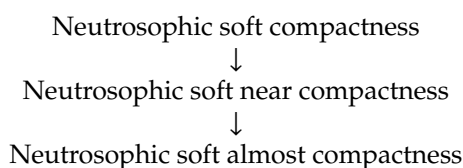
Theorem 4.16. *If a neutrosophic soft topological space (X, τ, E) is neutrosophic soft nearly compact, then (X, τ_S, E) is neutrosophic soft compact.*

Proof. Straightforward. $\square$

Theorem 4.17. *A neutrosophic soft topological space (X, τ, E) is neutrosophic soft nearly compact if (X, τ_S, E) is neutrosophic soft nearly compact.*

Proof. For a neutrosophic soft open cover

$$\Gamma = \left\{\left(\widetilde{U}_\alpha, E\right) : \alpha \in I\right\} \text{ in } (X, \tau, E),\ \left\{\tau - NSint\left(\tau - NScl\left(\widetilde{U}_\alpha, E\right)\right) : \alpha \in I\right\}$$

is a neutrosophic soft open cover in (X, τ_S, E). Since (X, τ_S, E) is neutrosophic soft nearly compact, there exists a finite subfamily

$$\Gamma_1 = \left\{ \tau - NSint\left(\tau - NScl\left(\widetilde{U}_{\alpha_i}, E\right)\right) = \left(\widetilde{U}_{\alpha_i}^S, E\right) : i = 1, 2, 3, ..., n \right\}$$

such that

$$1_{(X,E)} = \bigcup_{i=1}^{n} \left[\tau_S - NSint\left(\tau_S - NScl\left(\widetilde{U}_{\alpha_i}^S, E\right)\right) \right] = \bigcup_{i=1}^{n} \left[\tau - NSint\left(\tau - NScl\left(\widetilde{U}_{\alpha_i}, E\right)\right) \right]$$

Hence (X, τ, E) is neutrosophic soft nearly compact. $\square$

Corollary 4.18. *(a) A neutrosophic soft topological space (X, τ, E) is neutrosophic soft nearly compact if and only if (X, τ_S, E) is a neutrosophic soft compact.*
(b) For a neutrosophic soft topological space (X, τ, E), (X, τ_S, E) is neutrosophic soft compact if and only if it is neutrosophic soft nearly compact.
(c) A neutrosophic soft semi-regular space is neutrosophic soft compact if and only if it is neutrosophic soft nearly compact.

Corollary 4.19. *Neutrosophic soft near compactness is a neutrosophic soft semi-regular property.*

Theorem 4.20. *Neutrosophic soft almost compactness is a neutrosophic soft semi-regular property.*

Proof. If (X, τ, E) is neutrosophic soft almost compact then clearly so is (X, τ_S, E), since $\tau_S \subset \tau$. Conversely, suppose (X, τ_S, E) is neutrosophic soft almost compact and let $\left\{ \left(\widetilde{G}_\alpha, E\right) : \alpha \in I \right\}$ is a neutrosophic soft open cover in (X, τ, E). Then $\left\{ \tau - NSint\left(\tau - NScl\left(\widetilde{G}_\alpha, E\right)\right) : \alpha \in I \right\}$ is a neutrosophic soft open cover in (X, τ_S, E). By neutrosophic soft almost compactness of (X, τ_S, E), there exists a finite subset I_0 of I such that

$$\bigcup \left\{ \tau_S - NScl\left(\tau - NSint\left(\tau - NScl\left(\widetilde{G}_\alpha, E\right)\right)\right) : \alpha \in I_0 \right\} = 1_{(X,E)}$$

By Theorem 3.10, $\tau_S - NScl\left(\tau - NSint\left(\tau - NScl\left(\widetilde{G}_\alpha, E\right)\right)\right)$

$$= \tau - NScl\left(\tau - NSint\left(\tau - NScl\left(\widetilde{G}_\alpha, E\right)\right)\right) = \tau - NScl\left(\widetilde{G}_\alpha, E\right)$$

and hence $\bigcup \left\{ \tau - NScl\left(\widetilde{G}_\alpha, E\right) : \alpha \in I_0 \right\} = 1_{(X,E)}$. $\square$

Definition 4.21. A neutrosophic soft topological space (X, τ, E) is said to be neutrosophic soft S-closed iff every neutrosophic soft cover in (X, τ, E). by neutrosophic soft semi-open sets has a finite neutrosophic soft proximate sub-cover.

Theorem 4.22. *A neutrosophic soft topological space (X, τ, E) is neutrosophic soft S-closed iff every neutrosophic soft cover in (X, τ, E) by neutrosophic soft regular closed sets has a finite sub-cover.*

Proof. It is omitted. $\square$

Theorem 4.23. *Neutrosophic soft S-closedness is a neutrosophic soft semi-regular property.*

Proof. Follows from Theorem 4.22 and the fact that (X, τ, E) and (X, τ_S, E) have the same set of neutrosophic soft regular closed sets. $\square$

Definition 4.24. A neutrosophic soft topological space (X, τ, E) is said to be neutrosophic soft extremely disconnected iff for every neutrosophic soft open set $\left(\widetilde{V}, E\right)$ in (X, τ, E), $\overline{\left(\widetilde{V}, E\right)}$ is neutrosophic soft open.

Example 4.25. Consider that $X = \{x, y\}$ is a universe, $E = \{a, b\}$ be a parametric set. Consider the neutrosophic soft sets $\left(\widetilde{F}, E\right)$ and $\left(\widetilde{G}, E\right)$ defined as

$\widetilde{F}(a) = \{\langle x, 0.2, 0.2, 0.8 \rangle, \langle y, 0.2, 0.2, 0.8 \rangle\}$, $\widetilde{F}(b) = \{\langle x, 0.2, 0.2, 0.8 \rangle, \langle y, 0.2, 0.2, 0.8 \rangle\}$,
$\widetilde{G}(a) = \{\langle x, 0.8, 0.8, 0.2 \rangle, \langle y, 0.8, 0.8, 0.2 \rangle\}$, $\widetilde{G}(b) = \{\langle x, 0.8, 0.8, 0.2 \rangle, \langle y, 0.8, 0.8, 0.2 \rangle\}$.

The family $\tau = \left\{ 0_{(X,E)}, 1_{(X,E)}, \left(\widetilde{F}, E\right), \left(\widetilde{G}, E\right) \right\}$ is a neutrosophic soft topology over X. Then, (X, τ, E) is a neutrosophic soft extremely disconnected topological space.

Theorem 4.26. *The property of a neutrosophic soft topological space being neutrosophic soft extremely disconnected is a neutrosophic soft semi-regular property.*

Proof. Consider that (X, τ, E) is neutrosophic soft extremely disconnected and let $\left(\widetilde{U}, E\right) \in \tau_S$. Then, $\tau_S - NScl\left(\widetilde{U}, E\right) = \tau - NScl\left(\widetilde{U}, E\right) = \left(\widetilde{V}, E\right)$ (say). Since (X, τ, E) is neutrosophic soft extremely disconnected, $\left(\widetilde{V}, E\right) \in \tau$ and consequently,

$$\tau_S - NScl\left(\widetilde{U}, E\right) = \tau - NSint\left(\tau - NScl\left(\widetilde{U}, E\right)\right) = \tau - NSint\left(\tau - NScl\left(\widetilde{V}, E\right)\right) \in \tau_S.$$

Hence (X, τ_S, E) is neutrosophic soft extremely disconnected.

Conversely, suppose (X, τ_S, E) is neutrosophic soft extremely disconnected. For any $\left(\widetilde{U}, E\right) \in \tau$, $\tau_S - NScl\left(\widetilde{U}, E\right) = \tau - NScl\left(\widetilde{U}, E\right)$ and $\tau - NSint\left(\tau - NScl\left(\widetilde{U}, E\right)\right) \in \tau_S$. Put $\tau - NSint\left(\tau - NScl\left(\widetilde{U}, E\right)\right) = \left(\widetilde{V}, E\right)$. Then

$$\tau - NScl\left(\widetilde{U}, E\right) = \tau - NScl\left(\widetilde{V}, E\right) \in \tau_S \subset \tau.$$

Hence (X, τ, E) is neutrosophic soft extremely disconnected. $\square$
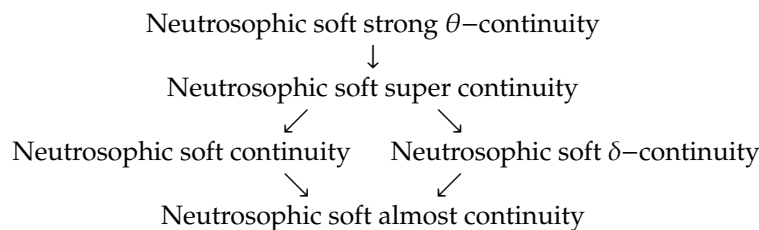
5. Neutrosophic Soft Semi-regularization Spaces and Mappings

Definition 5.1. A function $f : (X, \tau_1, E_1) \rightarrow (Y, \tau_2, E_2)$ is called neutrosophic soft almost continuous (neutrosophic soft δ-continuous) if corresponding to any neutrosophic soft point $x^e_{(\alpha, \beta, \gamma)}$ in (X, τ_1, E_1) and any neutrosophic soft regular open q-neighbourhood $\left(\widetilde{V}, E_2\right)$ of $f\left(x^e_{(\alpha, \beta, \gamma)}\right)$, there is a neutrosophic soft open q-neighbourhood (resp. neutrosophic soft regular open q-neighbourhood) $\left(\widetilde{U}, E_1\right)$ of $x^e_{(\alpha, \beta, \gamma)}$, such that $f\left(\left(\widetilde{U}, E_1\right)\right) \subseteq \left(\widetilde{V}, E_2\right)$.

Equivalently, $f : (X, \tau_1, E_1) \rightarrow (Y, \tau_2, E_2)$ is neutrosophic soft almost continuous iff for every neutrosophic soft regular open (neutrosophic soft regular closed) set $\left(\widetilde{V}, E_2\right)$ in (Y, τ_2, E_2), $f^{-1}\left(\left(\widetilde{V}, E_2\right)\right)$ is neutrosophic soft open (resp. neutrosophic soft closed) in (X, τ_1, E_1).

Definition 5.2. A function $f : (X, \tau_1, E_1) \rightarrow (Y, \tau_2, E_2)$ is said to be neutrosophic soft super continuous (neutrosophic soft strongly θ-continuous) if for every neutrosophic soft point $x^e_{(\alpha, \beta, \gamma)}$ in (X, τ_1, E_1) and for any neutrosophic soft open q- neighbourhood $\left(\widetilde{M}, E_2\right)$ of $f\left(x^e_{(\alpha, \beta, \gamma)}\right)$, there is a neutrosophic soft open q-neighbourhood $\left(\widetilde{N}, E_1\right)$ of $x^e_{(\alpha, \beta, \gamma)}$ such that $f\left(\left[\overline{\left(\widetilde{N}, E_1\right)}\right]^{\circ}\right) \subseteq \left(\widetilde{M}, E_2\right)$ (resp. $f\left(\overline{\left(\widetilde{N}, E_1\right)}\right) \subseteq \left(\widetilde{M}, E_2\right)$.

Then, we obtain the diagram below:

<div align="center">

Neutrosophic soft strong θ−continuity

$\downarrow$

Neutrosophic soft super continuity

$\swarrow \qquad \searrow$

Neutrosophic soft continuity $\qquad$ Neutrosophic soft δ−continuity

$\searrow \qquad \swarrow$

Neutrosophic soft almost continuity

</div>

These implications cannot be reversed as seen in the following examples.

Example 5.3. Consider that $X = \{x, y\}$ is a universe, $E = \{a, b\}$ be a parametric set. Consider the neutrosophic soft sets $\left(\widetilde{F}, E\right)$ defined as $\widetilde{F}(a) = \{\langle x, 0.3, 0.3, 0.7\rangle, \langle y, 0.3, 0.3, 0.7\rangle\}$, $\widetilde{F}(b) = \{\langle x, 0.3, 0.3, 0.7\rangle, \langle y, 0.3, 0.3, 0.7\rangle\}$. The family $\tau = \left\{0_{(X,E)}, 1_{(X,E)}, \left(\widetilde{F}, E\right)\right\}$ is a neutrosophic soft topology over X. So, (X, τ, E) is a neutrosophic soft topological space. Then, the identity map $id_X : (X, \tau, E) \rightarrow (X, \tau, E)$ is neutrosophic soft super continuous but not neutrosophic soft strongly $\theta-$continuous.

Example 5.4. Consider that $X = \{x, y\}$ is a universe, $E = \{a, b\}$ be a parametric set. Consider the neutrosophic soft sets $\left(\widetilde{F}, E\right)$ and $\left(\widetilde{G}, E\right)$ defined as

$\widetilde{F}(a) = \{\langle x, 0.3, 0.3, 0.7\rangle, \langle y, 0.3, 0.3, 0.7\rangle\}$, $\widetilde{F}(b) = \{\langle x, 0.3, 0.3, 0.7\rangle, \langle y, 0.3, 0.3, 0.7\rangle\}$,
$\widetilde{G}(a) = \{\langle x, 0.4, 0.4, 0.6\rangle, \langle y, 0.4, 0.4, 0.6\rangle\}$, $\widetilde{G}(b) = \{\langle x, 0.4, 0.4, 0.6\rangle, \langle y, 0.4, 0.4, 0.6\rangle\}$.

The families $\tau_1 = \left\{0_{(X,E)}, 1_{(X,E)}, \left(\widetilde{G}, E\right), \left(\widetilde{F}, E\right)\right\}$ and $\tau_2 = \left\{0_{(X,E)}, 1_{(X,E)}, \left(\widetilde{F}, E\right)\right\}$ are neutrosophic soft topologies over X. So, (X, τ_1, E_1) and (X, τ_2, E_2) are neutrosophic soft topological spaces. Then, the identity map $id_X : (X, \tau_1, E_1) \rightarrow (X, \tau_2, E_2)$ is neutrosophic soft continuous but not neutrosophic soft super continuous.

Example 5.5. Consider that $X = \{x, y\}$ is a universe, $E = \{a, b\}$ be a parametric set. Consider the neutrosophic soft sets $\left(\widetilde{F}, E\right)$ and $\left(\widetilde{G}, E\right)$ defined as

$\widetilde{F}(a) = \{\langle x, 0.3, 0.3, 0.7\rangle, \langle y, 0.3, 0.3, 0.7\rangle\}$, $\widetilde{F}(b) = \{\langle x, 0.3, 0.3, 0.7\rangle, \langle y, 0.3, 0.3, 0.7\rangle\}$,
$\widetilde{G}(a) = \{\langle x, 0.4, 0.4, 0.6\rangle, \langle y, 0.4, 0.4, 0.6\rangle\}$, $\widetilde{G}(b) = \{\langle x, 0.4, 0.4, 0.6\rangle, \langle y, 0.4, 0.4, 0.6\rangle\}$.

The family $\tau = \left\{0_{(X,E)}, 1_{(X,E)}, \left(\widetilde{G}, E\right), \left(\widetilde{F}, E\right)\right\}$ is a neutrosophic soft topology over X. So, (X, τ, E) is a neutrosophic soft topological space. Then, the identity map $id_X : (X, \tau_1, E_1) \rightarrow (X, \tau_2, E_2)$ is neutrosophic soft $\delta-$continuous but not neutrosophic soft super continuous.

Example 5.6. Consider that $X = \{x, y\}$ is a universe, $E = \{a, b\}$ is a parametric set. Consider the neutrosophic soft sets $\left(\widetilde{F}, E\right)$ and $\left(\widetilde{G}, E\right)$ defined as

$\widetilde{F}(a) = \{\langle x, 0.3, 0.3, 0.7\rangle, \langle y, 0.3, 0.3, 0.7\rangle\}$, $\widetilde{F}(b) = \{\langle x, 0.3, 0.3, 0.7\rangle, \langle y, 0.3, 0.3, 0.7\rangle\}$,
$\widetilde{G}(a) = \{\langle x, 0.4, 0.4, 0.6\rangle, \langle y, 0.4, 0.4, 0.6\rangle\}$, $\widetilde{G}(b) = \{\langle x, 0.4, 0.4, 0.6\rangle, \langle y, 0.4, 0.4, 0.6\rangle\}$.

The families $\tau_1 = \left\{0_{(X,E)}, 1_{(X,E)}, \left(\widetilde{G}, E\right)\right\}$ and $\tau_2 = \left\{0_{(X,E)}, 1_{(X,E)}, \left(\widetilde{G}, E\right), \left(\widetilde{F}, E\right)\right\}$ are neutrosophic soft topologies over X. So, (X, τ_1, E_1) and (X, τ_2, E_2) are neutrosophic soft topological spaces. Then, the identity map $id_X : (X, \tau_1, E_1) \rightarrow (X, \tau_2, E_2)$ isneutrosophic soft almost continuous but not neutrosophic soft continuous.

Example 5.7. Consider that $X = \{x, y\}$ is a universe, $E = \{a, b\}$ is a parametric set. Consider the neutrosophic soft sets $\left(\widetilde{F}, E\right)$, $\left(\widetilde{G}, E\right)$ and $\left(\widetilde{H}, E\right)$ defined as

$\widetilde{F}(a) = \{\langle x, 0.3, 0.3, 0.7\rangle, \langle y, 0.3, 0.3, 0.7\rangle\}$, $\widetilde{F}(b) = \{\langle x, 0.3, 0.3, 0.7\rangle, \langle y, 0.3, 0.3, 0.7\rangle\}$,
$\widetilde{G}(a) = \{\langle x, 0.4, 0.4, 0.6\rangle, \langle y, 0.4, 0.4, 0.6\rangle\}$, $\widetilde{G}(b) = \{\langle x, 0.4, 0.4, 0.6\rangle, \langle y, 0.4, 0.4, 0.6\rangle\}$.

The families $\tau_1 = \left\{0_{(X,E)}, 1_{(X,E)}, \left(\widetilde{G}, E\right), \left(\widetilde{F}, E\right)\right\}$ and $\tau_2 = \left\{0_{(X,E)}, 1_{(X,E)}, \left(\widetilde{F}, E\right)\right\}$ are neutrosophic soft topologies over X. So, (X, τ_1, E_1) and (X, τ_2, E_2) are neutrosophic soft topological spaces. Then, the identity map $id_X : (X, \tau_1, E_1) \rightarrow (X, \tau_2, E_2)$ is neutrosophic soft almost continuous but not neutrosophic soft $\delta-$continuous.

The identity map defined in Example 5.6. is neutrosophic soft $\delta-$continuous but not neutrosophic soft continuous and the identity map defined in Example 5.7. is neutrosophic soft continuous but not neutrosophic soft $\delta-$continuous. This implies that neutrosophic soft $\delta-$ continuity and neutrosophic soft continuity are independent notions.

Theorem 5.8. *Consider that $f : (X, \tau, E_1) \rightarrow (Y, \vartheta, E_2)$ is a neutrosophic soft function. Then*

(a) Neutrosophic soft almost continuity of the function f defined from (X, τ, E_1) to (Y, ϑ, E_2) implies neutrosophic soft continuity of the function f defined from (X, τ, E_1) to (Y, ϑ_S, E_2),

(b) Neutrosophic soft almost continuity of the function f defined from (X, τ, E_1) to (Y, ϑ_S, E_2) implies neutrosophic soft almost continuity of the function f defined from (X, τ, E_1) to (Y, ϑ, E_2),

(c) Neutrosophic soft δ–continuity of the function f defined from (X, τ, E_1) to (Y, ϑ, E_2) implies neutrosophic soft super continuity of the function f defined from (X, τ_S, E_1) to (Y, ϑ_S, E_2),

(d) Neutrosophic soft almost continuity of the function f defined from (X, τ_S, E_1) to (Y, ϑ_S, E_2) implies neutrosophic soft δ–continuity of the function f defined from (X, τ, E_1) to (Y, ϑ, E_2) ,

(e) Neutrosophic soft super continuity of the function f defined from (X, τ, E_1) to (Y, ϑ, E_2) implies neutrosophic soft super continuity of the function f defined from (X, τ_S, E_1) to (Y, ϑ_S, E_2),

(f) Neutrosophic soft continuity of the function f defined from (X, τ_S, E_1) to (Y, ϑ, E_2) implies neutrosophic soft super continuity of the function f defined from (X, τ, E_1) to (Y, ϑ, E_2).

Proof. The proofs, being straightforward, are omitted. □

It is clear that many more results of the type found in Theorem 5.8. can be achieved if the spaces X and/or Y are themselves neutrosophic soft semi-regular or neutrosophic soft almost regular or neutrosophic soft regular. We consider here one of these cases.

Lemma 5.9. *Consider that $f : (X, \tau, E_1) \to (Y, \vartheta, E_2)$ is a mapping. Then, f is a neutrosophic soft almost continuous mapping if and only if $f^{-1}\left(\left(\widetilde{F}, E\right)\right) \subseteq \left[f^{-1}\left(\left[\overline{\left(\widetilde{F}, E\right)}\right]^{\circ}\right)\right]^{\circ}$ for every neutrosophic soft open set $\left(\widetilde{F}, E\right)$ in (Y, ϑ, E_2).*

Proof. ($\Rightarrow$) Since $\left(\widetilde{F}, E\right)$ is a (Y, ϑ, E_2), $\left(\widetilde{F}, E\right) \subseteq \left[\overline{\left(\widetilde{F}, E\right)}\right]^{\circ}$. So, $f^{-1}\left(\left(\widetilde{F}, E\right)\right) \subseteq f^{-1}\left(\left[\overline{\left(\widetilde{F}, E\right)}\right]^{\circ}\right)$. Since $\left[\overline{\left(\widetilde{F}, E\right)}\right]^{\circ}$ is neutrosophic soft regular open, $f^{-1}\left(\left[\overline{\left(\widetilde{F}, E\right)}\right]^{\circ}\right)$ is neutrosophic soft open set in (X, τ, E_1). Thus, $f^{-1}\left(\left(\widetilde{F}, E\right)\right) \subseteq f^{-1}\left(\left[\overline{\left(\widetilde{F}, E\right)}\right]^{\circ}\right) = \left[f^{-1}\left(\left[\overline{\left(\widetilde{F}, E\right)}\right]^{\circ}\right)\right]^{\circ}$.

($\Leftarrow$) Consider that $\left(\widetilde{F}, E\right)$ is a neutrosophic soft regular open set in (Y, ϑ, E_2), then we have $f^{-1}\left(\left(\widetilde{F}, E\right)\right) \subseteq \left[f^{-1}\left(\left[\overline{\left(\widetilde{F}, E\right)}\right]^{\circ}\right)\right]^{\circ} = \left[f^{-1}\overline{\left(\widetilde{F}, E\right)}\right]^{\circ}$. Thus $f^{-1}\left(\left(\widetilde{F}, E\right)\right) = \left[f^{-1}\overline{\left(\widetilde{F}, E\right)}\right]^{\circ}$. This shows that $f^{-1}\left(\left(\widetilde{F}, E\right)\right)$ is neutrosophic soft open set. So, f is a neutrosophic soft almost continuous. □

Theorem 5.10. *If $f : (X, \tau, E_1) \to (Y, \vartheta, E_2)$ is neutrosophic soft almost continuous, where (Y, ϑ, E_2) is neutrosophic soft regular, then $f : (X, \tau_S, E_1) \to (Y, \vartheta, E_2)$ is neutrosophic soft strongly θ–continuous.*

Proof. Suppose that (Y, ϑ, E_2) is neutrosophic soft semi-regular and $f : (X, \tau, E_1) \to (Y, \vartheta, E_2)$ is neutrosophic soft almost continuous. Consider that $\left(\widetilde{F}, E\right)$ be a neutrosophic soft open set in (Y, ϑ, E_2). From the semi-regularity of (Y, ϑ, E_2), $\left(\widetilde{F}, E\right) = \bigcup_{i \in I}\left(\widetilde{F}_i, E\right)$ where $\left(\widetilde{F}_i, E\right)$ is a neutrosophic soft regular open set in (Y, ϑ, E_2) for every $i \in I$. From Lemma 5.9., we get

$$f^{-1}\left(\left(\widetilde{F}, E\right)\right) = f^{-1}\left(\bigcup_{i \in I}\left(\widetilde{F}_i, E\right)\right) \subseteq \bigcup_{i \in I}\left[f^{-1}\left(\left[\overline{\left(\widetilde{F}_i, E\right)}\right]^{\circ}\right)\right]^{\circ}$$
$$= \bigcup_{i \in I}\left[f^{-1}\left(\left(\widetilde{F}_i, E\right)\right)\right]^{\circ} \subseteq \left[\bigcup_{i \in I} f^{-1}\left(\left(\widetilde{F}_i, E\right)\right)\right]^{\circ} = \left[f^{-1}\left(\left(\widetilde{F}_i, E\right)\right)\right]^{\circ}.$$

This implies that f is neutrosophic soft continuous. Consider that $x^e_{(\alpha,\beta,\gamma)}$ be any neutrosophic soft point in (X, τ, E_1) and $\left(\widetilde{U}, E_2\right)$ any neutrosophic soft open q-neighbourhood of $f\left(x^e_{(\alpha,\beta,\gamma)}\right)$. By neutrosophic soft regularity of (Y, ϑ, E_2), there is a neutrosophic soft open q-neighbourhood $\left(\widetilde{V}, E_2\right)$ of $f\left(x^e_{(\alpha,\beta,\gamma)}\right)$ in (Y, ϑ, E_2) such that $\vartheta - NScl\left(\widetilde{V}, E_2\right) \subseteq \left(\widetilde{U}, E_2\right)$. By neutrosophic soft continuity of f, there exists a neutrosophic soft open q- neighbourhood $\left(\widetilde{W}, E_1\right)$ of $x^e_{(\alpha,\beta,\gamma)}$ in (X, τ, E_1) such that $f\left(\left(\widetilde{W}, E_1\right)\right) \subseteq \left(\widetilde{V}, E_2\right)$. Thus

$$f\left(\tau - NScl\left(\widetilde{W}, E_1\right)\right) \subseteq \theta - NScl\, f\left(\left(\widetilde{W}, E_1\right)\right) \subseteq \vartheta - NScl\left(\widetilde{V}, E_2\right) \subseteq \left(\widetilde{U}, E_2\right).$$

If we put $\left(\widetilde{G}, E_1\right) = \tau - NSint\left(\tau - NScl\left(\widetilde{W}, E_1\right)\right)$, then $\left(\widetilde{G}, E_1\right)$ is a neutrosophic soft open q-neighbourhood of $x^e_{(\alpha,\beta,\gamma)}$ in (X, τ_S, E_1). Now,

$$\tau_S - NScl\left(\widetilde{G}, E_1\right) = \tau - NScl\left(\widetilde{G}, E_1\right) = \tau - NScl\left(\widetilde{W}, E_1\right).$$

Hence $f\left(\tau_S - NScl\left(\widetilde{G}, E_1\right)\right) \subseteq \left(\widetilde{U}, E_2\right)$ and $f : (X, \tau, E_1) \to (Y, \vartheta, E_2)$ is neutrosophic soft strongly θ−continuous. $\quad\square$

Definition 5.11. A function $f : (X, \tau, E_1) \to (Y, \vartheta, E_2)$ is called neutrosophic soft weakly continuous if for each neutrosophic soft open set $\left(\widetilde{B}, E_2\right)$ in (Y, ϑ, E_2), $f^{-1}\left(\left(\widetilde{B}, E_2\right)\right) \subseteq \left[f^{-1}\left(\overline{\left(\widetilde{B}, E_2\right)}\right)\right]^\circ$.

Equivalently, a function $f : (X, \tau, E_1) \to (Y, \vartheta, E_2)$ is neutrosophic soft weakly continuous if for any neutrosophic soft point $x^e_{(\alpha,\beta,\gamma)}$ in (X, τ, E_1) and any neutrosophic soft open set $\left(\widetilde{V}, E_2\right)$ containing $f\left(x^e_{(\alpha,\beta,\gamma)}\right)$, there is a neutrosophic soft open set $\left(\widetilde{U}, E_1\right)$, containing $x^e_{(\alpha,\beta,\gamma)}$ such that $f\left(\left(\widetilde{U}, E_1\right)\right) \subseteq \overline{\left(\widetilde{V}, E_2\right)}$.

It is known that a neutrosophic soft almost continuous function is always neutrosophic soft weakly continuous. But, the converse may not be true as seen in example below.

Example 5.12. Consider that $X = \{x, y\}$ is a universe, $E = \{a, b\}$ is a parametric set, the neutrosophic soft sets $\left(\widetilde{F}, E\right)$ and $\left(\widetilde{G}, E\right)$ are defined as

$\widetilde{F}(a) = \{\langle x, 0.3, 0.3, 0.7\rangle, \langle y, 0.3, 0.3, 0.7\rangle\}$, $\widetilde{F}(b) = \{\langle x, 0.3, 0.3, 0.7\rangle, \langle y, 0.3, 0.3, 0.7\rangle\}$,

$\widetilde{G}(a) = \{\langle x, 0.4, 0.4, 0.6\rangle, \langle y, 0.4, 0.4, 0.6\rangle\}$, $\widetilde{G}(b) = \{\langle x, 0.4, 0.4, 0.6\rangle, \langle y, 0.4, 0.4, 0.6\rangle\}$.

The families $\tau_1 = \left\{0_{(X,E)}, 1_{(X,E)}, \left(\widetilde{G}, E\right)\right\}$ and $\tau_2 = \left\{0_{(X,E)}, 1_{(X,E)}, \left(\widetilde{F}, E\right)\right\}$ are neutrosophic soft topologies over X. So, (X, τ_1, E_1) and (X, τ_2, E_2) are neutrosophic soft topological spaces.

Then, the identity map $id_X : (X, \tau_1, E_1) \to (X, \tau_2, E_2)$ is neutrosophic soft weakly continuous but not neutrosophic soft almost continuous.

6. Conclusion

We have introduced the concepts of neutrosophic soft semi-regularization topology, neutrosophic soft semi-regularization space, neutrosophic soft semi-regular space and neutrosophic soft sub-maximal space. The definitions of neutrosophic soft quasi-coincidence and neutrosophic soft q-neighbourhood have been also given. Using these definitions, we have defined the concepts of neutrosophic soft almost regular space and neutrosophic soft almost regular space, and their properties have been analysed. A new approach to the concept of neutrosophic soft separation axioms has been made, and the characteristics of neutrosophic soft semi-regularization spaces related to neutrosophic soft separation axioms have been examined. Additionally, the notions of neutrosophic soft near compactness and neutrosophic soft almost compactness have been introduced, and relationships of them have been shown with a diagram. Furthermore, some certain types of continuous mappings defined in general and fuzzy topological spaces have been adapted to neutrosophic soft topological spaces, and a diagram has been obtained which shows the relations of these mappings. Some examples have been given to show that the converse implications are not true in general. The properties of neutrosophic soft compactness, neutrosophic soft near compactness and neutrosophic soft almost compactness have been analyzed under these mappings. Since several mathematicians focused on topological structures of neutrosophic soft sets, some terms have been generalized to the neutrosophic soft sets which may be beneficial in different fields. We hope that many researchers will benefit from the findings in this document to further their studies on neutrosophic soft topology to carry out a general framework for their applications in real life problems.

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