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A Hilbert's Type Inequality with Four Parameters

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Abstract. In this paper, by introducing four parameters A, B, α, β and using the Euler-Maclaurin expansion for the Riemann zeta function, we establish an inequality of a weight coefficient. Using this inequality, we derive generalizations of a Hilbert's type inequality.

1. Introduction

If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $b_n \geq 0$, for $n \geq 1, n \in N$ and $0 < \sum\limits_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum\limits_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1}$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < pq \left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{2}$$

where the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ and pq is best possible for each inequality respectively. Inequality (1) is Hardy-Hilbert's inequality. Inequality (2) is a Hilbert's type inequality [1].

In [4], [10] and [9], Krnić, Pečarić and Yang gave some generalization and reinforcement of inequality (1). In [2], Kuang and Debnath gave a reinforcement of inequality (2):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < \left\{\sum_{n=1}^{\infty} [pq - G(p,n)] a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} [pq - G(q,n)] b_n^q\right\}^{\frac{1}{q}}, \tag{3}$$

where $G(r,n) = \frac{r + \frac{1}{3r} - \frac{4}{3}}{(2n+1)^{\frac{1}{r}}} > 0$ $(r = p, q)$.

In [8] ,Xi gave a generalization and reinforcement of inequalities (2) and (3):

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$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max(m^\lambda,n^\lambda)} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3qn^{\frac{q+\lambda-2}{q}}}\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3pn^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \quad (4)$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}>0$, $2-\min\{p,q\}<\lambda\le 2$.

In [5] and [7], Zhang and Xi gave two generalizations of inequalities (4):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda+A,n^\lambda+B\}} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{B}{1+B}\right)\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{A}{1+A}\right)\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}} \quad (5)$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}+\alpha} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{\alpha}{1+\alpha}\right)\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{\alpha}{1+\alpha}\right)\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \quad (6)$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}>0$, $2-\min\{p,q\}<\lambda\le 2$, $0\le A\le B\le\min\{\frac{1}{3p-1},\frac{1}{3q-1}\}$, $0\le\alpha\le\min\{\frac{1}{3p-1},\frac{1}{3q-1}\}$.

In [6], Xi gave a generalizations of inequalities (5) and (6):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda+A,n^\lambda+B\}+\alpha} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{B+\alpha}{1+B+\alpha}\right)\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{A+\alpha}{1+A+\alpha}\right)\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \quad (7)$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}>0$, $2-\min\{p,q\}<\lambda\le 2$,$A,B,\alpha\ge 0$, and $A+\alpha\le B+\alpha\le\min\{\frac{1}{3p-1},\frac{1}{3q-1}\}$.

In this paper, by introducing four parameters A,B,α,β and using the Euler-Maclaurin expansion for the Riemann zeta function, we establish an inequality for a weight coefficient. Using this inequality, we derive a generalization of inequalities (5), (6) and (7).

2. A Lemma

First, we need the following formula of the Riemann-ζ function (see [3], [12] and [11]):

$$\zeta(\sigma) = \sum_{k=1}^{n}\frac{1}{k^\sigma}-\frac{n^{1-\sigma}}{1-\sigma}-\frac{1}{2n^\sigma}-\sum_{k=1}^{l-1}\frac{B_{2k}}{2k}\binom{-\sigma}{2k-1}\frac{1}{n^{\sigma+2k-1}} -\frac{B_{2l}}{2l}\binom{-\sigma}{2l-1}\frac{\varepsilon}{n^{\sigma+2l-1}}, \quad (8)$$

where $\sigma > 0$, $\sigma \neq 1$, $n,\ l \geq 1$, $n,\ l \in N$, $0 < \varepsilon = \varepsilon(\sigma, l, n) < 1$. The numbers $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $\cdots$ are Bernoulli numbers. In particular, $\zeta(\sigma) = \sum_{k=1}^{\infty} \frac{1}{k^\sigma}$ $(\sigma > 1)$.

Lemma 2.1. *Since $\zeta(0) = -1/2$, then the formula of the Riemann-ζ function (8) is also true for $\sigma = 0$. If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - min\{p, q\} < \lambda \leq 2$, $A, B, \alpha, \beta \geq 0$, and $A + \alpha \leq B + \alpha \leq \min\{\frac{\beta}{3p-1}, \frac{\beta}{3q-1}\}$, $n \geq 1$ and $n \in N$, then*

$$\begin{aligned} \omega(n, \lambda, p, A, B, \alpha, \beta) &= \sum_{k=1}^{\infty} \frac{1}{\max\{\beta k^\lambda + A, \beta n^\lambda + B\} + \alpha} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\ &< \frac{n^{1-\lambda}}{\beta} \left[\kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}} \left(\frac{1}{3p} - \frac{A+\alpha}{\beta + A + \alpha}\right)\right], \end{aligned} \tag{9}$$

and

$$\begin{aligned} \omega(n, \lambda, q, B, A, \alpha, \beta) &= \sum_{k=1}^{\infty} \frac{1}{\max\{\beta k^\lambda + B, \beta n^\lambda + A\} + \alpha} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} \\ &< \frac{n^{1-\lambda}}{\beta} \left[\kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}} \left(\frac{1}{3q} - \frac{B+\alpha}{\beta + B + \alpha}\right)\right], \end{aligned} \tag{10}$$

where $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$. When $\lambda = 1$, we have following the stronger inequality:

$$\begin{aligned} \omega(n, 1, p, A, B, \alpha, \beta) &= \sum_{k=1}^{\infty} \frac{1}{\max\{\beta k + A, \beta n + B\}} \left(\frac{n}{k}\right)^{\frac{1}{p}} \\ &< \frac{1}{\beta} \left[pq - \frac{1}{n^{\frac{1}{q}}} \left(\frac{12q^2 + 3q + 5p}{12pq} - \frac{A+\alpha}{\beta + A + \alpha}\right)\right], \end{aligned} \tag{11}$$

and

$$\begin{aligned} \omega(n, 1, q, B, A, \alpha, \beta) &= \sum_{k=1}^{\infty} \frac{1}{\max\{\beta k + A, \beta n + B\}} \left(\frac{n}{k}\right)^{\frac{1}{q}} \\ &< \frac{1}{\beta} \left[pq - \frac{1}{n^{\frac{1}{p}}} \left(\frac{12p^2 + 3p + 5q}{12pq} - \frac{B+\alpha}{\beta + B + \alpha}\right)\right]. \end{aligned} \tag{12}$$

Proof. Equalities (9) and (10) define the weight coefficient. When $2 - min\{p, q\} < \lambda \leq 2$, taking $\sigma = \frac{2-\lambda}{p} \geq 0$, $l = 1$, in (8), we obtain

$$\zeta\left(\frac{2-\lambda}{p}\right) = \sum_{k=1}^{n} \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2} - \frac{1}{2n^{\frac{2-\lambda}{p}}} + \frac{2-\lambda}{12pn^{1+\frac{2-\lambda}{p}}}\varepsilon_1, \tag{13}$$

where $0 < \varepsilon_1 < 1$.

Taking $\sigma = \frac{2}{p} + \frac{\lambda}{q}$, $l = 1$, we obtain

$$\zeta\left(\frac{2}{p} + \frac{\lambda}{q}\right) = \sum_{k=1}^{n-1} \frac{1}{k^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{p\lambda + 2q}{12pqn^{1+\frac{2}{p}+\frac{\lambda}{q}}}\varepsilon_2, \tag{14}$$

where $0 < \varepsilon_2 < 1$.

In addition,

$$\begin{aligned}\omega(n,\lambda,p,A,B,\alpha,\beta) &= \sum_{k=1}^{\infty}\frac{1}{\max\{\beta k^{\lambda}+A,\beta n^{\lambda}+B\}+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}\\ &= \sum_{k=1}^{n}\frac{1}{\max\{\beta k^{\lambda}+A,\beta n^{\lambda}+B\}+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}\\ &\quad+\sum_{k=n+1}^{\infty}\frac{1}{\max\{\beta k^{\lambda}+A,\beta n^{\lambda}+B\}+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}.\end{aligned}$$

For $n \geq 1$, $n \in N$, $2-\min\{p,q\} < \lambda \leq 2$, $A,B,\alpha,\beta \geq 0$, and $A+\alpha \leq B+\alpha \leq \min\{\frac{\beta}{3p-1},\frac{\beta}{3q-1}\}$, we have $A+\alpha \leq B+\alpha \leq \min\{\frac{\beta}{3p-1},\frac{\beta}{3q-1}\} \leq \frac{\beta}{2} < \beta$. So

$$\begin{aligned}\omega(n,\lambda,p,A,B,\alpha,\beta) &= \sum_{k=1}^{n}\frac{1}{\max\{\beta k^{\lambda}+A,\beta n^{\lambda}+B\}+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}\\ &\quad+\sum_{k=n+1}^{\infty}\frac{1}{\beta k^{\lambda}+A+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}\\ &= \sum_{k=1}^{n}\frac{1}{\max\{\beta k^{\lambda}+A,\beta n^{\lambda}+B\}+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}-\frac{1}{\beta n^{\lambda}+A+\alpha}\\ &\quad+\sum_{k=n}^{\infty}\frac{1}{\beta k^{\lambda}+A+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}\\ &= \sum_{k=1}^{n}\frac{1}{\beta n^{\lambda}+B+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}-\frac{1}{\beta n^{\lambda}+A+\alpha}+\sum_{k=n}^{\infty}\frac{1}{\beta k^{\lambda}+A+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}\\ &\leq \sum_{k=1}^{n}\frac{1}{\beta n^{\lambda}}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}-\frac{1}{\beta n^{\lambda}+A+\alpha}+\sum_{k=n}^{\infty}\frac{1}{\beta k^{\lambda}}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}\\ &= \frac{1}{\beta n^{\frac{(p+1)\lambda-2}{p}}}\sum_{k=1}^{n}\frac{1}{k^{\frac{2-\lambda}{p}}}-\frac{1}{\beta n^{\lambda}+A+\alpha}+\frac{n^{\frac{2-\lambda}{p}}}{\beta}\sum_{k=n}^{\infty}\frac{1}{k^{\frac{2}{p}+\frac{\lambda}{q}}}.\end{aligned}$$

By (13) and (14)

$$\begin{aligned}\omega(n,\lambda,p,A,B,\alpha,\beta) &< \frac{1}{\beta n^{\frac{(p+1)\lambda-2}{p}}}\left[\zeta\left(\frac{2-\lambda}{p}\right)+\frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2}+\frac{1}{2n^{\frac{2-\lambda}{p}}}\right]-\frac{1}{\beta n^{\lambda}+A+\alpha}\\ &\quad+\frac{n^{\frac{2-\lambda}{p}}}{\beta}\left[\frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2}+\frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}}+\frac{p\lambda+2q}{12p\,qn^{1+\frac{2}{p}+\frac{\lambda}{q}}}\right]\\ &= \frac{1}{\beta n^{\frac{(p+1)\lambda-2}{p}}}\zeta\left(\frac{2-\lambda}{p}\right)+\frac{pn^{\,1-\lambda}}{\beta(p+\lambda-2)}+\frac{1}{2\beta n^{\lambda}}-\frac{1}{\beta n^{\lambda}+A+\alpha}+\frac{qn^{\,1-\lambda}}{\beta(q+\lambda-2)}\\ &\quad+\frac{1}{2\beta n^{\lambda}}+\frac{p\lambda+2q}{12\beta p\,qn^{1+\lambda}}\end{aligned}$$

$$= \frac{1}{\beta n^{\frac{(p+1)\lambda-2}{p}}}\zeta\left(\frac{2-\lambda}{p}\right) + \frac{pq\lambda\, n^{1-\lambda}}{\beta(p+\lambda-2)(q+\lambda-2)} + \frac{p\lambda+2q}{12\beta pqn^{1+\lambda}} + \frac{A+\alpha}{\beta n^{\lambda}(\beta n^{\lambda}+A+\alpha)}$$

$$= \frac{n^{1-\lambda}}{\beta}\left\{\kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}}\left[-\zeta\left(\frac{2-\lambda}{p}\right) - \frac{p\lambda+2q}{12pqn^{\frac{p-\lambda+2}{p}}} - \frac{A+\alpha}{n^{\frac{2-\lambda}{p}}(\beta n^{\lambda}+A+\alpha)}\right]\right\}.$$

In (13), taking $n = 1$, by $2 - \min\{p, q\} < \lambda \leq 2$, we obtain

$$\begin{aligned}\zeta\left(\frac{2-\lambda}{p}\right) &= 1 - \frac{p}{p+\lambda-2} - \frac{1}{2} + \frac{(2-\lambda)\varepsilon_1}{12p}\\ &< \frac{1}{2} - \frac{p}{p+\lambda-2} + \frac{2-\lambda}{12p}\\ &= -\frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)}\\ &< 0.\end{aligned}$$

So for $n \geq 1$, $n \in N$, $2 - \min\{p, q\} < \lambda \leq 2$, $A, B, \alpha, \beta \geq 0$, and $A + \alpha \leq B + \alpha \leq \min\{\frac{\beta}{3p-1}, \frac{\beta}{3q-1}\}$, we have

$$\begin{aligned}&-\zeta\left(\frac{2-\lambda}{p}\right) - \frac{p\lambda+2q}{12pqn^{\frac{p-\lambda+2}{p}}} - \frac{A+\alpha}{n^{\frac{2-\lambda}{p}}(\beta n^{\lambda}+A+\alpha)}\\ &> \frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)} - \frac{p\lambda+2q}{12pq} - \frac{\alpha+A}{\beta+\alpha+A}\\ &= \frac{q(\lambda-2-3p)(\lambda-2-2p) - (p\lambda+2q)(p+\lambda-2)}{12pq(p+\lambda-2)} - \frac{A+\alpha}{\beta+A+\alpha}\\ &> \frac{-p(p\lambda+2q)+6p^2q}{12pq(p+\lambda-2)} - \frac{A+\alpha}{\beta+A+\alpha}\\ &\geq \frac{-(2p+2q)+6pq}{12q(p+\lambda-2)} - \frac{A+\alpha}{\beta+A+\alpha}\\ &> \frac{1}{3(p+\lambda-2)} - \frac{A+\alpha}{\beta+A+\alpha}\\ &> \frac{1}{3p} - \frac{A+\alpha}{\beta+A+\alpha}\\ &\geq 0.\end{aligned}$$

Using the last result and the inequality for $\omega(n, \lambda, p, A, B, \alpha, \beta)$ above, we obtain (9).

When $\lambda = 1$, we have

$$\begin{aligned}&-\zeta\left(\frac{2-\lambda}{p}\right) - \frac{p\lambda+2q}{12pqn^{\frac{p-\lambda+2}{p}}} - \frac{A+\alpha}{n^{\frac{2-\lambda}{p}}(\beta n^{\lambda}+A+\alpha)}\\ &> \frac{q(\lambda-2)^2 + (p\lambda+5pq+2q)(2-\lambda) - p(p\lambda+2q) + 6p^2q}{12pq(p+\lambda-2)} - \frac{A+\alpha}{\beta+A+\alpha}\\ &= \frac{p+3q-p^2+3pq+6p^2q}{12pq(p-1)} - \frac{A+\alpha}{\beta+A+\alpha}\end{aligned}$$

$$
\begin{aligned}
&= \frac{5p^2+10p+12q}{12pq(p-1)} - \frac{A+\alpha}{\beta+A+\alpha} \\
&= \frac{(5p^2+10p+12q)(q-1)}{12pq} - \frac{A+\alpha}{\beta+A+\alpha} \\
&= \frac{12q^2+3q+5p}{12pq} - \frac{A+\alpha}{\beta+A+\alpha}.
\end{aligned}
$$

Using the last result and the inequality for $\omega(n,\lambda,p,A,B,\alpha,\beta)$ above, we obtain (11).

In a similar way, we have

$$
\begin{aligned}
\omega(m,\lambda,q,B,A,\alpha,\beta) &= \sum_{k=1}^{\infty} \frac{1}{\max\{\beta m^\lambda + A, \beta k^\lambda + B\} + \alpha}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}} \\
&= \sum_{k=1}^{m} \frac{1}{\max\{\beta m^\lambda + A, \beta k^\lambda + B\} + \alpha}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}} \\
&\quad + \sum_{k=m+1}^{\infty} \frac{1}{\max\{\beta m^\lambda + A, \beta k^\lambda + B\} + \alpha}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}} \\
&= \sum_{k=1}^{m} \frac{1}{\max\{\beta m^\lambda + A, \beta k^\lambda + B\} + \alpha}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}} \\
&\quad + \sum_{k=m+1}^{\infty} \frac{1}{\beta k^\lambda + B + \alpha}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}} \\
&= \sum_{k=1}^{m} \frac{1}{\max\{\beta m^\lambda + A, \beta k^\lambda + B\} + \alpha}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{\beta m^\lambda + B + \alpha} \\
&\quad + \sum_{k=m}^{\infty} \frac{1}{\beta k^\lambda + B + \alpha}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}} \\
&\le \sum_{k=1}^{m} \frac{1}{\beta m^\lambda + A + \alpha}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{\beta m^\lambda + B + \alpha} + \sum_{k=m}^{\infty} \frac{1}{\beta k^\lambda + B + \alpha}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}} \\
&\le \sum_{k=1}^{m} \frac{1}{\beta m^\lambda}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{\beta m^\lambda + B + \alpha} + \sum_{k=m}^{\infty} \frac{1}{\beta k^\lambda}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}} \\
&= \frac{1}{\beta m^{\frac{(q+1)\lambda-2}{q}}} \sum_{k=1}^{m} \frac{1}{k^{\frac{2-\lambda}{q}}} - \frac{1}{\beta m^\lambda + B + \alpha} + \frac{m^{\frac{2-\lambda}{q}}}{\beta} \sum_{k=m}^{\infty} \frac{1}{k^{\frac{2}{q}+\frac{\lambda}{p}}}.
\end{aligned}
$$

By (13) and (14)

$$
\begin{aligned}
\omega(m,\lambda,q,B,A,\alpha,\beta) &< \frac{1}{\beta m^{\frac{(q+1)\lambda-2}{q}}}\left[\zeta\left(\frac{2-\lambda}{q}\right) + \frac{q m^{\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2m^{\frac{2-\lambda}{q}}}\right] - \frac{1}{\beta m^\lambda + B + \alpha} \\
&\quad + \frac{m^{\frac{2-\lambda}{q}}}{\beta}\left[\frac{p m^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2m^{\frac{2}{q}+\frac{\lambda}{p}}} + \frac{q\lambda+2p}{12q\, p m^{1+\frac{2}{q}+\frac{\lambda}{p}}}\right] \\
&= \frac{m^{1-\lambda}}{\beta}\left\{\kappa(\lambda) - \frac{1}{m^{\frac{q+\lambda-2}{q}}}\left[-\zeta\left(\frac{2-\lambda}{q}\right) - \frac{q\lambda+2p}{12q\, p m^{\frac{q-\lambda+2}{q}}} - \frac{B+\alpha}{m^{\frac{2-\lambda}{q}}(\beta m^\lambda + B + \alpha)}\right]\right\}.
\end{aligned}
$$

Since for $m \geq 1$, $m \in N$, $2 - \min\{p, q\} < \lambda \leq 2$, $A, B, \alpha \geq 0$, and $A + \alpha \leq B + \alpha \leq \min\{\frac{\beta}{3p-1}, \frac{\beta}{3q-1}\}$, we have

$$\begin{aligned}
&-\zeta\left(\frac{2-\lambda}{q}\right) - \frac{q\lambda + 2p}{12q\,pm^{\frac{q-\lambda+2}{q}}} - \frac{B+\alpha}{m^{\frac{2-\lambda}{q}}(\beta m^{\lambda} + B + \alpha)} \\
&\quad > \frac{(\lambda - 2 - 3q)(\lambda - 2 - 2q)}{12q(q + \lambda - 2)} - \frac{q\lambda + 2p}{12pq} - \frac{B+\alpha}{\beta + B + \alpha} \\
&\quad > \frac{1}{3q} - \frac{B+\alpha}{\beta + B + \alpha} \\
&\quad \geq 0.
\end{aligned}$$

Using the last result and the inequality for $\omega(m, \lambda, q, A, B, \alpha, \beta)$ above, we obtain (10).

When $\lambda = 1$, we have

$$\begin{aligned}
&-\zeta\left(\frac{2-\lambda}{q}\right) - \frac{q\lambda + 2p}{12q\,pm^{\frac{q-\lambda+2}{q}}} - \frac{B+\alpha}{m^{\frac{2-\lambda}{q}}(\beta m^{\lambda} + B + \alpha)} \\
&\quad > \frac{p(\lambda-2)^2 + (q\lambda + 5pq + 2p)(2-\lambda) - q(q\lambda + 2p) + 6q^2 p}{12pq(q + \lambda - 2)} - \frac{B+\alpha}{\beta + B + \alpha} \\
&\quad = \frac{12p^2 + 3p + 5q}{12pq} - \frac{B+\alpha}{\beta + B + \alpha}.
\end{aligned}$$

Using the last result and the inequality for $\omega(m, \lambda, q, A, B, \alpha, \beta)$ above, we obtain (12). □

3. Main Results

Theorem 3.1. *If* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \leq 2$, $A, B, \alpha, \beta \geq 0$, *and* $A + \alpha \leq B + \alpha \leq \min\{\frac{\beta}{3p-1}, \frac{\beta}{3q-1}\}$, $a_n \geq 0$, $b_n \geq 0$, *for* $n \geq 1, n \in N$ *and* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then*

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{\beta m^{\lambda} + A, \beta n^{\lambda} + B\} + \alpha} &< \frac{1}{\beta}\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q} - \frac{B+\alpha}{\beta + B + \alpha}\right)\right] n^{1-\lambda} a_n^p\right\}^{\frac{1}{p}} \\
&\quad \times \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p} - \frac{A+\alpha}{\beta + A + \alpha}\right)\right] n^{1-\lambda} b_n^q\right\}^{\frac{1}{q}}, \qquad (15)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{m=1}^{\infty} m^{(1-p)(\lambda-1)}\left(\sum_{n=1}^{\infty} \frac{a_n}{\max\{\beta m^{\lambda} + A, \beta n^{\lambda} + B\} + \alpha}\right)^{p} \\
&\quad < \frac{\kappa(\lambda)^{p-1}}{\beta^p}\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q} - \frac{A+\alpha}{\beta + A + \alpha}\right)\right] n^{1-\lambda} a_n^p, \qquad (16)
\end{aligned}$$

where $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$. *When* $\lambda = 1$, *we have*

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{\beta m + A, \beta n + B\} + \alpha} &< \frac{1}{\beta}\left\{\sum_{n=1}^{\infty}\left[pq - \frac{1}{n^{\frac{1}{p}}}\left(\frac{12p^2 + 3p + 5q}{12pq} - \frac{B+\alpha}{\beta + B + \alpha}\right)\right]\right. \\
&\quad \left.\times\, a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty}\left[pq - \frac{1}{n^{\frac{1}{q}}}\left(\frac{12q^2 + 3q + 5p}{12pq} - \frac{A+\alpha}{\beta + A + \alpha}\right)\right] b_n^q\right\}^{\frac{1}{q}}. \qquad (17)
\end{aligned}$$

Proof. By Hölder's inequality, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{\beta m^\lambda+A,\beta n^\lambda+B\}+\alpha}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m}{(\max\{\beta m^\lambda+A,\beta n^\lambda+B\}+\alpha)^{\frac{1}{p}}}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}}\right]$$
$$\times\left[\frac{b_n}{(\max\{\beta m^\lambda+A,\beta n^\lambda+B\}+\alpha)^{\frac{1}{q}}}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{pq}}\right]$$
$$\leq\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m^p}{\max\{\beta m^\lambda+A,\beta n^\lambda+B\}+\alpha}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{q}}\right]\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{b_n^q}{\max\{\beta m^\lambda+A,\beta n^\lambda+B\}+\alpha}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{p}}\right]\right\}^{\frac{1}{q}}$$
$$=\left\{\sum_{m=1}^{\infty}\omega(m,\lambda,q,B,A,\alpha,\beta)a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega(n,\lambda,p,A,B,\alpha,\beta)b_n^q\right\}^{\frac{1}{q}}.$$

By (9), (10), (11) and (12), we obtain (15) and (17).
By Hölder's inequality and Lemma 2.1, we have

$$\sum_{n=1}^{\infty}\frac{a_n}{\max\{\beta m^\lambda+A,\beta n^\lambda+B\}+\alpha}=\sum_{n=1}^{\infty}\left[\frac{1}{(\max\{\beta m^\lambda+A,\beta n^\lambda+B\}+\alpha)^{\frac{1}{p}}}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{pq}}\right.$$
$$\left.\times a_n\frac{1}{(\max\{\beta m^\lambda+A,\beta n^\lambda+B\}+\alpha)^{\frac{1}{q}}}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}}\right]$$
$$\leq\left\{\sum_{n=1}^{\infty}\left[\frac{1}{\max\{\beta m^\lambda+A,\beta n^\lambda+B\}+\alpha}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}}a_n^p\right]\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=1}^{\infty}\left[\frac{1}{\max\{\beta m^\lambda+A,\beta n^\lambda+B\}+\alpha}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{p}}\right]\right\}^{\frac{1}{q}}$$
$$=\left\{\sum_{n=1}^{\infty}\left[\frac{1}{\max\{\beta m^\lambda+A,\beta n^\lambda+B\}+\alpha}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}}a_n^p\right]\right\}^{\frac{1}{p}}[\omega(m,\lambda,p,B,A,\alpha,\beta)]^{\frac{1}{q}}$$
$$<\left\{\sum_{n=1}^{\infty}\left[\frac{1}{\max\{\beta m^\lambda+A,\beta n^\lambda+B\}+\alpha}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}}a_n^p\right]\right\}^{\frac{1}{p}}[\frac{m^{1-\lambda}}{\beta}\kappa(\lambda)]^{\frac{1}{q}}.$$

So

$$\sum_{m=1}^{\infty}m^{(1-p)(\lambda-1)}\left(\sum_{n=1}^{\infty}\frac{a_n}{\max\{\beta m^\lambda+A,\beta n^\lambda+B\}+\alpha}\right)^p$$
$$<\kappa(\lambda)^{p-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{\max\{\beta m^\lambda+A,\beta n^\lambda+B\}+\alpha}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}}a_n^p\right]$$
$$<[\frac{\kappa(\lambda)}{\beta}]^{p-1}\sum_{n=1}^{\infty}\omega(n,\lambda,q,A,B,\alpha,\beta)a_n^p.$$

By Lemma 2.1, the proof of the theorem is completed. □

In inequality (17), taking $p=q=2$, we have:

Corollary 3.2. *Let* $a_n \geq 0$, $b_n \geq 0$, $A, B, \alpha, \beta \geq 0, A + \alpha \leq B + \alpha \leq \frac{\beta}{5}$, *and* $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$, $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{\beta m + A, \beta n + B\} + \alpha} < \frac{4}{\beta}\left\{\sum_{n=1}^{\infty}\left[1 - \frac{1}{3\sqrt{n}}\left(1 - \frac{3B + 3\alpha}{4\beta + 4B + 4\alpha}\right)\right] a_n^2\right\}^{\frac{1}{2}} \times\left\{\sum_{n=1}^{\infty}\left[1 - \frac{1}{3\sqrt{n}}\left(1 - \frac{3A + 3\alpha}{4\beta + 4A + 4\alpha}\right)\right] b_n^2\right\}^{\frac{1}{2}}. \tag{18}$$

In inequality (15), taking $\beta = 1$, we obtain:

Corollary 3.3. *If* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \leq 2$,$A, B, \alpha \geq 0$, *and* $A + \alpha \leq B + \alpha \leq \min\{\frac{1}{3p-1}, \frac{1}{3q-1}\}$, $a_n \geq 0$, $b_n \geq 0$, *for* $n \geq 1, n \in N$ *and* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda + A, n^\lambda + B\} + \alpha} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q} - \frac{B + \alpha}{1 + B + \alpha}\right)\right] n^{1-\lambda} a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p} - \frac{A + \alpha}{1 + A + \alpha}\right)\right] n^{1-\lambda} b_n^q\right\}^{\frac{1}{q}}. \tag{19}$$

In inequality (15), taking $\beta = 1, \alpha = 0$, we obtain:

Corollary 3.4. *If* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \leq 2$, *and* $0 \leq A \leq B \leq \min\{\frac{1}{3p-1}, \frac{1}{3q-1}\}$, $a_n \geq 0$, $b_n \geq 0$, *for* $n \geq 1, n \in N$ *and* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda + A, n^\lambda + B\}} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q} - \frac{B}{1 + B}\right)\right] n^{1-\lambda} a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p} - \frac{A}{1 + A}\right)\right] n^{1-\lambda} b_n^q\right\}^{\frac{1}{q}}. \tag{20}$$

In inequality (15), taking $\beta = 1, A = 0, B = 0$, we obtain:

Corollary 3.5. *If* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \leq 2$, $0 \leq \alpha \leq \min\{\frac{1}{3p-1}, \frac{1}{3q-1}\}$, $a_n \geq 0$, $b_n \geq 0$, *for* $n \geq 1, n \in N$ *and* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\} + \alpha} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q} - \frac{\alpha}{1 + \alpha}\right)\right] n^{1-\lambda} a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p} - \frac{\alpha}{1 + \alpha}\right)\right] n^{1-\lambda} b_n^q\right\}^{\frac{1}{q}}. \tag{21}$$

Apparently, inequality (20) is inequality (5). inequality (21) is inequality (6). inequality (19) is inequality (7). So, inequality (15) is a generalization of inequalities (5), (6) and (7).

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