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On the Relationship and the Norm Discretization for Lizorkin-Triebel Spaces with Generalized Smoothness

Vedaste Mutarutinya[a], **Marco Mpimbo**[b], **Gabriel Nshizirungu**[b]

[a] *Department of Mathematics, Science and Physical Education, University of Rwanda - College of Education, Kayonza, Rwanda.*
[b] *Department of Mathematics, University of Dar es Salaam, Dar es Salaam, Tanzania.*

Abstract. Homogeneous and non-homogeneous Lizorkin-Triebel spaces with generalized smoothness $\dot{F}^{\lambda(.)}_{pq}(\mathbb{R}^n)$ and $F^{\lambda(.)}_{pq}(\mathbb{R}^n)$ have been considered. In particular, under some assumptions of the function $\lambda(t) : \mathbb{R}^+ \to \mathbb{R}^+, \lambda(1) = 1$ determining the generalized smoothness properties, the discretization procedure is realized and the relationship is established between these spaces on $\mathbb{R}^n$ and their discrete analogues.

1. Introduction

In this paper, homogeneous and non-homogeneous Lizorkin-Triebel spaces with generalized smoothness $\dot{F}^{\lambda(.)}_{pq}(\mathbb{R}^n)$ and $F^{\lambda(.)}_{pq}(\mathbb{R}^n)$ are discussed. Under some assumptions of the function $\lambda(t) : \mathbb{R}^+ \to \mathbb{R}^+, \lambda(1) = 1$ determining the generalized smoothness properties, the discretization procedure is realized and the relationship is established between these spaces on $\mathbb{R}^n$ and their discrete analogues. In modern mathematical analysis, the function spaces play the key role. The most important among these spaces is the class of spaces containing smooth functions (in general functions with non entire order). One can cite the widely used spaces that are Nikoliski-Bezov $[\beta^{\alpha}_{pq}]$ and Lizorkin- Triebel $[F^{\alpha}_{pq}]$ spaces (Frazier and Jawerth 1988, Frazier et al.1991, Mutarutinya 1996, Mutarutinya 1999). The authors Farkas and Leopold 2006, studied the function spaces of generalized smoothness of Besov and Triebel– Lizorkin type.They established the equivalent quasi-norms in terms of maximal functions. A more recent study by Ullrich 2012 investigates the continuous Characterizations of Besov-Lizorkin-Triebel Spaces whereby characterizations for homogeneous and inhomogeneous Besov-Lizorkin-Triebel spaces in terms of continuous local means for the full range of parameters are established. A paper by Moura et al.2014 considers the Spaces of generalized smoothness in the critical case: Optimal embeddings, continuity envelopes and approximation numbers. In this paper, the necessary and sufficient conditions for embeddings of Besov spaces of generalized smoothness $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$ into generalized Holder spaces $\wedge^{\mu(.)}_{\infty,r}(\mathbb{R}^n)$ are established and the analogous results for the Triebel–Lizorkin spaces of generalized smoothness $F^{\sigma,N}_{p,q}(\mathbb{R}^n)$ are given. In this study, we consider the Lizorkin-Triebel spaces of differential functions with generalized smoothness on $\mathbb{R}^n$. The topicality of this study of functions with generalized smoothness is most found in applications of inclusion and approximation theories. The importance of using this generalization is the transit from the real number parameters to generalized parameters,

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Email addresses: mutavedik@gmail.com (Vedaste Mutarutinya), kmpimbo33@gmail.com (Marco Mpimbo), gabrielnshizirungu@gmail.com (Gabriel Nshizirungu)

functions or sequences with minimum limitations on them. We start by giving notations of basic concepts and background tools, then proceed with defining homogeneous and non-homogeneous Lizorkin-Triebel spaces denoted by $\dot{F}_{pq}^{\lambda(.)}(\mathbb{R}^n)$ and $F_{pq}^{\lambda(.)}(\mathbb{R}^n)$ respectively. Next, we deal with discretization of norms in homogeneous Lizorkin-Triebel spaces using so called φ-transform. Finally, the relationship between these spaces and their discrete analogues generalizing the known Frazier-Jawerth results is established (Frazier et al.1991, Bownik 2000).

1.1. *Notations and some background tools*

As usual, the notations $\mathbb{Z}$,$\mathbb{R}$ and $\mathbb{C}$ will denote the set of integers, real numbers and complex numbers respectively. Consider the function $\varphi : (\mathbb{R}^n) \to \mathbb{C}$. Let

$$t > 0,\ \varphi_{(t)}(x) = t^{-n}\varphi(t^{-1}x) \tag{1}$$

Choosing $\varphi \in S(\mathbb{R}^n)$ such that

$$\sup F_\varphi \subset \left\{\xi : \frac{1}{2} \le |\xi| \le 2\right\} \tag{2}$$

and

$$\left|F_\varphi(\xi)\right| \ge C_0 > 0, \frac{3}{5}|\xi| < \frac{5}{3}, \tag{3}$$

then $\forall f \in S'$ and $\forall t > 0$, we can define the convolution

$$\left(\varphi_{(t)} * f\right)(x) \overset{def}{=} (2\pi)^{-\frac{n}{2}}\left(f_{|y|}, \varphi_t(x-y)\right).$$

From the theory of generalized functions, it is known that

$$\left(\varphi_{(t)} * f\right) \in C^\infty(\mathbb{R}^n) \cap S'(\mathbb{R}^n).$$

In addition, the formula

$$F\left(\varphi_{(t)} * f\right)(x) = (F\varphi_{(t)})(\xi)(Ff)(\xi)$$

or

$$(\varphi_{(t)} * f)(x) = F^{-1}\left[(F\varphi_{(t)})(\xi)(Ff(\xi))\right](x)$$

holds. Hence, we see that

$$\sup F(\varphi_{(t)} * f) \subseteq \sup F\varphi_{(t)} \subseteq \left\{\xi : \frac{t^{-1}}{2} \le |\xi| \le 2t^{-1}\right\} \tag{4}$$

In addition, if $t > 0$, then

$$\begin{aligned} |\varphi_{(t)}|_{L1}(\mathbb{R}^n) &= t^{-n}\int_{\mathbb{R}^n} |\varphi(t^{-1}x)|dx, \\ &= \int_{\mathbb{R}^n} |\varphi(y)|dy, \\ &= c < \infty \end{aligned}$$

This norm does not depend on t, Consequently, from generalized Minkowiski inequality if $1 \leq p \leq \infty$, then $\forall g \in L_p$,

$$\begin{aligned} \|\varphi_{(t)} * g\|_{L_p} &\leq \|\varphi_{(t)}\|_{L_1}\|g\|_{L_P} \\ &= c\|g\|_{L_P} \end{aligned} \tag{5}$$

where $c = \|\varphi\|_{L_1}$ and does not depend on t and on the function g.

Now, Let's consider the function λ defining the generalized smoothness:

$$\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^+; \lambda(1) = 1;$$

$$0 < \lambda_0 \leqslant \frac{\lambda(\tau)}{\lambda(t)} < \Lambda_0 < \infty, \forall t > 0, \tau \in [t, 2t] \tag{6}$$

Thus, for example if $\exists \alpha, \beta \in \mathbb{R} : \frac{\lambda(t)}{t^\alpha} \uparrow$ (increases)$\frac{\lambda(t)}{t^\beta} \downarrow$ (decreases); then, $\forall t > 0, \tau \in [t, 2t]$, we have

$$\begin{aligned} \frac{\lambda(\tau)}{\lambda(t)} &= \frac{\lambda(\tau)}{t^\alpha} \cdot \frac{t^\alpha}{\lambda(t)} \\ &\geq \frac{\lambda(t)}{t^\alpha} \cdot \frac{t^\alpha}{\lambda(t)} \cdot \left(\frac{\tau}{t}\right)^\alpha \\ &= \left(\frac{\tau}{t}\right)^\alpha \geq \min\{1; 2^\alpha\} \end{aligned}$$

If $\alpha \geq 0$, then $\frac{\lambda(\tau)}{\lambda(t)} \leq 1$ (here $\Lambda_0 = 1$); while if $\alpha < 0$, then $(\frac{\tau}{t})^\alpha \geq \frac{\lambda(\tau)}{\lambda(t)} = 2^\alpha > 0$ (here $\Lambda_0 = 2^\alpha$). So, if $\frac{\lambda(t)}{t^\alpha} \uparrow$, then $\frac{\lambda(\tau)}{\lambda(t)} \geq \Lambda_0 > 0, \forall \tau \in [t, 2t]$. Analogically, $\forall \tau \in [t, 2t]$ we have

$$\begin{aligned} \frac{\lambda(\tau)}{\lambda(t)} &= \frac{\lambda(\tau)}{t^\beta} \cdot \frac{t^\beta}{\lambda(t)} \\ &\geq \frac{\lambda(t)}{t^\beta} \cdot \frac{t^\beta}{\lambda(t)} \cdot \left(\frac{\tau}{t}\right)^\alpha \\ &= \left(\frac{\tau}{t}\right)^\beta \leq \max\{1; 2^\beta\} \end{aligned}$$

If $\beta \geq 0$, then $\frac{\lambda(\tau)}{\lambda(t)} \leq 1$ (here $\Lambda_0 = 1$); while if $\beta < 0$, then $\frac{\lambda(\tau)}{\lambda(t)} < 2^\beta$ (here $\Lambda_0 = 2^\beta$). Hence, if $\frac{\lambda(t)}{t^\beta} \downarrow$, then $\frac{\lambda(\tau)}{\lambda(t)} \geq \Lambda_0 < \infty, \forall \tau \in [t, 2t]$.

1.2. Generalized Lizorkin-Triebel spaces

For $\nu \in \mathbb{Z}, \varphi : \mathbb{R}^n \to \mathbb{C} \subset S$, we have $\varphi_\nu(x) = \varphi(2^{-\nu})(x) = 2^{\nu n}\varphi(2^\nu x)$, and we suppose the system $\{\Phi, \varphi_\nu, \nu \geq 1\}$ forms the Fourier expansion of 1, that is,

$$F\phi(\xi) + \sum_{\nu=1}^{\infty} F\varphi_\nu(\xi) \equiv 1, \forall \xi \in \mathbb{R}^n.$$

Definition 1.1. *Let $1 \leq p < \infty$ and $0 \leq q < \infty$. Then,*

$$\dot{F}_{pq}^{\lambda(\cdot)}(\mathbb{R}^n) = \left\{ f \in S' : \|f\|_{\dot{F}_{pq}^{\lambda(\cdot)}} = \left\| \left\{ \sum_{\nu \in \mathbb{Z}} \left[\lambda(2^\nu)|\varphi_\nu * f| \right]^q \right\}^{\frac{1}{q}} \right\|_{L_p} < \infty \right\} \tag{7}$$

is the homogeneous generalized Lizorkin-Triebel space. Note that if $\lambda(t) = t^\alpha, \alpha > 0$, then $\dot{F}_{pq}^{\lambda(\cdot)}(\mathbb{R}^n) = \dot{F}_{pq}^{t^\alpha}(\mathbb{R}^n)$ which is a usual Lizorkin-Triebel space (Frazier et al. 1991, Mutarutinya 1999). The condition such that $\dot{F}_{pq}^{\lambda(\cdot)}(\mathbb{R}^n) = 0$ means that $f \in P$, the class of all polynomials.

Definition 1.2. *Let* $1 \leq p < \infty$ *and* $0 \leq q < \infty$. *The space*

$$F_{pq}^{\lambda(\cdot)}(\mathbb{R}^n) = \left\{ f \in S' : \|f\|_{F_{pq}^{\lambda(\cdot)}} = \left\| \left\{ \left| \Phi * f \right|^q + \sum_{\nu \geq 1} \left[\lambda(2^\nu) |\varphi_\nu * f| \right]^q \right\}^{\frac{1}{q}} \right\|_{L_p} < \infty \right\} \tag{8}$$

is called the non-homogeneous generalized Lizorkin-Triebel space, where $\Phi \in S(\mathbb{R}^n)$ and

$$\sup F\varphi \subset \{\xi : \|\xi\| \leq 2\}, \left| F\Phi(\xi) \right| \geq 0, \text{ for } |\xi| < \frac{5}{3}. \tag{9}$$

For non-homogeneous space

$$\|f\|_{F_{pq}^{\lambda(\cdot)}} = 0 \Leftrightarrow f = 0 \text{ (in } S')$$

and we have to note that

$$\|f\|_{F_{pq}^{\lambda(\cdot)}} \approx \|\Phi * f\|_{L_p} + \left\| \left\{ \sum_{\nu \geq 1} \left[\lambda(2^\nu) |\varphi_\nu * f| \right]^q \right\} \right\|_{L_p} \Bigg\} \tag{10}$$

Likewise in the case of homogeneous spaces, if $\lambda(t) = t^\alpha$ then $F_{pq}^{\lambda(\cdot)}(\mathbb{R}^n) = F_{pq}^{\alpha}(\mathbb{R}^n)$ is a usual non-homogeneous Lizorkin-Triebel space.

1.3. Relationship between homogeneous and non-homogeneous Lizorkin-Triebel spaces

The following theorem gives the relationship between homogeneous and non-homogeneous Lizorkin-Triebel spaces.

Theorem 1.3. *Let* $1 \leq p < \infty$ *and* $0 \leq q \leq \infty$, $\lambda(\cdot) \uparrow$ *satisfy the condition* $\lambda(2t) \geq \lambda_0 \lambda(t), \forall t > 0$, *where* $\lambda_0 > 1$, *then*

$$f \in F_{pq}^{\lambda(\cdot)}(\mathbb{R}^n) \Leftrightarrow \{f \in L_p\} \cap \left\{ f \in \dot{F}_{pq}^{\lambda(\cdot)}(\mathbb{R}^n) \right\}$$

and

$$\|f\|_{F_{pq}^{\lambda(\cdot)}} \approx \|\phi\|_{L_1} + \|f\|_{\dot{F}_{pq}^{\lambda(\cdot)}}$$

Note that $\lambda(t) = t^\alpha \ln^\gamma(2+t), \alpha > 0, \gamma \in \mathbb{R}$ satisfies the condition of this theorem, while $\lambda(t) = t^\alpha \ln^\gamma(2+t), \forall \alpha \in \mathbb{R}$ doesn't.

2. Main Results

Our first result is about the discretization of the norm in Lizorkin-Triebel space using $\varphi-$ transform.

Let the functions φ and ψ satisfy the conditions $(1) - (3)$ and such that

$$\sum_{\nu \in \mathbb{Z}} \overline{F\varphi(2^\nu \xi)} F\psi(2^\nu \xi) = 1, \forall \xi \neq 0 \tag{11}$$

As above, Let

$$\varphi_\nu(x) = 2^{\nu n}(2^\nu x), \psi_\nu(x) = 2^{\nu n}\psi(2^\nu x), \nu \in \mathbb{Z} \tag{12}$$

We introduce the dyadic cubes: $\forall \nu \in \mathbb{Z}, K \in \mathbb{Z}^n$, we define

$$Q \equiv Q_{\nu k} = \left\{ (x_1, x_2, ..., x_n) \in \mathbb{R}^n : 2^{-\nu} k_j \leq x_j \leq 2^{-\nu}(k_j + 1), j = 1, 2, ..., n \right\} \tag{13}$$

The point $2^{-\nu}k = x_Q$ is the left lower vertex of cube $Q_{\nu k}$ and $\ell(Q) = 2^{-\nu}$ is the side of length of the cube. Next, suppose for the cube

$$Q : |Q| = \ell(Q)^n = 2^{-n\nu} \text{ (the volume of cube)}$$

we define

$$\begin{aligned}\varphi_Q(x) &= |Q|^{-\frac{1}{2}}\varphi(2^\nu x - k)\\ &= |Q|^{-\frac{1}{2}}\varphi_\nu(x - x_Q)\end{aligned} \tag{14}$$

Similarly, we define for π

$$\begin{aligned}\psi_Q(x) &= |Q|^{-\frac{1}{2}}\psi(2^\nu x - k)\\ &= |Q|^{-\frac{1}{2}}\psi_\nu(x - x_Q)\end{aligned}$$

For the $\varphi-$ transform and its inverse transform we proceed as follows:
Let's introduce the operators: S_φ (φ − transform) and S_ψ (inverse ψ − transform). Note that the function $\tilde{\varphi} = \overline{\varphi(-x)}$ also satisfies the conditions (2) and (3). For the dyadic cube $Q = Q_{\nu k}, f \in S^{'}(\mathbb{R}^n)/P$ (the space of tempered distributions modulo polynomials). We have

$$\left(S_\varphi f\right)_Q \equiv \langle f, \varphi_Q\rangle \equiv 2^{-\frac{\nu n}{2}}\left(\tilde{\varphi}_\nu * f\right)(x_Q) = 2^{-\frac{\nu n}{2}}\left(\tilde{\varphi}_\nu * f\right)(2^{-\nu}k), \tag{15}$$

knowing that

$$(\tilde{\varphi}_\nu * f)(x) \in C^\infty(\mathbb{R}^n) \cap S^{'}(\mathbb{R}^n) \text{ see (1)}$$

Thus, the value of the function at $x_Q = 2^{-\nu}k$ is defined. Finally, applying the operator S_Q to the function $f \in S^{'}(\mathbb{R}^n)/P$, we obtain the collection of numbers $S_\varphi f = \left\{(S_\varphi f)_Q\right\}_Q$ satisfying the dyadic cubes Q. Now, let $S = \{S_Q\}_Q$ be a given collection of numbers satisfying all dyadic cubes Q. Then, the operator T_ψ is defined by

$$(T_\psi)(S)(x) = \sum_Q S_Q\psi_Q(x), \tag{16}$$

where the sum is taken over all dyadic cubes Q, i.e.,

$$S = \{S_{\nu,k}\}_{\nu\in\mathbb{Z}}, k \in \mathbb{Z}^n$$

and

$$\begin{aligned}(T_\psi S)(x) &= \sum_Q S_Q\Psi_Q(x)\\ &= \sum_{\nu\in\mathbb{Z}} 2^{-\frac{\nu n}{2}} \sum_{\nu\in\mathbb{Z}} S_{\nu,k}\Psi_\nu(x - 2^{-\nu}k).\end{aligned} \tag{17}$$

Finally, we define the discrete analogous space of $\dot{F}^{\lambda(\cdot)}_{pq}(\mathbb{R}^n)$, for $S = \{S_Q\}_Q$ and denote it by

$$\|S\|_{\dot{f}^{\lambda(\cdot)}_{pq}} = \left\|\left\{\sum_Q\left[\lambda(2^\nu)|S_Q|\bar{\chi}_Q(x)\right]^q\right\}^{\frac{1}{q}}\right\|_{L_p(\mathbb{R}^n)} \tag{18}$$

where $\bar{\chi}_Q(x) = |Q|^{-\frac{1}{2}}\chi_Q(x)$ is the normed characteristic function of dyadic cube in L_2 and $\dot{f}^{\lambda(\cdot)}_{pq}(\mathbb{R}^n)$ is the set of all sequences $S = \{S_Q\}_Q$, such that $\|S\|_{\dot{f}^{\lambda(\cdot)}_{pq}} < \infty$..
This leads us to our next result.

Theorem 2.1. *(Discrete analogous of homogeneous Lizorkin-Triebel space $\dot{F}_{pq}^{\lambda(\cdot)}(\mathbb{R}^n)$).*
Let $1 \leq p < \infty$ and $1 < q \leq \infty$; each of the functions φ and ψ satisfy the conditions (1) - (3) and in addition, φ and ψ satisfy the relation (11). Moreover, if the function $\lambda(\cdot) > 0$ is such that

$$\lambda(t) \approx \lambda(\tau) \text{ for } \tau \in [t, 2t] \text{ and } t > 0,$$

then, under the assigned conditions on p, q, φ and ψ, the operators

$$S_\varphi : \dot{F}_{pq}^{\lambda(\cdot)} \longrightarrow \dot{f}_{pq}^{\lambda(\cdot)}$$

and

$$T_\psi : \dot{f}_{pq}^{\lambda(\cdot)} \longrightarrow \dot{F}_{pq}^{\lambda(\cdot)}$$

are bounded. Furthermore,

$$T_\psi \ o \ S_\varphi = id : \dot{F}_{pq}^{\lambda(\cdot)} \longrightarrow \dot{F}_{pq}^{\lambda(\cdot)}.$$

In particular,

$$\|f\|_{\dot{F}_{pq}^{\lambda(\cdot)}(\mathbb{R}^n)} \approx \|S_\varphi f\|_{\dot{F}_{pq}^{\lambda(\cdot)}(\mathbb{R}^n)} \tag{19}$$

and $\dot{F}_{pq}^{\lambda(\cdot)}(\mathbb{R}^n)$ can be identified with complement space in $\dot{F}_{pq}^{\lambda(\cdot)}(\mathbb{R}^n)$.

Remark 2.2. *The operator S_φ identifies $\dot{F}_{pq}^{\lambda(\cdot)}(\mathbb{R}^n)$ with space $S_\varphi\Big(\dot{F}_{pq}^{\lambda(\cdot)}\Big) \subset \dot{f}_{pq}^{\lambda(\cdot)}$ and thus the operator*

$$P_r = S\varphi \ o \ T_\psi : \dot{f}_{pq}^{\lambda(\cdot)} \longrightarrow S_\varphi\Big(\dot{F}_{pq}^{\lambda(\cdot)}\Big)$$

is the projector in $\dot{f}_{pq}^{\lambda(\cdot)}$.

Indeed,

$$\begin{aligned} P_r^2 &= (S_\varphi \ o \ T_\psi)(S_\varphi \ o \ T_\psi) \\ &= S_\varphi \ o \ (S_\varphi \ o \ T_\psi) \ o \ T_\psi \\ &= S_\varphi \ o \ Id \ o \ T_\psi \\ &= S_\varphi \ o \ T_\psi \\ &= P_r \end{aligned}$$

Thus, there exists a bounded projection P_r such that

$$\|P_r\| \leq \|S_\varphi\| \cdot \|T_\psi\|$$

To prove Theorem 2.1, we need two lemmas. First of all, let us introduce the following notations.
Let $S = \{S_Q\}_Q$ be a family / set of numbers corresponding to dyadic cubes $Q, \ell(Q)$ the side's length of $Q, x_Q = 2^{-\nu}k$ the left lower angle of the cube $Q = Q_{\nu,k}, 0 < \nu \leq \infty$, $\delta > 0$ fixed, $S_Q^* = \Big\{(S_r^*)_Q\Big\}_Q$ where

$$(S_r^*)_Q = \left\{ \sum_{p:\ell(p)=\ell(Q)} \frac{|S_p|^r}{1 + \ell(Q)^{-1}|x_p - x_q|^\delta} \right\}^{\frac{1}{r}} \tag{20}$$

It is clear that

$$(S_\varphi^*)_Q \geq |S_Q|$$

The sum from the right hand side in the relation (20) is greater than one term where $P = Q$.

Lemma 2.3. *Let* $1 < p < \infty,\ 0 < q < \infty,\ \delta > n$, *and the function* $\lambda(\cdot) > 0$. *Then*

$$r = \min\{p, q\} \cdot \|S\|_{\dot{f}_{pq}^{\lambda(\cdot)}}$$
$$\approx |S_r^*|_{\dot{f}_{pq}^{\lambda(\cdot)}}$$

Furthermore, for $f \in S'/P$ and $Q = Q_{\nu,k}(i.e.\ell(Q) = 2^{-\nu}, |Q| = 2^{-\nu n})$. By letting

$$\sup(f) = \left\{\sup_Q(f)\right\}_Q$$

we see that $\sup(f)$ is a sequence of the form

$$\sup_Q(f) = |Q|^{\frac{1}{2}} \sup_{y\in Q}\left|(\tilde{\varphi}\nu * f)(y)\right| \tag{21}$$

Lemma 2.4. *Let* $1 < p < \infty,\ 0 < q < \infty,\ \delta > n$, *and the function* $\lambda(\cdot) > 0$ *satisfy the condition*

$$\lambda(\tau) \backsimeq \lambda(t), \forall t > 0, \tau \in [t, 2t].$$

Then,

$$\|f\|_{\dot{F}_{pq}^{\lambda(\cdot)}} \approx \| \sup(f)\|_{\dot{f}_{pq}^{\lambda(\cdot)}}$$

First, let us prove the theorem basing on Lemma 2.3 and Lemma 2.4 and thereafter, we will prove the lemmas.

Proof of theorem 2.3

1. For the dyadic cube $Q = Q_{\nu,k,}$ we have

$$|Q|^{\frac{1}{2}} = 2^{-\frac{\nu n}{2}},$$

$$\begin{aligned}|(S_\varphi f)_Q| &\overset{(15)}{=} 2^{-\frac{\nu n}{2}}\left|(\tilde{\varphi}_\psi * f)(2^{-\nu}k)\right| \\ &\le 2^{-\frac{\nu n}{2}} \sup_{y\in Q}\left|(\tilde{\varphi}_\psi * f)(y)\right| \\ &\overset{(21)}{=} \sup_Q(f).\end{aligned}$$

So,

$$\begin{aligned}\|S_\varphi f\|_{F_{pq}^{\lambda(\cdot)}} &\overset{(15)}{=} \left\|\left[\sum_Q |\lambda(2^\nu)|(S_\varphi f)_Q|\tilde{\chi}Q(x)|^q\right]^{\frac{1}{q}}\right\|_{L_p} \\ &\le \left\|\left[\sum_Q |\lambda(2^\nu)| \sup_Q |(f)\tilde{\chi}_Q(x)|^q\right]^{\frac{1}{q}}\right\|_{L_p(\mathbb{R}^n)} \\ &= \left\|\sup_Q(f)\right\|_{f_{pq}^{\lambda(\cdot)}} \\ &\overset{Lemma\ 2.4}{=} c_1\|f\|_{f_{pq}^{\lambda(\cdot)}}\end{aligned}$$

Thus,

$$S_\varphi : \dot{F}^{\lambda(\cdot)}_{pq} \longrightarrow \dot{f}^{\lambda(\cdot)}_{pq}$$

is a bounded operator.

2. Now, let us prove the boundness of the operator:

$$T_\varphi : \dot{f}^{\lambda(\cdot)}_{pq} \longrightarrow \dot{F}^{\lambda(\cdot)}_{pq}$$

Let $S = \{S_K\}_k,\ K$ be a dyadic cube, $f = T_\psi S$ given by the formulas (16) and (17), i.e.$f = \sum_k S_k \psi_k$. Then, we have

$$\begin{aligned}(\tilde{\varphi}_\nu * f) &= \tilde{\varphi}_\mu * \Big(\sum_k S_k\psi_k\Big)\\ &= \sum_{\mu=\nu-1}^{\nu+1}\Big[\sum_{k:k(\ell)=2^{-\mu}} S_k(\tilde{\varphi}_\nu * \psi_k)\Big] \end{aligned}\tag{22}$$

The relation (22) means that if $K = Q_{\nu,k}$, then

$$\sup F\psi_k \subset \{\zeta \in \mathbb{R}^n : 2^{\mu-1} \leq |\zeta| \leq 2^{\mu+1}\}$$

(see (11) – (14) for $t = 2^{-\nu}$). Then,

$$\sup F_\varphi \subset \left\{\zeta \in \mathbb{R}^n : 2^{\nu-1} \leq |\zeta| \leq 2^{\nu+1}\right\}$$

follows for

$$\tilde{\varphi}_\nu * \psi_k = F^{-1}[F\varphi_\nu \cdot F\psi_k].$$

We have, for $\mu < \nu - 1$ and for $\mu > \nu + 1$, that

$$\tilde{\varphi}_\nu * \psi_k = 0.$$

So why from this sum $\sum_k$ in (22) remains only sum for $K = Q_{\mu,i},$ where $\nu - 1 < \mu < \nu + 1$ so that $\sum_{i\in\mathbb{Z}^n}$ for fixed μ exists $\sum_{k:l(k)=2^{-\nu}}$.

3. Next, Let us prove that for $Q = Q_{\nu,k}$ and $r = \min\{p, q\}$, we have

$$|(\tilde{\varphi}_\nu * f)|(x) \leq c_1|Q|^{-\frac{1}{2}}\Big[(S^*_r)_{Q^*} + (S^*_r)_{Q^{**}}\Big],\tag{23}$$

where,

$$x \in Q^*, Q^* \subset Q \subset Q^{**} \textit{ and } \ell(Q^*) = 2^{-(\nu+1)}\ell(Q) = 2^{-(\nu)}\ell(Q^{**}) = 2^{-(\nu-1)},$$

such that

$$|Q^*| = 2^{-n}|Q|, |Q^{**}| = 2^n|Q|$$

For that the expression/assertion (23) is true $\forall x \in Q$ and such that for this x we can identify $Q^* \subset Q$, such that $x \in Q^*$ From (23), we see that $\forall x \in \mathbb{R}^n$ and for $Q = Q_{\nu,k}$ such that $x \in Q_{\nu,k},$

$$|(\tilde{\varphi}_\nu * f)| \leq c_2\Big[\sum_{Q^*\subset Q} |Q^*|^{-\frac{1}{2}}(S^*_r)_{Q^*}\tilde{\chi}^*_Q(x) + |Q|^{-\frac{1}{2}}(S^*_r)_Q\tilde{\chi}_Q(x) + |Q^{**}|^{-\frac{1}{2}}(S^*_r)Q^{**}\tilde{\chi}^*_Q(x)\Big]$$

Consequently,

$$\|T_\psi S'\|_{\dot F_{pq}^{\lambda(\cdot)}} = \|f\|_{\dot F_{pq}^{\lambda(\cdot)}}$$
$$= \left\| \left\{ \sum_{\nu\in\mathbb{Z}} \left[\lambda(2^\nu)(S_r^*)_{Q^*}\tilde{\chi}_{Q^*}\right]^q \right\}^{\frac{1}{q}} + \left\{ \sum_{Q} \left[\lambda(2^\nu)(S_r^*)_{Q}\tilde{\chi}_{Q}\right]^q \right\}^{\frac{1}{q}} + \left\{ \sum_{Q^{**}} \left[\lambda(2^\nu)(S_r^*)_{Q^{**}}\tilde{\chi}_{Q^{**}}\right]^q \right\}^{\frac{1}{q}} \right\|_{L_p}$$

Using triangular inequality and considering that

$$\lambda(2^\nu) = \lambda(2^{\nu+1})$$
$$\lambda(2^\nu) = \lambda(2^{\nu-1})$$
$$= C_4 \left\| \left\{ \sum_{Q^*} \left[\lambda(2^\nu+1)(S_r^*)_{Q^*}\tilde{\chi}_{Q^*}\right]^q \right\}^{\frac{1}{q}} \right\|_{L_p} + C_3 \left\| \left\{ \sum_{Q} \left[\lambda(2^\nu)(S_r^*)_{Q}\tilde{\chi}_{Q}\right]^q \right\}^{\frac{1}{q}} \right\|_{L_p}$$
$$+ \left\| \left\{ \sum_{Q^{**}} \left[\lambda(2^\nu-1)(S_r^*)_{Q^{**}}\tilde{\chi}_{Q^{**}}\right]^q \right\}^{\frac{1}{q}} \right\|_{L_p}$$
$$= C_5 \left\| S_r^* \right\|_{\dot f_{pq}^{\lambda(\cdot)}}$$

Now, applying lemma 2.3, we obtain that

$$\|T_\psi S\|_{\dot F_{pq}^{\lambda(\cdot)}} \leq C_6 \|T_\psi S\|_{\dot f_{pq}^{\lambda(\cdot)}}$$

and this proves the boundness of the operator

$$T_\psi : \dot f_{pq}^{\lambda(\cdot)} \longrightarrow \dot F_{pq}^{\lambda(\cdot)}$$

Finally, the identity

$$T_{\psi\ell} \; o \; S_\ell = id$$

is proved by Frezier-Jawerth on S'/P (Bownik 2000). In addition, it is also verified for $\dot F_{pq}^{\lambda(\cdot)} \subset S'/P$. The theorem is therefore proved (if lemmas 2.3 and 2.4 are verified).

Proof of lemma 2.3

This lemma is based on two results:

Lemma 2.5. ***(Feffermann-Stein).*** *Let* $1 < p < \infty$, $0 < q < \infty$, *then*

$$\left\| \left(\sum_{i=1}^{\infty} |Mf_i|^q \right)^{\frac{1}{q}} \right\|_{L_p} \leq C_{p,q} \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{L_p}$$

where M is maximum operator Hardi-Littlewood, defined by

$$Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)| dy$$

where the upper corner is taken for all cubes (not necessarily dyadic cubes) containing points x and having side $\|-\ell\|$ to axes of coordinates.

Lemma 2.6. ***(Frazier-Jawerth).*** *Let* $1 < a \leq r \leq \infty$, $\delta > \frac{nr}{a}$. *Let us fixe* $\mu, \nu \in \mathbb{Z}, \mu \leq \nu$, *for any dyadic cube* Q *with* $\ell(Q) = 2^{-\nu}$ *and* $\forall x \in Q$. *then*

$$\left\{ \sum_{p:\ell(p)=2^{-1}} \frac{|S_p|^r}{1+\ell(p)^{-1}|x_p - x_q|^\delta} \right\}^{\frac{1}{r}} \leq C \left[M\left(\sum_{\ell(p)=2^{-p}} |S_p|^a \chi_p \right)(x) \right]^{\frac{1}{a}}$$

where C depends on n and $\delta - \frac{nr}{a}$.

Let us now start to prove Lemma 2.3.

Proof of lemma 2.3.
Let us put $r = \min\{p, q\},\ \varepsilon = -1 + \frac{\delta}{n} > 0$. If $a = \frac{r}{1+\frac{\varepsilon}{2}} \Longrightarrow 0 < a < r,\ \delta > \frac{nr}{a}$
and according to Frazier-Jawerth lemma (and $\mu = \nu$), we have that

$$\Big\{\sum_{Q:d(Q)=2^{-\nu}}\Big[(S_r)_Q^{(\ell)}\tilde{\chi}(x)\Big]^q\Big\}^{\frac{1}{q}} \overset{by\ definition\ (S^*_\nu)_Q}{\leq} 2^{\nu\frac{n}{2}}\Big[\sum_{p:d(p)=2^{-\nu}}\frac{|S_p|^r}{1+\ell(p)^{-1}|x_p - x_Q|}\Big]^{\frac{q}{r}}$$

$$\leq ((11)\forall x \in \mathbb{R},\ except\ one\ term\ with\ Q : Q \ni x \neq 0)$$

$$\leq (by\ Frazier - Jawerth\ lemma)$$

$$\leq C2^{\nu\frac{n}{2}}\Big[M\Big(\sum_{\ell(p)=2^{-\nu}}|S_p|^q\chi_p\Big)(x)\Big]^{\frac{q}{a}}$$

$$= C\Big[M\Big(\sum_{\ell(p)=2^{-\nu}}|S_p|^q\tilde{\chi}_p\Big)^a\Big]^{\frac{q}{a}}$$

The last inequality comes from the case that the sum $p : \ell(p) = 2^{-\nu}$ for any x only one term is not zero such that

$$2^{\nu\frac{n}{2}}\Big(\sum_{\ell(p)=2^{-\nu}}|S_p|\chi_p(x)\Big)^a = 2^{\nu\frac{n}{2}}\sum_{\ell(p)=2^{-\nu}}|S_p|^a\chi_p(x)$$

Consider that

$$2^{\nu\frac{n}{2}}\chi_p(x) = \tilde{\chi}_p,\ for\ \ell(p) = 2^{-\nu}$$

So,

$$\Big\|S^*_p\Big\|_{\dot{f}^{\lambda(\cdot)}_{pq}} = \Big\|\Big\{\sum_{\nu\in\mathbb{Z}}[X(2^\nu)]^q\sum_{Q:\ell(Q)2^{-\nu}}\Big[(S^*_p)_Q\tilde{\chi}_Q\Big]^q\Big\}^{\frac{1}{q}}\Big\|_{L_p}$$

$$\leq c\Big\|\Big\{\sum_{\nu\in\mathbb{Z}}\Big[M\Big(\sum_{p:\ell(p)=2^{-\nu}}|S_p|\tilde{\chi}_p\Big)^a\Big]^{\frac{q}{a}}\Big\}^{\frac{1}{q}}\Big\|_{L_p}$$

$$= c\Big\|\Big\{\sum_{\nu\in\mathbb{Z}}\Big[M\Big(\sum_{p:\ell(p)=2^{-\nu}}\lambda(2^\nu)[|S_p|\tilde{\chi}_p]\Big)^a\Big]^{\frac{q}{a}}\Big\}^{\frac{1}{q}}\Big\|_{L_p}$$

Noting

$$f_\nu = \sum_{p:d(p)=2^{-\nu}}[\lambda(2^\nu)|S_p|\tilde{\chi}_p]^a. \tag{24}$$

Then,

$$\Big\|S^*_p\Big\|_{\dot{f}^{\lambda(\cdot)}_{pq}} \leq c\Big\|\Big\{\sum_{\nu\in\mathbb{Z}}\Big[Mf_\nu\Big]^{\frac{q}{a}}\Big\}^{\frac{a}{q}}\Big\|_{L_p}$$

$$= c\Big\|\Big\{\sum_{\nu\in\mathbb{Z}}\Big[Mf_\nu\Big]^{\frac{q}{a}}\Big\}^{\frac{a}{q}}\Big\|_{L_p}$$

Applying theorem Feffermann-Stein ($q/a > 1$ replaces q and p/a replaces p), then

$$\begin{aligned}\left\|S_p^*\right\|_{\dot{f}_{pq}^{\lambda(\cdot)}} &\leq c_1\left\|\left\{\sum_{\nu\in\mathbb{Z}}\left[f_\nu\right]^{\frac{q}{a}}\right\}^{\frac{a}{q}}\right\|_{L_p}\\ &= c_1\left\|\left\{\sum_{\nu\in\mathbb{Z}}\left[f_\nu\right]^{\frac{q}{a}}\right\}^{\frac{1}{q}}\right\|_{L_p}\\ &\overset{from\ (24)}{=} c_1\left\|\left\{\sum_{y\in\mathbb{Z}}\Big(\sum_{p:d(p)=2^{-\nu}}[\lambda(2^\nu)|S_p|\tilde{\chi}_p]\Big)^q\right\}^{\frac{1}{q}}\right\|_{L_p}\\ &= c_1\left\|S\right\|_{\dot{f}_{pq}^{\lambda(\cdot)}}\end{aligned}$$

In summary, we get that

$$\left\|S_r^*\right\|_{\dot{f}_{pq}^{\lambda(\cdot)}} \leq c_1\left\|S\right\|_{\dot{f}_{pq}^{\lambda(\cdot)}}$$

The inverse inequality is clear and automatic because $(S_r^*) \geq |S_Q|$ for all dyadic cubes Q. Thus, Lemma 2.3 is proven.

Proof of Lemma 2.4.
This proof is based on two supplementary lemmas.

Lemma 2.7. *Let $f \in S'$, $\sup Ff(\xi) \subset \{\xi : |\xi| \leq 2\}$. Let also $\gamma \in \mathbb{Z}, \gamma \geq 0$. For dyadic cube let put $a = \{a_Q\}_Q, b = \{b_Q\}_Q$, where*

$$a_Q = \sup_{y\in Q}[f(y)],\ b_Q = \max\{\inf_{y\in Q}|f(y)| : \bar{Q} \subset Q, \ell(\bar{Q}) = 2^{-\gamma}\ell(Q)\}$$

Let $0 < r < \infty : \ell(Q) = 1$ and γ sufficient close. Then,

$$(a_r^*)_Q \simeq (b_r^*)_Q$$

with constant which does not depend on f and Q (Bownik 2000). Let us denote for $f \in S'/P$ and dyadic cube Q,

$$\inf_{Q\ni\gamma} = |Q|\frac{1}{2}\max\left\{\inf_{y\in Q}|(\tilde{\varphi}_\nu * f)(y)| : \tilde{Q} \subseteq Q\ell(\tilde{Q}) = 2^{-\gamma}\ell(Q)\right\}$$

Then, let's deduct the sequence of numbers:

$$\inf_{y\in Q} = \left\{\inf_{y\in Q}(f)\right\}_Q$$

responding all cubes Q.

Lemma 2.8. *For $f \in S'/P, 0 < p < \infty, \lambda(t) > 0,\ \lambda(\tau) \approx \lambda(t),\ \forall t > 0,\ \tau \in [t, 2t]$. Then*

$$\left\|\inf_\gamma(f)\right\|_{\dot{f}_{pq}^{\lambda(\cdot)}} \leq C_{\gamma,n,p}\left\|f\right\|_{\dot{F}_{pq}^{\lambda(\cdot)}}$$

Proof. Let K be a dyadic cube, $\ell = \{t_k\}_k$. Let define by formula

$$\ell_k = |K|^{\frac{1}{2}}\inf_{y\in K}\left|\tilde{\varphi}_{\mu-\gamma} * f\right|(y),\ for\ \ell(k) = 2^{-\mu} \tag{25}$$

Then, for $0 < r < \infty$, for dyadic cube Q, we have

$$\inf_{Q,\gamma}(f) \cdot \tilde{\chi}_Q(x) \leq C_{n,\gamma} 2^{\frac{\gamma a}{r}} \sum_{K \subseteq Q} (\ell_r^*) \tilde{\chi} K(x), \tag{26}$$

where (26) means that

$$\ell(k) = 2^{-\gamma} \ell(Q) \tag{27}$$

(See definition (20), δ from (20), but for ℓ on the place S'). Let put $r = \min\{p, q\}$ and get

$$\begin{aligned}
\left\| \inf_{\gamma}(f) \right\|_{\dot{f}_{pq}^{\lambda(\cdot)}} &\overset{by\ def}{=} \left\| \left\{ \sum_{y \in \mathbb{Z}} [\lambda(2^{\nu})]^q \sum_{Q:\ell(Q)2^{-\nu}} \left[\inf_{Q,\gamma}(f) \tilde{\chi}_Q(x) \right]^q \right\}^{\frac{1}{q}} \right\|_{L_p} \\
&\overset{by\ (27)}{=} C_{n,r} 2^{\gamma\delta/r} \left\| \left\{ \sum_{y \in \mathbb{Z}} [\lambda(2^{\nu})]^q \sum_{\ell(a)2^{-\nu}} \left[\sum_{K \subseteq Q:\ell(k)=2^{-\gamma-\nu}} \ell_r^* K(*) \tilde{\chi}_J(x) \right]^q \right\}^{\frac{1}{q}} \right\|_{L_p} \\
&= ((*) \text{ for } x \text{ only one term is different from } 0) \\
&= C_{n,r} 2^{\gamma\delta/r} \left\| \left\{ \sum_{y \in \mathbb{Z}} [\lambda(2^{\nu})]^q \sum_{\ell(k)2^{-\gamma-\nu}} (\ell_r^*)_k^q \tilde{\chi}^q k(x) \right\}^{\frac{1}{q}} \right\|_{L_p} \\
&= C_{n,r} 2^{\gamma\delta/r} \left\| \left\{ \sum_{\mu \in \mathbb{Z}} [\lambda(2^{\mu-\gamma})]^q \sum_{\ell(k)2^{-\nu}} (\ell_r^*)_k^q \tilde{\chi}^q k(x) \right\}^{\frac{1}{q}} \right\|_{L_p} \\
&\overset{(\lambda(2^{\mu-\gamma}) \approx \lambda(2^{\mu}), \forall \mu \in \mathbb{Z})}{\leq} \bar{C} \left\| \left\{ \sum_{\mu \in \mathbb{Z}} [\lambda(2^{\mu})]^q \sum_{\ell(k)2^{-\mu}} (t_r^*)_k^q \tilde{\chi}^q k(x) \right\}^{\frac{1}{q}} \right\|_{L_p} \\
&= \bar{C} \left\| t_r^* \right\|_{\dot{f}_{pq}^{\lambda(\cdot)}} \\
&\leq (by\ lemma\ 2.3) \\
&\leq \bar{C} \left\| t \right\|_{\dot{f}_{pq}^{\lambda(\cdot)}}
\end{aligned}$$

So,

$$\begin{aligned}
\left\| \inf_{\gamma}(f) \right\|_{\dot{f}_{pq}^{\lambda(\cdot)}} &\leq \bar{C} \left\| t \right\|_{\dot{f}_{pq}^{\lambda(\cdot)}} \\
&= \left\| \left\{ \sum_{\mu \in \mathbb{Z}} [\lambda(2^{\mu})]^q \left| (\tilde{\varphi}_{\mu-\gamma} * f)(x) \right|^q \right\}^{\frac{1}{q}} \right\|_{L_p} \\
&\simeq \left(\lambda(2^{\mu-\gamma}) \right) \\
&\simeq 2(2^{\mu}), \forall \mu \in \mathbb{Z} \\
&\simeq \left\| \left\{ \sum_{\mu \in \mathbb{Z}} [\lambda(2^{\mu-\gamma})]^q \left| (\tilde{\varphi}_{\mu-\gamma} * f)(x) \right|^q \right\}^{\frac{1}{q}} \right\|_{L_p} \\
&\overset{\mu-\gamma=\nu}{=} \left\| \left\{ \sum_{\mu \in \mathbb{Z}} [\lambda(2^{\nu})]^q \left| (\tilde{\varphi}_{\nu} * f)(x) \right|^q \right\}^{\frac{1}{q}} \right\|_{L_p} \\
&\simeq \left\| f \right\|_{\dot{F}_{pq}^{\lambda(\cdot)}}
\end{aligned}$$

□

Hence, Lemma 2.8 is proved. Now, let us prove Lemma 2.4.

Proof of Lemma 2.4

The inequality

$$\left\| f \right\|_{\dot{F}_{pq}^{\lambda(\cdot)}} \leq \left\| \sup(f) \right\|_{\dot{f}_{pq}^{\lambda(\cdot)}}$$

follows immediately from definition. Let us prove the inverse inequality. Applying Lemma 2.3 to each function $(\tilde{\varphi} * f)(2^{\nu}x)$, we have the inequality

$$\left(\sup(f)_r^* \right)_Q < C_1 \left(\inf_{\gamma}(f)_r^* \right)_Q, \ for\ r = \min(p,q),\ \ell(Q) = 2^{-\nu} \tag{28}$$

But then, we have

$$\begin{aligned} \left\| \sup(f) \right\|_{\dot{f}_{pq}^{\lambda(\cdot)}} &\leq \left\| \sup(f)_r^* \right\|_{\dot{f}_{pq}^{\lambda(\cdot)}} \\ &\overset{by\ (28)}{\leq} C_1 \left\| \inf(f)_r^* \right\|_{\dot{f}_{pq}^{\lambda(\cdot)}} \\ &\overset{by\ Lemma\ 1}{\approx} \left\| \inf_{\gamma} f(x) \right\|_{\dot{f}_{pq}^{\lambda(\cdot)}} \end{aligned}$$

By applying Lemma 2.8, we obtain

$$\left\| \sup(f) \right\|_{\dot{f}_{pq}^{\lambda(\cdot)}} \leq \left\| f \right\|_{\dot{F}_{pq}^{\lambda(\cdot)}}$$

which proves lemma 2.4.

3. Conclusion

This paper has dealt with homogeneous and non-homogeneous Lizorkin-Triebel spaces with generalized smoothness $\dot{F}_{pq}^{\lambda(.)}(\mathbb{R}^n)$ and $F_{pq}^{\lambda(.)}(\mathbb{R}^n)$. In particular the Lizorkin-Triebel spaces of differential functions with generalized smoothness on $\mathbb{R}^n$ have been considered. We have discussed the discretization of norms in homogeneous Lizorkin-Triebel spaces and established the relationship between these spaces and their discrete analogues by generalizing the known results of Frazier-Jawerth and Bownik(Frazier et al.1991, Bownik 2000).

References

[1] M. Bownik, Extrapolation of discrete Triebel-Lizorkin spaces, University of Oregon, USA, 2000.
[2] M. Frazier, B.Jawerth, The φ-transform and applications to distribution spaces, Lecture Notes in Math. Springer-Verlangas, 1988.
[3] M. Frazier, B.Jawerth, A discrete transform, D.M.S. November, 1988.
[4] M. Frazier, B. Jawerth, G. Weiss, Littlewood-Paley theory and the study of functions spaces, CBMS, Regional conference. Ser. American Maths Society, 1991.
[5] H. Q. Bui , M. Paluszynsk , On the φ and ψ transforms of Frazier and Jawerth, Report Number UCDMS2004/1, 2004.
[6] W. Farkas, H. G. Leopold, Characterizations of function spaces of generalized smoothness, Annali di Matematica Pura ed Applicata, 185:1 (2006) 1-62.
[7] S. D. Moura, J. S. Neves, C. Schneider, Spaces of generalized smoothness in the critical case: optimal embeddings, continuity envelopes and approximation numbers, Journal of Approximation Theory 187,(2014) 82-117.
[8] V. Mutarutinya, Atomic expansion of functions from Lizorkin-Triebel spaces, XXXII Maths Conference, People's Feindship University of Russia, Moscow (1996) 123 - 126
[9] V. Mutarutinya , On discrete analogous of homogeneous Lizorkin - Triebel space with generalized smoothness, XXXV Scientific conference on problems of Physics, Mathematics, Chemistry, ICT and Teaching methodology. People's Feindship University of Russia, Mathematics Section, Edition PFUR (1999) 55 - 56.
[10] T. Ullrich, Continuous characterizations of Besov-Lizorkin-Triebel spaces and new interpretations as coorbits, J. Funct. Spaces Appl. 47 (2012)163213.