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On Discontinuity Problem with an Application to Threshold Activation Function

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Abstract. In this paper, some discontinuity results are obtained using the number $M_C(t,t^*)$ defined as

$$M_C(t,t^*) = \max\left\{\begin{array}{c} d(t,t^*), ad(t,Tt) + (1-a)d(t^*,St^*), \\ (1-a)d(t,Tt) + ad(t^*,St^*), \frac{b}{2}\left[d(t,St^*) + d(t^*,Tt)\right] \end{array}\right\},$$

at the common fixed point. Our results provide a new and distinct solution to an open problem "What are the contractive conditions which are strong enough to generate a fixed point but which do not force the map to be continuous at fixed point?" given by Rhoades [33]. To do this, we investigate a new discontinuity theorem at the common fixed point on a complete metric space. Also an application to threshold activation function is given.

1. Introduction and preliminaries

Rhoades studied many contractive conditions and compared 250 contractive definitions in a nice paper [32]. He also mentioned different power contractions in these contractive definitions. It was indicated that some contractive definitions do not require the mapping to be continuous in the entire domain, but this given mapping is continuous at the fixed point in all the cases (for example, see [10, 11, 18, 22] for more details). Therefore, Rhoades raised the following open question [33]:

What are the contractive conditions which are strong enough to generate a fixed point but which do not force the map to be continuous at fixed point?

Then some authors have been studying on this open question. For example, Pant obtained a first solution to this problem as follows:

Theorem 1.1. *[27] Let (X,d) be a complete metric space and T a self-mapping on X such that for any t, t^* in X*
(i) $d(Tt,Tt^*) \leq \phi(m(t,t^*))$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ denote a function such that $\phi(t^{**}) < t^{**}$ for each $t^{**} > 0$ and

$$m(t,t^*) = \max\{d(t,Tt), d(t^*,Tt^*)\}.$$

(ii) Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon < m(t,t^*) < \varepsilon + \delta \Longrightarrow d(Tt,Tt^*) \leq \varepsilon.$$

Then T has a unique fixed point α. Also, T is continuous at α if and only if $\lim\limits_{t\to\alpha} m(t,\alpha) = 0$.

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After, this result is generalized using different techniques and numbers. For example, Bisht and Pant studied on this open question using the Jachymski technique (see [16, 17]) and the number

$$M(t,t^*) = \max\left\{d(t,t^*), d(t,Tt), d(t^*,Tt^*), \frac{d(t,Tt^*)+d(t^*,Tt)}{2}\right\},$$

on a complete metric space [2]. More recently, new results have been obtained by many researchers (see [2–6, 26, 28–31, 34, 38] for more details).

In this paper, we investigate a new solution of the above open question using different numbers. For this purpose, in Section 2, we investigate a new theorem gives us a discontinuity result at the common fixed point on a complete metric space. Also we give an illustrative example and obtain some important corollaries. The obtained results generalize some known discontinuity theorems. So our results gain an importance for this problem. In Section 3, we mention some applications of fixed-point theory and discontinuity results. We give an application to threshold activation function at the fixed point and the common fixed point. The obtained results generalize some known fixed-point theorems in the literature.

2. Main results

In this section, we investigate some discontinuity results at the common fixed point on a complete metric space.

Theorem 2.1. *Let (X,d) be a complete metric space and $T,S : X \to X$ two self-mappings on X such that for any $t,t^* \in X$,*

(i) There exists a function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\xi(t^{**}) < t^{**}$ for each $t^{**} > 0$ and

$$d(Tt,St^*) \le \xi(M_C(t,t^*)),$$

where

$$M_C(t,t^*) = \max\left\{\begin{array}{c} d(t,t^*), ad(t,Tt)+(1-a)d(t^*,St^*), \\ (1-a)d(t,Tt)+ad(t^*,St^*), \frac{b}{2}\left[d(t,St^*)+d(t^*,Tt)\right] \end{array}\right\},$$

$0 \le a,b < 1$.

(ii) For a given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < M_C(t,t^*) < \varepsilon + \delta$ implies $d(Tt,St^*) \le \varepsilon$.

Then T and S have a unique common fixed point α and $T^n t \to \alpha$, $S^n t \to \alpha$ for each $t \in X$. Also, at least one of T and S is discontinuous if and only if

$$\lim_{t\to\alpha} M_C(t,\alpha) \ne 0 \text{ or } \lim_{t^*\to\alpha} M_C(\alpha,t^*) \ne 0.$$

Proof. Let t_0 be any point in X with $t_0 \ne Tt_0$, $t_0 \ne St_0$. Let us define a sequence $\{t_n\}$ in X such that

$$t_1 = Tt_0,\ t_2 = St_1,\ t_3 = Tt_2, \dots,$$

that is,

$$t_{2n+1} = Tt_{2n} \text{ and } t_{2n+2} = St_{2n+1},$$

for $n = 0,1,2,3\dots$. Suppose that $c_n = d(t_n,t_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$. Using the condition (i), we get

$$\begin{aligned} c_1 &= d(t_1,t_2) = d(Tt_0,St_1) \le \xi(M_C(t_0,t_1)) \\ &= \xi\left(\max\left\{\begin{array}{c} d(t_0,t_1), ad(t_0,Tt_0)+(1-a)d(t_1,St_1), \\ (1-a)d(t_0,Tt_0)+d(t_1,St_1), \\ \frac{b}{2}\left[d(t_0,St_1)+d(t_1,Tt_0)\right] \end{array}\right\}\right) \\ &= \xi\left(\max\left\{\begin{array}{c} d(t_0,t_1), ad(t_0,t_1)+(1-a)d(t_1,t_2), \\ (1-a)d(t_0,t_1)+ad(t_1,t_2), \\ \frac{b}{2}\left[d(t_0,t_2)+d(t_1,t_1)\right] \end{array}\right\}\right) \\ &= \xi(d(t_0,t_1)) = \xi(c_0) \end{aligned}$$

and

$$\begin{aligned}
c_2 &= d(t_2,t_3) = d(St_1,Tt_2) = d(Tt_2,St_1) \le \xi(M_C(t_2,t_1)) \\
&= \xi\left(\max\left\{\begin{array}{c} d(t_2,t_1), ad(t_2,Tt_2)+(1-a)d(t_1,St_1), \\ (1-a)d(t_2,Tt_2)+d(t_1,St_1), \\ \frac{b}{2}\left[d(t_2,St_1)+d(t_1,Tt_2)\right] \end{array}\right\}\right) \\
&= \xi\left(\max\left\{\begin{array}{c} d(t_2,t_1), ad(t_2,t_3)+(1-a)d(t_1,t_2), \\ (1-a)d(t_2,t_3)+ad(t_1,t_2), \\ \frac{b}{2}\left[d(t_2,t_2)+d(t_1,t_3)\right] \end{array}\right\}\right) \\
&= \xi(d(t_2,t_1)) = \xi(d(t_1,t_2)) \le \xi^2(c_0).
\end{aligned}$$

Since $\xi(t^{**}) \le t^{**}$ for each $t^{**} > 0$, using the similar arguments and the mathematical induction, we obtain

$$c_n \le \xi^n(c_0).$$

Therefore c_n is a strictly decreasing sequence in $\mathbb{R}^+$ and so tends to a limit $c \ge 0$. Assume that $c > 0$. Then there exists a positive integer k such that $n \ge k$ implies

$$c < c_n < c + \delta(c).$$

From the condition (ii) and $c_n < c_{n-1}$, we have

$$c_n < c \ (n \ge k),$$

which is a contradiction. Hence it should be $c = 0$.

We prove that $\{t_n\}$ is a Cauchy sequence. If we consider $m > n$, then we have

$$\begin{aligned}
d(t_n,t_m) &\le d(t_n,t_{n+1}) + d(t_{n+1},t_{n+2}) + \cdots + d(t_{m-1},t_m) \\
&\le \xi^n(d(t_0,t_1)) + \xi^{n+1}(d(t_0,t_1)) + \cdots + \xi^{m-1}(d(t_0,t_1)). \qquad (1)
\end{aligned}$$

If we put

$$R_n = \sum_{k=0}^{n} \xi^k(d(t_0,t_1)),$$

then using the inequality (1), we have

$$d(t_n,t_m) \le R_{m-1} - R_{n-1}. \qquad (2)$$

Since $\xi(t^{**}) < t^{**}$ for each $t^{**} > 0$, there exists $R \in [0,\infty)$ such that

$$\lim_{n\to\infty} R_n = R.$$

Using the inequality (2), we get

$$\lim_{n,m\to\infty} d(t_n,t_m) = 0.$$

So $\{t_n\}$ is Cauchy. From the completeness hypothesis, there exists a point $\alpha \in X$ such that $t_n \to \alpha$ as $n \to \infty$. Also $T^n t_n \to \alpha$ and $S^n t_n \to \alpha$.

Now we show that α is a common fixed point of T and S, that is,

$$T\alpha = S\alpha = \alpha.$$

Suppose that $S\alpha \neq \alpha$. Using the condition (i) and the triangle inequality, we get

$$\begin{aligned}
d(\alpha, S\alpha) &\leq d(\alpha, t_{2n+1}) + d(t_{2n+1}, S\alpha) \\
&= d(\alpha, t_{2n+1}) + (Tt_{2n}, S\alpha) \\
&\leq d(\alpha, t_{2n+1}) + \xi(M_C(t_{2n}, \alpha)) \\
&= d(\alpha, t_{2n+1}) + \xi\left(\max\left\{\begin{array}{c} d(t_{2n}, \alpha), ad(t_{2n}, Tt_{2n}) + (1-a)d(\alpha, S\alpha), \\ (1-a)d(t_{2n}, Tt_{2n}) + ad(\alpha, S\alpha), \\ \frac{b}{2}\left[d(t_{2n}, S\alpha) + d(\alpha, Tt_{2n})\right] \end{array}\right\}\right) \\
&= d(\alpha, t_{2n+1}) + \xi\left(\max\left\{\begin{array}{c} d(t_{2n}, \alpha), ad(t_{2n}, t_{2n+1}) + (1-a)d(\alpha, S\alpha), \\ (1-a)d(t_{2n}, t_{2n+1}) + ad(\alpha, S\alpha), \\ \frac{b}{2}\left[d(t_{2n}, S\alpha) + d(\alpha, t_{2n+1})\right] \end{array}\right\}\right)
\end{aligned}$$

and so taking limit for $n \to \infty$, we obtain

$$d(\alpha, S\alpha) \leq \xi\left(\max\left\{0, (1-a)d(\alpha, S\alpha), ad(\alpha, S\alpha), \frac{b}{2}d(\alpha, S\alpha)\right\}\right). \tag{3}$$

Therefore we have three cases.

Case 1: Let

$$\max\left\{0, (1-a)d(\alpha, S\alpha), ad(\alpha, S\alpha), \frac{b}{2}d(\alpha, S\alpha)\right\} = (1-a)d(\alpha, S\alpha).$$

Then using the inequality (3), we get

$$d(\alpha, S\alpha) \leq \xi((1-a)d(\alpha, S\alpha)) < (1-a)d(\alpha, S\alpha),$$

which is contradiction because of $0 < 1-a \leq 1$. Hence it should be $S\alpha = \alpha$.

Case 2: Let

$$\max\left\{0, (1-a)d(\alpha, S\alpha), ad(\alpha, S\alpha), \frac{b}{2}d(\alpha, S\alpha)\right\} = ad(\alpha, S\alpha).$$

Then using the inequality (3), we get

$$d(\alpha, S\alpha) \leq \xi(ad(\alpha, S\alpha)) < ad(\alpha, S\alpha),$$

which is contradiction because of $0 \leq a < 1$. Hence it should be $S\alpha = \alpha$.

Case 3: Let

$$\max\left\{0, (1-a)d(\alpha, S\alpha), ad(\alpha, S\alpha), \frac{b}{2}d(\alpha, S\alpha)\right\} = \frac{b}{2}d(\alpha, S\alpha).$$

Then using the inequality (3), we get

$$d(\alpha, S\alpha) \leq \xi\left(\frac{b}{2}d(\alpha, S\alpha)\right) < \frac{b}{2}d(\alpha, S\alpha),$$

which is contradiction because of $0 \leq \frac{b}{2} < \frac{1}{2}$. Hence it should be $S\alpha = \alpha$.

Now assume that $T\alpha \neq \alpha$. Using the condition (i), the symmetry condition and the triangle inequality, we find

$$\begin{aligned} d(\alpha, T\alpha) &\leq d(\alpha, t_{2n+2}) + d(t_{2n+2}, T\alpha) = d(\alpha, t_{2n+2}) + d(St_{2n+1}, T\alpha) \\ &= d(\alpha, t_{2n+2}) + d(T\alpha, St_{2n+1}) \leq d(\alpha, t_{2n+2}) + \xi(M_C(\alpha, t_{2n+1})) \\ &= d(\alpha, t_{2n+2}) + \xi\left(\max\left\{\begin{array}{c} d(\alpha, t_{2n+1}), ad(\alpha, T\alpha) + (1-a)d(t_{2n+1}, St_{2n+1}), \\ (1-a)d(\alpha, T\alpha) + ad(t_{2n+1}, St_{2n+1}), \\ \frac{b}{2}\left[d(\alpha, St_{2n+1}) + d(t_{2n+1}, T\alpha)\right] \end{array}\right\}\right) \\ &= d(\alpha, t_{2n+2}) + \xi\left(\max\left\{\begin{array}{c} d(\alpha, t_{2n+1}), ad(\alpha, T\alpha) + (1-a)d(t_{2n+1}, t_{2n+2}), \\ (1-a)d(\alpha, T\alpha) + ad(t_{2n+1}, t_{2n+2}), \\ \frac{b}{2}\left[d(\alpha, t_{2n+2}) + d(t_{2n+1}, T\alpha)\right] \end{array}\right\}\right) \end{aligned}$$

and so taking limit for $n \to \infty$, we get

$$d(\alpha, T\alpha) \leq \xi\left(\max\left\{0, ad(\alpha, T\alpha), (1-a)d(\alpha, T\alpha), \frac{b}{2}d(\alpha, T\alpha)\right\}\right). \tag{4}$$

From the similar arguments used in the above and the inequality (4), we obtain

$$d(\alpha, T\alpha) = 0, \text{ that is, } \alpha = T\alpha.$$

Hence we have

$$T\alpha = S\alpha = \alpha.$$

We show that α is the unique common fixed point of T and S. On the contrary, let α^* be another common fixed point of T and S. Using the condition (i), we get

$$\begin{aligned} d(\alpha, \alpha^*) &= d(T\alpha, S\alpha^*) \leq \xi(M_C(\alpha, \alpha^*)) \\ &= \xi\left(\max\left\{\begin{array}{c} d(\alpha, \alpha^*), ad(\alpha, T\alpha) + (1-a)d(\alpha^*, S\alpha^*), \\ (1-a)(\alpha, T\alpha) + ad(\alpha^*, S\alpha^*), \\ \frac{b}{2}\left[d(\alpha, S\alpha^*) + d(\alpha^*, T\alpha)\right] \end{array}\right\}\right) \\ &= \xi\left(\max\left\{d(\alpha, \alpha^*), 0, 0, bd(\alpha, \alpha^*)\right\}\right) = \xi(d(\alpha, \alpha^*)) < d(\alpha, \alpha^*), \end{aligned}$$

which is a contradiction. It should be $\alpha = \alpha^*$. Therefore, α is the unique common fixed point of T and S.

Finally, we prove that at least one of the self-mappings T and S is discontinuous at the point α if and only if

$$\lim_{t \to \alpha} M_C(t, \alpha) \neq 0 \text{ or } \lim_{t^* \to \alpha} M_C(\alpha, t^*) \neq 0.$$

At first, we show that $\lim\limits_{t \to \alpha} M_C(t, \alpha) = 0$ and $\lim\limits_{t^* \to \alpha} M_C(\alpha, t^*) = 0$ then both T and S are continuous at the unique common fixed point α. Assume that $\lim\limits_{t \to \alpha} M_C(t, \alpha)$ and $\lim\limits_{t^* \to \alpha} M_C(\alpha, t^*) = 0$. Using the definition of the numbers $M_C(t, \alpha)$ and $M_C(\alpha, t^*)$, we have

$$\lim_{t \to \alpha}\left\{\max\left\{\begin{array}{c} d(t, \alpha), ad(t, Tt) + (1-a)(\alpha, S\alpha), \\ (1-a)d(t, Tt) + ad(\alpha, S\alpha), \\ \frac{b}{2}\left[d(t, S\alpha) + d(\alpha, Tt)\right] \end{array}\right\}\right\} = 0$$

and

$$\lim_{t^* \to \alpha}\left\{\max\left\{\begin{array}{c} d(\alpha, t^*), ad(\alpha, T\alpha) + (1-a)(t^*, St^*), \\ (1-a)d(\alpha, T\alpha) + ad(t^*, St^*), \\ \frac{b}{2}\left[d(\alpha, St^*) + d(t^*, T\alpha)\right] \end{array}\right\}\right\} = 0,$$

which implies that

$$\lim_{t\to\alpha} Tt = \alpha = T\alpha \text{ and } \lim_{t\to\alpha} St = \alpha = S\alpha,$$

that is, the self-mappings T and S are continuous at the point α. For the converse statement, suppose that both T and S are continuous at the point α, that is,

$$\lim_{t\to\alpha} Tt = T\alpha \text{ and } \lim_{t\to\alpha} St = S\alpha.$$

Then we have

$$\lim_{t\to\alpha} M_C(t,\alpha) = \lim_{t\to\alpha}\left\{\max\left\{\begin{array}{c} d(t,\alpha), ad(t,Tt) + (1-a)(\alpha,S\alpha), \\ (1-a)d(t,Tt) + ad(\alpha,S\alpha), \\ \frac{b}{2}\left[d(t,S\alpha) + d(\alpha,Tt)\right] \end{array}\right\}\right\} = 0$$

and

$$\lim_{t^*\to\alpha} M_C(\alpha,t^*) = \lim_{t^*\to\alpha}\left\{\max\left\{\begin{array}{c} d(\alpha,t^*), ad(\alpha,T\alpha) + (1-a)(t^*,St^*), \\ (1-a)d(\alpha,T\alpha) + ad(t^*,St^*), \\ \frac{b}{2}\left[d(\alpha,St^*) + d(t^*,T\alpha)\right] \end{array}\right\}\right\} = 0.$$

□

Remark 2.2. *The last part of Theorem 2.1 can be considered as follows:*
Both T and S are continuous if and only if

$$\lim_{t\to\alpha} M_C(t,\alpha) = 0 \textit{ and } \lim_{t^*\to\alpha} M_C(\alpha,t^*) = 0.$$

Now we give the following illustrative example.

Example 2.3. *Let $X = [0,2]$ equipped with the Euclidian metric. Define $T, S : X \to X$ by*

$$Tt = \left\{\begin{array}{ccc} 1 & , & t \in [0,1] \\ 0.8 & , & t \in (1,2] \end{array}\right.$$

and

$$St = \left\{\begin{array}{ccc} 1 & , & t \in [0,1] \\ 0.85 & , & t \in (1,2] \end{array}\right. .$$

Then the point $t = 1$ is the unique common fixed point of the self-mappings T and S and both of the self-mappings are discontinuous at the point $t = 1$. In this example, it can be verified that

$$t, t^* \in [0,1] \Longrightarrow d(Tt, St^*) = 0 \textit{ and } 0 < M_C(t,t^*) \le 1, \tag{5}$$

$$t, t^* \in (1,2] \Longrightarrow d(Tt, St^*) = 0.05 \textit{ and } 0 < M_C(t,t^*) < 1, \tag{6}$$

$$t \in [0,1], t^* \in (1,2] \Longrightarrow d(Tt, St^*) = 0.2 \textit{ and } 0 < M_C(t,t^*) \le 2 \tag{7}$$

and

$$t \in (1,2], t^* \in [0,1] \Longrightarrow d(Tt, St^*) = 0.15 \textit{ and } 0 < M_C(t,t^*) \le 2. \tag{8}$$

The self-mappings T and S satisfy the condition (i) given in Theorem 2.1 with

$$\xi(t^{**}) = \begin{cases} 0.2 & , \quad t^{**} > 0.2 \\ 0.15 & , \quad 0.15 < t^{**} \leq 0.2 \\ \frac{t}{2} & , \quad 0 < t^{**} \leq 0.15 \end{cases}$$

and both of the self-mappings satisfy the condition (ii) given in Theorem 2.1 with

$$\delta(\varepsilon) = \begin{cases} 3 & , \quad \varepsilon \geq 0.2 \\ 3-\varepsilon & , \quad \varepsilon < 0.2 \end{cases},$$

for $a = 1$ and $b = \frac{1}{2}$. Indeed, for $t, t^* \in (1, 2]$, we have

$$Tt = Tt^* = 0.8, St = St^* = 0.85$$

and

$$M_C(t, t^*) = \max\left\{|t - t^*|, |t - 0.8|, |t^* - 0.85|, \frac{1}{4}(|t - 0.85| + |t^* - 0.8|)\right\} > 1.$$

From the definition of ξ, we see

$$\xi(t^{**}) < t^{**},$$

for each $t^{**} > 0$. Then we obtain

$$d(Tt, St^*) = 0.0.5 \leq \xi(M_C(t, t^*)) = 0.2$$

and so the self-mappings T and S satisfy the condition (i) given in Theorem 2.1. If we consider the inequalities (5), (6), (7), (8) and $\delta(\varepsilon)$, we show that the condition (ii) given in Theorem 2.1 is satisfied. Also, using the definition of the number $M_C(t, t^*)$, it can be easily seen that

$$\lim_{t\to 1} M_C(t, 1) \neq 0 \text{ and } \lim_{t^*\to 1} M_C(1, t^*) \neq 0.$$

From Theorem 2.1, we obtain the following corollary.

Corollary 2.4. *Let (X, d) be a complete metric space and $T, S : X \to X$ two self-mappings on X such that for any $t, t^* \in X$,*

(i) $d(Tt, St^*) < M_C(t, t^*)$ whenever $M_C(t, t^*) > 0$.

(ii) For a given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < M_C(t, t^*) < \varepsilon + \delta$ implies $d(Tt, St^*) \leq \varepsilon$.

Then T and S have a unique common fixed point α and $T^n t \to \alpha$, $S^n t \to \alpha$ for each $t \in X$. Also, at least one of T and S is discontinuous if and only if

$$\lim_{t\to\alpha} M_C(t, \alpha) \neq 0 \text{ or } \lim_{t^*\to\alpha} M_C(\alpha, t^*) \neq 0.$$

If we take $a = 0$, $b = 1$ in Theorem 2.1 then we get the following corollary.

Corollary 2.5. *Let (X, d) be a complete metric space and $T, S : X \to X$ two self-mappings on X such that for any $t, t^* \in X$,*

(i) There exists a function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\xi(t^{**}) < t^{**}$ for each $t^{**} > 0$ and

$$d(Tt, St^*) \leq \xi(M'_C(t, t^*)),$$

where

$$M'_C(t, t^*) = \max\left\{d(t, t^*), d(t^*, St^*), d(t, Tt), \frac{d(t, St^*) + (t^*, Tt)}{2}\right\}.$$

(ii) For a given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < M'_C(t,t^*) < \varepsilon + \delta$ implies $d(Tt, St^*) \leq \varepsilon$.

Then T and S have a unique common fixed point α and $T^n t \to \alpha$, $S^n t \to \alpha$ for each $t \in X$. Also, at least one of T and S is discontinuous if and only if

$$\lim_{t\to\alpha} M'_C(t,\alpha) \neq 0 \text{ or } \lim_{t^*\to\alpha} M'_C(\alpha,t^*) \neq 0.$$

Remark 2.6. *Corollary 2.5 coincides with Theorem 2.1 given in [31] on page 91 for the number $M'_C(t,t^*) = M_2(t,t^*)$.*

If we take $T = S$ in Corollary 2.4 then we get the following corollary.

Corollary 2.7. *Let (X,d) be a complete metric space and $T : X \to X$ a self-mapping on X such that for any $t, t^* \in X$,*

(i) $d(Tt, Tt^*) < M^*_C(t,t^*)$ whenever $M^*_C(t,t^*) > 0$, where

$$M^*_C(t,t^*) = \max\left\{ \begin{array}{c} d(t,t^*), ad(t,Tt) + (1-a)d(t^*,Tt^*), \\ (1-a)d(t,Tt) + ad(t^*,Tt^*), \frac{b}{2}\left[d(t,Tt^*) + (t^*,Tt)\right] \end{array} \right\},$$

(ii) For a given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < M^*_C(t,t^*) < \varepsilon + \delta$ implies $d(Tt, Tt^*) \leq \varepsilon$.

Then T has a unique fixed point α and $T^n t \to \alpha$ for each $t \in X$. Also, T is discontinuous if and only if $\lim_{t\to\alpha} M^*_C(t,\alpha) \neq 0$.

Remark 2.8. *Corollary 2.7 coincides with Theorem 2.1 given in [5] on page 3 for the number $M^*_C(t,t^*) = m_6(t,t^*)$. Consequently, Theorem 2.1 generalize the known discontinuity result at the fixed point.*

In the following theorem, we prove that the power contraction allows the possibility of discontinuity at the common fixed point for the commutative self-mappings (T and S are called commutative self-mappings if $TSt = STt$ for all $t \in X$). The commutative property is used in the proof of the following theorem (see 9 for more details).

Theorem 2.9. *Let (X,d) be a complete metric space and $T, S : X \to X$ two commutative self-mappings on X such that for any $t, t^* \in X$,*

(i) There exists a function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\xi(t^{**}) < t^{**}$ for each $t^{**} > 0$ and

$$d(T^p t, S^q t^*) \leq \xi(M^\star_C(t,t^*)),$$

where

$$M^\star_C(t,t^*) = \max\left\{ \begin{array}{c} d(t,t^*), ad(t,T^p t) + (1-a)d(t^*,S^q t^*), \\ (1-a)d(t,T^p t) + ad(t^*,S^q t^*), \frac{b}{2}\left[d(t,S^q t^*) + (t^*,T^p t)\right] \end{array} \right\},$$

$0 \leq a, b < 1$ and $p, q \in \mathbb{N}$.

(ii) For a given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < M^\star_C(t,t^*) < \varepsilon + \delta$ implies $d(T^p t, S^q t^*) \leq \varepsilon$.

Then T and S have a unique common fixed point α and $T^n t \to \alpha$, $S^n t \to \alpha$ for each $t \in X$. Also, at least one of T and S is discontinuous if and only if

$$\lim_{t\to\alpha} M^\star_C(t,\alpha) \neq 0 \text{ or } \lim_{t^*\to\alpha} M^\star_C(\alpha,t^*) \neq 0.$$

Proof. By Theorem 2.1, T^p and S^q have a unique common fixed point α, that is,

$$T^p\alpha = S^q\alpha = \alpha.$$

Then we obtain

$$T\alpha = T(T^p\alpha) = T^p(T\alpha)$$

and using the fact that commutative property, we get

$$T\alpha = T(S^q\alpha) = S^q(T\alpha).$$

So $T\alpha$ is a common fixed point of T^p and S^q. From the uniqueness of the common fixed point, we find $T\alpha = \alpha$. On the other hand, we have

$$S\alpha = S(S^q\alpha) = S^q(S\alpha)$$

and using the fact that commutative property, we get

$$S\alpha = S(T^p\alpha) = T^p(S\alpha). \tag{9}$$

Hence $S\alpha$ is a common fixed point of T^p and S^q. From the uniqueness of the common fixed point, we find $S\alpha = \alpha$. Therefore, we obtain

$$T\alpha = S\alpha = \alpha,$$

that is, α is a common fixed point of T and S.

Now we prove the uniqueness part of the proof. Suppose that α^* is another common fixed point of T and S. Then $T\alpha^* = S\alpha^* = \alpha^*$ and so $T^p\alpha^* = S^q\alpha^* = \alpha^*$. By the uniqueness of the common fixed point of T^p and S^q, it should be $\alpha = \alpha^*$. Consequently, α is a unique common fixed point of T and S. □

Remark 2.10. *In Theorem 2.9, we note that p and q do not have to be the same but especially, we can take $p = q$.*

3. An Application to Threshold activation function

Fixed-point theory has some applications in mathematics and some other science branches such as engineering, economy, biology etc. (see [7–9, 12–14, 24, 36]). Recently, discontinuous activation functions play an important role in neural networks (see [1, 15, 19–21, 23]). Therefore, some applications have been investigated to discontinuous activation functions. For example, fixed-circle problem and some discontinuity results at fixed point (or common fixed point) were used to obtain an application to discontinuous activation functions (see [25, 26, 29–31, 34, 35] for more details).

In 2017, Zhang studied on three types of activation functions using fixed-point data format [37]. In [37], it was compared single neuron models designed with bipolar ramp, threshold and sigmoid activation functions.

Using activation functions, the accumulated synaptic inputs of neuron is transferred to output. Also threshold, ramp and sigmoid functions are included in common activation functions. The mentioned functions can be used represent either unipolar or bipolar neuron model. For example, the unipolar and bipolar ramp activation functions are shown for a threshold activation function as follows:

$$f_u(t) = \begin{cases} 1.0 & , \quad t \geq t_{th} \\ 0.0 & , \quad t < t_{th} \end{cases} \text{ and } f_b(t) = \begin{cases} 1.0 & , \quad t \geq t_{th} \\ -1.0 & , \quad t < t_{th} \end{cases}. \tag{10}$$

Let $X = [-1, 1]$ equipped with the Euclidean metric and $t_{th} = 1$. Using the equation (10), let us define the functions $f_u, f_b : X \to X$ as

$$f_u(t) = \begin{cases} 1 & , \quad t \geq 1 \\ 0 & , \quad t < 1 \end{cases} \text{ and } f_b(t) = \begin{cases} 1 & , \quad t \geq 1 \\ -1 & , \quad t < 1 \end{cases}.$$

Then the point $\alpha = 1$ is the unique common fixed point of the self-mappings f_u and f_b and both of the self-mappings are discontinuous at the point $\alpha = 1$.

Notice that the self-mapping f_u has two fixed points $t_1 = 1$ and $t_2 = 0$. The self-mapping f_u is discontinuous at the fixed point $t_1 = 1$ but it is continuous at the fixed point $t_2 = 0$ since

$$\lim_{t\to 1} M_C(t, 1) \neq 0 \text{ and } \lim_{t\to 0} M_C(t, 0) = 0.$$

Also the self-mapping f_b has two fixed points $t_1 = 1$ and $t_2 = -1$. The self-mapping f_b is discontinuous at the fixed point $t_1 = 1$ but it is continuous at the fixed point $t_2 = -1$ since

$$\lim_{t\to 1} M_C(t, 1) \neq 0 \text{ and } \lim_{t\to 0} M_C(t, -1) = 0.$$

Therefore we say that the self-mappings f_u and f_b are discontinuous at the unique common fixed point $x = 1$.

Using the above techniques, it can be determined the common discontinuity points for more complicate activation functions.

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