



Study of Delay Induced Eco-Epidemiological Model Incorporating a Prey Refuge

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Abstract. The present study deals with the dynamical response of an eco-epidemiological model consisting of prey and predator species having infection in prey population. The inclusion of prey refugia is taken into account to avoid predator attack. The entire prey population is divided into two parts: healthy prey which are capable of reproducing following the logistic law and infected prey which is removed by predation or death before having the possibility of reproducing. This dynamical system assumes that predators form a dense colony or school in a single (possibly moving) location to encounter a prey and due to that an encounter between the prey and a single predator is immediately converted into an encounter between the prey and all the predators. This special type of interaction is approximated by a response function which is ratio dependent at high predator density. The dynamical responses in terms of boundedness, the local stability and bifurcation are studied in detail. Numerical simulations are performed at the end in order to exhibit the dynamical behavior of the present system for the occurrence of limit cycle and bifurcation based on the analytical results and choice of parameters involved in it.

1. Introduction

In mathematical ecology, one of the newest branch in which simultaneously both the ecological and epidemiological issues are taken under consideration is eco-epidemiology. Since the beginning of the ecological studies conducted by Lotka [27] and Volterra [37] and the revolutionary work of Kermack and McKendrick [24] in epidemiology, many researchers have been motivated to pursue quite a good number of research both in case of theoretical ecology and also in epidemiology. Bailey [5] and subsequently Anderson and May [4] were there among the pioneers who combined these fields and developed a model of predator and prey where prey species were affected by some diseases. Later on different types of models of prey and predator were studied by many researchers [cf. Hader and Freedman [15]; Xiao and Chen [39]; Hethcote et. al [20]] when the disease is present. Chattopadhyay and Arino [7] carried out an investigation of a three-dimensional eco-epidemiological model system and converted it to a two dimensional one based upon some basic properties and assumptions.

2020 Mathematics Subject Classification. 34C23, 92D25

Keywords. Eco-epidemiological model, refugia, boundedness, time-delay, global stability, hopf-bifurcation

Received: 16 September 2020; Revised: 10 December 2020; Accepted: 19 December 2020

Communicated by Maria Alessandra Ragusa

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One major issue seems to be more realistic is the consideration of the effect of refugia. In nature, some prey populations try to access to the areas where they can able to save themselves from their predators. This type of refugia play two important roles, by providing both to reduce the chance of disappearance due to predation and to dissipate the fluctuations between prey and predators. "In the predator-prey interactions, sometimes the effects of refuge afford the prey some degrees of protection from predation and reduce the chance of extinction due to predation" [Kar[23]; Huang et. al [21]]. Hassel [19] showed that "by adding a large portion of refuge to a model, which exhibited divergent oscillations in the absence of refuge, replaced the oscillatory behavior with a stable equilibrium". "In eco-epidemic model it is obvious that if the predator consumes the infected prey it will surely allow the susceptible prey to survive but may drive the infected prey to extinction due to over exploitation. Thus, infected prey can decrease their predation risk by using refuge. The effect of constant proportion of refuge on eco-epidemic models has been studied in" Sharma and Samanta [35], Pal and Samanta [32] and Kar [22].

The most interesting thing in a prey-predator system is that the choice of transmission rate of disease and the functional response of predator population which describes the number of prey consumed per predator per unit time. So the transmission rate of disease plays a key role in mathematical model of infectious disease. It is continued by numerous fields and laboratory experiments and observations that in many situations, especially where the predators have to search for foods, a more realistic general predator-prey theory should be based on the so called ratio dependent theory [Adriti et. al [1]; Adriti and Saiah[2]; Hanski [18]]. Dubey and Upadhyay [10] suggested a "two predator one prey system with ratio dependent predator growth rate to look into the cases: local stability, instability and global stability of the non-negative equilibriums together with the discussion of permanent co-existence of the three species". It is not the actual biological scenario that one predator will always attack or consume one prey only. Sometimes a group of predators, foraging in line abreast formation, might combined upon contact with a school or herd of preys. Cosner[9] proposed a response function, keeping the said type of interactions in mind which converts the encounter between the prey and a single predator into an encounter between prey and all predators so that the total number of encounters picks up another factor of P as $E = e_0NP^2$.

It is always very much desirable to consider a critical factor time delay for any biological and ecological systems to make the model convenient more to the reality of the system [Pulvirenti et. al [33], Ferrauto et. al [13], Chuan et. al [8] and Demou [11]]. Due to Kuang[25], consideration of time delay is necessary for digestion of the food of the animals before taking their next activities. Detailed arguments on the importance and usefulness of time-delays in realistic models may be found in the classical books of Macdonald[28], Gopalsamy[14], and Kuang[26]. Zhu et. al[42] analysed the effect of time delay on an eco-epidemic model. Shuxue et. al [36] investigated a delayed eco-epidemiological model with disease in predator and saturation incidence. Agaba et. al[3] analysed the dynamics of infectious disease with a concurrent spread of disease awareness. They also investigated the effects of time delay in response of unaware individuals to available information on the epidemic dynamics by establishing conditions for the Hopf bifurcation of the endemic steady state of the model. Wanjun et. al[38] studied Hopf-bifurcation of a delayed eco-epidemic model with ratio dependent transmission rate. Zhang et. al[41] also analysed permanence and hopf-bifurcation of a delayed eco-epidemic model with Leslie-Gower Holling Type III functional response. Mondal et. al[29] studied a delayed eco-epidemiological pest plant model with infected pest. Keeping all these studies in mind, it is felt that "Ordinary Differential Equation", which is the heart of Mathematical biology, should be replaced by "Delay Differential Equation".

In this paper the prey population has been classified into two categories, namely susceptible and infected prey. this investigation is addressed the dynamical behaviour of a ratio-dependent predator-prey system with infection within the prey population. Pal and Samanta [31] have assumed the effect of refuge in the infected prey population only and they have taken Michaelis-Menten functional response to describe the predator-prey interactions. But the current study includes the effect of refuge in the infected as well as susceptible population also. Holling type- II response functions with consideration of a group of predators foraging in line abreast formation (used by Cosner[9]) have been made use of in the present study. The

boundedness of the system along with the existence and stability analysis have been explained elaborately and also the condition for co-existence of all three types of species are studied. The Hopf-bifurcation criterion has been derived in terms of the prey refuge effect and encounter rate per predator per prey unit time as bifurcation parameter. The current epidemiological model will be discussed with the effect of discrete time delay. At the end the numerical results are produced to support the analytical outcomes of the present model.

2. The basic mathematical model

Before introducing the eco-epidemiological model, we shall first describe the assumptions we have taken into considerations:

1. Our proposed system consists of a single prey and a predator species. The prey species is divided into two classes in the presence of disease: one is sound or susceptible prey whose population biomass at time T is $S(T)$ and another one is infected prey whose population biomass at time T is $I(T)$. The predator is of population biomass Y at time T .
2. Susceptible prey population grows according to a logistic law with carrying capacity $K(K > 0)$ involving the whole prey population(susceptible and infected).
3. We assume the disease is spread among the prey populations only and therefore the disease isn't genetically inherited. The infected prey populations don't recover or become immune. Many researchers have assumed a classical mass-action principle incidence to explain the infection mechanism. it's going to be noted that if the degree of the infectivity increases, many other mechanisms often inherit the image. Therefore, it's more reasonable to vary the straightforward mass-action principle term by a Holling type- II term so as to possess a transparent insight of the infection.
4. The predators consume both susceptible and infected prey population according to Cosner[9] type of functional response. This is a special type of functional response which derived the fact that it increases with predator population. If predator acted for foraging individually, the functional response could follow a higher rate of predation of prey per predator.
5. To protect the prey populations (susceptible and infected, both), we construct our model by incorporating a refuge protecting mS and mI of the susceptible prey and infected prey respectively, where $m \in [0, 1)$ is constant. This leaves $(1 - m)S$ and $(1 - m)I$ of the susceptible and infected preys available to the predator.
6. It is also admissible to assume that the incubation period or response time for the predator mass will not be instantaneous, but arbitrated by some discrete time lag τ necessary for incubation.

On the basis of the above assumptions a three species model involving discrete time delay is considered:

$$\begin{aligned}
 \frac{dS}{dT} &= S \left\{ R \left(1 - \frac{S+I}{K} \right) - \frac{C_1(1-m)P^2}{A + (1-m)(S+I)P} - \frac{\lambda I}{B+S} \right\}, \\
 \frac{dI}{dT} &= I \left\{ \frac{\lambda S}{B+S} - \frac{C_2(1-m)P^2}{A + (1-m)(S+I)P} - \gamma \right\}, \\
 \frac{dP}{dT} &= P \left\{ \frac{e(1-m)(C_1S(T-\tau) + C_2I(T-\tau))P(T-\tau)}{A + (1-m)(S(T-\tau) + I(T-\tau))P(T-\tau)} - D \right\}
 \end{aligned} \tag{1}$$

where, S is the number of susceptible prey, I the number of infected prey population, P the number of predator population, R is the intrinsic growth rate of susceptible prey, K is the ecosystem support or environmental carrying capacity, λ is the rate of transmission from susceptible prey population to infected prey populations, γ is the death rate of infected species due to disease, C_1 and C_2 are extreme predation rates of S and I , A is the half saturation constant, $C_1(1 - m)$ and $C_2(1 - m)$ are the conversion rates of S and I , $m(0 < m < 1)$ is the prey refugia, D is the death rate of the predator, B is the half saturation constant and e is the encounters rate predator per prey unit time in the above mentioned model and τ signifies the reaction time or maturation period of the predator P .

Now we make the model dimensionless by using $s = \frac{S}{K}; i = \frac{I}{K}; y = \frac{P}{K}; t = KT$.

Using this substitutions in our epidemiological model system we get

$$\begin{aligned} \frac{ds}{dt} &= sf(s, i, y) = s\left\{r(1 - (s + i)) - \frac{c_1(1 - m)y^2}{a + (1 - m)(s + i)y} - \frac{li}{b + s}\right\}, \\ \frac{di}{dt} &= ig(s, i, y) = i\left\{\frac{ls}{b + s} - \frac{c_2(1 - m)y^2}{a + (1 - m)(s + i)y} - n\right\}, \\ \frac{dy}{dt} &= yh(s, i, y) = y\left\{\frac{e(1 - m)(c_1s(t - \tau) + c_2i(t - \tau))y(t - \tau)}{a + (1 - m)(s(t - \tau) + i(t - \tau))y(t - \tau)} - d\right\} \end{aligned} \tag{2}$$

along with the dimensionless initial data $s(0) \geq 0, i(0) \geq 0, y(0) \geq 0$.

The remaining dimensionless parameters are defined by

$$r = \frac{R}{K}; c_1 = \frac{C_1}{K}; c_2 = \frac{C_2}{K}; a = \frac{A}{K^2}; l = \frac{\lambda}{K}; b = \frac{B}{K}; n = \frac{\gamma}{K}; d = \frac{D}{K};$$

In the above mentioned model, we denote by C the Banach space of continuous functions $\psi : [-\tau, 0] \rightarrow \mathfrak{R}^3$ with

$$\|\psi\| = \sup_{-\tau \leq \phi \leq 0} \{|\psi_1|, |\psi_2|, |\psi_3|\}$$

where $\tau = \max\{\tau_1, \tau_2\}$ and $\psi = (\psi_1, \psi_2, \psi_3)$. As usual, the initial conditions of the system (2) are given as

$$s(\phi) = \psi_1(\phi), i(\phi) = \psi_2(\phi), y(\phi) = \psi_3(\phi), \phi \in [-\tau, 0] \tag{3}$$

where the initial function $\psi = (\psi_1, \psi_2, \psi_3)^T$ belongs to the Banach space $C = C([-\tau, 0], \mathfrak{R}^3)$ of continuous functions mapping the initial $[-\tau, 0]$ into \mathfrak{R}^3 . For biological reasons, the initial functions are assumed as

$$\psi_i(\phi) \geq 0, \phi \in [-\tau, 0], i = 1, 2, 3 \tag{4}$$

By the fundamental theory of functional differential equation Hale ([16], [17]), it can be proved that there is a unique solution $(s(t), i(t), y(t))$ to system (2) with initial condition (3). We further assume that $\psi_i(0) > 0$ for $i = 1, 2, 3$ for a biological feasibility.

3. Positivity and Boundedness of the system (2)

Here, we shall discourse the positivity, boundedness of solutions of the system (2) and study of equilibrium of the corresponding system. The subsequent theorem ensures the positivity and boundedness of the underlying system.

Theorem 3.1: Each compartment of the solution of the delayed system (2) with initial conditions (3) and (4) is positive and bounded in the interval $[0, +\infty)$.

Proof: Since the right hand side of the system (2) is completely continuous and locally Lipschitzian on C , the solution $(s(t), i(t), y(t))$ of (2) exists and is unique on $[0, \xi)$, where $0 < \xi \leq \infty$ (Hale [17]).

The model (2) can be written in the following form:

$$S = (s, i, y)^T \in \mathfrak{R}_+^3, (\psi_1(\phi), \psi_2(\phi), \psi_3(\phi)) \in C_+ = ([-\tau, 0], \mathfrak{R}_+^3), \psi_1(0), \psi_2(0), \psi_3(0) > 0.$$

$$F(S) = \begin{pmatrix} F_1(s, i, y) \\ F_2(s, i, y) \\ F_3(s, i, y) \end{pmatrix} = \begin{pmatrix} s(t)\left\{r(1 - (s + i)) - \frac{c_1(1 - m)y^2}{a + (1 - m)(s + i)y} - \frac{li}{b + s}\right\} \\ i(t)\left\{\frac{ls}{b + s} - \frac{c_2(1 - m)y^2}{a + (1 - m)(s + i)y} - n\right\} \\ y(t)\left\{\frac{e(1 - m)(c_1s(t - \tau) + c_2i(t - \tau))y(t - \tau)}{a + (1 - m)(s(t - \tau) + i(t - \tau))y(t - \tau)} - d\right\} \end{pmatrix}$$

Then system (2) becomes

$$\dot{S} = F(S) \tag{5}$$

Whenever $S(\phi) \in C_+$ such that $s = 0, i = 0, y = 0$, and letting $S_t \equiv S(\phi + t) \equiv (s(\phi + t), i(\phi + t), y(\phi + t))^T$, then it is easy to check that

$$F_1(S)|_{s=0, S_t \in C_+} \geq 0, \quad F_2(S)|_{i=0, S_t \in C_+} \geq 0, \quad F_3(S)|_{y=0, S_t \in C_+} \geq 0$$

Using the lemma in Yang et. al [40] any solution of (5) with $S(\phi) \in C_+$, say $S(t) = S(t, S(\phi))$, is such that $S(t) \in \mathfrak{R}_+^3$ for all $t \geq 0$. Therefore the solution of the system of equations (5) exists within the region \mathfrak{R}_+^3 and each solution remains non-negative for each $t > 0$. Consequently, the positive orthant \mathfrak{R}_+^3 is an invariant region.

Now we assume, $W_1(t) = es(t) + ei(t)$ and $\eta_1 > 0$ be a constant.

Then

$$\begin{aligned} \frac{dW_1(t)}{dt} + \eta_1 W_1(t) &= es'(t) + ei'(t) + \eta_1 es(t) + \eta_1 ei(t) \\ &= es(r + \eta_1) - ers^2 - ersi - ei(n - \eta_1) \\ &\leq es(r + \eta_1) - ers^2 \\ &= -er\left\{s^2 - \frac{r + \eta_1}{r}s + \left(\frac{r + \eta_1}{2r}\right)^2\right\} + \left(\frac{r + \eta_1}{2r}\right)^2 er \\ &= -er\left(s - \frac{r + \eta_1}{2r}\right)^2 + \frac{e(r + \eta_1)^2}{4r} \end{aligned}$$

Hence, $W_1'(t) + \eta_1 W_1(t) \leq \frac{e(r + \eta_1)^2}{4r} = \mu_1$ (say)

Applying a theorem on differential inequalities Birkhoff and Rota([6]), we obtain

$$0 < W_1(s(t), i(t)) \leq \frac{\mu_1}{\eta_1} + \frac{W_1(s(0), i(0))}{e^{\eta_1 t}}$$

and for $t \rightarrow \infty, 0 < W_1 \leq \frac{\mu_1}{\eta_1}$.

Let us define a function $W_2(t) = es(t - \tau) + ei(t - \tau) + y(t)$ and $\eta_2 > 0$ be a constant. Calculating the time derivative of $W_2(t)$ along the solution of (2) we obtain

$$\begin{aligned} \frac{dW_2(t)}{dt} + \eta_2 W_2(t) &= es'(t - \tau) + ei'(t - \tau) + y'(t) + \eta_2 es(t - \tau) + \eta_2 ei(t - \tau) + \eta_2 y(t) \\ &= e(r + \eta_2)s(t - \tau) - ers^2(t - \tau) - ers(t - \tau)i(t - \tau) - e(n - \eta_2)i(t - \tau) - (d - n)y(t) \\ &\leq e(r + \eta_2)s(t - \tau) - ers^2(t - \tau) \\ &= -er\left(s(t - \tau) - \frac{r + \eta_2}{2r}\right)^2 + \frac{e(r + \eta_2)^2}{4r} \\ &\leq \frac{e(r + \eta_2)^2}{4r} = \mu_2 \text{ (say)} \end{aligned}$$

$\frac{dW_2(t)}{dt} \leq \mu_2 - \eta_2 W_2(t)$ (for large time t) Therefore, as $t \rightarrow \infty, 0 < W_2 \leq \frac{\mu_2}{\eta_2}$.

Thus, solutions of system (2) enter into the region

$$B = \{(s, i, y) : 0 < es + ei + y < \frac{\mu_2}{\eta_2} + \epsilon, \text{ for any } \epsilon > 0\}$$

Hence system (2) is ultimately bounded.

4. Equilibria

In this section, we discuss the existence of various equilibrium points and stability behaviour of system (2) at those equilibrium points to determine the persistence of a biological system. System (2) has the following non-negative equilibrium points:

1. Two equilibrium points $E_1(0, 0, 0)$ and $E_2(1, 0, 0)$ always exist.
2. One predator free equilibrium: $E_3(s_3, i_3, 0)$ where $s_3 = \frac{nb}{l-n}$ and $i_3 = \frac{br(l-n-nb)}{(l-n)(l-n+br)}$. E_3 is feasible, provided $l > n(b + 1)$ which means that the rate of transmission from susceptible to infected prey population must exceed some threshold value which depends on the death rate of infected prey.
3. One infected prey free equilibrium: $E_4(s_4, 0, y_4)$ where $s_4 = \frac{ad}{(1-m)(ec_1-d)y_4}$ and $y_4^3 + Hy + G = 0$ such that $H = -\frac{aer}{(1-m)(ec_1-d)}$ and $G = \frac{a^2erd}{(1-m)^2(ec_1-d)^2}$. E_4 is feasible provided $ec_1 > d$ and $4er(1-m)(ec_1-d) < 27d^2$ which means that capturing rate must exceed a threshold value which depend on the death rate of the predator.
4. One coexistence equilibrium point: $E_*(s_*, i_*, y_*)$
Here s_*, i_* and y_* are the positive solutions of the system of algebraic equations given below :

$$\begin{aligned}
 r\{1 - (s_* + i_*)\} - \frac{c_1(1-m)y_*^2}{a + (1-m)(s_* + i_*)y_*} - \frac{li_*}{b + s_*} &= 0 \\
 \frac{ls_*}{b + s_*} - \frac{c_2(1-m)y_*^2}{a + (1-m)(s_* + i_*)y_*} - n &= 0 \\
 \frac{e(1-m)(c_1s_* + c_2i_*)y_*}{a + (1-m)(s_* + i_*)y_*} - d &= 0
 \end{aligned} \tag{6}$$

Solving (6) we get $y_* = \frac{e}{d}\{rs_*(1 - s_*) - i_*(rs_* + n)\}$, and $s_*^2 + As_* + B = 0$ where $A = \frac{c_1(l-n)-c_2r(1-i_*-b)}{c_2r}$, $B = \frac{c_2\{(l+br)i_*-br\}-bnc_1}{c_2r}$. Now, $A > 0$ and $B < 0$ must hold to get the feasible S_* and this condition implies $(1 - b) - \frac{c_1(l-n)}{c_2r} < i_* < \frac{br}{l+br} + \frac{bnc_1}{c_2(l+br)}$. Clearly, the interior equilibrium point E_* is feasible provided $l < n + \frac{c_2}{c_1}r(1 - b)$ and $rs_*(1 - s_*) > i_*(rs_* + n)$

4.1. Dynamical Behavior

Here, we investigate the local behavior of system (2) around the steady states defined above.

At the trivial equilibrium points $E_1(0, 0, 0)$ and $E_2(1, 0, 0)$:

Theorem 4.1.1 E_1 is always a saddle node and there cannot be total extinction of the system (2) for positive initial conditions. Suppose $l < n(1 + b)$, then the equilibrium point E_2 is asymptotically stable in $s-i-y$ space.

Proof: The characteristic equation of the Jacobian around the trivial equilibrium E_1 $(\lambda - r)(\lambda + n)(\lambda + d) = 0$. This implies that at this equilibrium point two of the eigen values are negative and one is positive. Hence E_1 acts as a saddle node.

Now, the Jacobian at the equilibrium $E_2(1, 0, 0)$ is given by

$$V(E_2) = \begin{pmatrix} -r & -\left(r + \frac{l}{b+1}\right) & 0 \\ 0 & \frac{l}{b+1} - n & 0 \\ 0 & 0 & -d \end{pmatrix}$$

The characteristic equation of the Jacobian matrix around E_2 is given by $(r + \lambda)(d + \lambda)\left(\frac{l}{b + 1} - (n + \lambda)\right) = 0$. Assumption of the theorem implies that all the three eigen values are negative and hence the theorem is proved.

At the predator free equilibrium $E_3(s_3, i_3, 0)$:

Theorem 4.1.2: Suppose $l > n(b + 1)$. Then boundary equilibrium $E_3(s_3, i_3, 0)$ is locally asymptotically stable.

Proof: The Jacobian at the boundary equilibrium E_3 is given by

$$V(E_3) = \begin{pmatrix} m_{11}^{(3)} & m_{12}^{(3)} & 0 \\ m_{21}^{(3)} & m_{22}^{(3)} & 0 \\ 0 & 0 & m_{33}^{(3)} \end{pmatrix}$$

where all $m_{ij}^{(3)}$ s are given as follows

$$m_{11}^{(3)} = s_3\left(-r + \frac{li_3}{(b + s_3)^2}\right), m_{12}^{(3)} = -s_3\left(r + \frac{l}{b + s_3}\right), m_{13}^{(3)} = 0, m_{21}^{(3)} = \frac{lb i_3}{(b + s_3)^2},$$

$$m_{22}^{(3)} = 0, m_{23}^{(3)} = 0, m_{31}^{(3)} = 0, m_{32}^{(3)} = 0, m_{33}^{(3)} = -d.$$

The characteristic equation of the Jacobian matrix around E_3 is given by

$$(m_{33}^{(3)} - \lambda)\{\lambda^2 - (m_{11}^{(3)} + m_{22}^{(3)})\lambda + (m_{11}^{(3)}m_{22}^{(3)} - m_{21}^{(3)}m_{12}^{(3)})\} = 0$$

Hence the eigen values are respectively λ_1, λ_2 and λ_3 such that

$$\lambda_1 = m_{33}^{(3)} = -d$$

$$\lambda_2 + \lambda_3 = m_{11}^{(3)} + m_{22}^{(3)} = s_3\left(-r + \frac{li_3}{(b + s_3)^2}\right)$$

$$\lambda_2\lambda_3 = m_{11}^{(3)}m_{22}^{(3)} - m_{21}^{(3)}m_{12}^{(3)} = \frac{lbs_3 i_3}{(b + s_3)^2}\left(r + \frac{l}{b + s_3}\right)$$

The assumption of the theorem implies that $r > \frac{li_3}{b + s_3}$ and hence all eigen values will have negative real part. Therefore the equilibrium point becomes asymptotically stable.

As stability of this boundary point implies extinction of the predator population, the rate of transmission from susceptible to infected prey population plays an important role for existence of predator.

At the infected prey free equilibrium $E_4(s_4, 0, y_4)$:

Theorem 4.1.3: If $0 < s_4 < \frac{27d^2}{e^2r^2}$ and $ae^2c_1rs_4 > d(ec_1 - d)\{ae + (1 - m)\}y_4^2$ hold good, then $E_4(s_4, 0, y_4)$

becomes asymptotically stable in $s - y$ plane. Moreover, suppose $\frac{ls_4}{b + s_4} < n + \frac{c_2(1 - m)y_4^2}{a + (1 - m)s_4y_4}$ holds then E_4

becomes asymptotically stable in $s - i - y$ space.

Proof: At the infected prey-free equilibrium $E_4(s_4, 0, y_4)$, the Jacobian matrix is

$$V(E_4) = \begin{pmatrix} v_{11}^{(4)} & v_{12}^{(4)} & v_{13}^{(4)} \\ 0 & v_{22}^{(4)} & 0 \\ v_{31}^{(4)} & v_{32}^{(4)} & v_{33}^{(4)} \end{pmatrix}$$

where all $v_{ij}^{(4)}$ s are given as follows

$$v_{11}^{(4)} = s_4\left[-r + \frac{c_1(1 - m)^2y_4^3}{\{a + (1 - m)s_4y_4\}^2}\right], v_{12}^{(4)} = s_4\left[-r + \frac{c_1(1 - m)^2y_4^3}{\{a + (1 - m)s_4y_4\}^2} - \frac{l}{b + s_4}\right],$$

$$v_{13}^{(4)} = -\frac{c_1(1-m)s_4y_4\{2a + (1-m)s_4y_4\}}{\{a + (1-m)s_4y_4\}^2}, v_{21}^{(4)} = 0,$$

$$v_{22}^{(4)} = \left[\frac{ls_4}{b + s_4} - \frac{c_2(1-m)y_4^2}{a + (1-m)s_4y_4} - n \right], v_{23}^{(4)} = 0, v_{31}^{(4)} = \frac{aec_1(1-m)y_4^2}{\{a + (1-m)s_4y_4\}^2},$$

$$v_{32}^{(4)} = \frac{e(1-m)\{ac_2 - (1-m)(c_1 - c_2)s_4y_4\}y_4^2}{\{a + (1-m)s_4y_4\}^2}, v_{33}^{(4)} = \frac{aec_1(1-m)s_4y_4}{\{a + (1-m)s_4y_4\}^2}.$$

The characteristic equation of the Jacobian matrix around E_4 is given by

$$(v_{22}^{(4)} - \lambda)\{\lambda^2 - (v_{11}^{(4)} + v_{33}^{(4)})\lambda + (v_{11}^{(4)}v_{33}^{(4)} - v_{13}^{(4)}v_{31}^{(4)})\} = 0$$

Clearly at this equilibrium point $E_4(s_4, 0, y_4)$ when inequalities $ae^2c_1rs_4 > d(ec_1 - d)\{ae + (1 - m)y_4^2\}$ and $0 < s_4 < \frac{27d^2}{e^2r^2}$ hold then two of the eigen values have negative real part and hence this equilibrium point becomes stable in $s - y$ plane. Moreover the last assumption of the theorem implies that E_4 is asymptotically stable in (s, i, y) space.

As stability of the boundary point implies extinction of infected prey population, prey refuge size plays a major role for existence of the infected prey.

At the coexistence equilibrium point $E_*(s_*, i_*, y_*)$:

We derive the following theorem to study the stability characteristics around the co-existence equilibrium E_* . The Jacobian matrix at the endemic equilibrium E_* is given by

$$V(E_*) = \begin{pmatrix} v_{11}^* & v_{12}^* & v_{13}^* \\ v_{21}^* & v_{22}^* & v_{23}^* \\ v_{31}^* & v_{32}^* & v_{33}^* \end{pmatrix}$$

where all v_{ij}^* values are given as follows:

$$v_{11}^* = -rs_* + \frac{c_1(1-m)^2s_*y_*^3}{\{a + (1-m)(s_* + i_*)y_*\}^2} + \frac{ls_*i_*}{(b + s_*)^2},$$

$$v_{12}^* = -rs_* + \frac{c_1(1-m)^2s_*y_*^3}{\{a + (1-m)(s_* + i_*)y_*\}^2} - \frac{ls_*}{b + s_*},$$

$$v_{13}^* = -\frac{c_1(1-m)\{2a + (1-m)(s_* + i_*)y_*\}s_*y_*}{\{a + (1-m)(s_* + i_*)y_*\}^2},$$

$$v_{21}^* = \frac{lbi_*}{(b + s_*)^2} + \frac{c_2(1-m)^2i_*y_*^3}{\{a + (1-m)(s_* + i_*)y_*\}^2},$$

$$v_{22}^* = \frac{c_2(1-m)^2i_*y_*^3}{\{a + (1-m)(s_* + i_*)y_*\}^2},$$

$$v_{23}^* = -\frac{c_2(1-m)\{2a + (1-m)(s_* + i_*)y_*\}i_*y_*}{\{a + (1-m)(s_* + i_*)y_*\}^2},$$

$$v_{31}^* = \frac{e(1-m)\{ac_1 + (1-m)(c_1 - c_2)i_*y_*\}y_*^2}{\{a + (1-m)(s_* + i_*)y_*\}^2},$$

$$v_{32}^* = \frac{e(1-m)\{ac_2 - (1-m)(c_1 - c_2)s_*y_*\}y_*^2}{\{a + (1-m)(s_* + i_*)y_*\}^2},$$

$$v_{33}^* = \frac{ae(1-m)(c_1s_* + c_2i_*)y_*}{\{a + (1-m)(s_* + i_*)y_*\}^2}$$

The characteristic equation of the Jacobian matrix around E_* is given by

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$$

where

$$\begin{aligned} A_1 &= -(v_{11}^* + v_{22}^* + v_{33}^*), \\ A_2 &= (v_{12}^*v_{21}^* + v_{22}^*v_{33}^* - v_{13}^*v_{31}^* - v_{23}^*v_{32}^*), \\ A_3 &= (v_{12}^*v_{23}^*v_{31}^* + v_{13}^*v_{22}^*v_{31}^* - v_{11}^*v_{22}^*v_{33}^* - v_{12}^*v_{21}^*v_{33}^* - v_{13}^*v_{21}^*v_{32}^*) \\ &= -\det(V(E_*)) \end{aligned}$$

Again, $\Delta = A_1A_2 - A_3$

Now, this endemic equilibrium point E_* becomes asymptotically stable by the application of Routh- Harwitz criterion.

Theorem 4.1.4: E_* is locally asymptotically stable if and only if $A_1 > 0, A_3 > 0$ and $\Delta > 0$.

4.2. Global stability analysis at $E_*(s_*, i_*, y_*)$

To show $E_*(s_*, i_*, y_*)$ is globally asymptotically stable in $s - i - y$ space we use the Lyapunov function L .

Theorem 4.2.1 Let $L = \frac{1}{2}(s - s_*)^2 + \frac{\alpha_1}{2}(i - i_*)^2 + \frac{\alpha_2}{2}(y - y_*)^2$ where $\alpha_1, \alpha_2 > 0$ are to be chosen properly such that $\frac{dL}{dT}|_{E_*} = 0$ where $E_*(s_*, i_*, y_*)$ and $L = L(s, i, y) > 0 \quad \forall (s, i, y) \in \{E\}$. The time derivative of L is $\frac{dL}{dt} \leq 0 \quad \forall (s, i, y) \in \Gamma^+$. It then follows that $\frac{dL}{dT} = 0, \quad \forall (s_*, i_*, y_*) \in \Gamma^+$ implies that E_* of the system is Lyapunov stable and $\frac{dL}{dT} < 0, \quad \forall (s, i, y) \in \Gamma^+$ near E_* implies that E_* is globally stable.

Proof: $\frac{dL}{dt} = (s - s_*)\frac{ds}{dt} + \alpha_1(i - i_*)\frac{di}{dt} + \alpha_2(y - y_*)\frac{dy}{dt}$
 Now by substituting equations of the model (2), we get

$$\begin{aligned} \frac{dL}{dt} &= (s - s_*)\left[rs\{1 - (s + i)\} - \frac{c_1(1 - m)sy^2}{a + (1 - m)(s + i)y} - \frac{lis}{b + s}\right] + \\ &\quad \alpha_1(i - i_*)\left[\frac{lsi}{b + s} - \frac{c_2(1 - m)iy^2}{a + (1 - m)(s + i)y} - ni\right] + \\ &\quad \alpha_2(y - y_*)\left[\frac{e(1 - m)(c_1s + c_2i)y^2}{a + (1 - m)(s + i)y} - dy\right] \end{aligned} \tag{7}$$

Then equation (7) becomes

$$\begin{aligned} \frac{dL}{dt} &= (s - s_*)\left[r\{1 - (s + i)\} - \frac{c_1(1 - m)y^2}{a + (1 - m)(s + i)y} - \frac{li}{b + s}\right](s - s_*) + \\ &\quad \alpha_1(i - i_*)\left[\frac{ls}{b + s} - \frac{c_2(1 - m)y^2}{a + (1 - m)(s + i)y} - n\right](i - i_*) + \\ &\quad \alpha_2(y - y_*)\left[\frac{e(1 - m)(c_1s + c_2i)y}{a + (1 - m)(s + i)y} - d\right](y - y_*) \end{aligned} \tag{8}$$

By rearranging, we obtain

$$\begin{aligned} \frac{dL}{dt} = & -(s - s_*)^2 \left[r\{-1 + (s + i)\} + \frac{c_1(1 - m)y^2}{a + (1 - m)(s + i)y} + \frac{li}{b + s} \right] - \\ & \alpha_1(i - i_*)^2 \left[-\frac{ls}{b + s} + \frac{c_2(1 - m)y^2}{a + (1 - m)(s + i)y} + n \right] - \\ & \alpha_2(y - y_*)^2 \left[-\frac{e(1 - m)(c_1s + c_2i)y}{a + (1 - m)(s + i)y} + d \right] \end{aligned} \tag{9}$$

Thus, it is possible to set α_1, α_2 such that $\frac{dL}{dt} \leq 0$ and the equilibrium point E_* is globally stable.

4.3. Hopf Bifurcation analysis at $E_*(s_*, i_*, y_*)$

For the above mentioned system the study of Hopf bifurcation is very much necessary. Basically bifurcation analysis deals with structurally unstable system and it is a branch of Mathematics that confirms a definite qualitative changes in the dynamical systems with respect to some parameters. A small change in parameter generates a topological change and therefore ensures the change in stability which means disease may be either endemic or fluctuating or controlled. Our main aim is to check whether Hopf bifurcation occurs in present dynamical system and to identify the parameter m which has a core role in varying the dynamics.

The characteristic equation of the Jacobian matrix around $E_*(s_*, i_*, y_*)$ is given by

$$\lambda^3 + A_1(m)\lambda^2 + A_2(m)\lambda + A_3(m) = 0 \tag{10}$$

where

$$A_1(m) = -(v_{11}^*(m) + v_{22}^*(m) + v_{33}^*(m))$$

$$A_2(m) = (v_{12}^*(m)v_{21}^*(m) + v_{22}^*(m)v_{33}^*(m) - v_{13}^*(m)v_{31}^*(m) - v_{23}^*(m)v_{32}^*(m))$$

and

$$A_3(m) = (v_{12}^*(m)v_{23}^*(m)v_{31}^*(m) + v_{13}^*(m)v_{22}^*(m)v_{31}^*(m) - v_{11}^*(m)v_{22}^*(m)v_{33}^*(m) - v_{12}^*(m)v_{21}^*(m)v_{33}^*(m) - v_{13}^*(m)v_{21}^*(m)v_{32}^*(m))$$

In order to observe the existence of instability of system (2), let us consider m as bifurcation parameter. For this purpose let us first state the following theorem:

Theorem 4.3.1(Hopf Bifurcation Theorem): According to Murray [30] $A_i(m), i = 1, 2, 3$ are smooth functions of m in an open interval about $m^* \in \mathfrak{R}$ such that the characteristic equation has

(i) a pair of complex eigenvalues $\lambda = \xi(m) \pm i\zeta(m)$ (with $\xi(m), \zeta(m) \in \mathfrak{R}$) so that they become purely imaginary at $m = m^*$ and $\left. \frac{d\xi}{dm} \right|_{m=m^*} \neq 0$

(ii) the other eigen value is negative at $m = m^*$

Then a Hopf bifurcation occurs around E_* at $m = m^*$ (i.e. a stability change of E_* accompanied by the creation of a limit cycle at $m = m^*$).

Theorem 4.3.2 System (2) undergoes a Hopf bifurcation around E_* when the number of prey refuge m crosses a critical value m^* provided $A_1(m^*) > 0, A_3(m^*) > 0$ and $A_1(m^*)A_2(m^*) = A_3(m^*)$.

Proof: For $m = m^*$, the characteristic equation of the system at E_{s^*iy} becomes

$$(\lambda^2 + A_2)(\lambda + A_1) = 0$$

providing roots $\lambda_1 = i\sqrt{A_2}, \lambda_2 = -i\sqrt{A_2}$ and $\lambda_3 = -A_1$. Thus there exists a pair of purely imaginary eigenvalues and a strictly negative real eigenvalue. Also $A_i(i = 1, 2, 3)$ are smooth functions of m . So, for m

in a neighbourhood of m^* , the roots have the form

$\lambda_1(m) = u_1(m) + iu_2(m), \lambda_2(m) = u_1(m) - iu_2(m), \lambda_3 = -u_3(m)(\lambda_3(m) = -A_1(m))$ where $u_i(m), i = 1, 2, 3$ are real.

Next we shall verify the transversality conditions: $\frac{d}{dm}(Re(\lambda_i(m)))|_{m=m^*} \neq 0, i = 1, 2$.

Substituting $\lambda = u_1(m) + iu_2(m)$ into the characteristic equation we get

$$(u_1 + iu_2)^3 + A_1(u_1 + iu_2)^2 + A_2(u_1 + iu_2) + A_3 = 0 \tag{11}$$

Now, taking derivative of (11) with respect to m , we get,

$$3(u_1 + iu_2)^2(\dot{u}_1 + i\dot{u}_2) + 2A_1(u_1 + iu_2)(\dot{u}_1 + i\dot{u}_2) + \dot{A}_1(u_1 + iu_2)^2 + A_2(\dot{u}_1 + i\dot{u}_2) + \dot{A}_2(u_1 + iu_2) + \dot{A}_3 = 0 \tag{12}$$

Comparing real and imaginary parts from both sides of (12):

$$P_1\dot{u}_1 - P_2\dot{u}_2 + P_3 = 0 \tag{13}$$

$$P_2\dot{u}_1 + P_2\dot{u}_2 + P_4 = 0 \tag{14}$$

where

$$P_1 = 3(u_1^2 - u_2^2) + 2A_1u_1 + A_2, \quad P_2 = 6u_1u_2 + 2A_1u_2, \\ P_3 = \dot{A}_1(u_1^2 - u_2^2) + \dot{A}_2u_1 + \dot{A}_3, \quad P_4 = 2\dot{A}_1u_1u_2 + \dot{A}_2u_2 \tag{15}$$

From (14) and (14), we get

$$\dot{u}_1 = \frac{P_2P_4 + P_1P_3}{P_1^2 + P_2^2} \tag{16}$$

Now, $P_3 = \dot{A}_1(u_1^2 - u_2^2) + \dot{A}_2u_1 + \dot{A}_3 \neq \dot{A}_1(u_1^2 - u_2^2) + \dot{A}_2P_1 + \dot{A}_1A_2 + \dot{A}_2A_1$, at $m = m^*$:

Case I: $u_1 = 0, u_2 = \sqrt{A_2}$

$$P_1 = -2A_2, P_2 = 2A_1\sqrt{A_2}, P_3 \neq \dot{A}_2A_1, P_4 = \dot{A}_2\sqrt{A_2}$$

$$P_2P_4 + P_1P_3 \neq 2A_1A_2\dot{A}_2 - 2A_1A_2\dot{A}_2 = 0$$

So, $P_2P_4 + P_1P_3 \neq 0$ at $m = m^*$, when $u_1 = 0, u_2 = \sqrt{A_2}$.

Case II: $u_1 = 0, u_2 = -\sqrt{A_2}$

$$P_1 = -2A_2, P_2 = -2A_1\sqrt{A_2}, P_3 \neq \dot{A}_2A_1, P_4 = -\dot{A}_2\sqrt{A_2}$$

$$P_2P_4 + P_1P_3 \neq 2A_1A_2\dot{A}_2 - 2A_1A_2\dot{A}_2 = 0$$

So, $P_2P_4 + P_1P_3 \neq 0$ at $m = m^*$, when $u_1 = 0, u_2 = -\sqrt{A_2}$.

Therefore, $\frac{d}{dm}(Re(\lambda_i(m)))|_{m=m^*} = -\frac{P_2P_4 + P_1P_3}{P_1^2 + P_2^2}|_{m=m^*} \neq 0$

and $u_3(m^*) = -A_1(m^*) < 0$

5. Stability of the model in presence of discrete delay

The system has the same equilibria as in the case of non-delay. The main purpose of this section is to study the stability behaviour of $E_*(s_*, i_*, y_*)$ in the presence of discrete delay ($\tau \neq 0$). We linearise our system by using the following transformation:

$$s_* = s_* + u, i_* = i_* + v, w_* = w_* + w$$

The linear system is given by

$$\frac{dU}{dt} = AU(t) + BU(t - \tau) \tag{17}$$

where $U(t) = [u \ v \ w]^T$, $A = (a_{ij})_{3 \times 3}$, $B = (b_{ij})_{3 \times 3}$ and

$$a_{11} = r(1 - 2s_* - i_*) - \frac{c_1}{P^2} y_*^2 \left(\frac{a}{1-m} + i_* y_* \right) - \frac{l b i_*}{(b + s_*)^2}, \quad a_{12} = \left(\frac{c_1}{P^2} y_*^3 - r - \frac{l}{b + s_*} \right) s_*,$$

$$a_{13} = -\frac{c_1}{P^2} s_* y_* \left(\frac{a}{1-m} + P \right), \quad a_{21} = i_* \left\{ \frac{l}{b + s_*} \left(1 - \frac{s_*}{b + s_*} \right) + \frac{c_2 y_*^3}{P^2} \right\},$$

$$a_{22} = \frac{l s_*}{b + s_*} - \frac{c_2 y_*^2}{P^2} \left(\frac{a}{1-m} + y_* s_* \right) - n, \quad a_{23} = -\frac{c_2 i_* y_*}{P^2} \left(\frac{a}{1-m} + P \right) \text{ and all other } a_{ij} = 0$$

$$b_{31} = \frac{e y_*^2}{P^2} \left(\frac{c_1 a}{1-m} + (c_1 - c_2) i_* y_* \right), \quad b_{32} = \frac{e y_*^2}{P^2} \left(\frac{c_2 a}{1-m} + (c_2 - c_1) y_* s_* \right),$$

$$b_{33} = \frac{e y_* a}{P(1-m)} (c_1 s_* + c_2 i_*) \text{ and all other } b_{ij} = 0$$

where $P = \frac{a}{1-m} + (s_* + i_*) y_*$

We look for the solution of the model (17) of the form $U(t) = \rho e^{\lambda t}$, $\rho \neq 0 \in \mathfrak{R}$. This leads to the following characteristic equation:

$$\lambda^3 + a_1 \lambda^2 + a_2 \lambda + (a_3 \lambda^2 + a_4 \lambda + a_5) e^{-\lambda \tau} = 0 \tag{18}$$

where, $a_1 = -a_{11} - a_{22}$; $a_2 = a_{11} a_{22} - a_{12} a_{21}$; $a_3 = -b_{33}$; $a_4 = a_{11} b_{33} + a_{22} b_{33} - a_{23} b_{32} - a_{13} b_{31}$;
 $a_5 = a_{11} a_{23} b_{32} + a_{12} a_{21} b_{33} + a_{13} a_{22} b_{31} - a_{11} a_{22} b_{33} - a_{12} a_{23} b_{31} - a_{13} a_{21} b_{32}$.

It is well known that the signs of the real parts of the solutions of (18) characterize the stability theorem of E_* . Therefore substituting $\lambda = \xi + i\eta$ in (18) we obtain real and imaginary parts, respectively as

$$\xi^3 - 3\xi\eta^2 + a_1\xi^2 - a_1\eta^2 + a_2\xi + e^{-\xi\tau} \left[(a_3\xi^2 - a_3\eta^2 + a_4\xi + a_5) \cos \eta\tau + (2a_3\xi\eta + a_4\eta) \sin \eta\tau \right] = 0 \tag{19}$$

and

$$3\xi^2\eta - \eta^3 + 2a_1\xi\eta + a_2\eta + e^{-\xi\tau} \left[(2a_3\xi\eta + a_4\eta) \cos \eta\tau - (a_3\xi^2 - a_3\eta^2 + a_4\xi + a_5) \sin \eta\tau \right] = 0 \tag{20}$$

A necessary condition for a stability change of E_* is that the characteristic equation (18) should have purely imaginary solutions. Hence to obtain the stability criterion, we get $\xi = 0$ in (19) and (20). Then we have,

$$a_1\eta^2 = (-a_3\eta^2 + a_5) \cos \eta\tau + a_4\eta \sin \eta\tau \tag{21}$$

$$-\eta^3 + a_2\eta = (-a_3\eta^2 + a_5) \sin \eta\tau - a_4\eta \cos \eta\tau \tag{22}$$

From (21) and (22) we have,

$$\tau_+^{(n)} = \frac{1}{\eta} \cos^{-1} \left[\frac{\eta^2 \{ (a_4 - a_1 a_3) \eta^2 + (a_1 a_5 - a_2 a_4) \}}{a_4^2 \eta^2 + (a_5 - a_3 \eta^2)^2} \right] + \frac{2n\pi}{\eta},$$

$n = 0, 1, 2, \dots$, where $\tau_+^{(n)}$ is the n th solution of τ . Eliminating τ from (21) and (22), we get

$$\eta^6 + d_1 \eta^4 + d_2 \eta^2 + d_3 = 0 \tag{23}$$

where $d_1 = a_1^2 - 2a_2 - a_3^2$; $d_2 = a_2^2 + 2a_3 a_5 - a_4^2$; $d_3 = -a_5^2$. Now, we put $\eta^2 = \sigma$ then the above equation becomes

$$\sigma^3 + d_1 \sigma + d_2 \sigma + d_3 = 0 \tag{24}$$

This is a cubic equation in σ . By Decarte’s rule of sign we can say that the equation has at least one positive real root. If σ_+ is the positive root, then $\eta = \pm \sqrt{\sigma_+}$. Let us state a lemma:

Lemma 5.1: Consider the exponential polynomial

$$P(\lambda, e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \dots, e^{-\lambda\tau_m}) = \lambda^n + p_1^{(0)}\lambda^{n-1} + \dots + p_{n-1}^{(0)}\lambda + p_n^{(0)} + [p_1^{(1)}\lambda^{n-1} + \dots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} + \dots + [p_1^{(m)}\lambda^{n-1} + \dots + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\tau_m}$$

where $\tau_j \geq 0 (j = 1, 2, \dots, m)$ and $p_i^{(j)} (j = 0, 1, \dots, m; i = 1, 2, \dots, n)$ are constants. As $(\tau_1, \tau_2, \dots, \tau_m)$ vary, the sum of the order of zero of $P(\lambda, e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \dots, e^{-\lambda\tau_m})$ in the open half plane can change only if a zero appears on or crosses the imaginary axis.

Proof: The lemma has been proved by Ruan and Wei[34].

Considering τ as a bifurcating parameter, a Hopf-bifurcation around the interior equilibrium point E_* is discussed in the next theorem:

Theorem 5.1: Suppose that E_* exists and is locally asymptotically stable for our system with $\tau = 0$. Also let $\sigma_+ = \eta^2$ be a positive root of (24).

1. Then there exists $\tau_+^{(0)} = \tau^*$ such that the interior equilibrium point E_* of the delay our system is asymptotically stable when $0 \leq \tau < \tau^*$ and unstable for $\tau > \tau^*$.
2. Furthermore, the system will undergo a Hopf bifurcation at E_* when $\tau = \tau^*$, provided $C(\eta)F(\eta) - D(\eta)G(\eta) > 0$.

Proof: Since η_0 is a solution of equation (23), the characteristic equation (18) has the pair of purely imaginary roots $\pm i\eta_0$. From equations (21) and (22), $\tau_+^{(n)}$ is a function of η_0 for $n = 0, 1, 2, \dots$, which is given by

$$\tau_+^{(n)} = \frac{1}{\eta_0} \arccos \left[\frac{\eta_0^2 \{(a_4 - a_1 a_3) \eta_0^2 + (a_1 a_5 - a_2 a_4)\}}{a_4^2 \eta_0^2 + (a_5 - a_3 \eta_0^2)^2} \right] + \frac{2n\pi}{\eta_0} \tag{25}$$

Now the system will be locally asymptotically stable around the interior equilibrium point E_{sly} for $\tau = 0$, if conditions of Theorem 4.5 holds. In that case by Lemma 5.1 E_{sly} will remain stable for $\tau < \tau^*$, such that

$$\tau^* = \min_{n \geq 0} \tau_+^{(n)}$$

Also it is easy to verify the following transversality condition

$$\frac{d}{d\tau} [Re\{\lambda(\tau)\}]_{\tau=\tau^*} > 0$$

Differentiating equations (19) and (20) with respect to τ and then putting $\xi = 0$ we obtain

$$\begin{aligned} C(\eta) \frac{d\xi}{d\tau} + D(\eta) \frac{d\eta}{d\tau} &= F(\eta) \\ -D(\eta) \frac{d\xi}{d\tau} + C(\eta) \frac{d\eta}{d\tau} &= G(\eta) \end{aligned} \tag{26}$$

where

$$\begin{aligned} C(\eta) &= -3\eta^2 + a_2 + a_4 \cos \eta\tau - a_4\eta\tau \sin \eta\tau + 2a_3\eta \sin \eta\tau + a_3\eta^2\tau \cos \eta\tau \\ D(\eta) &= -2a_1\eta - 2a_3\eta \cos \eta\tau + a_4 \sin \eta\tau + a_4\eta\tau \cos \eta\tau + a_3\eta^2\tau \sin \eta\tau - a_5\tau \sin \eta\tau \\ F(\eta) &= -a_3\eta^3 \sin \eta\tau + a_5\eta \sin \eta\tau - a_4\eta^2 \cos \eta\tau \\ G(\eta) &= a_4\eta^2 \sin \eta\tau + a_5\eta \cos \eta\tau - a_3\eta^3 \cos \eta\tau \end{aligned} \tag{27}$$

Solving the above system,

$$\frac{d}{d\tau} [Re\{\lambda(\tau)\}]_{\tau=\tau^*, \eta=\eta_0} = \left[\frac{C(\eta)F(\eta) - D(\eta)G(\eta)}{C^2(\eta) + D^2(\eta)} \right]_{\tau=\tau^*, \eta=\eta_0},$$

which shows that

$$\frac{d}{d\tau} [Re\{\lambda(\tau)\}]_{\tau=\tau^*, \eta=\eta_0} > 0$$

if

$$C(\eta)F(\eta) - D(\eta)G(\eta) > 0$$

Therefore, the transversality condition is satisfied and hence Hopf bifurcation occurs at $\tau = \tau^*$. This completes the proof of the theorem.

6. Estimation of the length of delay to preserve stability

We examine the system (2) and the space of all real valued continuous functions defined on $[\tau, \infty]$ satisfying the initial conditions on $[-\tau, 0]$, we linearize the system (2) about its interior equilibrium $E_*(s_*, i_*, y_*)$ and get

$$\begin{aligned} \frac{du}{dt} &= \Phi_1 u + \Psi_1 v + \Omega_1 w \\ \frac{dv}{dt} &= \Phi_2 u + \Psi_2 v + \Omega_2 w \\ \frac{dw}{dt} &= \Phi_3 u(t - \tau) + \Psi_3 v(t - \tau) + \Omega_3 w(t - \tau) + \Upsilon w \end{aligned} \tag{28}$$

where $\Phi_1 = r - 2rs_* - ri_* - \frac{c_1(1-m)y_*^2(a + (1-m)i_*y_*)}{(a + (1-m)(s_* + i_*)y_*)^2} - \frac{lb i_*}{(b + s_*)^2}$,

$$\Psi_1 = -rs_* + \frac{c_1(1-m)^2 y_*^3}{(a + (1-m)(s_* + i_*)y_*)^2} - \frac{ls_*}{b + s_*},$$

$$\Omega_1 = \frac{c_1(1-m)y_*(2a + y_*(1-m)(s_* + i_*))}{(a + (1-m)(s_* + i_*)y_*)^2},$$

$$\Phi_2 = \frac{lb}{(b + s_*)^2} + \frac{c_2(1-m)^2 y_*^3}{(a + (1-m)(s_* + i_*)y_*)^2},$$

$$\Psi_2 = \frac{ls_*}{b + s_*} - \frac{c_2(1-m)y_*^2(a + (1-m)s_*y_*)}{(a + (1-m)(s_* + i_*)y_*)^2} - n,$$

$$\Omega_2 = \frac{c_2(1-m)y_*(2a + y_*(1-m)(s_* + i_*))}{(a + (1-m)(s_* + i_*)y_*)^2},$$

$$\Phi_3 = \frac{e(1-m)(c_1 a + (c_1 - c_2)(1-m)i_*y_*)}{(a + (1-m)(s_* + i_*)y_*)^2},$$

$$\Psi_3 = \frac{e(1-m)(c_2 a + (c_2 - c_1)(1-m)s_*y_*)}{(a + (1-m)(s_* + i_*)y_*)^2},$$

$$\Omega_3 = \frac{e(1-m)ay_*(c_1 s_* + c_2 i_*)}{(a + (1-m)(s_* + i_*)y_*)^2},$$

$$\Upsilon = \frac{e(1-m)(c_1s_* + c_2i_*)}{a + (1-m)(s_* + i_*)y_*} - d,$$

$$s(t) = s_* + u(t), i(t) = i_* + v(t), y(t) = y_* + w(t).$$

Now considering Laplace Transform of the system(27), we get,

$$\begin{aligned} (p - \Phi_1)\bar{u}(p) &= \Psi_1\bar{v}(p) + \Omega_1\bar{w}(p) + u(0) \\ (p - \Psi_2)\bar{v}(p) &= \Phi_2\bar{u}(p) + \Omega_2\bar{w}(p) + v(0) \\ (p - \Upsilon - \Omega_3e^{-s\tau})\bar{w}(p) &= \Phi_3\bar{u}(p)e^{-p\tau} + \Psi_3\bar{v}(p)e^{-p\tau} + K_1e^{-p\tau}k_1(p) + K_2e^{-p\tau}k_2(p) + K_3e^{-p\tau}k_3(p) \end{aligned} \tag{29}$$

$$\text{where } K_1 = \frac{e(1-m)(c_1a + (c_1 - c_2)(1-m)i_*y_*)}{(a + (1-m)(s_* + i_*)y_*)^2},$$

$$K_2 = \frac{e(1-m)(c_2a + (c_2 - c_1)(1-m)s_*y_*)}{(a + (1-m)(s_* + i_*)y_*)^2},$$

$$K_3 = \frac{e(1-m)ay_*(c_1s_* + c_2i_*)}{(a + (1-m)(s_* + i_*)y_*)^2},$$

$$k_1(p) = \int_{-\tau}^0 e^{-p\tau}u(t)dt, \quad k_2(p) = \int_{-\tau}^0 e^{-p\tau}v(t)dt, \quad k_3(p) = \int_{-\tau}^0 e^{-p\tau}w(t)dt$$

and $\bar{u}(p), \bar{v}(p), \bar{w}(p)$ are the Laplace Transform of $u(t), v(t), w(t)$ respectively.

Now we apply the "Nyquist Theorem" which states that if s the arc length along a curve encircling the proper half of the plane, then a curve $\bar{x}(p)$ will encircle the origin variety of times adequate to the difference between the amount of poles and therefore the number of zeros of $\bar{x}(p)$ within the right half the plane. From the above equation and using "Nyquist Theorem", it are often shown that the conditions for local asymptotic stability of $E_*(s_*, i_*, y_*)$ are given by

$$ImH(i\eta_0) > 0 \tag{30}$$

$$ReH(i\eta_0) = 0 \tag{31}$$

where $H(p) = p^3 + a_1p^2 + a_2p + (a_3p^2 + a_4p + a_5)e^{-p\tau}$ and η_0 is that the smallest positive root of equation (30).

It is already presented that $E_*(s_*, i_*, y_*)$ is stable in absenteeism of delay. Hence, by continuity, all eigenvalues will still have negative real parts for sufficiently small $\tau > 0$ only if one can warranty that no eigenvalues with positive real parts bifurcates from infinity as τ increases from zero. This will be proved by using Butler's lemma (Erbe et. al[12]).

In this case conditions (29) and (30) give

$$a_2\eta_0 - \eta_0^3 > (a_5 - a_3\eta_0^2)\sin(\eta_0\tau) - a_4\eta_0\cos(\eta_0\tau) \tag{32}$$

$$a_1\eta_0^2 = (a_5 - a_3\eta_0^2)\cos(\eta_0\tau) + a_4\eta_0\sin(\eta_0\tau) \tag{33}$$

Now, (31) and (32), if satisfied at the same time, are sufficient conditions to ensure stability. We shall utilize them to urge an estimate on the length of delay. Our aim is to repair an boundary η_+ on η , independent of τ in order that (31) holds for all values of $\eta, 0 \leq \eta \leq \eta_+$ and hence especially at $\eta = \eta_0$.

Now maximizing,

$$(a_5 - a_3\eta_0^2)\cos(\eta_0\tau) + a_4\eta_0\sin(\eta_0\tau) \tag{34}$$

subject to $|\sin(\eta_0\tau)| \leq 1, |\cos(\eta_0\tau)| \leq 1$, we obtain

$$a_1\eta_0^2 \leq |a_5| + |a_3|\eta_0^2 + |a_4|\eta_0 \tag{35}$$

Hence, if

$$\eta_+ = \frac{|a_4| + \sqrt{|a_4|^2 + 4|a_5|(a_1 - |a_3|)}}{2(a_1 - |a_3|)} \tag{36}$$

Then clearly from (35), we have $\eta_0 \leq \eta_+$.
From the inequality (31), we get

$$\eta_0^2 < a_2 + a_4 \cos(\eta_0 \tau) - \frac{(a_5 - a_3 \eta_0^2)}{\eta_0} \sin(\eta_0 \tau) \tag{37}$$

Since $E_*(s_*, i_*, y_*)$ is locally asymptotically stable in the absence of delay ($\tau = 0$), (36) will continue to hold for sufficiently small $\tau > 0$. From (32) and (36) we get,

$$(a_5 - a_1 a_4 - a_3 \eta_0^2)(\cos(\eta_0 \tau) - 1) + \left\{ a_4 \eta_0 + \frac{a_1(a_5 - a_3 \eta_0^2)}{\eta_0} \right\} \sin(\eta_0 \tau) < a_1 a_2 - a_5 + a_1 a_4 + a_3 \eta_0^2 \tag{38}$$

Using the bounds

$$(a_5 - a_1 a_4 - a_3 \eta_0^2)(1 - \cos(\eta_0 \tau)) = 2(a_5 - a_1 a_4 - a_3 \eta_0^2) \sin^2\left(\frac{\eta_0 \tau}{2}\right) < \frac{1}{2} |a_5 - a_1 a_4 - a_3 \eta_+^2| \eta_+^2 \tau^2$$

and

$$\left\{ a_1 \eta_0 + \frac{a_1(a_5 - a_3 \eta_0^2)}{\eta_0} \right\} \sin(\eta_0 \tau) \leq ((a_4 - a_1 a_3) \eta_+^2 + a_1 |a_5|) \tau$$

So we obtain from (37)

$$A \tau^2 + B \tau < C$$

where

$$\begin{aligned} A &= \frac{1}{2} |a_5 - a_1 a_4 - a_3 \eta_+^2| \eta_+^2, \\ B &= (a_4 - a_1 a_3) \eta_+^2 + a_1 |a_5|, \\ C &= a_1 a_2 - a_5 + a_1 a_4 + a_3 \eta_+^2. \end{aligned}$$

Hence, if

$$\tau_+ = \frac{1}{2A} (-B + \sqrt{B^2 + 4AC}) \tag{39}$$

Then stability will be preserved for $\tau \in [0, \tau_+)$.

Theorem 6.1: The delayed model (2) will be locally asymptotically stable at $E_*(s_*, i_*, y_*)$ if the delay τ lies within the interval $(0, \tau_+)$ where τ_+ is given by (38).

7. Numerical Simulation

Analytical studies can never be completed without numerical verification of the results. In this section we present computer simulation of some solutions of the system. Beside verification of our analytical findings, these numerical solutions are very important from the practical point of view.

To illustrate the results numerically choose parameters as $r = 0.85; m = 0.7; a = 0.25; l = 0.65; b = 0.65; d = 0.1; e = 0.8; c_1 = 0.2; c_2 = 0.95, n = 0.1$. Then the conditions of Theorem 4.2 are satisfied. So the point $E_2(1, 0, 0)$ is locally asymptotically stable (see Figure 1).

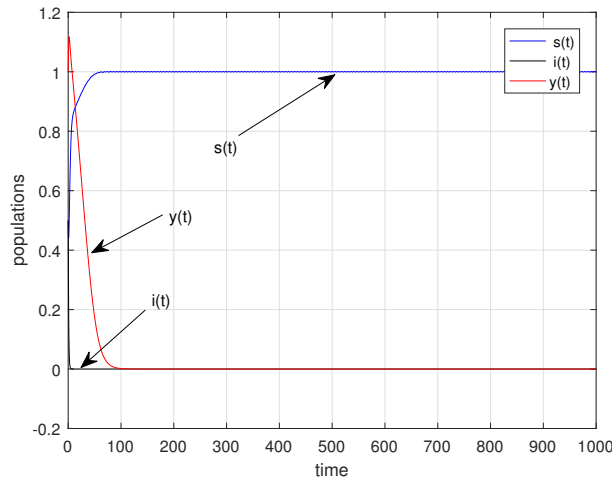


Figure 1: Here $r = 0.85; m = 0.7; a = 0.25; l = 0.65; b = 0.65; d = 0.1; e = 0.8; c_1 = 0.2; c_2 = 0.95, n = 0.1$ and $(s(0); i(0); y(0)) = (1.0; 0; 0)$. So the point $E_2(1, 0, 0)$ is locally asymptotically stable.

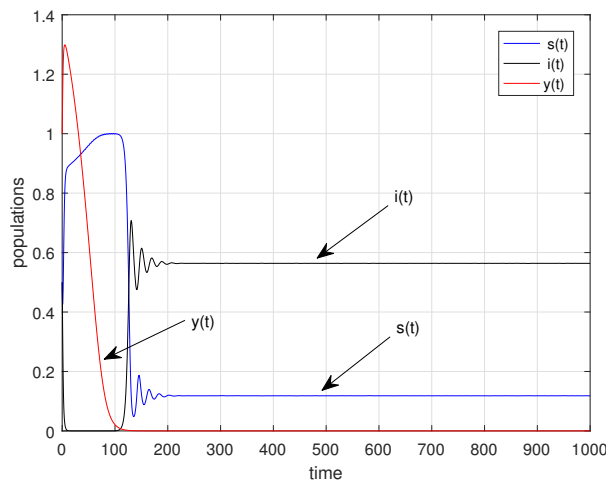


Figure 2: Here $r = 0.85; m = 0.7; a = 0.25; l = 0.65; b = 0.65; d = 0.1; e = 0.8; c_1 = 0.2; c_2 = 0.95, n = 0.1$ and $(s(0); i(0); 0) = (0.1; 0.5; 0)$. So the point $E_3(s_3, i_3, 0)$ is asymptotically stable.

Then we take the parameters as $r = 0.85; m = 0.7; a = 0.25; l = 0.65; b = 0.65; d = 0.1; e = 0.8; c_1 = 0.2; c_2 = 0.95, n = 0.1$. Then the conditions of Theorem 4.3 are satisfied and consequently $E_3(s_3, i_3, 0)$ is asymptotically stable (see Figure 2).

Then we take the parameters as $r = 0.85; m = 0.7; a = 0.25; l = 0.65; b = 0.65; d = 0.1; e = 0.8; c_1 = 0.2; c_2 = 0.95, n = 0.1$. Then the conditions of Theorem 4.4 are satisfied and consequently $E_4(s_4, 0, y_4)$ is stable (see Figure 3).

Then we take the parameters as $r = 0.85; m = 0.7; a = 0.25; l = 0.65; b = 0.65; d = 0.1; e = 0.8; c_1 = 0.2; c_2 = 0.95, n = 0.1$. Then the conditions of Theorem 4.1.4 are satisfied and consequently $E_*(s_*, i_*, y_*)$ is locally stable (see Figure 4).

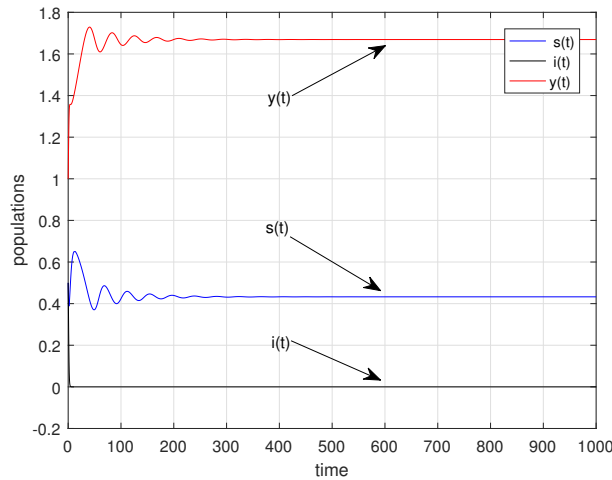


Figure 3: Here $r = 0.85; m = 0.7; a = 0.25; l = 0.65; b = 0.65; d = 0.1; e = 0.8; c_1 = 0.2; c_2 = 0.95, n = 0.1$ and $(s(0); 0; y(0)) = (0.1; 0; 0.5)$. So the point $E_4(s_4, 0, y_4)$ is asymptotically stable.

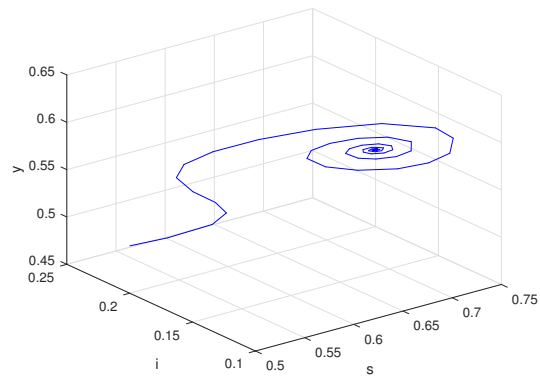
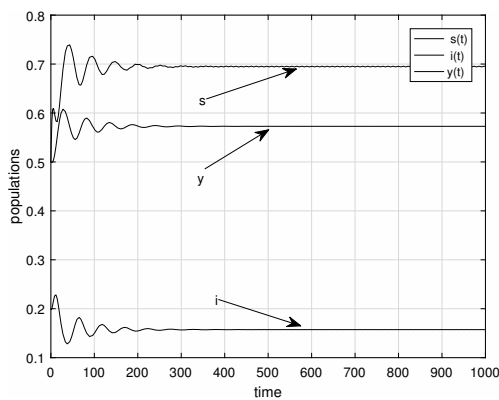


Figure 4: Stable behavior and phase portrait of the point $E_*(s_*, i_*, y_*)$

The parameters m and e have been found to have quite significant effect biologically in the formation of bifurcation diagrams. A chaotic attractor has been made by choosing an initial condition $(0.6952, 0.1594, 0.5737)$ and related phase plane diagram are presented in Figure 5. The equivalent time series representation for all the species plotted in Figure 5 demonstrating steady oscillations of some fixed amplitudes over the entire period of time. Such dynamic performance of the system undertaken ensures the existence of stable limit cycle enclosing the interior equilibrium point. In Figures 6, the respective bifurcation diagrams for the infective, susceptible and the predator populations are also plotted against the biologically significant prey refuge parameter m of the present model system in the range of m as $0.1 \leq m \leq 0.9$. Here, the chaotic region is observed in the range $0.1 \leq m \leq 0.6308$; in almost all the cases followed by a stability region for rest of the region $0.6308 < m \leq 0.9$.

In another attempt, the bifurcation diagrams as a function of the parameter e for both the preys and the predator populations have been plotted in Figures 8 with the range of values of $0.65 \leq e \leq 0.95$. The chaotic region is observed for the range $0.8055 < e \leq 0.95$ in the case of both the preys and the predator populations. Before such chaotic region, the system stability prevails over the rest of the region. Figure 7 shows the phase portrait and unstable behaviour of the dynamical system at the equilibrium point $(0.6952, 0.1594, 0.5737)$.

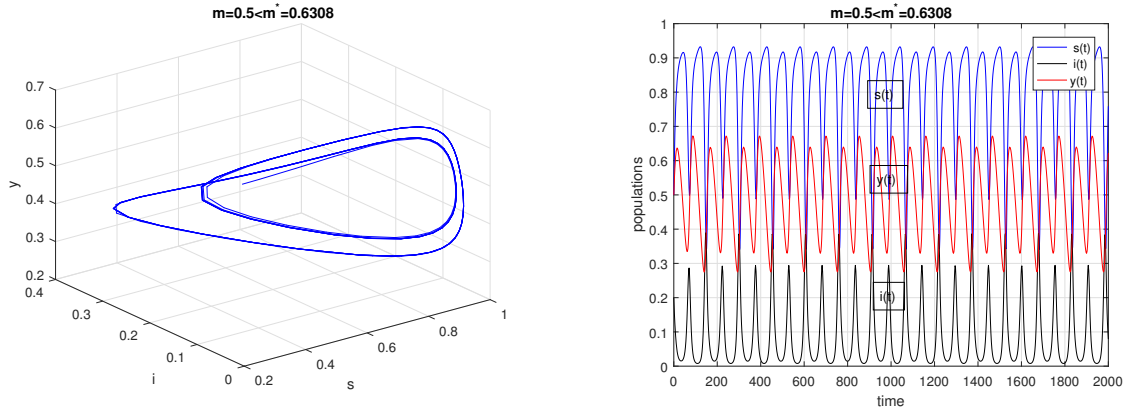


Figure 5: Phase portrait and unstable behaviour of $E_*(s_*, i_*, y_*)$ when $m = 0.5 < m^* = 0.6308$

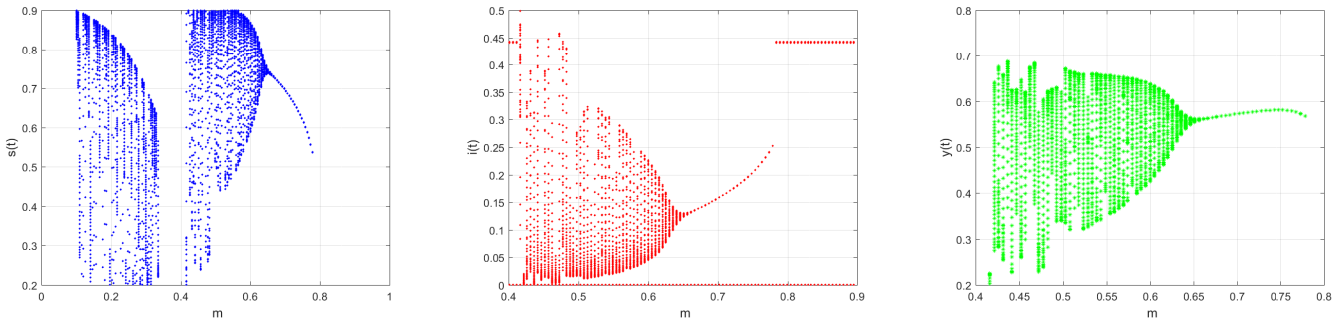


Figure 6: Hopf bifurcation diagram for the parameter $m \in [0, 1]$ and the other parameters are fixed.

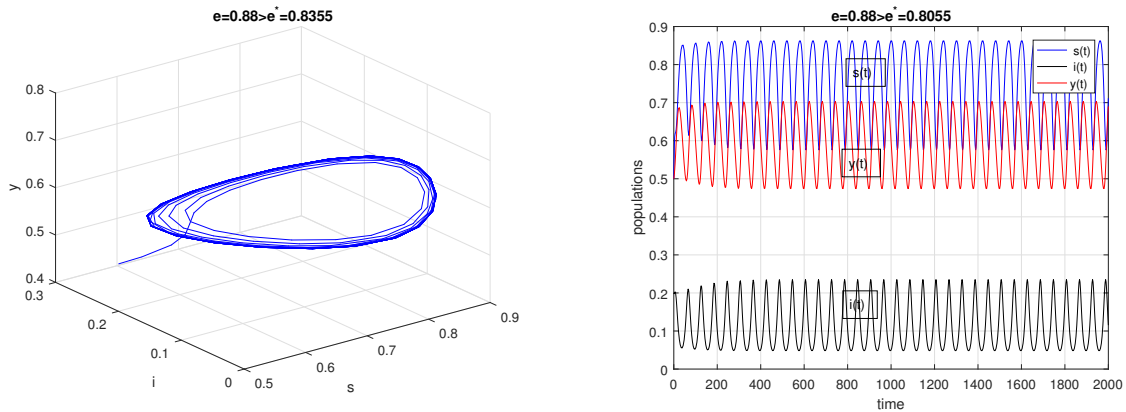


Figure 7: Phase portrait and unstable behaviour of $E_*(s_*, i_*, y_*)$ where $(s_*, i_*, y_*) = (0.6952, 0.1594, 0.5737)$ when $e = 0.88 > e^* = 0.8055$

It is already known that the stability criterion in the absence of delay ($\tau = 0$) will not necessarily guarantee the stability of the system in presence of delay ($\tau \neq 0$). For the above choice of parameters, from **Theorem 5.1** we see that there is a unique positive root of (24) given by $\sigma^* = \eta^{*2}$ for which $C(\eta^*)F(\eta^*) - D(\eta^*)G(\eta^*) > 0$ and Hopf-bifurcation occurs at $\tau = \tau^* = 0.8652$. We see that for $\tau = 0.75 < \tau^* = 0.8652$, $E_*(s_*, i_*, y_*) = (0.6952, 0.1594, 0.5737)$ is locally asymptotically stable and for $\tau = 0.95$ and $\tau = 1.2 > \tau^* = 0.8652$ respectively, E_* becomes unstable which is shown in Figure 9 and there is a bifurcating periodic solution near $E_*(s_*, i_*, y_*)$

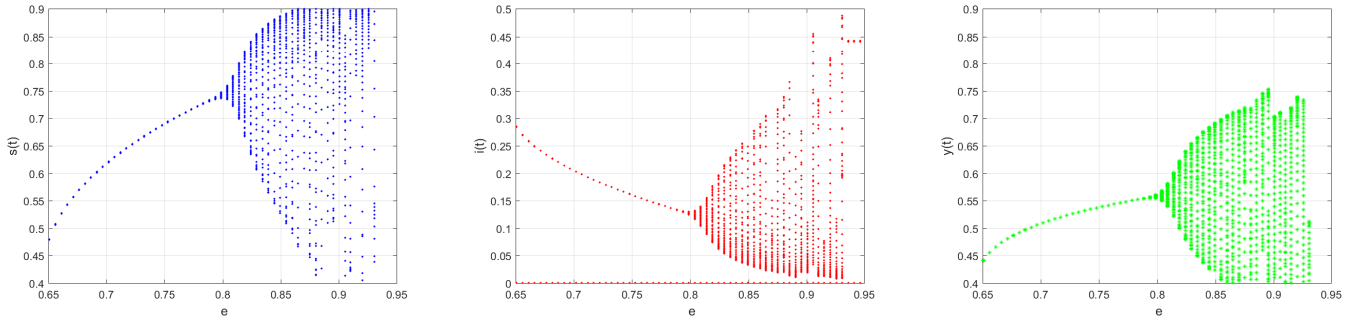


Figure 8: Hopf bifurcation diagram for the parameter $e \in [0.65, 0.95]$ and the other parameters are fixed.

(see Figure 10).

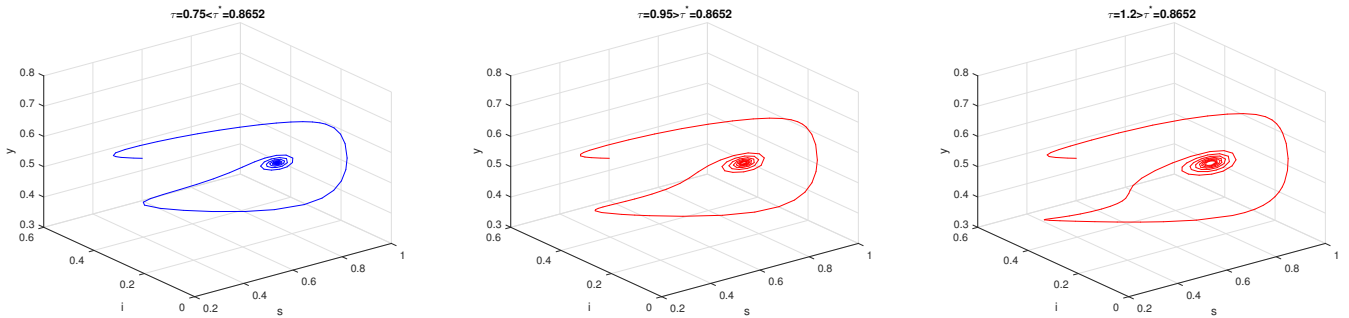


Figure 9: Phase portrait and stability of $E_*(s_*, i_*, y_*)$ when $\tau = 0.75 < \tau^* = 0.8652$ and $\tau = 0.95$ and $1.2 > \tau^* = 0.8652$ respectively

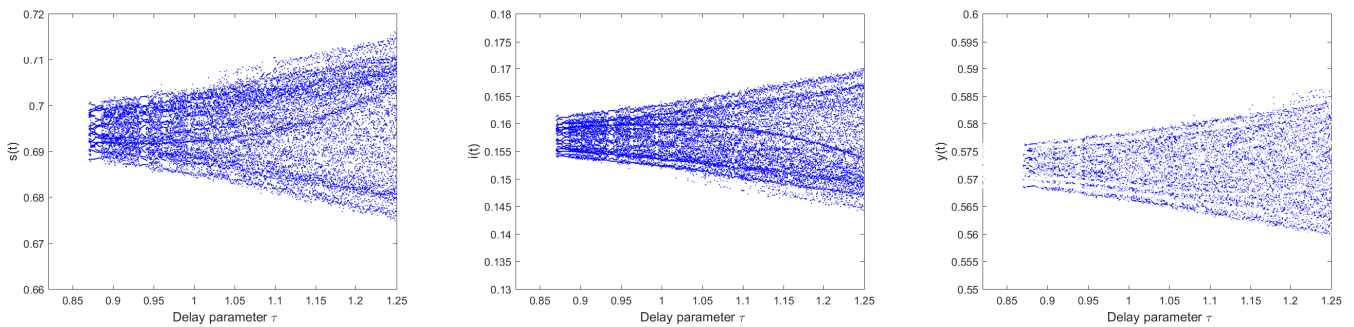


Figure 10: Hopf bifurcation diagram for the parameter $\tau \in [0, 1]$ and the other parameters are fixed.

8. Discussions

An eco-epidemic model with infection in prey has been considered in the current model. This system consists of two types of prey: one is susceptible prey and the other is infected prey and one predator. The model has been framed successfully based upon some ecological assumptions; it uses the nonlinear incidence rate and assume according to the Holling type- II response functions with consideration of a group of predators foraging in the abreast formation. Three autonomous non-linear ordinary differential

equations for the different classes of populations are considered in present paper. We have explained the boundedness of the system along with the existence and stability analysis elaborately. It is observed that the predator free equilibrium exists if and only if the rate of transmission of disease from susceptible to infected prey population exceeds some threshold value which depends on the death rate of the infected prey population. Once this predator free equilibrium exists under this condition it becomes asymptotically stable in $s - i - y$ space. Again, the existence of the infected prey-free equilibrium depends on a biological condition which states that the capturing rate must exceeds a threshold value which depends on the death rate of the predator. As stability of this boundary point implies extinction of infected prey population, prey refuge size plays a major role for existence of the infected prey. In the space point all three types of species were co-existing. We have applied the Routh-Hurwitz for satisfying the criterion for local stability around $E_*(s_*, i_*, y_*)$. The Hopf-bifurcation condition has been derived in terms of m (the prey refuge effect) and e (encounter rate per predator per prey unit time) as bifurcation parameter. Here, it is noticed that when the value of m is less than its critical value then the system is unstable in nature whereas when the value of e is greater than its critical value then the system is unstable in nature. Basically the present results further reveal that the specific ranges of these significant parameters appear to be $0.65 \leq e \leq 0.95$ and $0 \leq m \leq 1$ within which the system exhibits bifurcation but a slight variation of the parameters beyond this range makes the bifurcation feature of the dynamical system disappear completely and the corresponding stability regions are emerged. Here the parameters e and m are found to be biologically significant for the onset of bifurcation with their individual influences. Analytical consequence of our study are summarized in the Table 1. It is revealed by several researchers that the effect of time-delay must be taken into account to have a biologically useful mathematical model [[28], [14], [26]]. From this perspective, we have considered our system where the delay may be considered as the gestation period or reaction time of the predator. Then a rigorous analysis leads us to Theorem 5.1 which remarks that the stability criteria in absence of delay are no longer enough to guarantee the stability in the presence of delay, rather there is a value τ^* of the delay τ such that the system is stable for $\tau < \tau^*$ and become, unstable for $\tau > \tau^*$. By the numerical simulation we have seen the phase portrait and the stable and unstable behavior of the system. All of the significant mathematical outcomes are depicted by using MATLAB.

Table 1. A brief sketch of the conditions of existence and stability of all equilibrium points

SI No.	Equilibrium Point	Existence condition	Stability	Sufficient Conditions
1	$E_1(0, 0, 0)$	Always exists	Unstable	Saddle node
2	$E_2(1, 0, 0)$	Always exists	Unstable	Saddle node
3	$E_3(s_3, i_3, 0)$	$l > n(b + 1)$	Stable	$l > n(b + 1)$
4	$E_4(s_4, 0, y_4)$	$ec_1 < d$ and $4er(1 - m)(ec_1 - d) < 27ad^2$	Stable	$0 < s_4 < \frac{27d_2}{e^2r^2}$ $ae^2c_1rs_4 > d(ec_1 - d)ae + (1 - m)y_4^2$
5	$E_*(s_*, i_*, y_*)$	(i) $l < n + \frac{c_2}{c_1}r(1 - b)$ (ii) $rs_*(1 - s_*) > i_*(rs_* + n)$	Stable	$A_1 > 0, A_3 > 0$ and $\Delta > 0$

Acknowledgment: We are grateful to the anonymous referees and Prof. Maria Alessandra RAGUSA, Editor for their careful reading, valuable comments and helpful suggestions which have helped us to improve the presentation of this work significantly.

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