Filomat 36:2 (2022), 539–556
https://doi.org/10.2298/FIL2202539M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Properties of Cumulative Extropy and its Dynamic Past Version

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Abstract. Extropy has been discussed in many works of literature as a complementary dual of Shannon's entropy function. In this paper, a replacement procedure of uncertainty of random variable, constructed on the cumulative distribution function F, called cumulative extropy is proposed. Some properties and features of the deemed measure are obtained. Moreover, the dynamic form of cumulative extropy is considered. Finally, non-parametric estimators for the proposed measure are included.

1. Introduction

A vintage measure of uncertainty is the Shannon entropy (Shannon [34]) for the non-negative continuous random variable (r.v.) X, with probability density function (PDF) $f(x)$, defined as

$$H(X) = -E(\log f(X)) = -\int_0^\infty f(x) \log f(x) dx, \tag{1}$$

which has been mightily used in many regions such communication theory, computer science, physical and biological sciences and fuzzy sets. With introducing some additional parameters, many generalizations of entropy are obtained in literature, which make these entropies ticklish to different shapes of probability distributions. Rao et al., [28] defined the cumulative residual entropy as

$$\varepsilon(X) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x) dx, \tag{2}$$

where $F(x)$ is the cumulative distribution function (CDF) of a r.v. X, $\bar{F}(x) = 1 - F(x)$ is the survival function. Some implementations of (2) are presented in Asadi and Zohrevand [2]. Di Crescenzo and Longobardi [7] proposed an information measure similar to (2) is the cumulative entropy, as follows

$$C\varepsilon(X) = -\int_0^\infty F(x) \log F(x) dx. \tag{3}$$

A renowned generalization of Shannon's entropy is Tsallis entropy of order α that was premiered introduced by Havrda and Charvat [9] in the status of the cybernetics concept. Then, Tsallis [35] exploited

2020 *Mathematics Subject Classification*. Primary 94A17, 60E15; Secondary 62G05, 62N05
Keywords. Extropy, mean inactivity time, non-parametric estimation, reversed hazard function, stochastic ordering.
Received: 10 March 2021; Revised: 06 June 2021; Accepted: 15 June 2021
Communicated by Biljana Popović
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its non-extensive features and placed it in a physical context. This measure is defined for continuous r.v. X as

$$T_\alpha(X) = \frac{1}{\alpha-1}E(1-(f(X))^{\alpha-1}) = \frac{1}{\alpha-1}\left(1-\int_0^\infty (f(x))^\alpha dx\right), \tag{4}$$

where $0 < \alpha \neq 1$. Clearly, when $\alpha \to 1$, $T_\alpha(X) \to H(X)$. Recently, Sati and Gupta [32] presented the cumulative residual Tsallis entropy (CRTE) of order α, which defined as

$$\eta_\alpha(X) = \frac{1}{\alpha-1}\left(1-\int_0^\infty (\bar{F}(x))^\alpha dx\right). \tag{5}$$

Rajesh and Sunoj [29] defined an alternate procedure of CRTE of order α as

$$\xi_\alpha(X) = \frac{1}{\alpha-1}\left(\int_0^\infty (\bar{F}(x)-(\bar{F}(x))^\alpha)dx\right), \alpha \neq 1, \alpha > 0. \tag{6}$$

Cali et al. [3] proposed the cumulative Tsallis entropy (CTE) of order α as

$$C\xi_\alpha(X) = \frac{1}{\alpha-1}\left(\int_0^\infty (F(x)-(F(x))^\alpha)dx\right), \alpha \neq 1, \alpha > 0, \tag{7}$$

for more details see [21].

The extropy defined by Lad et al. [16] is an accomplishment to notions of information based on entropy. They exhibited that the information measure called "extropy" is a complementary dual function of Shannon's entropy function. In the view of extropy in discrete density, the extropy measure $-\sum_{i=1}^N(1-\theta_i)\log(1-\theta_i)$ can be closely approximated by $\frac{-1}{2}\sum_{i=1}^N \theta_i^2$ when the possibilities for X increases (as a result of larger N). Therefore, to realize extropy for a continuous density, extropy of a non-negative continuous r.v. X with PDF $f(x)$ is defined as

$$J(X) = \frac{-1}{2}\int_0^\infty f^2(x)dx. \tag{8}$$

Based on extropy of record values and order statistics, Qiu [24] further studied some monotone properties, characterization results, lower bounds and symmetric properties. For a non-negative r.v. X, Qiu and Jia [25] investigated residual extropy as follows

$$J(X;t) = \frac{-1}{2\bar{F}^2(t)}\int_t^\infty f^2(x)dx, t \geq 0. \tag{9}$$

Qiu et al. [27] presented a mixed systems lifetime via extropy and obtained some features and bounds of it. Recently, based on of k-records, Jose and Sathar [[12], [13]] exploited the residual and past extropy, respectively, emerging from any continuous distribution. For extra researches on extropy, see Qiu and Jia [26], Yang et al. [36], Noughabi and Jarrahiferiz [23], Raqab and Qiu [30] and Lad et al. [17].

Jahanshahi et al. [10] proposed cumulative residual extropy (CREX). Let X be a non-negative r.v. with continuous survival function $\bar{F}$, the CREX is given by

$$\zeta(X) = \frac{-1}{2}\int_0^\infty \bar{F}^2(x)dx. \tag{10}$$

It is simple to note that the model proposed in (9) is permanently negative. An analogy with, Jahanshahi et al. [10], Abdul Sathar and Dhanya [1] introduced the CREX and refers to it as survival extropy. Moreover, they conduct the dynamic survival extropy of the r.v. $[X-t|X \geq t]$ as

$$\zeta(X;t) = \frac{-1}{2\bar{F}^2(t)}\int_t^\infty \bar{F}^2(x)dx, t \geq 0. \tag{11}$$

Let X represents the random lifetime of an item survived up to time t. Therefore, the residual lifetime of the system at age t is $X_t = [X - t|X > t]$, which its mean residual life is denoted by $m(t) = \mathbb{E}(X_t)$. Di Crescenzo and Longobardi [6] defined the past entropy of $X_{(t)} = [X|X \le t]$, which characterize the past lifetime of the system at age t, with mean past lifetime $\mu(t) = \mathbb{E}[X_{(t)}]$. The mean inactivity time of the r.v. $[t - X|X \le t]$, $t > 0$, is

$$\tilde{\mu}(t) = \mathbb{E}[t - X|X \le t] = \frac{1}{F(t)} \int_0^t F(x)dx. \tag{12}$$

The derivative of $\tilde{\mu}(t)$ is given by

$$\tilde{\mu}'(t) = 1 - \tau_X(t)\tilde{\mu}(t), t > 0, F(t) > 0, \tag{13}$$

which conducted in term of the reversed hazard rate (if existing) $\tau(t) = \frac{f(t)}{F(t)}$. For more consequences about these notions in reliability theory see Ebrahimi [8], Kayid and Ahmad [14] and Misra et al. [18].

Krishnan et al. [15] presented the past extropy for past lifetime of r.v. $X_t = [t - X|X \le t]$ as follows

$$J_P(X; t) = \frac{-1}{2F^2(t)} \int_0^t f^2(x)dx. \tag{14}$$

Throughout this paper, we propose some properties and features of a measure of uncertainty based on the CDF called cumulative extropy (CEX). The proposed measures have some additional features and relationships with other important information and reliability measures. The purpose of using CDF in this model is that it is more interest than PDF, because the PDF is determined as the derivative of CDF. The rest of the article is ordered as follows: Section 2 holds the definition of CEX and illustrates the case when it exists. Meanwhile, several theorems discussed the description of its properties. Besides, we study upper and lower bounds and inequalities concerning CEX. In Section 3, we introduce the dynamic form of the CEX and provide some stochastic ordering and Characterization in terms of this model. Finally, the problem of estimating the CEX by exploring two different empirical estimators of CDF is considered in Section 4.

2. Cumulative extropy properties

In this section, we define CEX and give some properties and relationships about it.

Definition 2.1. *Let X be a non-negative r.v. with support in [a, b], $0 \le a < b < \infty$, having a CDF F. Then, the CEX of the r.v. X is defined as*

$$C\zeta(X) = \frac{-1}{2} \int_a^b F^2(x)dx. \tag{15}$$

Based on families and distributions with an infinite range, such as Pareto, exponential, and Weibull distributions, we can easily find that $C\zeta(X)$ doesn't exist. But for some distributions like power or uniform distributions which its range has an upper and lower limit, we can see that $C\zeta(X)$ exists, i.e. the r.v. X defined with support in $[a, b]$, $a, b \in \mathbb{R}^+$. Meanwhile, $\zeta(X)$ given in (10) exists, where X is defined in its support. Based on this notation, we provide the following applications and features for those measures.

Example 2.1. *In this example we will present some values of $C\zeta(X)$ based on some distributions as follows:*

1. *If X follows uniform distribution with CDF $F(x) = \frac{x-a}{b-a}$, $a \ge 0$, $b > 0$, $a \le x \le b$, using (15) and (10), then $C\zeta(X) = \zeta(X) = \frac{1}{2}(\frac{a}{3} - \frac{b}{3})$.*
2. *If X follows power function distribution with CDF $F(x) = (\frac{x}{b})^k$, $0 \le x \le b$, $b, k > 0$, using (15) and (10), then $C\zeta(X) = \frac{-b}{2(1+2k)}$ and $\zeta(X) = \frac{-b\, k^2}{(1+k)(1+2k)} = (\frac{2k^2}{1+k})\, C\zeta(X)$.*

3. *If X follows finite range distribution with CDF* $F(x) = 1 - (1 - a\,x)^b$, $a, b > 0$, $0 < x < \frac{1}{a}$, *using (15) and (10), then* $C\zeta(X) = \frac{-b^2}{a(1+b)(1+2b)}$ *and* $\zeta(X) = \frac{-1}{2a(1+2b)} = \left(\frac{1+b}{2b^2}\right) C\zeta(X)$.
4. *If X follows triangular Distribution with CDF*

$$F(x) = \begin{cases} 0, & x \le a, \\ \frac{(x-a)^2}{(b-a)(c-a)}, & a < x \le c, \\ 1 - \frac{(b-x)^2}{(b-a)(b-c)}, & c < x < b, \\ 1, & b \le x, \end{cases}$$

$\infty < a < -\infty$, $b:\ b > a$, $c:\ a \le c \le b$, *using (15), then*

$$C\zeta(X) = \begin{cases} \frac{(a-c)^3}{10(a-b)^2}, & a < x \le c, \\ \frac{-(((b-c)(15a^2+8b^2+4bc+3c^2-10a(2b+c)))}{30(a-b)^2}, & c < x < b. \end{cases}$$

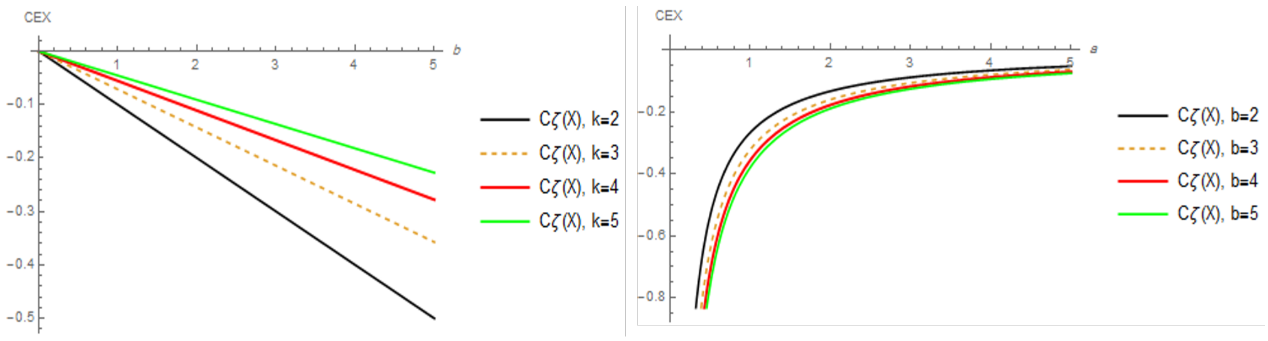

Figure 1: CEX of power function distribution (left panel), and finite range distribution (right panel).

Remark 2.1. *Figure 1 shows the CEX of power function and finite range distributions, respectively. Therefore, from Example 2.1, we can note the following:*

1. *Based on power function distribution, by increasing k with fixed b, we can see that CEX increases. Furthermore, by increasing b with fixed k, we can see that CEX decreases.*
2. *Based on finite range distribution, by increasing b with fixed a, we can see that CEX decreases. Furthermore, by increasing a with fixed b, we can see that CEX increases.*

The following theorem discuss the sufficient case for CEX to be limited.

Theorem 2.1. *Let X be a non-negative r.v. with support in [a, b], with a, b finite. If the moment generating function* $M_X(\phi) < +\infty$, *for every* $\phi < 0$, *then* $C\zeta(X) \in (-\infty, 0]$.

Proof. Since $\int_a^b F^2(x)dx < +\infty$. Using Chernoff bound, we can obtain

$$\begin{aligned} \int_a^b F^2(x)dx &\le \int_a^b \left[\frac{\mathbb{E}(e^{\phi X})}{e^{\phi x}}\right]^2 dx \\ &= M_X(\phi)^2\, \frac{e^{-2a\phi} - e^{-2b\phi}}{2\phi}. \end{aligned} \tag{16}$$

Thus, the result follows.

Remark 2.2. *The existence of the moment generating function $M_X(\phi)$ is the sufficient case for the convergence of CEX $C\zeta(X)$.*

The following proposition discuss the effect of linear transformations on CEX.

Proposition 2.1. *Let X be a non-negative r.v. with support in [a, b], such that a and b are finite. If $\psi(x) = a_1\, x + b_1$, $a_1 > 0$ and $b_1 \geq 0$, then $C\zeta(\psi(X)) = a_1\, C\zeta(X)$.*

Theorem 2.2. *Suppose that X_n be a sequence of N-dimensional random vectors converging in distribution to a random vector X which supported in [a, b], such that a and b are finite. If all the X_n are bounded, then*

$$\lim_{n\to+\infty} C\zeta(X_n) = X_n \text{ (Weak convergence)}. \tag{17}$$

Proof. Since X_n converges to X in distribution, we get

$$\lim_{n\to+\infty} F^2_{|X_n|}(x) = F^2_{|X|}(x), x \in \mathbb{R}^N_+.$$

Moreover, using (16), we can obtain

$$\begin{aligned} F^2_{|X_n|}(x) &\leq \prod_{i=1}^{N} F^{\frac{2}{N}}_{|X_i|}(x_i) \\ &\leq \prod_{i=1}^{N} \left[e^{-\phi x_i}\, I_{[a,b]}(x_i)\, \mathbb{E}(e^{\phi |X_{n_i}|}) \right]^{\frac{2}{N}} . \end{aligned}$$

Therefore, $F^2_{|X_n|}(x)$ is bounded by a function which is integrable. In the sequel, the proof is completed by the dominated of the convergence theorem.

Theorem 2.3. *Let X be a non-negative r.v. which supported in [a, b], such that a and b are finite. From the cumulative entropy $C\varepsilon$, $\zeta(X)$ and $C\zeta(X)$ defined in (2), (15) and (10), respectively. Then,*

1. $C\zeta(X) \geq \frac{-1}{2}(b - \mathbb{E}(X))$.
2. $C\zeta(X) \geq \zeta(X)$ *if* $0 \leq F(x) \leq \frac{1}{2}$ *and* $C\zeta(X) < \zeta(X)$ *if* $\frac{1}{2} < F(x) \leq 1$.
3. $C\zeta(X) \leq \frac{1}{2}[C\varepsilon - (b - \mathbb{E}(X))]$.

Proof. 1. Since $F^2(x) \leq F(x)$ and recalling the definition of CEX, the result follows. The proof of 2. can be obtained from the condition $x \geq (1 + \log x)$, for $x > 0$.

From (15) and (7), we can find the relation between the CEX and the CTE according to the following lemma.

Lemma 2.1. *If X is a r.v. defined in its support, then*

$$\begin{aligned} C\zeta(X) &= \frac{1}{2}\left[C\xi_2(X) - \int_a^b F(x)dx \right] \\ &= \frac{1}{2}\left[E(\tilde{\mu}(x)F(x)) - \int_a^b F(x)dx \right] \\ &\leq \frac{-1}{2}\left[\int_a^b (1 + \log F(x))F(x)dx \right], \end{aligned} \tag{18}$$

where $\tilde{\mu}(x)$ is defined in (12).

Proof. The proof is directly follows from (15), (7) and 3. in Theorem (2.3). Moreover, using the results obtained in Cali et al. [3] that $C\xi_2(X) = E(\tilde{\mu}(x)F(x)) \leq C\varepsilon$.

Remark 2.3. *If X is a r.v. defined in its support, then*

1. $C\xi_2(X) = 0$ *if and only if X is degenerate.*
2. $C\zeta(X) = 0$ *if and only if* $F(x) = 0$.

Theorem 2.4. *Let X be a non-negative r.v. which supported in [a, b], such that a and b are finite, with mean inactivity time* $\tilde{\mu}(t)$, *mean* μ *and CRX* $C\zeta(X)$. *We have:*

1. *For* $b - \mu \geq 1$, *then*

$$\frac{-1}{2}(b-\mu) \leq C\zeta(X) \leq \frac{-1}{2}\, e^{-\mathbb{E}(\tilde{\mu}(t))}, \tag{19}$$

2. *For* $b - \mu < 1$, *then*

$$\begin{aligned} C\zeta(X) &\geq \frac{-1}{2}(b-\mu) \\ &\geq \frac{-1}{2}\, e^{-\mathbb{E}(\tilde{\mu}(t))}, \end{aligned} \tag{20}$$

where $\tilde{\mu}(t)$ *is defined in (12).*

Proof. Based on the fact that $F^2(x) \leq F(x)$, integrating both sides with respect to x, we obtain $\frac{-1}{2}(b-\mu) \leq C\zeta(X)$. on the other hand, for $b - \mu \geq 1$ $(b - \mu < 1)$

$$\begin{aligned} \mathbb{E}(\tilde{\mu}(t)) &= \int_a^b \left[\frac{1}{F(x)} \int_a^x F(u)du\right] f(x)dx \\ &= -\int_a^b F(x)\log(F(x))dx. \end{aligned}$$

From log-sum inequality

$$\begin{aligned} \mathbb{E}(\tilde{\mu}(t)) &\geq \left(\int_a^b F(x)dx\right)\log\left[\frac{\int_a^b F(x)dx}{\int_a^b F^2(x)dx}\right] \\ &= (b-\mu)\log(b-\mu) - (b-\mu)\log\left(\int_a^b F^2(x)dx\right) \\ &\geq (\leq) - \log\left(\int_a^b F^2(x)dx\right), \end{aligned}$$

and the result follows.

Example 2.2. *Let X be a r.v. with a power function distribution defined in Example (2.1). We get:*

1. *For* $b = 3, k = 4$, *then* $b - \mu = 0.6$, $C\zeta(X) = -0.16667$, $\frac{-1}{2}(b-\mu) = -0.3$, $\frac{-1}{2}\, e^{-\mathbb{E}(\tilde{\mu}(t))=-0.3093}$,
2. *For* $b = 5, k = 4$, *then* $b - \mu = 1$, $C\zeta(X) = -0.277$, $\frac{-1}{2}(b-\mu) = -0.5$, $\frac{-1}{2}\, e^{-\mathbb{E}(\tilde{\mu}(t))=-0.224}$,
3. *For* $b = 7, k = 4$, *then* $b - \mu = 1.4$, $C\zeta(X) = -0.38889$, $\frac{-1}{2}(b-\mu) = -0.7$, $\frac{-1}{2}\, e^{-\mathbb{E}(\tilde{\mu}(t))=-0.16314}$,

which ensure the previous theorem.

2.1. Stochastic ordering of cumulative extropy

Suppose that X and Y denote r.v.'s with CDF's F and G. Now, we will present the following definitions to illustrate some results on the CEX ordering of r.v.'s, see Shaked and Shanthikumar [33].

Definition 2.2. *X is smaller than Y in the stochastic order, symbolized by $X \leq_{st} Y$ if and only if $F(t) \geq G(t)$, for all $t \in \mathbb{R}$.*

Definition 2.3. *X is smaller than Y in the dispersive order, symbolized by $X \leq^{disp} Y$, if $G^{-1}(F(x)) - x$ is increasing in $x \geq 0$.*

Definition 2.4. *X is smaller than Y in the increasing concave order, symbolized by $X \leq^{incv} Y$ if $\mathbb{E}[\varphi(X)] \leq \mathbb{E}[\varphi(Y)]$ such that $\mathbb{E}[.]$ exist and for all increasing concave function φ.*

Theorem 2.5. *Let X and Y be two non-negative continuous r.v.'s defined in its support, with CDF's F and G and finite mean $\mathbb{E}(X)$ and $\mathbb{E}(Y)$, respectively. If $X \leq_{st} Y$, then,*

1. *$C\zeta(X) \leq C\zeta(Y)$,*
2. *we have*

$$C\zeta(X) - C\zeta(Y) \leq \frac{-1}{2}[\mathbb{E}(Y) - \mathbb{E}(X)]. \tag{21}$$

Proof. Suppose $X \leq_{st} Y$ then, from (15), we have

$$\begin{aligned} C\zeta(X) - C\zeta(Y) &= \frac{-1}{2}\left[\int_a^b (F^2(x) - G^2(x))dx\right] \\ &\leq \frac{-1}{2}\left[\int_a^b (F(x) - G(x))dx\right] \\ &= \frac{-1}{2}[\mathbb{E}(Y) - \mathbb{E}(X)]. \end{aligned}$$

Now, we will discuss the connection between CEX and increasing concave ordering by the following theorem.

Theorem 2.6. *Let X and Y be two non-negative continuos r.v.'s supported in [a, b], such that a and b are finite, with increasing concave CDF's F and G, respectively. If $X \leq^{incv} Y$ then $C\zeta(X) \geq C\zeta(Y)$.*

Proof. Since the concave function $\int_t^b F(x)dx$ is increasing. Meanwhile, from (24), $C\zeta(X) = \frac{-1}{2}\mathbb{E}\left(\int_t^b F(x)dx\right)$ and recalling the concept of increasing concave order, we have

$$\int_t^b F(x)dx \leq \int_t^b G(x)dx \;\Rightarrow C\zeta(X) \geq C\zeta(Y).$$

Let $X_1, X_2, ..., X_n$ be n iid non-negative r.v.'s having CDF F, then the lifetime of a series system is determined by $X_{1:n} = \min\{X_1, X_2, ..., X_n\}$ and the lifetime of a parallel system is determined by $X_{n:n} = \max\{X_1, X_2, ..., X_n\}$ with CDF's $F_{1:n}$ and $F_{n:n}$, respectively. Based on the mean lifetime, the following proposition express lower bounds for CEX of series and parallel systems

Proposition 2.2. *Let $X_1, X_2, ..., X_n$ be iid non-negative continuous r.v.'s supported in [a, b], such that a and b are finite, with common CDF F. Then*

1. $C\zeta(X_{n:n}) \leq \frac{-b}{2} + a(\frac{1}{2} - n) + \mathbb{E}(X)$
2. $C\zeta(X_{n:n}) \geq C\zeta(X)$
3. $C\zeta(X_{n:n}) \geq \frac{-1}{2}(b - \mathbb{E}(X))$

4. *We have*

$$\begin{aligned}C\zeta(X_{1:n}) &\geq n^2 C\zeta(X)\\ &\geq \frac{-n^2}{2}(b-\mathbb{E}(X))\end{aligned}$$

Proof. 1. From (15), utilizing Bernoulli's inequality, we have

$$\begin{aligned}C\zeta(X_{n:n}) &= \frac{-1}{2}\left[\int_a^b F^{2n}(x)dx\right]\\ &\leq \frac{-1}{2}\left[\int_a^b (1-2n(1-F(x)))dx\right]\\ &= \frac{-1}{2}(b-a)+n\left[\int_a^b (1-F(x))dx\right]\\ &= \frac{-b}{2}+a(\frac{1}{2}-n)+\mathbb{E}(X).\end{aligned}$$

Since $F^{2n}(x) \leq F^2(x) \leq F(x)$. Thus, from (15), the proof for both (2.) and (3.) is simple to obtain. To prove 4., from (15) and utilizing Bernoulli's inequality, we have

$$\begin{aligned}C\zeta(X_{1:n}) &= \frac{-1}{2}\left[\int_a^b (1-\bar{F}^n(x))^2 dx\right]\\ &\geq \frac{-1}{2}\left[\int_a^b (nF(x))^2 dx\right] = n^2 C\zeta(X)\\ &\geq \frac{-n^2}{2}\int_a^b F(x)dx = \frac{-n^2}{2}(b-\mathbb{E}(X)).\end{aligned}$$

Example 2.3. *Let $X_1, X_2, ..., X_n$ be iid non-negative continuous r.v.'s with standard uniform distribution supported in (0, 1). We get $\mathbb{E}(X) = \frac{1}{2}$, $C\zeta(X) = \frac{-1}{6}$, $C\zeta(X_{1:n}) = \frac{-n^2}{2n^2+3n+1}$, $C\zeta(X_{n:n}) = \frac{-1}{2(2n+1)}$, for $n \geq 1$ which assure the previous theorem.*

Corollary 2.1. 1. *Let $X_{i:n}$ and $Y_{i:n}$ be the ith order statistic from samples of size n, $X_1, ..., X_n$ and $Y_1, ..., Y_n$ respectively. If $X \leq^{st} Y$, then $X_{i:n} \leq^{st} Y_{i:n}$ and we get $C\zeta(X_{i:n}) \leq C\zeta(Y_{i:n})$.*

2. *Suppose that U_n and V_n denotes the nth record of two sequences of r.v.'s $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ respectively. If $X \leq^{st} Y$, then $U_n \leq^{st} V_n$ and we get $C\zeta(U_n) \leq C\zeta(V_n)$.*

Theorem 2.7. *Let X and Y be two independent non-negative r.v.'s with right-end support points $u_X = u_Y < +\infty$. If X and Y have log-concave PDF's, thus*

1. $C\zeta(X+Y) \leq \max\{C\zeta(X), C\zeta(Y)\}$
2. $C\zeta(X+Y) \leq C\zeta(X) + C\zeta(Y)$.

Proof. Let X have a log-concave function. Then, $X \leq^{disp} X+Y$ for any r.v. Y independent of X, see Theorem 3.B.7 of Shaked and Shanthikumar [33]. Since $u_X = u_Y < +\infty$, we have, $X \leq^{st} X+Y$. Therefore, from Theorem (2.5) we get $C\zeta(X+Y) \leq C\zeta(X)$. Similarly when Y has a log-concave PDF i.e. $C\zeta(X+Y) \leq C\zeta(Y)$. Noting that the CEX of a r.v. is always non-positive.

2.2. Cumulative stop-loss transform

Let X be a non-negative r.v. which supported in [a, b], such that a and b are finite. The stop-loss transform $Z_F(t)$ of the r.v. X is defined as

$$Z_F(t) = \mathbb{E}(\max\{X-t, 0\}) = \int_t^b \bar{F}(x)dx.$$

Hence, the Cumulative stop-loss transform can be derived as

$$CZ_F(t) = \int_a^t F(x)dx. \tag{22}$$

Theorem 2.8. *Let X be a non-negative r.v. which supported in [a, b], such that a and b are finite, with $C\zeta(X)$. Then, we have*

$$C\zeta(X) = \frac{-1}{2}[b - \mathbb{E}(X) - \mathbb{E}(CZ_F(t))]. \tag{23}$$

Proof.

$$\begin{aligned} C\zeta(X) &= \frac{-1}{2}\int_a^b F^2(x)dx = \frac{-1}{2}\int_a^b F(x)\left(\int_a^x f(t)dt\right)dx \\ &= \frac{-1}{2}\int_a^b f(t)\left(\int_t^b F(x)dx\right)dt. \end{aligned} \tag{24}$$

Since

$$\int_t^b F(x)dx = \int_a^b F(x)dx - \int_a^t F(x)dx = b - \mathbb{E}(X) - \mathbb{E}(CZ_F(t)),$$

then the result follows.

Remark 2.4. *From (12), we can use the relation $\tilde{\mu}(t) = \frac{CZ_F(t)}{F(t)}$ to express the CEX in terms of mean inactivity time:*

$$C\zeta(X) = \frac{-1}{2}[b - \mathbb{E}(X) - \mathbb{E}(\tilde{\mu}(t)F(t))].$$

2.3. Proportional reversed hazard model and Gini index

If $F_{\theta^*}(x)$ and $F(x)$ denote the CDF of the r.v.'s X_{θ^*} and X, respectively. Then the proportional reversed hazard rate (PRHR) is given as follows

$$F_{\theta^*}(x) = [F(x)]^{\theta}, x \in \mathbb{R}, \tag{25}$$

where θ is a real number. Now, the following result is to compare $C\zeta(X)$, $C\zeta(X_{\theta^*})$ and $C\zeta(\theta X)$.

Proposition 2.3. *For the CEX, given in (15) the following statements hold:*

$$C\zeta(X_{\theta^*}) \geq (\leq) C\zeta(X) \geq (\leq) C\zeta(\theta X), \theta \geq 1 (0 < \theta \leq 1). \tag{26}$$

Proof. By using the relation $[F(x)]^{2\theta} \leq [F(x)]^2$, for $\theta > 1$ and $[F(x)]^{2\theta} \geq [F(x)]^2$, for $0 < \theta < 1$ and the result follows.

Corollary 2.2. *Let $X_1, ..., X_n$ be i.i.d non-negative continuous r.v.'s with common CDF F, with n a positive integer, then from Proposition (2.3)*

$$C\zeta(nX_n) \leq C\zeta(X_{n:n}),$$

where $X_{n:n} = \max\{X_1, ..., X_n\}$.

Definition 2.5. *(Gini coefficient) Let X be an independent r.v. which supported in [a, b], such that a and b are finite. The Gini index is*

$$\begin{aligned} G_{index} &= \frac{1}{\mathbb{E}(X)}\int_a^b F(x)(1 - F(x))dx \\ &= \frac{1}{\mathbb{E}(X)}[b - \mathbb{E}(X) - \int_a^b F^2(x)dx]. \end{aligned} \tag{27}$$

See Wang [37] for more details. Using (27), the CEX in terms of Gini index:

$$C\zeta(X) = \frac{1}{2}[\mathbb{E}(X)(G_{index} + 1) - b]. \tag{28}$$

From (28) and Proposition (2.3), we have the following result.

Proposition 2.4.

$$G_{index}(X_{\theta^*}) \geq (\leq) G_{index}(X) \geq (\leq) G_{index}(\theta X), \theta \geq 1 (0 < \theta \leq 1).$$

3. Dynamic past cumulative extropy

Abdul Sathar and Dhanya [1] proposed the dynamic cumulative extropy in (11). The r.v. $X_t = [X|X \leq t]$ is the past lifetime of the system at age t with CDF F_{X_t}. Then, motivated by (11), the dynamic past cumulative extropy (DCEX) of a non-negative r.v. X which supported in [a, b], such that a and b are finite, is

$$\begin{aligned} C\zeta(X;t) = C\zeta(t) &= \frac{-1}{2}\int_a^t [F_{X_t(x)}]^2 dx \\ &= \frac{-1}{2F^2(t)}\int_a^t F^2(x)dx, t \geq 0, \end{aligned} \tag{29}$$

and that $C\zeta(t)$ is always negative.

Proposition 3.1. *Let X be a non-negative r.v. with support in [a, b], such that a and b are finite. If $\psi(x) = a_1\, x + b_1$, $a_1 > 0$ and $b_1 \geq 0$, then $C\zeta(\psi(X);t) = a_1\, C\zeta(X; \frac{t-b}{a})$, $t \geq 0$.*

The following theorem is extension of Theorem (2.4) in the case of DCEX. The proof omitted by the same way of the proof of Theorem (2.4).

Theorem 3.1. *Let X be a non-negative r.v. which supported in [a, b], such that a and b are finite, with mean inactivity time $\tilde{\mu}(t)$, mean μ and DCRX $C\zeta(X;t)$. We have:*

1. *For $b - \mu \geq 1$, then*

$$\frac{-1}{2}\tilde{\mu}(t) \leq C\zeta(X;t) \leq \frac{-1}{2}\, e^{-\mathbb{E}(\tilde{\mu}(t)|X \leq t)}, \tag{30}$$

2. *For $b - \mu < 1$, then*

$$\begin{aligned} C\zeta(X;t) &\geq \frac{-1}{2}\tilde{\mu}(t) \\ &\geq \frac{-1}{2}\, e^{-\mathbb{E}(\tilde{\mu}(t)|X \leq t)}, \end{aligned} \tag{31}$$

where $\tilde{\mu}(t)$ is defined in (12).

Definition 3.1. *The CDF F is called increasing (decreasing) in DCEX, IDCEX (DDCEX), if $C\zeta(X;t)$ is an increasing (decreasing) function of t.*

Theorem 3.2. *Let X be a non-negative r.v. which supported in [a, b], such that a and b are finite, with reversed hazard rate $\tau(t) = \frac{f(t)}{F(t)}$. The CDF F(x) is IDCEX (DDCEX) if and only if, for all $t \geq 0$*

$$C\zeta(X;t) \geq (\leq) \frac{-1}{4\tau(t)}. \tag{32}$$

Proof. From (29), we have

$$-2\, C\zeta(X;t)F^2(t) = \int_a^t F^2(x)dx. \tag{33}$$

Differentiating (33) with respect to t, we get

$$\frac{d}{dt}C\zeta(X;t) = -2C\zeta(X;t)\tau(t) - \frac{1}{2}, \tag{34}$$

and the result follows.

The following theorem shows that the DCEX uniquely determines the distribution.

Theorem 3.3. *Let X and Y be two non-negative continuous r.v.'s supported in [a, b], such that a and b are finite, with CDF's F and G respectively and with reversed hazard rate functions $\tau_F(t)$ and $\tau_G(t)$ respectively. Let $C\zeta(X;t)$ and $C\zeta(Y;t)$ be the DCEX's corresponding to X and Y respectively. If, for all $t \geq 0$, $C\zeta(X;t) = C\zeta(Y;t)$ then $F(t) = G(t)$.*

Proof. Let $C\zeta(X;t) = C\zeta(Y;t)$ and using (34), we get

$$-2C\zeta(X;t)\tau_F(t) - \frac{1}{2} = -2C\zeta(Y;t)\tau_G(t) - \frac{1}{2},$$

which implies that $\tau_F(t) = \tau_G(t)$ or equivalently $F(t) = G(t)$.

The following theorem gives an identity between the dynamic cumulative extropy $\zeta(X;t)$ and DCEX $C\zeta(X;t)$.

Theorem 3.4. *Let X be a r.v. with support in $[0,b]$ and symmetric about $\frac{b}{2}$, i.e. $F(x) = \bar{F}(b-x)$, $0 \leq x \leq b$. Therefore, from (11) and (29), we have*

$$C\zeta(X;t) = \zeta(X;b-t), 0 \leq t \leq b. \tag{35}$$

Proof. Recalling (29), we have

$$\begin{aligned}
C\zeta(X;t) &= \frac{-1}{2}\frac{\int_0^t F^2(x)dx}{F^2(t)} \\
&= \frac{-1}{2}\frac{\int_0^t \bar{F}^2(b-x)dx}{\bar{F}^2(b-t)} \\
&= \frac{1}{2}\frac{\int_b^{b-t} \bar{F}^2(y)dy}{\bar{F}^2(b-t)} \\
&= \frac{-1}{2}\frac{\int_{b-t}^{b} \bar{F}^2(y)dy}{\bar{F}^2(b-t)} \\
&= \zeta(X;b-t).
\end{aligned}$$

Example 3.1. *If the r.v. X has uniform distribution supported in $[0,b]$, $0 \leq t \leq b$, we have*

$$\begin{aligned}
C\zeta(X;t) &= \frac{-t}{6}, \\
\zeta(X;t) &= \frac{-(b-t)}{6},
\end{aligned}$$

which is in agreement with the previous theorem.

3.1. Stochastic ordering of dynamic cumulative extropy

We discuss the ordering of DCEX of two r.v.'s. Let X and Y be two r.v.'s with CDF's F and G respectively and with reversed hazard rate functions $\tau_F(t)$ and $\tau_G(t)$ respectively.

Definition 3.2. *The r.v. X is less than or equal to Y in the*

1. *reversed hazard rate ordering, symbolized by* $X \geq^{rhr} Y$, *if* $\tau_F(x) \geq \tau_G(x)$, *for all* $x \geq 0$.
2. *cumulative extropy ordering, symbolized by* $X \leq^{CEX} Y$, *if* $C\zeta(X) \leq C\zeta(Y)$.

Theorem 3.5. *Let X and Y be two non-negative continuous r.v.'s supported in [a, b], such that a and b are finite, with CDF's F and G respectively. Thus:*

1. *If* $X \leq^{st} Y$, *then* $X \leq^{CEX} Y$.
2. *If* $X \geq^{rhr} Y$, *then* $C\zeta(X;t) \geq C\zeta(Y;t)$. *In particular,* $C\zeta(X) \geq C\zeta(Y)$.

Proof. 1. Since $X \leq^{st} Y \Rightarrow F(x) \geq G(x) \Rightarrow \frac{-1}{2}\int_a^b F^2(x)dx \leq \frac{-1}{2}\int_a^b G^2(x)$. Therefore, $X \leq^{CEX} Y$.
2. Since $X \geq^{rhr} Y \Rightarrow \tau_F(x) \geq \tau_G(x) \Rightarrow F(x) \leq G(x)$. Therefore, $X \geq^{st} Y$ and the result follows.

3.2. Characterization

In this subsection, we provide some characterizations of a r.v. based on these new measures.

Theorem 3.6. *Let X be a r.v. supported in* $[0, b]$, *b is finite, with mean inactivity time* $\tilde{\mu}(t)$ *defined in (12). For all* $t \in [0, b]$, *we get*

1. $C\zeta(X;t) = q\ \tilde{\mu}(t)$ *if and only if* $F(t) = (\frac{t}{b})^{\frac{k}{1-k}}$, $k = -(1+\frac{1}{2q})$, $\frac{-1}{2} < q < \frac{-1}{4}$,
2. $C\zeta(X;t) = q\ \mu(t) - \frac{t}{2}$ *if and only if* $F(t) = (\frac{t}{b})^{\frac{1-k}{k}}$, $k = 2 - \frac{1}{q}, \frac{1}{2} < q < 1$.

proof. 1. Let $C\zeta(X;t) = q\ \tilde{\mu}(t)$ for all $t \in [0, b]$. Differentiating both side with respect to t and from (34) and (13) we have

$$\begin{aligned}
&C\zeta'(X;t) = q\tilde{\mu}'(t)\\
&\Rightarrow -2C\zeta(X;t)\tau(t) - \frac{1}{2} = q[1 - \tau(t)\tilde{\mu}(t)]\\
&\Rightarrow \tau(t)\tilde{\mu}(t) = -(1 + \frac{1}{2q})\\
&\Rightarrow \tau(t)\tilde{\mu}(t) = k,
\end{aligned}$$

where $k = -(1 + \frac{1}{2q})$, such that $\frac{-1}{2} < q < \frac{-1}{4}$. Note that (13) gives

$$\tau(t) = \frac{1 - \tilde{\mu}'(t)}{\tilde{\mu}(t),}$$

then we have

$$\tilde{\mu}'(t) = 1 - k.$$

Which yields $\tilde{\mu}(t) = (1 - k)t$, where $\tilde{\mu}(0) = 0$. Moreover, we get

$$\tau(t) = \frac{k}{1-k}\frac{1}{t}.$$

Thus

$$F(t) = \left(\frac{t}{b}\right)^{\frac{k}{1-k}}, 0 \leq t \leq b.$$

2. Since the mean inactivity time can be written as

$$\tilde{\mu}(t) = t - \mu(t).$$

Let $C\zeta(X;t) = q\ \mu(t) - \frac{t}{2}$ for all $t \in [0, b]$. Differentiating both side with respect to t and from (34) and (13) we have

$$\begin{aligned} &C\zeta'(X;t) = q\mu'(t) - \frac{1}{2} \\ &\Rightarrow -2C\zeta(X;t)\tau(t) - \frac{1}{2} = q[1 - \tilde{\mu}'(t)] - \frac{1}{2} \\ &\Rightarrow \tilde{\mu}(t) = \left(2 - \frac{1}{q}\right)t \\ &\Rightarrow \tilde{\mu}(t) = k\ t, \end{aligned}$$

where $k = 2 - \frac{1}{q}$, such that $\frac{1}{2} < q < 1$. So, we have

$$\tau(t) = \frac{1-k}{k}\frac{1}{t}.$$

Thus

$$F(t) = \left(\frac{t}{b}\right)^{\frac{1-k}{k}}, 0 \le t \le b.$$

The converse for both (1.) and (2.) is simple to obtain.

4. Non-parametric estimation

Let $X_1, X_2, ..., X_n$ be a random sample of size n drawn from a population with a CDF F support in [a, b], $0 \le a < b < \infty$, and its order statistic denoted by $a \le X_{(1)} \le X_{(2)} \le ... \le X_{(n)} \le b$. Consequently, we use two different empirical estimators of the CDF to estimate CEX by means of the its empirical. We define the empirical CEX, from (15), as

$$\begin{aligned} C\zeta(F_n) &= \frac{-1}{2}\int_a^b F_n^2(x)dx \\ &= \frac{-1}{2}\sum_{j=1}^{n-1}\int_{X_{(j)}}^{X_{(j+1)}} F_n^2(x), \end{aligned} \tag{36}$$

where F_n is an empirical estimator of F and the empirical CEX converges to CEX of X i.e. $C\zeta(F_n) \to C\zeta(F)$ $a.s.$ as $n \to \infty$.

By replacing empirical CDF in (36) we can be obtain the first estimator ($C\zeta_1(F_n)$) as follows

$$C\zeta_1(F_n) = \frac{-1}{2}\sum_{j=1}^{n-1}(X_{(j+1)} - X_{(j)})\left(\frac{j}{n}\right)^2, \tag{37}$$

where $F_n(x) = \frac{j}{n}$, $j = 1, 2, ..., n-1$.

The second estimator (kernel-smoothed estimator) ($C\zeta_2(F_n)$) can be accomplished by replacing empirical kernel-smoothed estimator in (36) as

$$C\zeta_2(F_n) = \frac{-1}{2}\sum_{j=1}^{n-1}(X_{(j+1)} - X_{(j)})(F_l(x_j))^2, \tag{38}$$

where

$$F_l(x_j) = \frac{1}{n}\sum_{i=1}^{n} h\left(\frac{x - X_i}{l}\right),$$

$h(x) = \int_{-\infty}^{x} K(t)dt$ and l is a bandwidth parameter, see Nadaraya [22].

In the following examples, we apply the proposed methods to illustrate the effectiveness of the empirical and kernel estimators. For the kernel estimation, the Gaussian kernel, $K(x) = \phi(x)$ is used as the kernel function, ϕ is the standard normal density function. Meanwhile, to estimate the bandwidth (l) we utilize the procedure proposed by Sarda [31].

Example 4.1. *Let $X_1, ..., X_n$ be a random sample of standard uniform distribution supported in (0, 1). From David and Nagaraja [5], the sample spacings are non-dependent, with $W_{(j)} = X_{(j)} - X_{(j-1)}$ which has a $Beta(1, n)$ distribution. Hence, from (37) and (38), we have:*

$$\mathbb{E}(C\zeta_1(F_n)) = \frac{-1}{2n^2(1+n)}\sum_{j=1}^{n-1} j^2,\ \ Var(C\zeta_1(F_n)) = \frac{1}{4n^3(2+n)(1+n)^2}\sum_{j=1}^{n-1} j^4, \tag{39}$$

$$\mathbb{E}(C\zeta_2(F_n)) = \frac{-1}{2(1+n)}\sum_{j=1}^{n-1} F_l^2(x_j),\ \ Var(C\zeta_2(F_n)) = \frac{n}{4(2+n)(1+n)^2}\sum_{j=1}^{n-1} F_l^4(x_j). \tag{40}$$

In Table (1), we use different values of sample size ($n = 5, 10, 20, 30, 40, 50, 100$) and we conclude that the values of mean and variance of the suggested estimators are decreased when the sample size increases.

Table (1): Variance and mean of the empirical CEX.

n	$\mathbb{E}(C\zeta_1(F_n))$	$Var(C\zeta_1(F_n))$	$\mathbb{E}(C\zeta_2(F_n))$	$Var(C\zeta_2(F_n))$
5	-0.1	0.002809524	-0.06805947	0.001305995
10	-0.1295455	0.002639979	-0.1048848	0.001643077
20	-0.1470238	0.001812339	-0.1302864	0.001394198
30	-0.1533154	0.001344086	-0.1430673	0.001123888
40	-0.1565549	0.001063473	-0.1488226	0.0009387412
50	-0.1585294	0.0008786068	-0.1521608	0.0007967567
100	-0.1625495	0.0004686039	-0.1582621	0.0004383877

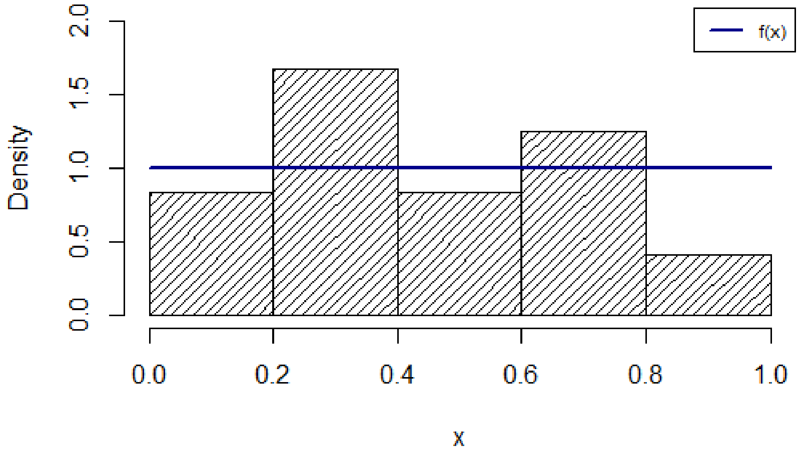

Figure 2: Histogram of the real data.

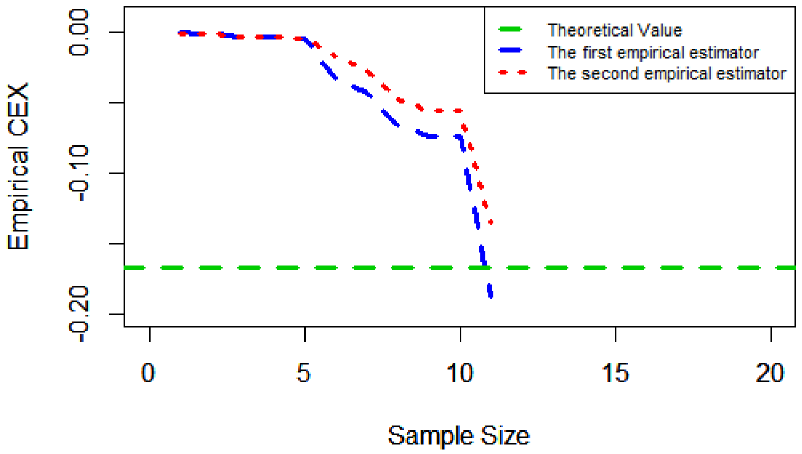

Figure 3: Empirical estimators for real data.

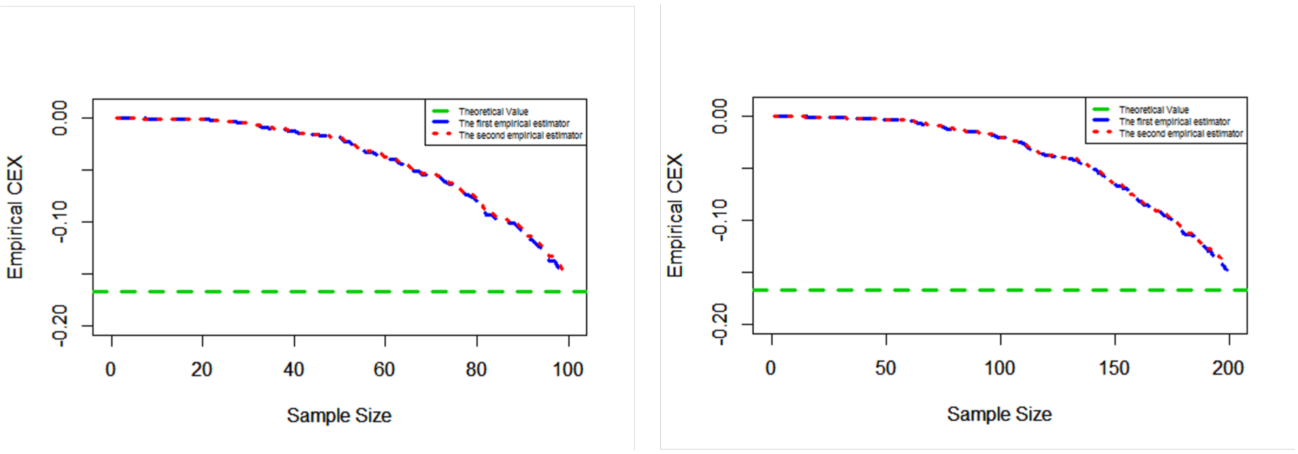

Figure 4: Empirical estimators for simulated data

Example 4.2. *We consider the data given in Crowder and Hand [4]. The original data consists of twelve hospital patients were given a special diet. Measurements of plasma ascorbic acid were taken twice before treatment, three times during, and twice after, at week numbers* $1, 2, 6, 10, 14, 15, 16$. *A question of interest is whether there is any treatment effect. We introduce here the data from weeks 1. The actual observations are recorded as follows* $0.22, 0.18, 0.73, 0.3, 0.54, 0.16, 0.3, 0.7, 0.31, 1, 0.6, 0.73$.

For fitting to data, we apply Kolmogorov-Smirnov (K-S) test to figure out the reasonableness of using standard uniform distribution. The K-S statistic is 0.19 and the corresponding p-value is 0.7792. Thus, it is feasible to use the uniform distribution to fit the data, see Figure 2. The theoretical value of CEX based on the standard uniform distribution is equal to -0.1666667 but empirical estimator ($C\zeta_1(F_n)$) and kernel-smoothed estimator ($C\zeta_2(F_n)$) are -0.1870486 and -0.1346251 respectively which $C\zeta_1(F_n)$ is more closer than $C\zeta_2(F_n)$ to the theoretical value. Figure 3 display that by increasing sample size, the empirical estimators become nearer to the theoretical value. Also, it can be noted that the first empirical estimator closest to the amount of the theory value more straight away than the second empirical estimator. Subsequently, we observe that the empirical estimator is more accurate than kernel-smoothed estimator.

We perform a simulation study by generating two samples of size $n = 100, 200$ from standard uniform distribution. Figure 4, gives an appropriate visional outline of comparing suggested estimators which are influenced by the sample size. In addition, it is evident that both of them are quite similar behave to be close to producing good accuracy to estimate the theoretical value by increasing the sample size.

5. Conclusion

In this dissertation, we introduced a new measure of information and its dynamic past version based on the CDF. We denoted them by CEX and DCEX respectively. Moreover, we mentioned the adequate family and distributions that the CEX defined in its support. In addition, we expressed the CEX in terms of other measures of information. Furthermore, we obtained several properties of the proposed measure and provide some results on the CEX and DCEX ordering of r.v.'s with applications on order statistics. Some features, such as cumulative stop-loss transform, Proportional reversed hazard model and Gini coefficient, are studied. In the last part, non-parametric estimation for the new measure is proposed. Moreover, we concluded that the suggested estimators are influenced by sample size and almost give the same behavior in the simulation procedure. In future work, we can extend the obtained models for ordered variables and their concomitants, for more details see [19, 20].

Conflict of interest

The author declare that he has no conflict of interest.

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