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Weakly J-Ideals of Commutative Rings

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Abstract. Let R be a commutative ring with non-zero identity. In this paper, we introduce the concept of weakly J-ideals as a new generalization of J-ideals. We call a proper ideal I of a ring R a weakly J-ideal if whenever $a, b \in R$ with $0 \neq ab \in I$ and $a \notin J(R)$, then $b \in I$. Many of the basic properties and characterizations of this concept are studied. We investigate weakly J-ideals under various contexts of constructions such as direct products, localizations, homomorphic images. Moreover, a number of examples and results on weakly J-ideals are discussed. Finally, the third section is devoted to the characterizations of these constructions in an amalgamated ring along an ideal.

1. Introduction

We assume throughout the whole paper, all rings are commutative with a non-zero identity. For any ring R, by $U(R)$, $N(R)$ and $J(R)$, we denote the set of all units in R, the nilradical and the Jacobson radical of R, respectively. In 2017, Tekir et al., [11] introduced the concept of n-ideals. A proper ideal I of a ring R is called an n-ideal if whenever $a, b \in R$ and $ab \in I$ such that $a \notin N(R)$, then $b \in I$. Recently, Khashan and Bani-Ata in [8] introduced the notion of J-ideals as a generalization of n-ideals in commutative rings, as follows: A proper ideal I of a ring R is called a J-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin J(R)$, then $b \in I$. In [9], the authors generalized J-ideals and defined quasi J-ideals as those ideals I for which $\sqrt{I} = \{x \in R : x^n \in I \text{ for some } n \in \mathbb{N}\}$ are J-ideals. In this paper, we define and study weakly J-ideals of commutative rings as a new generalization of J-ideals. We call a proper ideal I of R a weakly J-ideal if $0 \neq ab \in I$ where $a, b \in R$ and $a \notin J(R)$ imply $b \in I$. Clearly, every J-ideal is a weakly J-ideal. However, the converse is not true in general. Indeed, (by definition) the zero ideal of any ring is weakly J-ideal but for example $\langle \bar{0} \rangle$ is not a J-ideal of the ring $\mathbb{Z}_6$. For a non-trivial example, we present Example 2.2.

Among many other results in this paper, in section 2, we start with a characterization for quasi-local rings in terms of weakly J-ideals (Theorem 2.3). Many equivalent characterizations of weakly J-ideals for any commutative ring are presented in Proposition 2.4 and Theorem 2.7. As a generalization of prime ideals the concept of weakly prime ideals was first introduced in [2] by Anderson et al. A proper ideal of a ring R is said to be a weakly prime ideal if whenever $a, b \in R$ with $0 \neq ab \in I$, then $a \in I$ or $b \in I$. We give examples to show that weakly prime and weakly J-ideals are not comparable. Then we justify the relationships between these two concepts in Proposition 2.9 and Corollary 2.10. Further, for two weakly

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J-ideals I_1 and I_2 of a ring R, we show that $I_1 \cap I_2$ and $I_1 + I_2$ are weakly J-ideals of R, but $I_1 I_2$ is not so (see Propositions 2.11, 2.26 and Example 3.11).

Recall from [4] (resp. [9]) that a ring R is called presimplifiable (resp. quasi presimplifiable) if whenever $a, b \in R$ with $a = ab$, then $a = 0$ or $b \in U(R)$ (resp. $a \in N(R)$ or $b \in U(R)$). It is well known from [1] that presimplifiable property does not pass in general to homomorphic images. However, we show that this holds under a certain condition: If I is a weakly J-ideal of a (quasi) presimplifiable ring R, then R/I is a (quasi) presimplifiable ring (see Corollary 2.18 and Proposition 2.20). Moreover, we investigate weakly J-ideals under various contexts of constructions such as direct products, localizations, homomorphic images (see Propositions 2.12, 2.14 and 2.15).

Let R be a commutative ring with identity and M an R-module. We recall that $R(+)M = \{(r, m) : r \in R, m \in M\}$ with coordinate-wise addition and multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ is a commutative ring with identity $(1, 0)$. This ring is called the idealization of M. For an ideal I of R and a submodule N of M, $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. Moreover, the Jacobson radical of $R(+)M$ is $J(R(+)M) = J(R)(+)M$, [3]. We clarify the relationships between weakly J-ideals in a ring R and in an idealization ring $R(+)M$ in Theorem 2.30. The idealization can be used to extend results about ideals to modules and to provide interesting examples of commutative rings.

In Section 3, for a ring R and an ideal J of R, we examine weakly J-ideals of an amalgamated ring $R \bowtie^f J$ along J. Some characterizations of (weakly) J-ideals of the form $I \bowtie^f J$ and $\bar{K}^f$ of the amalgamation $R \bowtie^f J$ where $J \subseteq J(S)$ are given (see Theorems 3.3, 3.6 and Corollaries 3.4, 3.7). Finally, we give various counter examples associated with the stability of weakly J-ideals in these algebraic structures (see Examples 3.8, 3.9, 3.10, 3.13).

2. Properties of weakly J-ideals

In this section, we discuss some of the basic definitions and fundamental results concerning weakly J-ideal. Among many other properties, we present a number of characterizations of such class of ideals.

Definition 2.1. *Let R be a ring. A proper ideal I of R is called a weakly J-ideal if whenever $a, b \in R$ such that $0 \neq ab \in I$ and $a \notin J(R)$, then $b \in I$.*

Clearly any J-ideal is a weakly J-ideal. The converse is not true. For a non trivial example we have the following:

Example 2.2. *Consider the idealization ring $R = \mathbb{Z}(+)(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and consider the ideal $I = 0(+)\langle(\bar{1}, \bar{0})\rangle$ of R. Then I is not a J-ideal of R since for example, $(2, (\bar{0}, \bar{0}))(0, (\bar{1}, \bar{1})) = (0, (\bar{0}, \bar{0})) \in I$ and $(2, (\bar{0}, \bar{0})) \notin J(R)$ but $(0, (\bar{1}, \bar{1})) \notin I$. On the other hand, I is a weakly J-ideal of R. Indeed, let $(r_1, (\bar{a}, \bar{b})), (r_2, (\bar{c}, \bar{d})) \in R$ such that $(0, (\bar{0}, \bar{0})) \neq (r_1, (\bar{a}, \bar{b}))(r_2, (\bar{c}, \bar{d})) \in I$ and $(r_1, (\bar{a}, \bar{b})) \notin J(R)$. Then $r_1 \neq 0$ and $(r_1r_2, r_1.(\bar{c}, \bar{d}) + r_2.(\bar{a}, \bar{b})) \in I$. It follows that $r_1r_2 = 0$ and $r_1.(\bar{c}, \bar{d}) + r_2.(\bar{a}, \bar{b}) \in \langle(1, 0)\rangle$ and so $r_2 = 0$ and $r_1.(\bar{c}, \bar{d}) \in \langle(1, 0)\rangle$. By assumption, we must also have $r_1.(\bar{c}, \bar{d}) \neq (\bar{0}, \bar{0})$. If $(\bar{c}, \bar{d}) = (\bar{1}, \bar{1})$ or $(\bar{0}, \bar{1})$, then $r_1.(\bar{c}, \bar{d}) \in \langle(1, 0)\rangle$ if and only if $r_1 \in \langle 2\rangle$ and so $r_1.(\bar{c}, \bar{d}) = (\bar{0}, \bar{0})$, a contradiction. Thus, $(\bar{c}, \bar{d}) \in \langle(\bar{1}, \bar{0})\rangle$ and I is a weakly J-ideal of R.*

However, the classes of J-ideals, quasi J-ideals and weakly J-ideals coincide in any quasi local ring.

Theorem 2.3. *For a ring R, the following statements are equivalent.*

1. R is a quasi-local ring.
2. Every proper ideal of R is a J-ideal.
3. Every proper ideal of R is a quasi J-ideal.
4. Every proper ideal of R is a weakly J-ideal.
5. Every proper principal ideal of R is a weakly J-ideal.

Proof. (1)⇔(2)⇔(3) [9, Theorem 3].

(2)⇒(4)⇒(5) Clear.

(5)⇒(1) Let M be a maximal ideal of R. If $M = 0$, the result follows clearly. Otherwise, let $0 \neq a \in M$. Now, $\langle a \rangle$ is a weakly J-ideal and $0 \neq a \cdot 1 \in \langle a \rangle$. If $a \notin J(R)$, then $1 \in \langle a \rangle$, a contradiction. Thus, $a \in J(R)$ and $M = J(R)$. Therefore, R is quasi-local ring. □

Let I be a proper ideal of R. We denote by $J(I)$, the intersection of all maximal ideals of R containing I. Next, we obtain the following characterization for weakly J-ideals of R.

Proposition 2.4. *For a proper ideal I of R, the following statements are equivalent.*

1. I is a weakly J-ideal of R.
2. $I \subseteq J(R)$ and whenever $a, b \in R$ with $0 \neq ab \in I$, then $a \in J(I)$ or $b \in I$.

Proof. (1)⇒(2) Suppose I is a weakly J-ideal of R. Let $0 \neq a \in I$. Since $0 \neq a \cdot 1 \in I$ and $1 \notin I$, then $a \in J(R)$. Hence, $I \subseteq J(R)$. The other claim of (2) follows clearly since $J(R) \subseteq J(I)$.

(2)⇒(1) Suppose that $0 \neq ab \in I$ and $a \notin J(R)$. Since $I \subseteq J(R)$, we conclude that $J(I) \subseteq J(J(R)) = J(R)$ and so we get $a \notin J(I)$. Thus, $b \in I$ and I is a weakly J-ideal of R. □

We recall that a ring R is called semiprimitive if $J(R) = 0$. By (2) of Proposition 2.4, we conclude that 0 is the only weakly J-ideal in any semiprimitive ring.

Next, we show that a weakly J-ideal I that is not a J-ideal of a ring R satisfies $I^2 = 0$.

Theorem 2.5. *Let I be a weakly J-ideal of a ring R that is not a J-ideal. Then* $I^2 = 0$.

Proof. Suppose $I^2 \neq 0$. We prove that I is a J-ideal. Let $a, b \in R$ such that $ab \in I$ and $a \notin J(R)$. If $ab \neq 0$, then $b \in I$ since I is a weakly J-ideal. Suppose $ab = 0$. If $aI \neq 0$, then $0 \neq ax$ for some $x \in I$ and so $0 \neq a(b + x) \in I$. Again, since I is a weakly J-ideal, $b + x \in I$ and so $b \in I$. If $bI \neq 0$, then $by \neq 0$ for some $y \in I \subseteq J(R)$. Since $0 \neq yb = (y + a)b \in I$ and clearly $y + a \notin J(R)$, then $b \in I$. So, we may assume that $aI = bI = 0$. Since $I^2 \neq 0$, then there exist $x, y \in I$ such that $yx \neq 0$. Hence, $0 \neq yx = (y + a)(x + b) \in I$ and $y + a \notin J(R)$ imply that $x + b \in I$. Therefore, $b \in I$ and I is a J-ideal of R. □

However an ideal I satisfies $I^2 = 0$ need not be a weakly J-ideal. For example, the ideal $I = 0(+)4\mathbb{Z}$ of $R = \mathbb{Z}(+)\mathbb{Z}$ satisfies $I^2 = 0$. But, I is not a weakly J-ideal of R as $(0, 0) \neq (2, 0)(0, 2) \in I$ with $(2, 0) \notin J(R)$ and $(0, 2) \notin I$.

As a corollary of Theorem 2.5, we have:

Corollary 2.6. *Let I be a weakly J-ideal of a ring R that is not a J-ideal. Then*

1. $I \subseteq N(R)$.
2. *Whenever M is an R-module and* $IM = M$, *then* $M = 0$.

Proof. (1) Clear by Theorem 2.5.

(2) If $IM = M$, then $M = IM = I^2M = 0$ by Theorem 2.5. □

In particular, if R is a reduced ring, then by Corollary 2.6, a non-zero proper ideal I is a weakly J-ideal if and only if I is a J-ideal.

In the following theorem, we give some other characterizations of weakly J-ideals.

Theorem 2.7. *Let I be a proper ideal of a ring R. Then the following are equivalent.*

1. I is a weakly J-ideal of R.
2. $(I : a) = I \cup (0 : a)$ for every $a \in R \backslash J(R)$.
3. $(I : a) \subseteq J(R) \cup (0 : a)$ for every $a \in R \backslash I$.
4. If $a \in R$ and K is an ideal of R with $0 \neq Ka \subseteq I$, then $K \subseteq J(R)$ or $a \in I$.

5. If K and L are ideals of R with $0 \neq KL \subseteq I$, then $K \subseteq J(R)$ or $L \subseteq I$.

Proof. (1)⇒(2) Let $a \in R \backslash J(R)$. It is clear that $I \cup (0 : a) \subseteq (I : a)$. Let $x \in (I : a)$ so that $ax \in I$. If $ax \neq 0$, then $x \in I$ as I is a weakly J-ideal. If $ax = 0$, then $x \in (0 : a)$. Thus, $(I : a) \subseteq I \cup (0 : a)$ and the equality holds.

(2)⇒(1) Let $a, b \in R$ such that $0 \neq ab \in I$ and $a \notin J(R)$. Since $b \notin (0 : a)$, then $b \in (I : a) \subseteq I$.

(1)⇒(3) Similar to the proof of (1)⇒(2).

(3)⇒(4) Suppose that $0 \neq aK \subseteq I$ and $a \notin I$. Then $(I : a) \neq (0 : a)$ and so $K \subseteq (I : a) \subseteq J(R)$, as needed.

(4)⇒(5) Assume on the contrary that there are ideals K and L of R such that $0 \neq KL \subseteq I$ but $K \nsubseteq J(R)$ and $L \nsubseteq I$. Since $KL \neq 0$, there exists $a \in L$ such that $0 \neq Ka \subseteq I$. Since $K \nsubseteq J(R)$, we have $a \in I$ by (4). Now, choose an element $x \in L \backslash I$. Similar to the previous argument, we conclude $Kx = 0$. (Indeed, if $Kx \neq 0$, then $x \in I$). Hence, $0 \neq K(a + x) \subseteq I$ and $K \nsubseteq J(R)$ imply that $(a + x) \in I$. Thus, $x \in I$, a contradiction.

(5)⇒(1) Let $a, b \in R$ with $0 \neq ab \in I$. Write $K =< a >$ and $L = \langle b \rangle$. Then the result follows directly by (5). □

Proposition 2.8. *Let S be a non-empty subset of R. If I and $(0 : S)$ are weakly J-ideals of R where $S \nsubseteq I$, then so is $(I : S)$.*

Proof. We first note that $(I : S)$ is proper in R since otherwise, $S \subseteq I$, a contradiction. Let $0 \neq ab \in (I : S)$ and $a \notin J(R)$. If $abS \neq 0$, then $bS \subseteq I$ and so $b \in (I : S)$ as I is weakly J-ideal. If $abS = 0$, then $0 \neq ab \in (0 : S)$ which implies $b \in (0 : S)$ as $(0 : S)$ is a weakly J-ideal of R. Thus, again $b \in (I : S)$ as required. □

Recall that a proper ideal P of a ring R is called weakly prime if whenever $a, b \in R$ such that $0 \neq ab \in P$, then $a \in P$ or $b \in P$. In general, weakly J-ideals and weakly prime ideals are not comparable. For example, any non-zero prime ideal in the domain of integers is weakly prime that is not a weakly J-ideal. On the other hand, the ideal $\langle 16 \rangle$ is a weakly J-ideal in the ring $\mathbb{Z}_{32}$ which is clearly not weakly prime. However, for ideals contained in the Jacobson radical, we clarify in the following proposition that weakly prime ideals are weakly J-ideals. The proof is straightforward.

Proposition 2.9. *If I is a weakly prime ideal of a ring R and $I \subseteq J(R)$, then I is a weakly J-ideal.*

The converse of the previous proposition holds under certain conditions:

Corollary 2.10. *Let I be an ideal of a ring R. Suppose I is maximal with respect to the property: I and $(0 : a)$ are weakly J-ideals for all $a \notin I$. Then I is weakly prime in R.*

Proof. Let $a, b \in R$ such that $0 \neq ab \in I$ and $a \notin I$. If we choose $S = \{a\}$ in Proposition 2.8, then we conclude that $(I : a)$ is a weakly J-ideal of R. Moreover, clearly $c \notin I$ for all $c \notin (I : a)$ and so $(0 : c)$ is a weakly J-ideals. By maximality of I, we must have $b \in (I : a) = I$ as required. □

Proposition 2.11. *If $\{I_i : i \in \Delta\}$ is a non-empty family of weakly J-ideals of a ring R, then $\bigcap_{i \in \Delta} I_i$ is a weakly J-ideal.*

Proof. Let $a, b \in R$ such that $0 \neq ab \in \bigcap_{i \in \Delta} I_i$ and $a \notin J(R)$. Since for all $i \in \Delta$, I_i is a weakly J-ideal of R, we have $b \in I_i$. Hence, $b \in \bigcap_{i \in \Delta} I_i$ and the result follows. □

In general, the converse of Proposition 2.11 is not true. For example, while $\langle \bar{0} \rangle = \langle \bar{2} \rangle \cap \langle \bar{3} \rangle$ is a weakly J-ideal of $\mathbb{Z}_6$, non of the ideals $\langle \bar{2} \rangle$ and $\langle \bar{3} \rangle$ are (weakly) J-ideals.

Next, we characterize weakly J-ideals of a Cartesian product of two rings.

Proposition 2.12. *Let $R = R_1 \times R_2$ be a decomposable ring and I be a non-zero proper ideal of R. Then the following statements are equivalent.*

1. *I is a weakly J-ideal of R.*
2. *$I = I_1 \times R_2$ where I_1 is a J-ideal of R_1 or $I = R_1 \times I_2$ where I_2 is a J-ideal of R_2.*

3. *I is a J-ideal of R.*

Proof. (1)⇒(2) Let $I = I_1 \times I_2$ be a non-zero weakly J-ideal of R. Assume I_1 and I_2 are proper in R_1 and R_2, respectively and choose $0 \neq (a,b) \in I$. Then $0 \neq (a,1)(1,b) \in I$ and neither $(a,1) \in J(R)$ nor $(1,b) \in I$, a contradiction. We may assume with no loss of generality that $I_1 \neq R_1$ and $I_2 = R_2$. Since clearly $I^2 \neq 0$, I is a J-ideal of R by Corollary 2.6 (2). Let $a, b \in R_1$ such that $ab \in I_1$ and $a \notin J(R_1)$. Then $(a,1)(b,1) \in I$ and $(a,1) \notin J(R)$ imply that $(b,1) \in I$ and so $b \in I_1$ as required.

(2)⇒(3) We may assume that $I = I_1 \times R_2$ where I_1 is a J-ideal of R_1. Suppose that $(a,b)(c,d) \in I$ and $(a,b) \notin J(R)$. Then clearly $a \notin J(R_1)$ and $ac \in I_1$ which imply $c \in I_1$. Thus, $(c,d) \in I$ and we are done.

(3)⇒(1) straightforward. □

Corollary 2.13. *Let $R = R_1 \times R_2$ be a decomposable ring. If I is a weakly J-ideal of R that is not a J-ideal, then $I = 0$.*

Let I be a proper ideal of R. In the following, the notation $Z_I(R)$ denotes the set of $\{r \in R | rs \in I$ for some $s \in R \backslash I\}$.

Proposition 2.14. *Let S be a multiplicatively closed subset of a ring R such that $J(S^{-1}R) = S^{-1}J(R)$.*

1. If I is a weakly J-ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a weakly J-ideal of $S^{-1}R$.
2. If $S^{-1}I$ is a weakly J-ideal of $S^{-1}R$ and $S \cap Z(R) = S \cap Z_I(R) = S \cap Z_{J(R)}(R) = \emptyset$, then I is a weakly J-ideal of R.

Proof. (1) Let $0 \neq \frac{a}{s_1}\frac{b}{s_2} \in S^{-1}I$ and $\frac{a}{s_1} \notin J(S^{-1}R)$ for some $\frac{a}{s_1}, \frac{b}{s_2} \in S^{-1}R$. Then $0 \neq uab \in I$ for some $u \in S$. Since clearly $a \notin J(R)$ and I is a weakly J-ideal, we have $ub \in I$. Hence $\frac{b}{s_2} = \frac{ub}{us_2} \in S^{-1}I$, as needed.

(2) Let $a, b \in R$ and $0 \neq ab \in I$. Then $\frac{a}{1}\frac{b}{1} \in S^{-1}I$. If $\frac{a}{1}\frac{b}{1} = 0$, then $uab = 0$ for some $u \in S$. Since $S \cap Z(R) = \emptyset$, we have $ab = 0$, a contradiction. Since $S^{-1}I$ is a weakly J-ideal of $S^{-1}R$, $0 \neq \frac{a}{1}\frac{b}{1} \in S^{-1}I$ implies either $\frac{a}{1} \in J(S^{-1}R) = S^{-1}J(R)$ or $\frac{b}{1} \in S^{-1}I$. If $\frac{a}{1} \in S^{-1}J(R)$, then there exists $u \in S$ with $ua \in J(R)$. Since $S \cap Z_{J(R)}(R) = \emptyset$, then $a \in J(R)$. If $\frac{b}{1} \in S^{-1}I$, then there exists $v \in S$ with $vb \in I$ and so $b \in I$ as $S \cap Z_I(R) = \emptyset$. Therefore, I is a weakly J-ideal of R. □

Note that in (2) of Proposition 2.14, such a multiplicatively closed subset S exists. For example, consider $S = \{\bar{1}, \bar{3}\} \subseteq \mathbb{Z}_4$ and $I = \langle \bar{2} \rangle = J(\mathbb{Z}_4)$. Then clearly $S \cap Z(R) = S \cap Z_I(R) = S \cap Z_{J(R)}(R) = \emptyset$.

Proposition 2.15. *Let $f : R_1 \to R_2$ be a ring homomorphism. Then the following statements hold.*

1. If f is a monomorphism and I_2 is a weakly J-ideal of R_2, then $f^{-1}(I_2)$ is a weakly J-ideal of R_1.
2. If f is an epimorphism and I_1 is a weakly J-ideal of R_1 with $Ker(f) \subseteq I_1$, then $f(I_1)$ is a weakly J-ideal of R_2.

Proof. (1) Suppose that $a, b \in R_1$ with $0 \neq ab \in f^{-1}(I_2)$ and $a \notin J(R_1)$. First, we show that $f(a) \notin J(R_2)$. Suppose $f(a) \in J(R_2)$ and let M be a maximal ideal of R_1. Then $f(M)$ is a maximal ideal of R_2 as f is a monomorphism. Thus, $f(a) \in f(M)$ and so $a \in M$. Hence, $a \in J(R_1)$ which is a contradiction. Since $Ker(f) = 0$, we have $0 \neq f(ab) = f(a)f(b) \in I_2$. Since I_2 is a weakly J-ideal of R_2, we get $f(b) \in I_2$, and so $b \in f^{-1}(I_2)$.

(2) Let $a, b \in R_2$ and $0 \neq ab \in f(I_1)$. Since f is onto, $a = f(x)$ and $b = f(y)$ for some $x, y \in R_1$. Hence, $0 \neq f(x)f(y) = f(xy) \in f(I_1)$. Since $Ker(f) \subseteq I_1$, we have $0 \neq xy \in I_1$ which implies $x \in J(R_1)$ or $y \in I_1$. Thus, $a = f(x) \in J(R_2)$ by [8, Lemma 2.22] or $b = f(y) \in f(I_1)$ and we are done. □

Corollary 2.16. *Let I and K be proper ideals of R with $K \subseteq I$.*

1. If I is a weakly J-ideal of R, then I/K is a weakly J-ideal of R/K.
2. If K is a J-ideal of R and I/K is a weakly J-ideal of R/K, then I is a J-ideal of R.
3. If K is a weakly J-ideal of R and I/K is a weakly J-ideal of R/I, then I is a weakly J-ideal of R.

Proof. (1) Follows by Proposition 2.15.

(2) Let $a, b \in R$ with $ab \in I$ and $a \notin J(R)$. If $ab \in K$, then $b \in K \subseteq I$. Now, suppose that $ab \notin K$. Since K is a J-ideal, $K \subseteq J(R)$ and so clearly $a + K \notin J(R/K)$. Since $K \neq (a + K)(b + K) = ab + K \in I/K$ and I/K is weakly J-ideal, we have $(b + K) \in I/K$. Thus, $b \in I$ and we are done.

(3) Similar to (2). □

Recall from [4] that a ring R is called presimplifiable if whenever $a, b \in R$ with $a = ab$, then $a = 0$ or $b \in U(R)$. Equivalently, R is presimplifiable if and only if $Z(R) \subseteq J(R)$. The next result states that in a presimplifiable ring, weakly J -ideals and J-ideals coincide.

Proposition 2.17. *Every weakly J-ideal of a presimplifiable ring is a J-ideal.*

Proof. Let R be a presimplifiable ring and I be a weakly J-ideal of R. Then 0 is a J-ideal by [9, Corollary 5]. So, the claim follows from Corollary 2.16 (2). □

It is well known that presimplifiable property does not pass in general to homomorphic images, [1]. In view of Proposition 2.17 and [9, Theorem 8], we prove that this holds under a certain condition.

Corollary 2.18. *If R is a presimplifiable ring and I is a weakly J-ideal of R, then R/I is presimplifiable.*

Recall from [9] that a proper ideal I of a ring R is said to be quasi J-ideal if $\sqrt{I}$ is a J-ideal of R. A ring R is called quasi presimplifiable if whenever $a, b \in R$ with $a = ab$, then $a \in N(R)$ or $b \in U(R)$. We need the following lemma which justifies the relation between these two concepts.

Lemma 2.19. *[9, Theorem 5] Let I be a proper ideal of a ring R. Then I is a quasi J-ideal of R if and only if $I \subseteq J(R)$ and R/I is quasi presimplifiable.*

Proposition 2.20. *Let R be a quasi presimplifiable ring and I a weakly J-ideal of R. Then R/I is a quasi presimplifiable ring.*

Proof. Suppose that $ab \in \sqrt{I}$ and $a \notin J(R)$. Then $a^n b^n \in I$ for some $n \in \mathbb{N}$. Suppose $a^n b^n = 0$. Since R is quasi presimplifiable, then 0 is a quasi J-ideal of R by Lemma 2.19 and so $N(R)$ is a J-ideal. Hence, $ab \in N(R)$ implies $b \in N(R) \subseteq \sqrt{I}$. Now, suppose that $0 \neq a^n b^n \in I$. Since clearly $a^n \notin J(R)$, we conclude that $b^n \in I$ and $b \in \sqrt{I}$. Hence $\sqrt{I}$ is a J-ideal and so I is a quasi J-ideal of R. Therefore, R/I is a quasi presimplifiable ring by Lemma 2.19. □

Proposition 2.21. *Let R be a Noetherian domain and I be a proper ideal of R. Then I is a J-ideal of R if and only if $I \subseteq J(R)$ and I/I^n is a weakly J-ideal of R/I^n for all positive integers n.*

Proof. Suppose I is a J-ideal of R. Then $I \subseteq J(R)$ by Proposition 2.4 and I/I^n is a weakly J-ideal of R/I^n by Corollary 2.16 (1). Conversely, suppose that for all $n \in \mathbb{N}$, I/I^n is a weakly J-ideal of R/I^n and let $ab \in I$. If $ab \notin I^n$ for some $n \geq 2$, then clearly $I^n \neq (a + I^n)(b + I^n) \in I/I^n$ which implies $(a + I^n) \in J(R/I^n)$ or $(b + I^n) \in I/I^n$. Since by assumption, $I^n \subseteq I \subseteq J(R)$, then $J(R/I^n) = J(R)/I^n$. Thus, $a \in J(R)$ or $b \in I$, as needed. Now, assume that $ab \in I^n$ for all n. Since R is Noetherian, the Krull's intersection theorem implies that $\bigcap\limits_{n=1}^{\infty} I^n = 0$. Therefore, $ab = 0$ and since R is a domain, we conclude $a = 0$ or $b = 0$ and we are done. □

Definition 2.22. *Let I be a non-zero ideal of a ring R. An element $a + I \in R/I$ is called a strong zero divisor in R/I if there exists $I \neq b + I \in R/I$ such that $(a + I)(b + I) = I$ and $ab \neq 0$.*

It is clear that any strong zero divisor in R/I is a zero divisor. The converse is not true since for example $\bar{2} + \langle \bar{4} \rangle$ is a zero divisor in $\mathbb{Z}_8 / \langle \bar{4} \rangle$ which is not a strong zero divisor.

For an ideal I of a ring R, we denote the set of strong zero divisors of R/I by $SZ(R/I)$. It is clear that if $I = 0$, (e.g. R is a field), then $SZ(R/I) = \phi$.

Let I be a non-zero ideal of a ring R. Analogous to the presimplifiable rings, we define a ring R/I to be S-presimplifiable if $SZ(R/I) \subseteq J(R/I)$. Next, we characterize non-zero weakly J-ideals in terms of S-presimplifiable quotient rings.

Theorem 2.23. *Let I be a non-zero ideal of a ring R. Then I is a weakly J-ideal of R if and only if $I \subseteq J(R)$ and R/I is S-presimplifiable.*

Proof. Suppose I is a weakly J-ideal of R and note that $I \subseteq J(R)$ by Proposition 2.4. Let $a + I \in SZ(R/I)$ and choose $I \neq b + I \in R/I$ such that $(a + I)(b + I) = I$ and $ab \neq 0$. Then $0 \neq ab \in I$ and $b \notin I$. Hence, $a \in J(R)$ as I is a weakly J-ideal. Therefore, $(a + I) \in J(R)/I = J(R/I)$ and we are done. Conversely, let $a, b \in R$ such that $0 \neq ab \in I$ and $b \notin I$. Then clearly, $a + I$ is a strong zero divisor in R/I and so $a + I \in J(R/I)$. It follows that $a \in J(R)$ and so I is a weakly J-ideal of R. □

It is clear that for a non-zero ideal I of a ring R, if R/I is presimplifiable, then it is S-presimplifiable. However, we have seen in Example 2.2 that $0(+)\left\langle(\bar{1},\bar{0})\right\rangle$ is a weakly J-ideal of $\mathbb{Z}(+)\left(\mathbb{Z}_2 \times \mathbb{Z}_2\right)$ that is not a J-ideal. In view of the above theorem and [9, Theorem 5], we conclude that $\mathbb{Z}(+)\left(\mathbb{Z}_2 \times \mathbb{Z}_2\right)/0(+)\left\langle(\bar{1},\bar{0})\right\rangle$ is an S-presimplifiable ring that is not presimplifiable.

It is well known that for any ring R, $J(R[|x|]) = J(R) + xR[|x|]$.

Proposition 2.24. *Let R be a ring. If I is a weakly J-ideal of $R[|x|]$ (resp., $R[x]$), then $I \cap R$ is a weakly J-ideal of R.*

Proof. (1) Suppose I is a weakly J-ideal of $R[|x|]$. Let $0 \neq ab \in I \cap R$ and $a \notin J(R)$ for $a, b \in R$. Then $0 \neq ab \in I$ and $a \notin J(R[|x|])$ imply that $b \in I$. Thus, $b \in I \cap R$ as needed. □

A proper ideal I in a ring R is called superfluous if whenever J is an ideal of R with $I + J = R$, then $J = R$.

Lemma 2.25. *If an ideal I of a ring R is a weakly J-ideal, then it is superfluous.*

Proof. Suppose $I + J = R$ for some ideal J of R. Then $1 = x + y$ for some $x \in I$ and $y \in J$ and so $1 - y \in I \subseteq J(R)$ by Proposition 2.4. Thus $y \in J$ is a unit and $J = R$. □

Proposition 2.26. *Let I_1 and I_2 be weakly J-ideals of a ring R. Then $I_1 + I_2$ is a weakly J-ideal of R.*

Proof. Suppose that I_1 and I_2 are weakly J-ideals. Then $I_1 + I_2$ is proper by Lemma 2.25. Since $I_1 \cap I_2$ is a weakly J-ideal by Proposition 2.11, then $I_1/(I_1 \cap I_2)$ is a weakly J-ideal of $R/(I_1 \cap I_2)$ by Corollary 2.16 (1). From the isomorphism $I_1/(I_1 \cap I_2) \cong (I_1 + I_2)/I_2$, we conclude that $(I_1 + I_2)/I_2$ is a weakly J-ideal of R/I_2. Thus, $I_1 + I_2$ is a weakly J-ideal of R by Corollary 2.16 (3). □

Next, we generalize the concept of J-multiplicatively closed subset of a ring R, [8, Definition 2.27].

Definition 2.27. *Let S be a non-empty subset of a ring R such that $R - J(R) \subseteq S$. Then S is called weakly J-multiplicatively closed if $ab \in S$ or $ab = 0$ for all $a \in R - J(R)$ and $b \in S$.*

Similar to the relation between J-ideals and J-multiplicatively closed subsets of rings, we have:

Proposition 2.28. *An ideal I is a weakly J-ideal of a ring R if and only if $R - I$ is a weakly J-multiplicatively closed subset of R.*

Proof. If I is a weakly J-ideal of R, then $I \subseteq J(R)$ and so $R - J(R) \subseteq R - I$. Let $a \in R - J(R)$ and $b \in R - I$. If $ab = 0$, then we are done. Otherwise, suppose $ab \neq 0$. Since I is a weakly J-ideal, then $ab \in R - I$ and so $R - I$ is a weakly J-multiplicatively closed subset of R. Conversely, suppose $R - I$ is a J-multiplicatively closed subset of R. Let $a, b \in R$ such that $0 \neq ab \in I$ and $a \notin J(R)$. If $b \in R - I$, Then $ab \in R - I$ as $R - I$ is a weakly J-multiplicatively closed subset. This contradiction implies $b \in I$ and so I is a weakly J-ideal of R. □

Proposition 2.29. *Let S be a weakly J-multiplicatively closed subset of a ring R such that $S \cap \bigcup\limits_{a \notin J(R)} (0 : a) = \phi$. If an ideal I of R is maximal with respect to the property $I \cap S = \phi$, then I is a weakly J-ideal of a ring R.*

Proof. Suppose I is not a weakly J-ideal of R. Then there are $a \notin J(R)$ and $b \notin I$ such that $ab \in I$ but $ab \neq 0$. Since $I \subsetneq (I : a)$, then $(I : a) \cap S \neq \phi$ and so there exists $s \in (I : a) \cap S$. Now, $as \in I$ and since S is weakly J-multiplicatively closed, we have either $as \in S$ or $as = 0$. If $as \in S$, then $S \cap I \neq \phi$, a contradiction. If $as = 0$, then $s \in S \cap \bigcup\limits_{a \notin J(R)} (0 : a)$ which is also a contradiction. Therefore, I is a weakly J-ideal of a ring R. □

Next, we justify the relation between weakly J-ideals of a ring R and those of the idealization ring $R(+)M$.

Theorem 2.30. *Let I be an ideal of a ring R and N be a submodule of an R-module M.*

1. *If $I(+)N$ is a weakly J-ideal of the idealization ring $R(+)M$, then I is a weakly J-ideal of R.*
2. *$I(+)M$ is a weakly J-ideal of $R(+)M$ if and only if I is a weakly J-ideal of R and for $x, y \in R$ with $xy = 0$ but $x \notin J(R)$ and $y \notin I$, $x, y \in Ann(M)$.*

Proof. (1) If $I = R$, then clearly $I(+)N = R(+)M$, a contradiction. Let $a, b \in R$ with $0 \neq ab \in I$ and $a \notin J(R)$. Then $(0, 0) \neq (a, 0)(b, 0) \in I(+)N$ and $(a, 0) \notin J(R(+)M) = J(R)(+)M$. Since $I(+)N$ is a weakly J-ideal, we have $(b, 0) \in I(+)N$ and $b \in I$ as needed.

(2) Suppose $I(+)M$ is a weakly J-ideal. Then I is so by (1). Now, for $x, y \in R$, suppose $xy = 0$ but $x \notin J(R)$ and $y \notin I$. If $x \notin Ann(M)$, then there exists $m \in M$ such that $xm \neq 0$. Hence, $(0, 0) \neq (x, 0)(y, m) \in I(+)M$ but $(x, 0) \notin J(R(+)M)$ and $(y, m) \notin I(+)M$, a contradiction. Therefore, $x \in Ann(M)$. Similarly, we can prove that $y \in Ann(M)$. Conversely, suppose $(0, 0) \neq (a, m_1)(b, m_2) \in I(+)M$ and $(a, m_1) \notin J(R(+)M)$ for $(a, m_1), (b, m_2) \in R(+)M$. Then $ab \in I$ and $a \notin J(R)$. If $ab \neq 0$, then $b \in I$ as I is a weakly J-ideal and so $(b, m_2) \in I(+)M$. Suppose $ab = 0$ but neither $a \in J(R)$ nor $b \in I$. By assumption, $a, b \in Ann(M)$ and so $(a, m_1)(b, m_2) = (0, 0)$, a contradiction. Thus, $ab = 0$ and clearly $I(+)M$ is a weakly J-ideal of $R(+)M$. □

However, in general, if I is a weakly J-ideal of R, then $I(+)N$ need not be so, where N is an R-submodule of M. For example, although 0 is a (weakly) J-ideal of $\mathbb{Z}$, the ideal $0(+)\langle 4\rangle$ of $\mathbb{Z}(+)\mathbb{Z}$ is not a weakly J-ideal. Indeed, $(0, 0) \neq (2, 2)(0, 2) \in 0(+)\langle 4\rangle$ but $(2, 2) \notin J(\mathbb{Z}(+)\mathbb{Z})$ and $(0, 2) \notin 0(+)\langle 4\rangle$.

3. (Weakly) J-ideals of Amalgamated Rings Along an Ideal

Let R and S be two rings, J be an ideal of S and $f : R \to S$ be a ring homomorphism. The set $R \bowtie^f J = \{(r, f(r) + j) : r \in R, j \in J\}$ is a subring of $R \times S$ (with identity element $(1_R, 1_S)$) called the amalgamation of R and S along J with respect to f. In particular, if $Id_R : R \to R$ is the identity homomorphism on R, then $R \bowtie J = R \bowtie^{Id_R} J = \{(r, r + j) : r \in R, j \in J\}$ is the amalgamated duplication of a ring along an ideal J. This construction has been first defined and studied by D'Anna and Fontana, [5]. Many properties of this ring have been investigated and analyzed over the last two decades, see for example [6], [7].

Let I be an ideal of R and K be an ideal of $f(R) + J$. Then $I \bowtie^f J = \{(i, f(i) + j) : i \in I, j \in J\}$ and $\bar{K}^f = \{(a, f(a) + j) : a \in R, j \in J, f(a) + j \in K\}$ are ideals of $R \bowtie^f J$, [7].

Lemma 3.1. *[7]Let R, S, J and f be as above. The set of all maximal ideals of $R \bowtie^f J$ is $Max(R \bowtie^f J) = \left\{M \bowtie^f J : M \in Max(R)\right\} \cup \left\{\bar{Q}^f : Q \in Max(S)\backslash V(J)\right\}$ where $V(J)$ denotes the set of all prime ideals containing J.*

In particular if $J \subseteq J(S)$ (e.g. S is quasi-local or J is a weakly J-ideal), then we conclude from Lemma 3.1 that $J(R \bowtie^f J) = J(R) \bowtie^f J$. The proof of the following proposition is straightforward by using Theorem 2.3.

Proposition 3.2. *Consider the amalgamation of rings R and S along the ideal J of S with respect to a homomorphism f. If R is a quasi-local ring and $J \subseteq J(S)$, then every ideal of $R \bowtie^f J$ is a (weakly) J-ideal.*

Next, we give a characterization of (weakly) J-ideals of the form $I \bowtie^f J$ and $\bar{K}^f$ of the amalgamation $R \bowtie^f J$ when $J \subseteq J(S)$.

Theorem 3.3. *Consider the amalgamation of rings R and S along the ideals J of S with respect to a homomorphism f. Let I be an ideal of R. Then*

1. *If $I \bowtie^f J$ is a J-ideal of $R \bowtie^f J$, then I is a J-ideal of R. Moreover, the converse is true if $J \subseteq J(S)$.*
2. *If $I \bowtie^f J$ is a weakly J-ideal of $R \bowtie^f J$, then I is a weakly J-ideal of R and for $a, b \in R$ with $ab = 0$, but $a \notin J(R)$, $b \notin I$, then $f(a)j + f(b)i + ij = 0$ for every $i, j \in J$. Moreover, the converse is true if $J \subseteq J(S)$.*

Proof. (1) Suppose $I \bowtie^f J$ is a J-ideal of $R \bowtie^f J$. Let $a, b \in R$ such that $ab \in I$ and $a \notin J(R)$. Then $(a, f(a))(b, f(b)) \in I \bowtie^f J$ and $(a, f(a)) \notin J(R \bowtie^f J)$ since otherwise $a \in M$ for each $M \in Max(R)$ by Lemma 3.1, a contradiction. It follows that $(b, f(b)) \in I \bowtie^f J$ and so $b \in I$ as needed.

Moreover, suppose $J \subseteq J(S)$ and I is a a J-ideal of R. Let $(a, f(a) + j_1)(b, f(b) + j_2) = (ab, f(ab) + f(a)j_2 + f(b)j_1 + j_1j_2) \in I \bowtie^f J$ for $(a, f(a) + j_1), (b, f(b) + j_1) \in R \bowtie^f J$. If $(a, f(a) + j_1) \notin J(R \bowtie^f J) = J(R) \bowtie^f J$, then $a \notin J(R)$. Since $ab \in I$, we conclude that $b \in I$ and so $(b, f(b) + j_2) \in I \bowtie^f J$. Thus, $I \bowtie^f J$ is a J-ideal of $R \bowtie^f J$.

(2) Suppose $I \bowtie^f J$ is a weakly J-ideal of $R \bowtie^f J$ and let $a, b \in R$ such that $0 \neq ab \in I$ and $a \notin J(R)$. Then $(0, 0) \neq (a, f(a))(b, f(b)) \in I \bowtie^f J$ and $(a, f(a)) \notin J(R \bowtie^f J)$ by Lemma 3.1. It follows that $(b, f(b)) \in I \bowtie^f J$ and so $b \in I$. For the second claim, suppose there exist $i, j \in J$ such whenever $a, b \in R$ with $ab = 0$, but $a \notin J(R)$, $b \notin I$ and $f(a)j + f(b)i + ij \neq 0$. Then $(0, 0) \neq (a, f(a) + i)(b, f(b) + j) = (ab, f(ab) + f(a)j + f(b)i + ij) = (0, f(a)j + f(b)i + ij) \in I \bowtie^f J$. This is a contradiction since $I \bowtie^f J$ is a weakly J-ideal, $(a, f(a) + i) \notin J(R \bowtie^f J)$ and $(b, f(b) + j) \notin I \bowtie^f J$. Now, we prove the converse under the assumption $J \subseteq J(S)$. Let $0 \neq (a, f(a) + j_1)(b, f(b) + j_2) = (ab, f(ab) + f(a)j_2 + f(b)j_1 + j_1j_2) \in I \bowtie^f J$ where $(a, f(a) + j_1) \notin J(R \bowtie^f J) = J(R) \bowtie^f J$. Then $ab \in I$ and we have two cases:

Case I: If $ab \neq 0$, then as clearly $a \notin J(R)$, we have $b \in I$. Therefore, $(b, f(b) + j_2) \in I \bowtie^f J$ and $I \bowtie^f J$ is a weakly J-ideal of $R \bowtie^f J$.

Case II: Suppose $ab = 0$. If $a \notin J(R)$ and $b \notin I$, then by assumption, $f(a)j + f(b)i + ij = 0$ for every $i, j \in J$. This implies $(a, f(a) + j_1)(b, f(b) + j_2) = (0, 0)$, a contradiction. Thus, $a \in J(R)$ or $b \in I$ and so $(a, f(a) + j_1) \in J(R \bowtie^f J)$ or $(b, f(b) + j_2) \in I \bowtie^f J$ as required. □

Corollary 3.4. *Consider the amalgamation of rings R and S along the ideal $J \subseteq J(S)$ of S with respect to a homomorphism f. The J-ideals of $R \bowtie^f J$ containing $\{0\} \times J$ are of the form $I \bowtie^f J$ where I is a J-ideal of R.*

Proof. First, we note that $I \bowtie^f J$ is a J-ideal of $R \bowtie^f J$ for any J-ideal I of R by Theorem 3.3. Let K be a J-ideal of $R \bowtie^f J$ containing $\{0\} \times J$. Consider the surjective homomorphism $\varphi : R \bowtie^f J \to R$ defined by $\varphi(a, f(a) + j) = a$ for all $(a, f(a) + j) \in R \bowtie^f J$. Since $Ker(\varphi) = \{0\} \times J \subseteq K$, then $I := \varphi(K)$ is a J-ideal of R by [8, Proposition 2.23 (1)] . Since $\{0\} \times J \subseteq K$, we conclude that K is of the form $I \bowtie^f J$. □

As a particular case of Theorem 3.3, we have the following immediate corollary.

Corollary 3.5. *Let R be a ring and I, J be be proper ideals of R. Then*

1. *If $I \bowtie J$ is a J-ideal of $R \bowtie J$, then I is a J-ideal of R. Moreover, the converse is true if $J \subseteq J(R)$.*
2. *If $I \bowtie J$ is a weakly J-ideal of $R \bowtie J$, then I is a weakly J-ideal of R and for $a, b \in R$ with $ab = 0$, but $a \notin J(R)$, $b \notin I$, then $aj + bi + ij = 0$ for every $i, j \in J$. Moreover, the converse is true if $J \subseteq J(R)$.*

Theorem 3.6. *Consider the amalgamation of rings R and S along the maximal ideal J of S with respect to an epimorphism f. Let K be an ideal of S.*

1. *If $\bar{K}^f$ is a J-ideal of $R \bowtie^f J$, then K is a a J-ideal of S. The converse is true if $f(J(R)) = J(S) + J$ and $Ker(f) \subseteq J(R)$.*
2. *If $\bar{K}^f$ is a weakly J-ideal of $R \bowtie^f J$, then K is a weakly J-ideal of S and when $f(a) + j \notin J(S)$, $f(b) + k \notin K$ with $a, b \in R$, $j, k \in J$ and $(f(a) + j)(f(b) + k) = 0$, then $ab = 0$. The converse is true if $f(J(R)) = J(S) + J$ and $Ker(f) \subseteq J(R)$.*

Proof. (1) Suppose $\bar{K}^f$ is a J-ideal of $R \bowtie^f J$. Let $x, y \in S$, say, $x = f(a)$ and $y = f(b)$ for $a, b \in R$. Suppose $xy \in K$ and $x \notin J(S)$. Then $(a, f(a)), (b, f(b)) \in R \bowtie^f J$ such that $(a, f(a))(b, f(b)) = (ab, (f(a))(f(b))) \in \bar{K}^f$. If $(a, f(a)) \in J(R \bowtie^f J)$, then $(a, f(a)) \in \bar{Q}^f$ for all $Q \in Max(S) \backslash V(J)$. Moreover, since J is maximal in S, then $f^{-1}(J)$ is maximal in R and so $(a, f(a)) \in f^{-1}(J) \bowtie^f J$. Thus, $f(a) \in J = V(J)$ and so $f(a) \in Q$ for all $Q \in Max(S)$, a contradiction. Therefore, $(a, f(a)) \notin J(R \bowtie^f J)$ and so $(b, f(b)) \in \bar{K}^f$ as $\bar{K}^f$ is a J-ideal of $R \bowtie^f J$. Hence, $y = f(b) \in K$ and we are done.

Now, suppose $f(J(R)) = J(S) + J$, $Ker(f) \subseteq J(R)$ and K is a J-ideal of S. Let $(a, f(a) + j), (b, f(b) + k) \in R \bowtie^f J$ such that $(a, f(a) + j)(b, f(b) + k) = (ab, (f(a) + j)(f(b) + k)) \in \bar{K}^f$ and $(a, f(a) + j) \notin J(R \bowtie^f J)$. We claim that $f(a) + j \notin J(S)$. Suppose not. Then $f(a) \in J(S) + J = f(J(R))$ and so $a \in J(R)$ as $Ker(f) \subseteq J(R)$. Thus, $(a, f(a) + j) \in \left\{M \bowtie^f J : M \in Max(R)\right\}$. Moreover, $f(a) + j \in Q$ for all $Q \in Max(S)$ implies that $(a, f(a) + j) \in \left\{\bar{Q}^f : Q \in Max(S)\backslash V(J)\right\}$. It follows by Lemma 3.1 that $(a, f(a) + j) \in J(R \bowtie^f J)$, a contradiction. Since $(f(a)+j)(f(b)+k) \in K$ and K is a J-ideal of S, then $f(b)+k \in K$. Hence, $(b, f(b)+k) \in \bar{K}^f$ and the result follows.

(2) Suppose $\bar{K}^f$ is a weakly J-ideal of $R \bowtie^f J$. Let $x = f(a), y = f(b) \in S$ for $a, b \in R$ such that $0 \neq xy \in K$ and $x \notin J(S)$. Then $(0, 0) \neq (a, f(a))(b, f(b)) = (ab, f(ab))$ and similar to the proof of (1), we have $(a, f(a)) \notin J(R \bowtie^f J)$. By assumption, we have $(b, f(b)) \in \bar{K}^f$ and so $y = f(b) \in K$. Moreover, let $a, b \in R$ and $j, k \in J$ such that $f(a) + j \notin J(S), f(b) + k \notin K$ and $(f(a) + j)(f(b) + k) = 0$. Suppose $ab \neq 0$. Then $(0, 0) \neq (a, f(a)+j)(b, f(b)+k) = (ab, 0) \in \bar{K}^f$. Similar to the proof of (1), we conclude that $(a, f(a)+j) \notin J(R \bowtie^f J)$ and $(b, f(b) + k) \notin \bar{K}^f$ which contradict the assumption that $\bar{K}^f$ is a weakly J-ideal of $R \bowtie^f J$. Thus, $ab = 0$ as needed.

Now, we assume $f(J(R)) = J(S)+J$, $Ker(f) \subseteq J(R)$ and K a weakly J-ideal of S. Let $(a, f(a)+j), (b, f(b)+k) \in R \bowtie^f J$ such that $(0, 0) \neq (a, f(a) + j)(b, f(b) + k) = (ab, (f(a) + j)(f(b) + k)) \in \bar{K}^f$ and $(a, f(a) + j) \notin J(R \bowtie^f J)$. Then $f(a) + j \notin J(S)$ as in the proof of (1) and $(f(a) + j)(f(b) + k) \in K$. We have two cases:

Case I: $(f(a) + j)(f(b) + k) \neq 0$. In this case we conclude directly that $(f(b) + k) \in K$. Thus, $(b, f(b) + k) \in \bar{K}^f$ and $\bar{K}^f$ is a weakly J-ideal of $R \bowtie^f J$.

Case II: $(f(a) + j)(f(b) + k) = 0$. If $f(a) + j \notin J(S)$ and $f(b) + k \notin K$, then by assumption we should have $ab = 0$. It follows that $(a, f(a) + j)(b, f(b) + k) = (0, 0)$, a contradiction. Therefore, again $\bar{K}^f$ is a weakly J-ideal of $R \bowtie^f J$. □

Corollary 3.7. *Let R be a ring, K a proper ideal of R and J a maximal ideal of R. Then*

1. *If $\bar{K} = \{(a, a + j) : a \in R, j \in J, a + j \in K\}$ is a J-ideal of $R \bowtie J$, then K is a J-ideal of R. Moreover, the converse is true if $J \subseteq J(R)$.*
2. *If $\bar{K}$ is a weakly J-ideal of $R \bowtie J$, then I is a weakly J-ideal of R and when $a + j \notin J(R), b + k \notin K$ with $a, b \in R$, $j, k \in J$ and $ab + ak + bj + jk = 0$, then $ab = 0$. Moreover, the converse is true if $J \subseteq J(R)$.*

In the following example, we prove that the condition $J \subseteq J(S)$ can not be discarded in the proof of the converses of (1) and (2) in Theorem 3.3.

Example 3.8. *Let $R = \mathbb{Z}(+)\mathbb{Z}_4$, $I = 0(+)\mathbb{Z}_4$ and $J = \langle 2\rangle(+)\mathbb{Z}_4 \nsubseteq J(R)$. Then I is a weakly J-ideal of R by Theorem 3.3 (2). Moreover, one can easily see that there are no $(r_1, m_1), (r_2, m_2) \in R$ with $(r_1, m_1)(r_2, m_2) = (0, \bar{0})$, but $(r_1, m_1) \notin J(R)$, $(r_2, m_2) \notin I$. Now, $((0, \bar{1}), (2, \bar{1}))$,$((1, \bar{0}), (1, \bar{0})) \in R \bowtie J$ with $((0, \bar{1}), (2, \bar{1}))((1, \bar{0}), (1, \bar{0})) = ((0, \bar{1}), (2, \bar{1})) \in I \bowtie J\backslash((0, \bar{0}), (0, \bar{0}))$. Moreover, $((0, \bar{1}), (2, \bar{1})) \notin J(R \bowtie J)$ since for example, $((0, \bar{1}), (2, \bar{1})) \notin \bar{Q}$ where $Q = \langle 3\rangle + \mathbb{Z}_4 \in Max(S)\backslash V(J)$. Since also clearly $((1, \bar{0}), (1, \bar{0})) \notin I \bowtie J$, then $I \bowtie J$ is not a (weakly) J-ideal of $R \bowtie J$.*

Similarly, we justify in the following example that if $J \nsubseteq J(R)$, then the converses of (1) and (2) of Corollary 3.7 are not true in general.

Example 3.9. *$R = \mathbb{Z}(+)\mathbb{Z}_4$, $K = 0(+)\mathbb{Z}_4$ and $J = \langle 2\rangle + \mathbb{Z}_4 \nsubseteq J(R)$. Then K is a (weakly) J-ideal of R. Moreover, if for $a, b \in R$, $j, k \in J$, $(a + j, m_1) \notin J(R)$ and $(b + k, m_2) \notin K$, then clearly $ab + ak + bj + jk \neq 0$. Take $((2, \bar{0}), (0, \bar{1})) = ((2, \bar{0}), (2, \bar{0}) + (-2, \bar{1}))$,$((0, \bar{0}), (1, \bar{0})) \in R \bowtie J$. Then $((2, \bar{0}), (0, \bar{1}))((0, \bar{0}), (1, \bar{0})) = ((0, \bar{0}), (0, \bar{1})) \in \bar{K}\backslash((0, \bar{0}), (0, \bar{0}))$ since $(0, \bar{1}) \in K$. But, clearly, $((2, \bar{0}), (0, \bar{1})) \notin J(R \bowtie J)$ and $((0, \bar{0}), (1, \bar{0})) \notin \bar{K}$. Hence, $\bar{K}$ is not a (weakly) J-ideal of $R \bowtie J$.*

Even if $Ker(f) \subseteq J(R)$, the converse of (1) of Theorem 3.6 need not be true if $f(J(R)) \neq J(S) + J$.

Example 3.10. *Let $R = \mathbb{Z}(+)\mathbb{Z}_4$, $S = \mathbb{Z}$ and $J = \langle 2\rangle$ the ideal of S. Consider the homomorphism $f : R \to S$ defined by $f((r, m)) = r$. Note that $J(S) = \langle 0\rangle$, $Ker(f) = 0(+)\mathbb{Z}_4 = J(R)$ and $J(S) + J = J \neq f(J(R))$. Now, $K = \langle 0\rangle$ is a (weakly) J-ideal of S. Moreover, for $(r_1, m_1), (r_2, m_2) \in R$, $j, k \in J$ whenever $f(r_1, m_1)+j \notin J(S), f(r_2, m_2)+k \notin K$, then $(f(r_1, m_1) + j)(f(r_2, m_2) + k) \neq 0$. Take $((-2, 0), 0)$, $((1, 0), 1) \in R \bowtie^f J$. Then $((-2, 0), 0)((1, 0), 1) = ((-2, 0), 0) \in \bar{K}^f$ but $((-2, 0), 0) \notin J(R \bowtie^f J)$ and $((1, 0), 1) \notin \bar{K}^f$. Therefore, $\bar{K}^f$ is not a (weakly) J-ideal of $R \bowtie^f J$.*

We have proved in section 2 that if I_1 and I_2 are weakly J-ideals of a ring R, then so is $I + J$. However, in the next example, we clarify that the product I_1I_2 need not be a weakly J-ideal.

Example 3.11. *Let $R = \mathbb{Z}(+)\mathbb{Z}_4$ and $I = J = 0(+)\mathbb{Z}_4$. Now, I is a weakly J-ideal of R by Theorem 3.3 (2) and clearly there are no $(r_1, m_1), (r_2, m_2) \in R$ with $(r_1, m_1)(r_2, m_2) = (0, \bar{0})$, but $(r_1, m_1) \notin J(R)$, $(r_2, m_2) \notin I$. Since also $J = J(R)$, then $I \bowtie J$ is a weakly J-ideal of $R \bowtie J$ by Corollary 3.5. On the other hand, $(I \bowtie J)^2 = I^2 \bowtie J = \langle(0, \bar{0})\rangle \bowtie J$ is not a weakly J-ideal. Indeed, $((2, \bar{1}), (2, \bar{0}))$,$((0, \bar{2}), (0, \bar{1})) \in R \bowtie J$ with $((2, \bar{1}), (2, \bar{0}))((0, \bar{2}), (0, \bar{1})) = ((0, \bar{0}), (0, \bar{2})) \in I \bowtie J \backslash ((0, \bar{0}), (0, \bar{0}))$ but clearly $((2, \bar{1}), (2, \bar{0})) \notin J(R \bowtie J)$ and $((0, \bar{2}), (0, \bar{1})) \notin I^2 \bowtie J$.*

Let R, S, J, f and I be as in Theorem 3.3 and let T be an ideal of $f(R) + J$. As a general case of $I \bowtie^f J$, one can verify that if $f(I)J \subseteq T \subseteq J$, then $I \bowtie^f T := \{(i, f(i) + t) : i \in I, t \in T\}$ is an ideal of $R \bowtie^f J$. The proof of the following result is similar to that of (1) of Theorem 3.3 and left to the reader.

Proposition 3.12. *Let R, S, J, f, I and T as above. If $I \bowtie^f T$ is a weakly J-ideal of $R \bowtie^f J$, then I is a weakly J-ideal of R.*

The following example shows that the converse of Proposition 3.12 is not true in general.

Example 3.13. *Let R, S, J and I be as in Example 3.8 and let $T = \langle 4\rangle (+)\mathbb{Z}_4$. Then $IJ \subseteq T \subseteq J$ and $I \bowtie T$ is not a weakly J-ideal of $R \bowtie^f J$. Indeed, $((0, \bar{1}), (4, \bar{1}))((1, \bar{0}), (1, \bar{0})) = ((0, \bar{1}), (4, \bar{1})) \in I \bowtie T \backslash ((0, \bar{0}), (0, \bar{0}))$ but clearly $((0, \bar{1}), (4, \bar{1})) \notin J(R \bowtie J)$ and $((1, \bar{0}), (1, \bar{0})) \notin I \bowtie T$.*

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