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Generalized Hermite-Hadamard-Mercer Type Inequalities via Majorization

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Abstract. The Hermite-Hadamard inequality has been recognized as the most pivotal inequality which has grabbed the attention of several mathematicians. In recent years, load of results have been established for this inequality. The main theme of this article is to present generalized Hermite-Hadamard inequality via the Jensen-Mercer inequality and majorization concept. We establish a Hermite-Hadamard inequality of the Jensen-Mercer type for majorized tuples. With the aid of weighted generalized Mercer's inequality, we also prove a weighted generalized Hermite-Hadamard inequality for certain tuples. The idea of obtaining the results of this paper, may explore a new way for derivation of several other results for Hermite-Hadamard inequality.

1. Introduction

For the past few decades, the field of mathematical inequalities has been developed vigorously and has a great impact in different branches of science like information theory [14], economics [26] and engineering [10] etc. It must be noted that convexity is the key concept used in this field which gave rise to many new ideas about research [5, 13, 33]. Furthermore, various classes of the convex functions were discovered and related inequalities were deduced [8, 10, 18]. Convex functions also have a major role in majorization theory and many results have been established in this direction. Over the past years, the theory of majorization has been used as a powerful tool for research in the field of mathematics [39]. The definition of majorization for two $m-$tuples is given below:

Definition 1.1 ([11, 39]). *Let* $\mathbf{a} = (a_1, \ldots, a_m)$ *and* $\mathbf{b} = (b_1, \ldots, b_m)$ *be two* $m-$ *tuples of real numbers and* $a_{[1]} \geq a_{[2]} \geq \cdots \geq a_{[m]}$, $b_{[1]} \geq b_{[2]} \geq \cdots \geq b_{[m]}$ *be their ordered components then* $\mathbf{a}$ *is said to majorize* $\mathbf{b}$ *(or* $\mathbf{b}$ *is to be majorize by* $\mathbf{a}$*, symbolically* $\mathbf{b} \prec \mathbf{a}$*), if*

$$\sum_{j=1}^{k} b_{[j]} \leq \sum_{j=1}^{k} a_{[j]} \qquad \text{for } k = 1, 2, \ldots, m-1, \tag{1}$$

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and

$$\sum_{j=1}^{m} a_j = \sum_{j=1}^{m} b_j. \tag{2}$$

Maojorization is a partial order relation of two $m-$tuples $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ which explains that the tuple **a** is "less spread out" or "more nearly equal" than the tuple **b**. The theory of majorization helps us to convert complicated problems of optimization into simple problems which can be easily solved [7, 21]. Some current applications of majorization theory in signal processing and communication can be accessed in [23, 34].

A lot of work is devoted to the theory of majorization. Khan et al. [5] extended the majorization inequality for the function defined on interval to the convex function defined on rectangle. In this extension, the authors considered certain monotonic tuples. Wu et al. [38] gave some refinement of the majorization inequalities with the help of Taylor's theorem. They used a convex function whose double derivative exists on interval and various types of monotonic tuples. Also, a new fractional inequality has been obtained as an application of the main result. Khan et al. [4] used Green function and Taylor's formula to generalize the majorization theorem for $n-$convex functions. The authors also deduced bounds for some related identities. In [24], the authors used generalized majorization inequalities to present generalized form of the Jensen and the Jensen-Steffensen inequalitites. They also gave generalization of a variant of Jensen's inequality. Khan et al. [1] extended majorization inequality from strongly convex functions defined on interval to the functions which are strongly convex defined on rectangle and also obtained its weighted version. Also, the authors presented Favard's type inequalities with the help of obtained results. Zaheer et al. [40] obtained integral inequalities related to strongly convex functions using majoriaztion. For more recent results, we refer the reader to [36, 37].

The following theorem is due to Hardy, Littlewood and Pólya [22], known as the majorization theorem. The monograph of Marshall and Olkin [31] also provide the proof of this theorem.

Theorem 1.2. *Let* $\mathbf{a} = (a_1, \ldots, a_m)$ *and* $\mathbf{b} = (b_1, \ldots, b_m)$ *be two real* $m-$*tuples such that* $a_j, b_j \in I$. *Then*

$$\sum_{j=1}^{m} \phi(b_j) \leq \sum_{j=1}^{m} \phi(a_j) \tag{3}$$

is valid for each continuous convex function $\phi : I \to \mathbb{R}$ *iff* $\mathbf{b} \prec \mathbf{a}$.

The above theorem in its weighted form is given below [27].

Theorem 1.3. *Suppose that* ϕ *is a real valued convex function on* I *and* $\mathbf{a} = (a_1, \ldots, a_m)$, $\mathbf{b} = (b_1, \ldots, b_m)$, $\mathbf{p} = (p_1, \ldots, p_m)$ *are three* $m-$*tuples such that* a_j, $b_j \in I$, $p_j \geq 0$ *for all* $j \in \{1, 2 \ldots, m\}$. *If* **b** *is decreasing* $m-$*tuple and*

$$\sum_{j=1}^{k} p_j b_j \leq \sum_{j=1}^{k} p_j a_j, \quad \text{for } k = 1, 2, \ldots, m-1, \tag{4}$$

$$\sum_{j=1}^{m} p_j a_j = \sum_{j=1}^{m} p_j b_j, \tag{5}$$

then

$$\sum_{j=1}^{m} p_j \phi(b_j) \leq \sum_{j=1}^{m} p_j \phi(a_j). \tag{6}$$

In [11] Dragomir proved weighted version of Theorem 1.2 without using condition (4).

Theorem 1.4. *Suppose that ϕ is a real valued convex function on I and $\boldsymbol{a} = (a_1, \dots, a_m)$, $\boldsymbol{b} = (b_1, \dots, b_m)$, $\boldsymbol{p} = (p_1, \dots, p_m)$ are three $m-$tuples such that $a_j, b_j \in I$, $p_j \geq 0$ with $P_m = \sum_{j=1}^{m} p_j > 0$ for all $j \in \{1, 2, \dots, m\}$. If $\mathbf{a} - \mathbf{b}$ and $\mathbf{b}$ are monotonic in the same sense and satisfying*

$$\sum_{j=1}^{m} p_j a_j = \sum_{j=1}^{m} p_j b_j$$

then

$$\sum_{j=1}^{m} p_j \phi(b_j) \leq \sum_{j=1}^{m} p_j \phi(a_j). \tag{7}$$

As our main results concern with the Mercer inequality. Therefore, in below theorem, we state the Mercer inequality.

Theorem 1.5. *([28]) Suppose that ϕ is a real valued convex function on I such that $[\delta_1, \delta_2] \subset I$, $x_j \in [\delta_1, \delta_2]$, $p_j \geq 0$ for all $j \in \{1, 2, \dots, m\}$ with $\sum\limits_{j=1}^{m} p_j = 1$, then*

$$\phi\Big(\delta_1 + \delta_2 - \sum_{j=1}^{m} p_j x_j\Big) \leq \phi(\delta_1) + \phi(\delta_2) - \sum_{j=1}^{m} p_j \phi(x_j). \tag{8}$$

Niezgoda [32] used the concept of majorization and extended the Jensen-Mercer inequality given as under:

Theorem 1.6. *Suppose that ϕ is a real valued convex function on I, (x_{ij}) is a $n \times m$ real matrix and $\mathbf{e} = (e_1, \dots, e_m)$ is $m-$tuple such that $e_j, x_{ij} \in I$ for all i, j, $w_i \geq 0$ for $i = 1, 2, \dots, n$ with $\sum\limits_{i=1}^{n} w_i = 1$. If $\mathbf{e}$ majorizes every row of (x_{ij}), then we have*

$$\phi\left(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i x_{ij}\right) \leq \sum_{j=1}^{m} \phi(e_j) - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i \phi(x_{ij}). \tag{9}$$

Another significant inequality is the Hermite-Hadamard inequality defined for convex functions as [19]: Suppose $\vartheta, \theta \in I$ such that $\vartheta < \theta$ and $\phi : I \to \mathbb{R}$ is a convex function then

$$\phi\Big(\frac{\vartheta + \theta}{2}\Big) \leq \frac{1}{\theta - \vartheta} \int_{\vartheta}^{\theta} \phi(u) du \leq \frac{\phi(\vartheta) + \phi(\theta)}{2} \tag{10}$$

holds. The inequality (10) was first proved by Hermite [20] in 1883 but his work was not commonly known in the literature of mathematics. According to a famous historian, Beckenbach [6], Hadamard rediscoverd this inequality ten years later. Afterwards Hermite's note was found by Mitrinović [29] in Mathesis. Therefore, the inequality given in (10), is widely known by the name Hermite-Hadamard inequality. It guaranties the integrability of convex function and gives estimate of integral mean of convex function.
Dragomir and Agarwal [15] formulated an integral identity using right hand part of Hermite-Hadamard inequality and presented some good results along with applications. Pearce and Pečarić [35] proved some more results by using the same integral identity given in [15]. They also gave some applications to trapezoidal and midpoint formulas. For further generalizations, extensions and refinements of Hermite-Hadamard inequality one can see [2, 3, 9, 19].

2. Main Results

In the underlying theorems we give Hermite-Hadamard inequality of the Jensen-Mercer type by using majorization concept.

Theorem 2.1. *Suppose that ϕ is a real valued convex function on I and $\mathbf{e} = (e_1, \ldots, e_m)$, $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m)$ are three m-tuples such that $e_j, x_j, y_j \in I$, $x_j \neq y_j$ for all $j \in \{1, \cdots, m\}$. If $\mathbf{x} \prec \mathbf{e}$ and $\mathbf{y} \prec \mathbf{e}$, then*

$$\begin{aligned}\phi\Big(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1}\Big(\frac{x_j + y_j}{2}\Big)\Big) &\leq \sum_{j=1}^{m}\phi(e_j) - \sum_{j=1}^{m-1}\frac{1}{y_j - x_j}\int_{x_j}^{y_j}\phi(u)du \\ &\leq \sum_{j=1}^{m}\phi(e_j) - \sum_{j=1}^{m-1}\phi\Big(\frac{x_j + y_j}{2}\Big).\end{aligned} \tag{11}$$

Proof. Let $t \in [0, 1]$, then we may write

$$\begin{aligned}&\phi\Big(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1}\Big(\frac{x_j + y_j}{2}\Big)\Big) \\ &\quad = \phi\Big(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1}\Big(\frac{tx_j + (1-t)y_j + ty_j + (1-t)x_j}{2}\Big)\Big).\end{aligned} \tag{12}$$

In order to apply Theorem 1.6 in (12), first we show that $\mathbf{e}$ majorizes $\mathbf{r}$ and $\mathbf{z}$, where $\mathbf{r} = (r_1, \cdots, r_m)$, $\mathbf{z} = (z_1, \cdots, z_m)$, $r_j = tx_j + (1-t)y_j$ and $z_j = ty_j + (1-t)x_j$ for $j = 1, 2, \ldots, m$.
For this, let $\sum_{j=1}^{k} x_{[j]} = \beta_{1k}$ and $\sum_{j=1}^{k} y_{[j]} = \beta_{2k}$, $\quad for \quad k = 1, \cdots, m-1$. Then

$$\sum_{j=1}^{k} r_{[j]} = t\sum_{j=1}^{k} x_{[j]} + (1-t)\sum_{j=1}^{k} y_{[j]} = t\beta_{1k} + (1-t)\beta_{2k}. \tag{13}$$

Since $\mathbf{x} \prec \mathbf{e}$ and $\mathbf{y} \prec \mathbf{e}$, therefore by definition of majorization, we have $\sum_{j=1}^{k} x_{[j]} \leq \sum_{j=1}^{k} e_{[j]}$ and $\sum_{j=1}^{k} y_{[j]} \leq \sum_{j=1}^{k} e_{[j]}$ i.e.

$$\beta_{1k} \leq \sum_{j=1}^{k} e_{[j]} \tag{14}$$

and

$$\beta_{2k} \leq \sum_{j=1}^{k} e_{[j]}. \tag{15}$$

Multiplying (14) by t and (15) by $1 - t$ and then adding the resulting inequalities, we get

$$t\beta_{1k} + (1-t)\beta_{2k} \leq \sum_{j=1}^{k} e_{[j]}. \tag{16}$$

Now using (13) in (16), we have

$$\sum_{j=1}^{k} r_{[j]} \leq \sum_{j=1}^{k} e_{[j]}.$$

Also,

$$\sum_{j=1}^{m} r_j = t\sum_{j=1}^{m} x_j + (1-t)\sum_{j=1}^{m} y_j. \tag{17}$$

But $\sum\limits_{j=1}^{m} e_j = \sum\limits_{j=1}^{m} x_j$ and $\sum\limits_{j=1}^{m} e_j = \sum\limits_{j=1}^{m} y_j$, therefore from (17) we have

$$\sum_{j=1}^{m} r_j = \sum_{j=1}^{m} e_j.$$

Hence, $\mathbf{r} \prec \mathbf{e}$. Similarly we can show that $\mathbf{z} \prec \mathbf{e}$. Therefore, using Theorem 1.6 in (12) for the case of $n = 2$ and $w_1 = w_2 = \frac{1}{2}$, we obtain

$$\phi\bigg(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1}\Big(\frac{x_j+y_j}{2}\Big)\bigg) \leq \sum_{j=1}^{m}\phi(e_j) - \frac{\sum\limits_{j=1}^{m-1}\Big(\phi\Big((tx_j+(1-t)y_j\Big) + \phi(ty_j+(1-t)x_j)\Big)}{2}. \tag{18}$$

Now, integration of (18) with respect to t, gives

$$\phi\bigg(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1}\Big(\frac{x_j+y_j}{2}\Big)\bigg) \leq \sum_{j=1}^{m}\phi(e_j) - \frac{1}{2}\sum_{j=1}^{m-1}\int_0^1\Big(\phi\Big((tx_j+(1-t)y_j\Big) + \phi(ty_j+(1-t)x_j)\Big)dt. \tag{19}$$

Since

$$\int_0^1 \phi\Big((tx_j+(1-t)y_j\Big)dt = \int_0^1 \phi(ty_j+(1-t)x_j)dt = \frac{1}{y_j-x_j}\int_{x_j}^{y_j}\phi(u)du. \tag{20}$$

Therefore, using (20) in (19), we get the left inequality in (11).

To obtain the right inequality in (11), we know from Hermite-Hadamard inequality that

$$\frac{-1}{\theta-\vartheta}\int_{\vartheta}^{\theta}\phi(u)du \leq -\phi\Big(\frac{\vartheta+\theta}{2}\Big). \tag{21}$$

Replacing ϑ, θ by x_j, y_j respectively and summing both sides over $j = 1, \cdots, m-1$, we get

$$-\sum_{j=1}^{m-1}\frac{1}{y_j-x_j}\int_{x_j}^{y_j}\phi(u)du \leq -\sum_{j=1}^{m-1}\phi\Big(\frac{x_j+y_j}{2}\Big). \tag{22}$$

Adding $\sum\limits_{j=1}^{m}\phi(e_j)$ to both sides of (22), we deduce the right inequality in (11). □

Remark 2.2. *In the above theorem, if $x_j = y_j$ for all $j \in \{1, 2, \ldots, m\}$ then by following the proof of Theorem 2.1, we obtain*

$$\phi\bigg(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1} x_j\bigg) \leq \sum_{j=1}^{m}\phi(e_j) - \sum_{j=1}^{m-1}\phi(x_j).$$

If $x_j = y_j$ for some j, then (11) reduces to the form

$$\phi\bigg(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1}\Big(\frac{x_j+y_j}{2}\Big)\bigg) \leq \sum_{j=1}^{m}\phi(e_j) - \sum_{j\in I}\phi(x_j) - \sum_{j\in I^c}\frac{1}{y_j-x_j}\int_{x_j}^{y_j}\phi(u)du,$$

where $I = \{j \in \{1, 2, \ldots, m\} : x_j = y_j\}$ and $I^c = \{1, 2, \ldots, m\} \setminus I$.

Theorem 2.3. *If all the hypotheses of Theorem* 2.1 *are valid and $x_m \neq y_m$, then*

$$\phi\Big(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1}\Big(\frac{x_j + y_j}{2}\Big)\Big) \leq \frac{1}{\sum\limits_{j=1}^{m-1}(y_j - x_j)} \int_{\sum\limits_{j=1}^{m-1} x_j}^{\sum\limits_{j=1}^{m-1} y_j} \phi\Big(\sum_{j=1}^{m} e_j - u\Big) du$$

$$\leq \sum_{j=1}^{m} \phi(e_j) - \frac{1}{2}\left\{\sum_{j=1}^{m-1}\phi(x_j) + \sum_{j=1}^{m-1}\phi(y_j)\right\}. \tag{23}$$

Proof. Clearly,

$$\phi\Big(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1}\Big(\frac{x_j + y_j}{2}\Big)\Big) = \phi\Big(\frac{1}{2}\Big(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1} x_j + \sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1} y_j\Big)\Big) \tag{24}$$

By second condition of majorization, we have

$$\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1} x_j = x_m \tag{25}$$

and

$$\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1} y_j = y_m. \tag{26}$$

Therefore $\sum\limits_{j=1}^{m} e_j - \sum\limits_{j=1}^{m-1} x_j, \sum\limits_{j=1}^{m} e_j - \sum\limits_{j=1}^{m-1} y_j \in I$.

Also, as $x_m \neq y_m$ so from (25) and (26), we get that $\sum\limits_{j=1}^{m-1} x_j \neq \sum\limits_{j=1}^{m-1} y_j$. Now, using Hermite-Hadamard inequality in (24), we obtain

$$\begin{aligned}
\phi\Big(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1}\Big(\frac{x_j + y_j}{2}\Big)\Big) \;&\leq\; \int_0^1 \phi\Big(t\Big(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1} x_j\Big) + (1-t) \\
&\qquad \times\Big(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1} y_j\Big)\Big) dt \\
&= \int_0^1 \phi\Big(\sum_{j=1}^{m} e_j - \Big(t\sum_{j=1}^{m-1} x_j + (1-t)\sum_{j=1}^{m-1} y_j\Big)\Big) dt \\
&= \frac{1}{\sum\limits_{j=1}^{m-1}(y_j - x_j)} \int_{\sum\limits_{j=1}^{m-1} x_j}^{\sum\limits_{j=1}^{m-1} y_j} \phi\Big(\sum_{j=1}^{m} e_j - u\Big) du. \qquad (27)
\end{aligned}$$

This completes the proof of the left inequality in (23).

Now, we derive the right inequality in (23). Since $\mathbf{x} \prec \mathbf{e}$ and $\mathbf{y} \prec \mathbf{e}$, that is all the hypotheses of Theorem 1.6 are satisfied for $n = 2$. Therefore using Theorem 1.6, for $n = 2$, $w_1 = t$ and $w_2 = 1 - t$, we can write

$$\begin{aligned}
\phi\Big(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1}(tx_j + (1-t)y_j)\Big) \\
&\leq \sum_{j=1}^{m} \phi(e_j) - \sum_{j=1}^{m-1}\Big(t\phi(x_j) + (1-t)\phi(y_j)\Big) \\
&= \sum_{j=1}^{m} \phi(e_j) - \Big(t\sum_{j=1}^{m-1} \phi(x_j) + (1-t)\sum_{j=1}^{m-1} \phi(y_j)\Big) \qquad (28)
\end{aligned}$$

Integration of (28) with respect to t, gives

$$\begin{aligned}
\int_0^1 \phi\left(\sum_{j=1}^{m} e_j - \sum_{j=1}^{m-1}(tx_j + (1-t)y_j)\right) dt \\
&\leq \sum_{j=1}^{m} \phi(e_j) - \int_0^1 \left(t\sum_{j=1}^{m-1} \phi(x_j) + (1-t))\sum_{j=1}^{m-1} \phi(y_j)\right) dt. \qquad (29)
\end{aligned}$$

Using integration and change of variable we get the right inequality in (23). □

Remark 2.4. *Following the suppositions of Theorem* 2.1 *and Theorem* 2.3*, if* $m = 2$*, then we get the following inequalities which have been given in [25]:*

$$\begin{aligned}
\phi\left(e_1 + e_2 - \frac{x_1 + y_1}{2}\right) &\leq \phi(e_1) + \phi(e_2) - \frac{1}{y_1 - x_1}\int_{x_1}^{y_1} \phi(u)du \\
&\leq \phi(e_1) + \phi(e_2) - \phi\left(\frac{x_1 + y_1}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\phi\left(e_1 + e_2 - \frac{x_1 + y_1}{2}\right) &\leq \frac{1}{y_1 - x_1}\int_{x_1}^{y_1} \phi\left(e_1 + e_2 - u\right) du \\
&\leq \phi(e_1) + \phi(e_2) - \frac{1}{2}\left\{\phi(x_1) + \phi(y_1)\right\}. \qquad (30)
\end{aligned}$$

We establish the following lemma which will help us to give our next generalized Hermite-Hadamard type inequality.

Lemma 2.5. *Assume that* $\phi : I \to \mathbb{R}$ *is a convex function and* $\mathbf{e} = (e_1, \ldots, e_m)$*,* $\mathbf{p} = (p_1, \ldots, p_m)$ *are two* $m-$ *tuples,* (x_{ij}) *is a* $n \times m$ *real matrix such that* $e_j,\ x_{ij} \in I$*,* $p_j \geq 0$ *with* $p_m \neq 0$ *for all* i, j *and* $\eta = \frac{1}{p_m}$*. Also,* $w_i \geq 0$ *for* $i = 1, 2, \ldots, n$ *with* $\sum_{i=1}^{n} w_i = 1$*. If* $(x_{i1}, \ldots, x_{im})$ *is a decreasing m-tuple for each* $i = 1, 2, \ldots, n$*, and satisfying*

$$\sum_{j=1}^{k} p_j x_{ij} \leq \sum_{j=1}^{k} p_j e_j \ \text{for}\ k = 1, 2, \ldots, m-1,\ \sum_{j=1}^{m} p_j e_j = \sum_{j=1}^{m} p_j x_{ij},$$

then

$$\phi\Big(\sum_{j=1}^{m} \eta p_j e_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n} \eta w_i p_j x_{ij}\Big) \leq \sum_{j=1}^{m} \eta p_j \phi(e_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} \eta w_i p_j \phi(x_{ij}). \qquad (31)$$

Proof. It is evident that

$$\begin{aligned}\phi\Big(\sum_{j=1}^{m}\eta p_je_j-\sum_{j=1}^{m-1}\sum_{i=1}^{n}\eta w_ip_jx_{ij}\Big) &= \phi\Big(\sum_{j=1}^{m}\eta p_je_j\sum_{i=1}^{n}w_i-\sum_{j=1}^{m-1}\sum_{i=1}^{n}\eta w_ip_jx_{ij}\Big)\\ &\leq \sum_{i=1}^{n}w_i\phi\Big(\eta\Big(\sum_{j=1}^{m}p_je_j-\sum_{j=1}^{m-1}p_jx_{ij}\Big)\Big). \qquad (32)\end{aligned}$$

Since $\sum\limits_{j=1}^{m}p_je_j=\sum\limits_{j=1}^{m}p_jx_{ij}$ for each $i=1,\cdots,n$, then we have

$$x_{im}=\eta\Big(\sum_{j=1}^{m}p_je_j-\sum_{j=1}^{m-1}p_jx_{ij}\Big). \qquad (33)$$

Using Theorem 1.3, for **e**, **p** and $(x_{i1},\dots,x_{im})$ for each $i=1,2,\dots,n$, we write

$$\begin{aligned}&\sum_{j=1}^{m}p_j\phi(x_{ij})\leq\sum_{j=1}^{m}p_j\phi(e_j)\\ \Rightarrow&\sum_{j=1}^{m-1}p_j\phi(x_{ij})+p_m\phi(x_{im})\leq\sum_{j=1}^{m}p_j\phi(e_j)\\ \Rightarrow&\, p_m\phi(x_{im})\leq\sum_{j=1}^{m}p_j\phi(e_j)-\sum_{j=1}^{m-1}p_j\phi(x_{ij})\\ \Rightarrow&\,\phi(x_{im})\leq\sum_{j=1}^{m}\eta p_j\phi(e_j)-\sum_{j=1}^{m-1}\eta p_j\phi(x_{ij}) \qquad (34)\end{aligned}$$

Using (33) in (34), we obtain

$$\phi\Big(\eta\Big(\sum_{j=1}^{m}p_je_j-\sum_{j=1}^{m-1}p_jx_{ij}\Big)\Big)\leq\sum_{j=1}^{m}\eta p_j\phi(e_j)-\sum_{j=1}^{m-1}\eta p_j\phi(x_{ij}) \qquad (35)$$

From (32) and (35), we get (31). □

In the following theorems we prove Hermite-Hadamard inequalities of the Jensen-Mercer type for one arbitrary and two monotonic tuples.

Theorem 2.6. *Suppose that ϕ is a real valued convex function on I and* $\mathbf{e}=(e_1,\dots,e_m)$, $\mathbf{x}=(x_1,\dots,x_m)$, $\mathbf{y}=(y_1,\dots,y_m)$, $\mathbf{p}=(p_1,\dots,p_m)$ *are four m-tuples such that $e_j,x_j,y_j\in I$, $x_j\neq y_j$, $p_j\geq 0$ with $p_m\neq 0$ for all $j\in\{1,\dots,m\}$ and $\eta=\frac{1}{p_m}$. If* **x** *and* **y** *are decreasing m-tuples and*

$$\sum_{j=1}^{k}p_jx_j\leq\sum_{j=1}^{k}p_je_j,\ \sum_{j=1}^{k}p_jy_j\leq\sum_{j=1}^{k}p_je_j \text{ for } k=1,\dots,m-1,$$

$$\sum_{j=1}^{m}p_je_j=\sum_{j=1}^{m}p_jx_j,\ \sum_{j=1}^{m}p_je_j=\sum_{j=1}^{m}p_jy_j,$$

then

$$\phi\Big(\sum_{j=1}^{m}\eta p_j e_j-\eta\sum_{j=1}^{m-1}\Big(\frac{p_jx_j+p_jy_j}{2}\Big)\Big)\leq\sum_{j=1}^{m}\eta p_j\phi(e_j)-\eta\sum_{j=1}^{m-1}\frac{p_j}{y_j-x_j}\int_{x_j}^{y_j}\phi(u)du$$
$$\leq\sum_{j=1}^{m}\eta p_j\phi(e_j)-\eta\sum_{j=1}^{m-1}p_j\phi\Big(\frac{x_j+y_j}{2}\Big).\tag{36}$$

Proof. Let $t\in[0,1]$, we have

$$\phi\Big(\sum_{j=1}^{m}\eta p_j e_j-\eta\sum_{j=1}^{m-1}\Big(\frac{p_jx_j+p_jy_j}{2}\Big)\Big)$$
$$=\phi\Big(\sum_{j=1}^{m}\eta p_j e_j-\eta\sum_{j=1}^{m-1}p_j\Big(\frac{tx_j+(1-t)y_j+ty_j+(1-t)x_j}{2}\Big)\Big).\tag{37}$$

Let $\mathbf{r}=(r_1,\ldots,r_m)$ and $\mathbf{z}=(z_1,\ldots,z_m)$ where $r_j=tx_j+(1-t)y_j$ and $z_j=ty_j+(1-t)x_j$ for $j=1,2,\ldots,m$. By similar idea as given in Theorem 2.1, we can show that $\mathbf{r}$ and $\mathbf{z}$ satisfy the conditions $\sum_{j=1}^{k}p_jr_j\leq\sum_{j=1}^{k}p_je_j$, $\sum_{j=1}^{k}p_jz_j\leq\sum_{j=1}^{k}p_je_j$ for $k=1,2,\ldots,m-1$ and $\sum_{j=1}^{m}p_jr_j=\sum_{j=1}^{m}p_je_j$, $\sum_{j=1}^{m}p_jz_j=\sum_{j=1}^{m}p_je_j$. Therefore, applying Lemma 2.5 in (37), we have

$$\phi\Big(\sum_{j=1}^{m}\eta p_j e_j-\eta\sum_{j=1}^{m-1}\Big(\frac{p_jx_j+p_jy_j}{2}\Big)\Big)$$
$$\leq\sum_{j=1}^{m}\eta p_j\phi(e_j)-\frac{1}{2}\eta\sum_{j=1}^{m-1}\Big(p_j\phi\big(tx_j+(1-t)y_j\big)+p_j\phi\big(ty_j+(1-t)x_j\big)\Big).\tag{38}$$

Integration of (38) with respect to t, delivers

$$\phi\Big(\sum_{j=1}^{m}\eta p_j e_j-\eta\sum_{j=1}^{m-1}\Big(\frac{p_jx_j+p_jy_j}{2}\Big)\Big)$$
$$\leq\sum_{j=1}^{m}\eta p_j\phi(e_j)-\frac{1}{2}\eta\sum_{j=1}^{m-1}p_j\int_0^1\Big(\phi\big(tx_j+(1-t)y_j\big)+\phi\big(ty_j+(1-t)x_j\big)\Big)dt.\tag{39}$$

Since

$$\int_0^1\phi\big((tx_j+(1-t)y_j\big)dt=\int_0^1\phi\big(ty_j+(1-t)x_j\big)dt=\frac{1}{y_j-x_j}\int_{x_j}^{y_j}\phi(u)du.\tag{40}$$

Using (40) in (39), we get the left inequality in (36).

Next we prove the right inequality in (36). For this, replace ϑ, θ by x_j, y_j respectively in left inequality of (10) and multiplying ηp_j, we get

$$\frac{-\eta p_j}{y_j-x_j}\int_{x_j}^{y_j}\phi(u)du\leq-\eta p_j\phi\Big(\frac{x_j+y_j}{2}\Big).\tag{41}$$

Taking summation on both sides over $j=1,\ldots,m-1$ and then adding $\sum_{j=1}^{m}\eta p_j\phi(e_j)$ to both sides, we obtain the right inequality in (36). □

Remark 2.7. *In the above theorem, if $x_j = y_j$ for all $j \in \{1, 2, \ldots, m\}$ then by following the proof of Theorem* 2.6, *we obtain*

$$\phi\left(\sum_{j=1}^{m} \eta p_j e_j - \eta \sum_{j=1}^{m-1} p_j x_j\right) \leq \sum_{j=1}^{m} \eta p_j \phi(e_j) - \eta \sum_{j=1}^{m-1} p_j \phi(x_j).$$

If $x_j = y_j$ for some j, then (36) *reduces to the form*

$$\phi\left(\sum_{j=1}^{m} \eta p_j e_j - \eta \sum_{j=1}^{m-1} \left(\frac{p_j x_j + p_j y_j}{2}\right)\right)$$
$$\leq \sum_{j=1}^{m} \eta p_j \phi(e_j) - \sum_{j \in I} \eta p_j \phi(x_j) - \eta \sum_{j \in I^c} p_j \frac{1}{y_j - x_j} \int_{x_j}^{y_j} \phi(u) du,$$

where $I = \{j \in \{1, 2, \ldots, m\} : x_j = y_j\}$ and $I^c = \{1, 2, \ldots, m\} \setminus I$.

Theorem 2.8. *If all the hypotheses of Theorem* 2.6 *are valid and $x_m \neq y_m$, then*

$$\phi\left(\sum_{j=1}^{m} \eta p_j e_j - \eta \sum_{j=1}^{m-1} \left(\frac{p_j x_j + p_j y_j}{2}\right)\right)$$
$$\leq \frac{1}{\sum\limits_{j=1}^{m-1} (\eta p_j y_j - \eta p_j x_j)} \int_{\sum_{j=1}^{m-1} \eta p_j x_j}^{\sum_{j=1}^{m-1} \eta p_j y_j} \phi\left(\sum_{j=1}^{m} \eta p_j e_j - u\right) du$$
$$\leq \sum_{j=1}^{m} \eta p_j \phi(e_j) - \frac{1}{2} \left\{\sum_{j=1}^{m-1} \eta p_j \phi(x_j) + \sum_{j=1}^{m-1} \eta p_j \phi(y_j)\right\}. \tag{42}$$

Proof. It can be observed that

$$\phi\left(\sum_{j=1}^{m} \eta p_j e_j - \eta \sum_{j=1}^{m-1} \left(\frac{p_j x_j + p_j y_j}{2}\right)\right)$$
$$= \phi\left(\frac{1}{2}\left(\sum_{j=1}^{m} \eta p_j e_j - \sum_{j=1}^{m-1} \eta p_j x_j + \sum_{j=1}^{m} \eta p_j e_j - \sum_{j=1}^{m-1} \eta p_j y_j\right)\right). \tag{43}$$

As $\sum_{j=1}^{m} p_j e_j = \sum_{j=1}^{m} p_j x_j$ and $\sum_{j=1}^{m} p_j e_j = \sum_{j=1}^{m} p_j y_j$, so we have

$$\sum_{j=1}^{m} \eta p_j e_j - \sum_{j=1}^{m-1} \eta p_j x_j = x_m \tag{44}$$

and

$$\sum_{j=1}^{m} \eta p_j e_j - \sum_{j=1}^{m-1} \eta p_j y_j = y_m. \tag{45}$$

Hence $\sum_{j=1}^{m} \eta p_j e_j - \sum_{j=1}^{m-1} \eta p_j x_j, \sum_{j=1}^{m} \eta p_j e_j - \sum_{j=1}^{m-1} \eta p_j y_j \in I$.

Also, as $x_m \neq y_m$ so from (44) and (45), we get that $\sum_{j=1}^{m-1} \eta p_j x_j \neq \sum_{j=1}^{m-1} \eta p_j y_j$. Now using Hermite-Hadamard inequality in right side of (43), we obtain

$$\begin{aligned}
&\phi\Big(\sum_{j=1}^{m} \eta p_j e_j - \eta \sum_{j=1}^{m-1}\Big(\frac{p_j x_j + p_j y_j}{2}\Big)\Big) \\
&\leq \int_0^1 \phi\Big(t\Big(\sum_{j=1}^{m} \eta p_j e_j - \sum_{j=1}^{m-1} \eta p_j x_j\Big) + (1-t) \\
&\times\Big(\sum_{j=1}^{m} \eta p_j e_j - \sum_{j=1}^{m-1} \eta p_j y_j\Big)\Big)dt \\
&= \int_0^1 \phi\Big(\sum_{j=1}^{m} \eta p_j e_j - \sum_{j=1}^{m-1}\Big(t\eta p_j x_j + (1-t)\eta p_j y_j\Big)\Big)dt \\
&= \frac{1}{\sum_{j=1}^{m-1}(\eta p_j y_j - \eta p_j x_j)} \int_{\sum_{j=1}^{m-1} \eta p_j x_j}^{\sum_{j=1}^{m-1} \eta p_j y_j} \phi\Big(\sum_{j=1}^{m} \eta p_j e_j - u\Big)du. \qquad (46)
\end{aligned}$$

Thus the left inequality in (42) is proved.

Now, we prove the right inequality in (42). Using Lemma 2.5 for $n = 2$, $w_1 = t$ and $w_2 = 1 - t$, we deduce

$$\begin{aligned}
&\phi\Big(\sum_{j=1}^{m} \eta p_j e_j - \sum_{j=1}^{m-1}\Big(t\eta p_j x_j + (1-t)\eta p_j y_j\Big)\Big) \\
&\qquad \leq \sum_{j=1}^{m} \eta p_j \phi(e_j) - \sum_{j=1}^{m-1}\Big(t\eta p_j \phi(x_j) + (1-t)\eta p_j \phi(y_j)\Big). \qquad (47)
\end{aligned}$$

Integration of (47) with respect to t, follows

$$\begin{aligned}
&\int_0^1 \phi\Big(\sum_{j=1}^{m} \eta p_j e_j - \sum_{j=1}^{m-1}\Big(t\eta p_j x_j + (1-t)\eta p_j y_j\Big)\Big)dt \\
&\qquad \leq \sum_{j=1}^{m} \eta p_j \phi(e_j) - \sum_{j=1}^{m-1} \int_0^1 \Big(t\eta p_j \phi(x_j) + (1-t)\eta p_j \phi(y_j)\Big)dt. \qquad (48)
\end{aligned}$$

Using integration and change of variable, we deduce the right inequality in (42).

For more results of Hermite-Hadamard inequality of the Jensen-Mercer type, we propose another lemma which is given as follows:

Lemma 2.9. *Assume that $\phi : I \to \mathbb{R}$ is a convex function and $\mathbf{e} = (e_1, \ldots, e_m)$, $\mathbf{p} = (p_1, \ldots, p_m)$ be two m-tuples, (x_{ij}) is a $n \times m$ real matrix such that $e_j, x_{ij} \in I$, $p_j \geq 0$, $p_m \neq 0$ for all i, j, $\eta = \frac{1}{p_m}$ and $w_i \geq 0$ for $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} w_i = 1$. If $(e_j - x_{ij})$ and x_{ij} are monotonic in the same sense for each $i = 1, \ldots, n$, and*

$$\sum_{j=1}^{m} p_j e_j = \sum_{j=1}^{m} p_j x_{ij},$$

then

$$\phi\Big(\sum_{j=1}^{m}\eta p_je_j-\sum_{j=1}^{m-1}\sum_{i=1}^{n}\eta w_ip_jx_{ij}\Big)\le\sum_{j=1}^{m}\eta p_j\phi(e_j)-\sum_{j=1}^{m-1}\sum_{i=1}^{n}\eta w_ip_j\phi(x_{ij}). \tag{49}$$

Proof. Using Jensen's inequality, we may write

$$\begin{aligned}\phi\Big(\sum_{j=1}^{m}\eta p_je_j-\sum_{j=1}^{m-1}\sum_{i=1}^{n}\eta w_ip_jx_{ij}\Big) &= \phi\Big(\sum_{j=1}^{m}\eta p_je_j\sum_{i=1}^{n}w_i-\sum_{j=1}^{m-1}\sum_{i=1}^{n}\eta w_ip_jx_{ij}\Big)\\ &\le \sum_{i=1}^{n}w_i\phi\Big(\eta\Big(\sum_{j=1}^{m}p_je_j-\sum_{j=1}^{m-1}p_jx_{ij}\Big)\Big).\end{aligned} \tag{50}$$

As $\sum\limits_{j=1}^{m}p_je_j=\sum\limits_{j=1}^{m}p_jx_{ij}$ for each $i=1,\dots,n$, so we may write

$$x_{im}=\eta\Big(\sum_{j=1}^{m}p_je_j-\sum_{j=1}^{m-1}p_jx_{ij}\Big). \tag{51}$$

Since ϕ is convex. Therefore $\forall\, a,b\in I$, we have

$$\phi(a)-\phi(b)\ge\phi'_+(b)(a-b).$$

For the selection $a=e_j$ and $b=x_{ij}$, we can write

$$\phi(e_j)-\phi(x_{ij})\ge\phi'_+(x_{ij})(e_j-x_{ij}). \tag{52}$$

Multiplying (52) by p_j and summing over j from 1 to m, we have

$$\sum_{j=1}^{m}p_j\phi(e_j)-\sum_{j=1}^{m}p_j\phi(x_{ij})\ge\sum_{j=1}^{m}p_j(e_j-x_{ij})\phi'_+(x_{ij}). \tag{53}$$

Using Čebyšev's inequality in right side of (53), we btain

$$\begin{aligned}\sum_{j=1}^{m}p_j(e_j-x_{ij})\phi'_+(x_{ij}) &\ge \frac{1}{\sum\limits_{j=1}^{m}p_j}\sum_{j=1}^{m}p_j(e_j-x_{ij})\cdot\sum_{j=1}^{m}p_j\phi'_+(x_{ij})\\ &= 0.\end{aligned} \tag{54}$$

Using (54) in (53), we have

$$\begin{aligned}&\sum_{j=1}^{m}p_j\phi(e_j)-\sum_{j=1}^{m}p_j\phi(x_{ij})\ge0\\ \Rightarrow\quad&\sum_{j=1}^{m}p_j\phi(e_j)-\sum_{j=1}^{m-1}p_j\phi(x_{ij})-p_m\phi(x_{im})\ge0\\ \Rightarrow\quad&\sum_{j=1}^{m}p_j\phi(e_j)-\sum_{j=1}^{m-1}p_j\phi(x_{ij})\ge p_m\phi(x_{im})\\ \Rightarrow\quad&\sum_{j=1}^{m}\eta p_j\phi(e_j)-\sum_{j=1}^{m-1}\eta p_j\phi(x_{ij})\ge\phi(x_{im}).\end{aligned} \tag{55}$$

Using (51) in (55), we obtain

$$\sum_{j=1}^{m}\eta p_j\phi(e_j)-\sum_{j=1}^{m-1}\eta p_j\phi(x_{ij})\geq\phi\Big(\eta\Big(\sum_{j=1}^{m}p_je_j-\sum_{j=1}^{m-1}p_jx_{ij}\Big)\Big). \tag{56}$$

From (56) and (50), we deduce (49). □

Remark 2.10. *It is important to note that (49) has been proved for separable sequences in [32].*

The following theorems give generalized Hermite-Hadamard type inequalities by imposing relax condition on weights.

Theorem 2.11. *Suppose that ϕ is a real valued convex function on I and $\mathbf{e}=(e_1,\dots,e_m)$, $\mathbf{x}=(x_1,\dots,x_m)$, $\mathbf{y}=(y_1,\dots,y_m)$, $\mathbf{p}=(p_1,\dots,p_m)$ are four $m-$ tuples such that $e_j,x_j,y_j\in I$, $x_j\neq y_j$, $p_j\geq 0$ with $p_m\neq 0$ for all $j\in\{1,\dots,m\}$ and $\eta=\frac{1}{p_m}$. If $\mathbf{e}-\mathbf{x}$, $\mathbf{x}$, $\mathbf{e}-\mathbf{y}$ and $\mathbf{y}$ are monotonic in the same sense and*

$$\sum_{j=1}^{m}p_je_j=\sum_{j=1}^{m}p_jx_j,\quad \sum_{j=1}^{m}p_je_j=\sum_{j=1}^{m}p_jy_j,$$

then

$$\phi\Big(\sum_{j=1}^{m}\eta p_je_j-\eta\sum_{j=1}^{m-1}\Big(\frac{p_jx_j+p_jy_j}{2}\Big)\Big)\leq\sum_{j=1}^{m}\eta p_j\phi(e_j)-\eta\sum_{j=1}^{m-1}\frac{p_j}{y_j-x_j}\int_{x_j}^{y_j}\phi(u)du$$
$$\leq\sum_{j=1}^{m}\eta p_j\phi(e_j)-\eta\sum_{j=1}^{m-1}p_j\phi\Big(\frac{x_j+y_j}{2}\Big). \tag{57}$$

Proof. Since we know that the tuples $\mathbf{r}=(r_1,\dots,r_m)$, $\mathbf{z}=(z_1,\dots,z_m)$ where $r_j=tx_j+(1-t)y_j$, $z_j=ty_j+(1-t)x_j$, $\mathbf{e}-\mathbf{r}$ and $\mathbf{e}-\mathbf{z}$ are monotonic in the same sense for $\forall\, t\in[0,1]$. Therefore, using Lemma 2.9 and then adopting the same procedure as given in the proof of Theorem 2.6, we obtain (57). □

Theorem 2.12. *If all the hypotheses of Theorem* 2.11 *are valid and $x_m\neq y_m$, then*

$$\phi\Big(\sum_{j=1}^{m}\eta p_je_j-\eta\sum_{j=1}^{m-1}\Big(\frac{p_jx_j+p_jy_j}{2}\Big)\Big)$$
$$\leq\frac{1}{\sum\limits_{j=1}^{m-1}(\eta p_jy_j-\eta p_jx_j)}\int_{\sum_{j=1}^{m-1}\eta p_jx_j}^{\sum_{j=1}^{m-1}\eta p_jy_j}\phi\Big(\sum_{j=1}^{m}\eta p_je_j-u\Big)du$$
$$\leq\sum_{j=1}^{m}\eta p_j\phi(e_j)-\frac{1}{2}\left\{\sum_{j=1}^{m-1}\eta p_j\phi(x_j)+\sum_{j=1}^{m-1}\eta p_j\phi(y_j)\right\}. \tag{58}$$

Proof. Using Lemma 2.9 and adopting the same procedure as given in the proof of Theorem 2.8, we obtain (58). □

Remark 2.13. *It may be noted that Theorem* 2.1 *and Theorem* 2.3 *have been proved for three $m-$tuples without monotonicity conditions while Theorem* 2.6 *and Theorem* 2.8 *have been proved by considering a decreasing $m-$tuple with non-negative weights. Moreover, Theorem* 2.11 *and Theorem* 2.12 *have been proved for monotonic $m-$tuples with strict conditions of monotonicity and relax conditions on weights.*

□

Now we present an illustrative example for our first main result.

Example 2.14. *Let $a, b, c \in \mathbb{R}$ such that $a \geq b \geq c$. Suppose that* $\mathbf{e} = (2a, 2b, 2c)$, $\mathbf{x} = (a+b, c+a, b+c)$ *and* $\mathbf{y} = (a + \frac{b}{2} + \frac{c}{2}, b + \frac{a}{2} + \frac{c}{2}, \frac{a}{2} + \frac{b}{2} + c)$ *are three tuples. First, we show that* $\mathbf{x} \prec \mathbf{e}$.
Clearly, $a + b \geq c + a \geq b + c$.
As $a \geq b$ *and* $b \geq c$ *therefore, we have*

$$a + b \leq a + a \Rightarrow a + b \leq 2a, \quad \text{and} \quad a + b + c + a = 2a + b + c \leq 2a + 2b.$$

$$\text{Also,} \quad a + b + c + a + b + c = 2a + 2b + 2c.$$

Hence, $\mathbf{x} \prec \mathbf{e}$. *Similarly, we can show that* $\mathbf{y} \prec \mathbf{e}$.
Now, applying Theorem 2.1, *for these tuples, we obtain*

$$\begin{aligned}\phi\Big(\frac{a+3b+4c}{4}\Big) &\leq \phi(2a) + \phi(2b) + \phi(2c) - \frac{2}{c-b}\int_{a+b}^{a+b/2+c/2} \phi(u)du \\ &\qquad - \frac{2}{2b-a-c}\int_{c+a}^{b+a/2+c/2} \phi(u)du \\ &\leq \phi(2a) + \phi(2b) + \phi(2c) - \phi\Big(\frac{4a+3b+c}{4}\Big) - \phi\Big(\frac{3c+3a+2b}{4}\Big).\end{aligned}$$

3. Conclusion

The Hermite-Hadamard inequality has been studied in several directions. It has been established for different generalized convex functions such as $\eta-$convex [16], $s-$convex [17], coordinate convex [12] and strongly convex function [30] etc. Several integral identities have been proved related to Hermite-Hadamard inequality which provide several bounds for the difference of Hermite-Hadamard inequality. Also, many applications have been presented for this inequality. In this article, we have initiated to link the results for majorization and the Jensen-Mercer inequality with the Hermite-Hadamard inequality. We have obtained generalized Hermite-Hadamard inequality using majorization without monotonicity conditions on the tuples. By utilizing generalized results of the Jensen-Mercer type for certain monotonic tuples, we obtained weighted generalized Hermite-Hadamard inequality. As particular cases we have deduced inequalities obtained earlier. The idea adopted in this article may explore further research for Hermite-Hadamard inequality.

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