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Two-Sided Quaternion Wave-Packet Transform and the Quantitative Uncertainty Principles

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Abstract. In this article, we introduce the notion of two-sided quaternion wave-packet transform which inherits the advantages of both the quaternion windowed Fourier and wavelet transforms with some additional promising features. The preliminary analysis encompasses the derivation of fundamental properties including, orthogonality relation, energy preserving relation, inversion formula and the range theorem by utilizing the machinery of two-sided quaternion Fourier transforms. Besides, we also derive the Heisenberg's and logarithmic uncertainty principles for the proposed transform. We culminate our investigation by presenting some illustrative examples.

1. Introduction

An utter representation of non-transient signals requires frequency analysis that is local in time, resulting in the time-frequency analysis. The major development in the realm of time-frequency analysis came in the form of short-time Fourier transform (STFT) or Gabor transform (see [12]), which is reliant upon analysing functions determined by the fundamental operations of translation and modulation acting on a given window function. Although the Gabor representations are quite handy, however, such representations are not adequate for signals having high frequency components for shorter durations and low frequency components for longer durations, leading to the birth of time-scale integral transform, often known as the wavelet transform [11, 26, 33, 36]. As of now, several generalizations of the classical wavelet transform have been reported in recent years including the fractional wavelet transform [32, 34, 35], linear canonical wavelet transform [28, 29], quadratic-phase and special affine wavelet transform [30]. Owing to the lucid nature and close resemblance with the conventional Fourier transform, the wavelet transforms have fascinated the mathematical, physical, chemical, biological and engineering communities with their versatile applicability [37, 38].

On the other hand, the quaternion algebra has attained respectable status in the realm of contemporary harmonic analysis as it offers a lucid representation of multi-dimensional signals, wherein several components are to be controlled simultaneously [18, 25]. Due to the non-commutativity of the elements in the field of

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quaternions, several integral transforms have been generalized in the quaternion settings [2, 16, 20, 24, 27]. As a consequence, these integral transforms have found numerous applications in diverse fields of science and engineering, including three-dimensional computer graphics, colour image processing, speech recognition, edge detection, data compression, texture classification, aerospace engineering and many more [17, 31, 39, 40].

Undoubtedly, the quaternion Fourier transforms (QFT) plays a significant role in the representation of quaternion-valued signals by transforming them into the quaternionic frequency domains, however, it is inadequate to provide local features of non-transient signals due to its global kernel [11, 12, 21]. To overcome this disadvantage, Bahri et al.[6] introduced the notion of quaternion windowed Fourier transform (QWFT) using the kernel of the right-sided QFT and have derived some Heisenberg-type uncertainty principles for the novel transform. Later on, Fu et al.[15] studied the Balian-Low theorem for the two-sided windowed quaternion Fourier transform, which asserts that the time-frequency concentration and non-redundancy are incompatible properties for quaternionic Gabor systems. Subsequently, the quaternionic Gabor frames were introduced and investigated in [8] by choosing some suitable versions of the translation and modulation operators. Besides, they studied some structural properties for the quaternionic Gabor frames including the Walnut-Janssen representation, Wexler-Rax biorthogonality and Ron-Shen duality using the machinery of operator theory and two-sided quaternion Fourier transforms. Very recently, Li and He [22] investigated some basic properties of the two-sided quaternion Gabor transforms, such as Parseval's formula, characterization of range and other boundedness results.

Although, the quaternion windowed Fourier transform has proved to be a valuable and powerful time-frequency analyzing tool in optics and signal processing, the rigidity of the quaternion window is not befitting for the non-transient signals. As such, many ramifications have been introduced to circumvent the limitations of the QWFT from time to time. For instance, Bahri et al.[4, 5] proposed a novel wavelet transform in the quaternion domain and derived the corresponding Heisenberg type uncertainty inequalities by means of the quaternion Fourier transforms. On the flipside, Ali and Thirulogasanthar [1] studied the continuous wavelet transforms for the quaternionic Hilbert spaces by invoking the unitary irreducible representations, whereas Hemmat et al.[19] provided a novel discretization scheme for the quaternionic wavelet transform, and derived a necessary and sufficient condition for the discrete quaternionic wavelet system to be a frame for $L^2(\mathbb{R}^2, \mathbb{H})$. Recently, Fashandi [13] generalized the results of [1] by defining a new quaternionic unitary representation from a LCAG to the unitary group of a quaternionic Hilbert space and establish the corresponding continuous wavelet transform.

Despite of the fact that quaternion wavelet transforms have rectified the limitations of both the quaternion Fourier and quaternion windowed Fourier transforms, however, they seem to be inadequate for representing those signals whose energy is not well concentrated in the frequency domain. The purpose of this paper is to address this issue by introducing a new time-frequency transform namely two-sided quaternion wave-packet transform (QWPT) which employs the generalized modulations, translations and localized quaternion window function for providing better time-frequency resolutions over high-frequency regions and capturing the geometric features of multi-dimensional signals in general.

The core objectives of the article are given as follows:

- To introduce a novel two-sided quaternion wave-packet transform by rectifying the limitations of quaternion windowed Fourier and wavelet transforms.
- To study the fundamental properties of two-sided quaternion wave-packet transform including the inner product relation, energy preserving relation, reconstruction formula and range theorem.
- To extend the scope of the study, we formulate Heisenberg-type uncertainty inequalities for the novel two-sided quaternion wave-packet transform.
- To demonstrate the validity of the proposed transform via illustrative examples.

The rest of the article is structured as follows: Section 2 is entirely devoted for an overview of the prerequisites including quaternion Fourier, quaternion windowed Fourier and quaternion wavelet transforms. In Section 3, we present the novel two-sided quaternion wave-packet transform and investigate its basic properties by virtue of two-sided quaternion Fourier transforms. In Section 4, we derive some Heisenberg's and logarithmic uncertainty principles for the proposed transform. Final Section is devoted to present some illustrative examples to demonstrate our study.

2. Preliminaries

In this section, we recall some basic definitions including the two-sided quaternion Fourier transform, quaternion windowed Fourier transform and the quaternion wavelet transform.

2.1. Basics of Quaternion Algebra

In 1843, W.R. Hamilton introduced the theory of quaternions while attempting to extend the complex numbers to 3-dimension [18]. As a consequence, the quaternion algebra provides an extension of the complex number system to an associative non-commutative four-dimensional algebra and is denoted by $\mathbb{H}$ in his honour. The quaternion algebra $\mathbb{H}$ over $\mathbb{R}$ is given by

$$\mathbb{H}=\Big\{f=a_0+i\,a_1+j\,a_2+k\,a_3\ :\ a_0,a_1,a_2,a_3\in\mathbb{R}\Big\}, \tag{2.1}$$

where i, j, k denote the three imaginary units, obeying the Hamilton's multiplication rules:

$$ij=k=-ji,\ jk=i=-kj,\ ki=j=-ik,\ \text{and}\ i^2=j^2=k^2=ijk=-1. \tag{2.2}$$

For quaternions $f_1=a_0+i\,a_1+j\,a_2+k\,a_3$ and $f_2=b_0+i\,b_1+j\,b_2+k\,b_3$, the addition is defined component-wise, whereas the multiplication is defined by

$$\begin{aligned} f_1f_2 &= (a_0b_0-a_1b_1-a_2b_2-a_3b_3)+i\,(a_1b_0+a_0b_1+a_2b_3-a_3b_2)\\ &\quad + j(a_0b_2+a_2b_0+a_3b_1-a_1b_3)+k\,(a_0b_3+a_3b_0+a_1b_2-a_2b_1). \end{aligned} \tag{2.3}$$

Moreover, the conjugate and norm of any quaternion $f=a_0+i\,a_1+j\,a_2+k\,a_3$, are given by $\overline{f}=a_0-i\,a_1-j\,a_2-k\,a_3$ and $\|f\|_{\mathbb{H}}^2={a_0}^2+{a_1}^2+{a_2}^2+{a_3}^2$, respectively. We also note that an arbitrary quaternion-valued function f can be represented as $f=(a_0+i\,a_1)+j\,(a_2-i\,a_3)=f_1+j\,f_2$, where $f_1,f_2\in\mathbb{C}$. Subsequently, the inner product of two quaternion-valued functions $f=f_1+\ jf_2$, and $g=g_1+j\,g_2$ in $\mathbb{H}$ can be defined as

$$\big\langle f,\,g\big\rangle_{\mathbb{H}}=f\overline{g}=\big(f_1\overline{g}_1+\overline{f}_2g_2\big)+j\big(f_2\overline{g}_1-\overline{f}_1g_2\big). \tag{2.4}$$

Denote $L^2(\mathbb{R}^2,\mathbb{H})$ as the space of all quaternion-valued functions f satisfying

$$\big\|f\big\|_{L^2(\mathbb{R}^2)}=\left\{\int_{\mathbb{R}^2}\Big(|f_1(\mathbf{x})|^2+|f_2(\mathbf{x})|^2\Big)\,d\mathbf{x}\right\}^{1/2}<\infty. \tag{2.5}$$

Consequently, the norm on $L^2(\mathbb{R}^2,\mathbb{H})$ is obtained via (2.4) as

$$\begin{aligned} \big\langle f,\,g\big\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} &= \int_{\mathbb{R}^2}\big\langle f,\,g\big\rangle_{\mathbb{H}}\,d\mathbf{x}\\ &= \int_{\mathbb{R}^2}\Big(\Big(f_1(\mathbf{x})\,\overline{g_1}(\mathbf{x})+\overline{f_2}(\mathbf{x})\,g_2(\mathbf{x})\Big)+j\Big(f_2(\mathbf{x})\,\overline{g_1}(\mathbf{x})-\overline{f_1}(\mathbf{x})\,g_2(\mathbf{x})\Big)\Big)d\mathbf{x}. \end{aligned} \tag{2.6}$$

Therefore, the quaternion version of Cauchy-Schwartz's inequality becomes

$$\left|\int_{\mathbb{R}^2}f(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x}\right|_{\mathbb{H}}\le\left\{\int_{\mathbb{R}^2}\big|f(\mathbf{x})\big|_{\mathbb{H}}^2\,d\mathbf{x}\right\}^{1/2}\left\{\int_{\mathbb{R}^2}\big|g(\mathbf{x})\big|_{\mathbb{H}}^2\,d\mathbf{x}\right\}^{1/2},\forall\,f,g\in L^2(\mathbb{R}^2,\mathbb{H}). \tag{2.7}$$

2.2. Time-Frequency Analysis in Quaternion Algebra

Due to the non-commutativity of the elements in the field of quaternions $\mathbb{H}$, different types of quaternion Fourier transforms have been introduced and investigated in recent years, including the right-sided, left-sided and two-sided quaternion Fourier transform [20]. However, throughout this article, we shall be focussed only on the two-sided quaternion Fourier transform.

Definition 2.1. For any quaternion-valued function $f \in L^2(\mathbb{R}^2, \mathbb{H})$, the two-sided quaternion Fourier transform (QFT) is denoted by $\mathscr{F}_q$ and is given by

$$\mathscr{F}_q\Big[f(\mathbf{x})\Big](\mathbf{w}) = \hat{f}(\mathbf{w}) = \int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1} f(\mathbf{x})\, e^{-2\pi j x_2 w_2}\, d\mathbf{x}, \tag{2.8}$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{w} = (w_1, w_2)$ and $e^{-2\pi x_1 w_1}$ and $e^{-2\pi x_2 w_2}$ are the quaternion Fourier kernels. The corresponding inversion formula is given by

$$f(\mathbf{x}) = \int_{\mathbb{R}^2} e^{2\pi i x_1 w_1} f(\mathbf{x})\, e^{2\pi j x_2 w_2}\, d\mathbf{w}, \tag{2.9}$$

whereas the Parseval formula for the two-sided quaternionic Fourier transform read as

$$\Big\langle \mathscr{F}_q[f],\, \mathscr{F}_q[g] \Big\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \Big\langle f,\, g \Big\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{2.10}$$

For $f = g$, relation (2.10) reduces to

$$\Big\| \mathscr{F}_q\Big[f(\mathbf{x})\Big](\mathbf{w}) \Big\|_{L^2(\mathbb{R}^2,\mathbb{H})} = \Big\| f \Big\|_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{2.11}$$

We now recall the two-sided quaternion windowed Fourier and wavelet transforms.

Definition 2.2 [6]. For any quaternion-valued function $f \in L^2(\mathbb{R}^2, \mathbb{H})$, the two-sided quaternion windowed Fourier transform of f is denoted by $\mathscr{G}_\psi^{\mathbb{H}}[f]$ and is given by

$$\mathscr{G}_\psi^{\mathbb{H}}\Big[f(\mathbf{x})\Big](\mathbf{w}, \mathbf{b}) = \int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1}\, f(\mathbf{x})\, \overline{\psi(\mathbf{x} - \mathbf{b})}\, e^{-2\pi j x_2 w_2}\, d\mathbf{x}, \tag{2.12}$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{w} = (w_1, w_2)$, $\mathbf{b} \in \mathbb{R}^2$, and $\psi \in L^2(\mathbb{R}^2, \mathbb{H})$ is the window function.

Definition 2.3 [5]. The continuous quaternion wavelet transform of any quaternion-valued function $f \in L^2\Big(\mathbb{R}^2,\, \mathbb{H}\Big)$ with respect to the analyzing function $\psi \in L^2\Big(\mathbb{R}^2,\, \mathbb{H}\Big)$, is defined by

$$\mathscr{W}_\psi^{\mathbb{H}}[f](a, \mathbf{b}, \theta) = \frac{1}{a} \int_{\mathbb{R}^2} f(\mathbf{x})\, \overline{\psi\left(\frac{R_{-\theta}(\mathbf{x} - \mathbf{b})}{a}\right)}\, d\mathbf{x}, \quad a \in \mathbb{R}^+, \mathbf{b} \in \mathbb{R}^2 \tag{2.13}$$

where $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2)$, is the special orthogonal group of rotations in $\mathbb{R}^2$.

3. Two-sided Quaternion Wave-packet Transform

In this section, we shall formally introduce a novel two-sided quaternion wave-packet transform which combines advantages of the well-known quaternion windowed Fourier and wavelet transforms. Subsequently, we shall investigate the fundamental properties including orthogonality relation, inversion formula and the range theorem.

Definition 3.1. The two-sided quaternion wave-packet transform of a quaternion-valued function $f \in L^2\left(\mathbb{R}^2, \mathbb{H}\right)$ is denoted by $\mathscr{P}_\psi^{\mathbb{H}}$ and is defined by

$$\mathscr{P}_\psi^{\mathbb{H}}[f]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big) = \frac{1}{a}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1}\, f(\mathbf{x})\, \overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)} e^{-2\pi j x_2 w_2}\, d\mathbf{x}, \tag{3.1}$$

where $a \in \mathbb{R}^+, \mathbf{b} \in \mathbb{R}^2, R_\theta \in SO(2), \mathbf{x} = (x_1, x_2), \mathbf{w} = (w_1, w_2)$, and $\psi \in L^2(\mathbb{R}^2, \mathbb{H})$.

Definition 3.1 allows us to make the following comments:

- The exponential terms appearing in the integrand of (3.1) cannot be interchanged due to the non-commutativity of quaternions.
- The left-sided and right-sided quaternion wave-packet transforms can similarly be formulated by placing the product $e^{-2\pi i x_1 w_1} e^{-2\pi j x_2 w_2}$ either on left side or right side of $f(\mathbf{x})\,\overline{\psi\left(R_{-\theta}(\mathbf{x}-\mathbf{b})/a\right)}$.
- For $a = 1$ and $R_{-\theta} = I$, Definition 3.1 boils down to the two-sided quaternion windowed Fourier transform as defined in (2.12).
- For $\mathbf{w} = (w_1, w_2) = (0, 0)$, Definition 3.1 reduces to the ordinary quaternion wavelet transform given by (2.13).

Next, we shall investigate the basic properties of the two-sided quaternion wave-packet transform (3.1) by means of the two-sided quaternion Fourier transforms.

Property-1 (Linearity). *Let $\mathscr{P}_\psi^{\mathbb{H}}[f_1]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big)$ and $\mathscr{P}_\psi^{\mathbb{H}}[f_2]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big)$ be the two-sided quaternion wave-packet transforms of the quaternion-valued functions f_1 and f_2, respectively. Then, for $\alpha_1, \alpha_2 \in \mathbb{R}$, we have*

$$\mathscr{P}_\psi^{\mathbb{H}}\Big[\alpha_1 f_1 + \alpha_2 f_2\Big]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big) = \alpha_1\, \mathscr{P}_\psi^{\mathbb{H}}\Big[f_1\Big]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big) + \alpha_2\, \mathscr{P}_\psi^{\mathbb{H}}\Big[f_2\Big]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big). \tag{3.2}$$

Proof. For the sake of brevity, we omit the proof of Property 1.

Property 2 (Time-shift). *Let ψ be a quaternion window function and $f \in L^2(\mathbb{R}^2, \mathbb{H})$. Then, we have*

$$\mathscr{P}_\psi^{\mathbb{H}}\Big[f(\mathbf{x}-\mathbf{k})\Big]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big) = e^{-2\pi i k_1 w_1}\, \mathscr{P}_\psi^{\mathbb{H}}\Big[f(\mathbf{x})\Big]\Big(a, \mathbf{b}-\mathbf{k}, \theta, \mathbf{w}\Big) e^{-2\pi j k_2 w_2}, \tag{3.3}$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{b} = (b_1, b_2)$, $\mathbf{w} = (w_1, w_2)$ *and* $\mathbf{k} = (k_1, k_2)$.

Proof. Using the Definition 3.1, we obtain

$$\begin{aligned}
\mathscr{P}_\psi^{\mathbb{H}}\Big[f(\mathbf{x}-\mathbf{k})\Big]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big) &= \frac{1}{a}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1}\, f(\mathbf{x}-\mathbf{k})\, \overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)} e^{-2\pi j x_2 w_2}\, d\mathbf{x}\\
&= \frac{1}{a}\int_{\mathbb{R}^2} e^{-2\pi i (z_1+k_1) w_1}\, f(\mathbf{z})\, \overline{\psi\left(\frac{R_{-\theta}(\mathbf{z}+\mathbf{k}-\mathbf{b})}{a}\right)} e^{-2\pi j (z_2+k_2) w_2}\, d\mathbf{z}\\
&= \frac{1}{a}\, e^{-2\pi i k_1 w_1}\int_{\mathbb{R}^2} e^{-2\pi i z_1 w_1}\, f(\mathbf{z})\, \overline{\psi\left(\frac{R_{-\theta}\Big(\mathbf{z}-(\mathbf{b}-\mathbf{k})\Big)}{a}\right)} e^{-2\pi j z_2 w_2}\, d\mathbf{z}\; e^{-2\pi j k_2 w_2}\\
&= e^{-2\pi i k_1 w_1}\, \mathscr{P}_\psi^{\mathbb{H}}\Big[f(\mathbf{x})\Big]\Big(a, \mathbf{b}-\mathbf{k}, \theta, \mathbf{w}\Big) e^{-2\pi j k_2 w_2}.
\end{aligned}$$

Property 3 (Scaling). *Let $\mathscr{P}_\psi^{\mathbb{H}}[f]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big)$ be the two-sided quaternion wave-packet transforms of any quaternion-valued function f. Then, for any $\lambda \in \mathbb{R}$, we have*

$$\mathscr{P}_\psi^{\mathbb{H}}\Big[f(\lambda\mathbf{x})\Big]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big) = \frac{1}{\lambda}\, \mathscr{P}_\psi^{\mathbb{H}}\, [f(\mathbf{x})]\left(\lambda a, \lambda\mathbf{b}, \theta, \frac{\mathbf{w}}{\lambda}\right). \tag{3.4}$$

Proof. Using (3.1), we have

$$\begin{aligned}\mathscr{P}_{\psi}^{\mathbb{H}}\Big[f(\lambda\mathbf{x})\Big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big) &= \frac{1}{a}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1}\, f(\lambda\mathbf{x})\,\overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)}\, e^{-2\pi j x_2 w_2}\, d\mathbf{x}\\ &= \frac{1}{a\lambda^2}\int_{\mathbb{R}^2} e^{-2\pi i z_1 \frac{w_1}{\lambda}}\, f(\mathbf{z})\,\overline{\psi\left(\frac{R_{-\theta}(\mathbf{z}-\lambda\mathbf{b})}{a\lambda}\right)}\, e^{-2\pi j x_2 \frac{w_2}{\lambda}}\, d\mathbf{z}\\ &= \frac{1}{\lambda}\,\mathscr{P}_{\psi}^{\mathbb{H}}\Big[f(\mathbf{x})\Big]\left(\lambda a,\lambda\mathbf{b},\theta,\frac{\mathbf{w}}{\lambda}\right).\end{aligned}$$

Property 4 (Parity). *Let $\psi \in L^2(\mathbb{R}^2,\mathbb{H})$ be a quaternion analyzing function. Then, we have*

$$\mathscr{P}_{P\psi}^{\mathbb{H}}\Big[Pf(\mathbf{x})\Big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big) = \mathscr{P}_{\psi}^{\mathbb{H}}\Big[f(\mathbf{x})\Big]\Big(a,-\mathbf{b},\theta,-\mathbf{w}\Big), \text{ where } Pf(\mathbf{x}) = f(-\mathbf{x}). \tag{3.5}$$

Proof. A direct calculation gives for every $f \in L^2(\mathbb{R}^2,\mathbb{H})$

$$\begin{aligned}\mathscr{P}_{P\psi}^{\mathbb{H}}\Big[Pf(\mathbf{x})\Big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big) &= \frac{1}{a}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1}\, Pf(\mathbf{x})\,\overline{P\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)}\, e^{-2\pi j x_2 w_2}\, d\mathbf{x}\\ &= \frac{1}{a}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1}\, f(-\mathbf{x})\,\overline{\psi\left(\frac{-R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)}\, e^{-2\pi j x_2 w_2}\, d\mathbf{x}\\ &= \frac{1}{a}\int_{\mathbb{R}^2} e^{2\pi i z_1 w_1}\, f(\mathbf{z})\,\overline{\psi\left(\frac{R_{-\theta}(\mathbf{z}+\mathbf{b})}{a}\right)}\, e^{2\pi j z_2 w_2}\, d\mathbf{z}\\ &= \frac{1}{a}\int_{\mathbb{R}^2} e^{-2\pi i z_1(-w_1)}\, f(\mathbf{z})\,\overline{\psi\left(\frac{R_{-\theta}\Big(\mathbf{z}-(-\mathbf{b})\Big)}{a}\right)}\, e^{-2\pi j z_2(-w_2)}\, d\mathbf{z}\\ &= \mathscr{P}_{\psi}^{\mathbb{H}}\Big[f(\mathbf{x})\Big]\Big(a,-\mathbf{b},\theta,-\mathbf{w}\Big).\end{aligned}$$

Property 5 (Anti-linearity). *For any quaternion-valued function f and $\psi_1,\psi_2 \in L^2(\mathbb{R}^2,\mathbb{H})$, we have*

$$\mathscr{P}_{\beta_1\psi_1+\beta_2\psi_2}^{\mathbb{H}}\Big[f(\mathbf{x})\Big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big) = \mathscr{P}_{\psi_1}^{\mathbb{H}}\Big[f(\mathbf{x})\Big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big)\cdot\overline{\beta}_1 + \mathscr{P}_{\psi_2}^{\mathbb{H}}\Big[f(\mathbf{x})\Big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big)\cdot\overline{\beta}_2,$$

where $\beta_s = c_s + jc'_s, c_s, c'_s \in \mathbb{R}$, $s = 1,2$.

Proof. This property follows in similar lines as that of Property 1.

Property 6 (Translation in ψ). *For a quaternion-valued function $f \in L^2(\mathbb{R}^2,\mathbb{H})$, and analyzing function $\psi \in L^2(\mathbb{R}^2,\mathbb{H})$, we have*

$$\mathscr{P}_{T_{\mathbf{k}}\psi}^{\mathbb{H}}\Big[f(\mathbf{x})\Big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big) = \mathscr{P}_{\psi}^{\mathbb{H}}\Big[f(\mathbf{x})\Big]\Big(a,\mathbf{b}+aR_{\theta}\mathbf{k},\theta,\mathbf{w}\Big), \text{ where } T_{\mathbf{k}}\psi(\mathbf{x}) = \psi(\mathbf{x}-\mathbf{k}),\ \mathbf{k} = (k_1,k_2). \tag{3.6}$$

Proof. The property follows immediately from the Definition 3.1 as

$$\begin{aligned}\mathscr{P}_{T_{\mathbf{k}}\psi}^{\mathbb{H}}\Big[f(\mathbf{x})\Big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big) &= \frac{1}{a}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1}\, f(\mathbf{x})\,\overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}-\mathbf{k}\right)}\, e^{-2\pi j x_2 w_2}\, d\mathbf{x}\\ &= \frac{1}{a}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1}\, f(\mathbf{x})\,\overline{\psi\left(\frac{R_{-\theta}\Big(\mathbf{z}-\Big(\mathbf{b}+aR_{\theta}\mathbf{k}\Big)\Big)}{a}\right)}\, e^{-2\pi j x_2 w_2} d\mathbf{x}\\ &= \mathscr{P}_{\psi}^{\mathbb{H}}\Big[f(\mathbf{x})\Big]\Big(a,\mathbf{b}+aR_{\theta}\mathbf{k},\theta,\mathbf{w}\Big).\end{aligned}$$

Property 7 (Dilation in ψ). *For $f, \psi \in L^2(\mathbb{R}^2, \mathbb{H})$, we have*

$$\mathscr{P}^{\mathbb{H}}_{D_c\psi}\Big[f(\mathbf{x})\Big]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big) = \mathscr{P}^{\mathbb{H}}_{\psi}\Big[f(\mathbf{x})\Big]\Big(ac, \mathbf{b}, \theta, \mathbf{w}\Big), \text{ where } D_c\psi(\mathbf{x}) = \frac{1}{c}\psi\left(\frac{\mathbf{x}}{c}\right). \tag{3.7}$$

Proof. We have

$$\begin{aligned}\mathscr{P}^{\mathbb{H}}_{D_c\psi}\Big[f(\mathbf{x})\Big]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big) &= \frac{1}{a}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1} f(\mathbf{x})\, \overline{\frac{1}{c}\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{ac}\right)} e^{-2\pi j x_2 w_2}\, d\mathbf{x}\\ &= \frac{1}{ac}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1} f(\mathbf{x})\, \overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{ac}\right)} e^{-2\pi j x_2 w_2}\, d\mathbf{x}\\ &= \mathscr{P}^{\mathbb{H}}_{\psi}\Big[f(\mathbf{x})\Big]\Big(ac, \mathbf{b}, \theta, \mathbf{w}\Big).\end{aligned}$$

We now formulate the inner product relation for the two-sided quaternion wave-packet transform by applying the cyclic multiplication symmetry, which resists the formula to scalar part only. As a consequence of this formula, we can deduce the energy preserving relation for the proposed transform (3.1). To facilitate the intent, we shall first define the admissibility condition of any quaternion-valued function.

Definition 3.2 (Admissibility). A quaternion-valued function $\psi \in L^2\left(\mathbb{R}^2, \mathbb{H}\right)$ is said to be admissible if

$$C_\psi = \int_{\mathbb{R}^+}\int_{\mathrm{SO}(2)}\int_{\mathbb{R}^2}\Big|\mathscr{F}_q\Big[\psi\Big]\Big(R_{-\theta}a\mathbf{w}\Big)\Big|^2 \frac{da\,d\theta\,d\mathbf{w}}{a} < \infty, \text{ real-valued positive constant.} \tag{3.8}$$

Theorem 3.3 (Inner Product Relation). *Let $\mathscr{P}^{\mathbb{H}}_{\psi}[f]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big)$ and $\mathscr{P}^{\mathbb{H}}_{\psi}[g]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big)$ be the two-sided quaternion wave-packet transforms of f and g, respectively. Then, we have*

$$\Big\langle \mathscr{P}^{\mathbb{H}}_{\psi}\Big[f\Big], \mathscr{P}^{\mathbb{H}}_{\psi}\Big[g\Big]\Big\rangle_{L^2(\mathscr{G},\mathbb{H})} = C_\psi\Big\langle f, g\Big\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}, \tag{3.9}$$

where C_ψ is given by (3.2) and $\mathscr{G} = \mathbb{R}^+ \times \mathbb{R}^2 \times SO(2) \times \mathbb{R}^2$ is the similitude group constituted by the dilation, translation, rotation and modulation operators with left Haar measure $d\eta = da\, d\mathbf{b} d\theta\, d\mathbf{w}/a^3$.

Proof. By virtue of Definition 3.1 and the Fubini's theorem, we have

$$\begin{aligned}&\Big\langle \mathscr{P}^{\mathbb{H}}_{\psi}\Big[f\Big], \mathscr{P}^{\mathbb{H}}_{\psi}\Big[g\Big]\Big\rangle_{L^2(\mathscr{G},\mathbb{H})}\\ &\quad= \int_{\mathscr{G}} \mathscr{P}^{\mathbb{H}}_{\psi}\Big[f\Big]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big)\, \overline{\mathscr{P}^{\mathbb{H}}_{\psi}\Big[g\Big]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big)}\, d\eta\\ &\quad= \int_{\mathscr{G}}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1} f(\mathbf{x})\overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)} e^{-2\pi j x_2 w_2}\, d\mathbf{x}\, \overline{\int_{\mathbb{R}^2} e^{-2\pi i z_1 w_1} g(\mathbf{z})\overline{\psi\left(\frac{R_{-\theta}(\mathbf{z}-\mathbf{b})}{a}\right)} e^{-2\pi j z_2 w_2}\, d\mathbf{z}}\, \frac{d\eta}{a^2}\\ &\quad= \int_{\mathscr{G}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1} f(\mathbf{x})\overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)} e^{-2\pi j x_2 w_2}\, e^{2\pi j z_2 w_2}\psi\left(\frac{R_{-\theta}(\mathbf{z}-\mathbf{b})}{a}\right)\overline{g(\mathbf{z})}\,\overline{e^{-2\pi i z_1 w_1}}\, d\mathbf{x}\, d\mathbf{z}\, \frac{d\eta}{a^2}\\ &\quad= \int_{\mathbb{R}^+\times\mathrm{SO}(2)\times\mathbb{R}^2\times\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\mathbf{x})\overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)} e^{-2\pi j x_2 w_2}\, e^{2\pi j z_2 w_2}\, \psi\left(\frac{R_{-\theta}(\mathbf{z}-\mathbf{b})}{a}\right)\overline{g(\mathbf{z})}\\ &\qquad\qquad\times e^{2\pi i z_1 w_1}\, e^{-2\pi i x_1 w_1}\, d\mathbf{x}\, d\mathbf{z}\, \frac{da\, d\mathbf{b}\, d\theta\, d\mathbf{w}}{a^5}\\ &\quad= \int_{\mathbb{R}^+\times\mathrm{SO}(2)\times\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\mathbf{x})\overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)}\int_{\mathbb{R}} e^{2\pi j(z_2-x_2)w_2}\, dw_2\, \psi\left(\frac{R_{-\theta}(\mathbf{z}-\mathbf{b})}{a}\right)\overline{g(\mathbf{z})}\\ &\qquad\qquad\times \int_{\mathbb{R}} e^{2\pi i(z_1-x_1)w_1}\, dw_1\, d\mathbf{x}\, d\mathbf{z}\, \frac{da\, d\mathbf{b}\, d\theta}{a^5}\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^+\times SO(2)\times\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\mathbf{x})\overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)}\delta(z_2-x_2)\,\psi\left(\frac{R_{-\theta}(\mathbf{z}-\mathbf{b})}{a}\right)\overline{g(\mathbf{z})}\,\delta(z_1-x_1)\,d\mathbf{x}\,d\mathbf{z}\,\frac{da\,d\mathbf{b}\,d\theta}{a^5}\\
&= \int_{\mathbb{R}^+\times SO(2)\times\mathbb{R}^2}\int_{\mathbb{R}^2} f(\mathbf{x})\overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)}\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)\overline{g(\mathbf{x})}\,d\mathbf{x}\,\frac{da\,d\mathbf{b}\,d\theta}{a^5}\\
&= \int_{\mathbb{R}^2} f(\mathbf{x})\int_{\mathbb{R}^+\times SO(2)\times\mathbb{R}^2}\overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)}\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)\frac{da\,d\mathbf{b}\,d\theta}{a^5}\,\overline{g(\mathbf{x})}\,d\mathbf{x}\\
&= \int_{\mathbb{R}^2} f(\mathbf{x})\int_{\mathbb{R}^+\times SO(2)}\int_{\mathbb{R}^2}\overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)}\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)d\mathbf{b}\,\frac{da\,d\theta}{a^5}\,\overline{g(\mathbf{x})}\,d\mathbf{x}\\
&= \int_{\mathbb{R}^2} f(\mathbf{x})\int_{\mathbb{R}^+\times SO(2)}\int_{\mathbb{R}^2}\overline{\psi\left(\frac{R_{-\theta}\mathbf{b}'}{a}\right)}\psi\left(\frac{R_{-\theta}\mathbf{b}'}{a}\right)d\mathbf{b}'\,\frac{da\,d\theta}{a^5}\,\overline{g(\mathbf{x})}\,d\mathbf{x}\\
&= \int_{\mathbb{R}^2} f(\mathbf{x})\int_{\mathbb{R}^+\times SO(2)}\overline{\left\langle\psi\left(\frac{R_{-\theta}\mathbf{b}'}{a}\right),\psi\left(\frac{R_{-\theta}\mathbf{b}'}{a}\right)\right\rangle}_{L^2(\mathbb{R}^2,\mathbb{H})}\frac{da\,d\theta}{a^5}\,\overline{g(\mathbf{x})}\,d\mathbf{x}\\
&= \int_{\mathbb{R}^2} f(\mathbf{x})\int_{\mathbb{R}^+\times SO(2)}\overline{\left\langle\mathscr{F}_q\left[\psi\left(\frac{R_{-\theta}\mathbf{b}'}{a}\right)\right],\mathscr{F}_q\left[\psi\left(\frac{R_{-\theta}\mathbf{b}'}{a}\right)\right]\right\rangle}_{L^2(\mathbb{R}^2,\mathbb{H})}\frac{da\,d\theta}{a^5}\,\overline{g(\mathbf{x})}\,d\mathbf{x}\\
&= \int_{\mathbb{R}^2} f(\mathbf{x})\int_{\mathbb{R}^+\times SO(2)}\int_{\mathbb{R}^2}\overline{\mathscr{F}_q\big[\psi\big]\left(\frac{R_{-\theta}\mathbf{w}}{a}\right)}\mathscr{F}_q\big[\psi\big]\left(\frac{R_{-\theta}\mathbf{w}}{a}\right)d\mathbf{w}\,\frac{da\,d\theta}{a^5}\,\overline{g(\mathbf{x})}\,d\mathbf{x}\\
&= \int_{\mathbb{R}^2} f(\mathbf{x})\int_{\mathbb{R}^+\times SO(2)}\int_{\mathbb{R}^2}\overline{\mathscr{F}_q\big[\psi\big]\big(R_{-\theta}a\mathbf{w}'\big)}\mathscr{F}_q\big[\psi\big]\big(R_{-\theta}a\mathbf{w}'\big)a^4\,d\mathbf{w}'\,\frac{da\,d\theta}{a^5}\,\overline{g(\mathbf{x})}\,d\mathbf{x}\\
&= \int_{\mathbb{R}^2} f(\mathbf{x})\left[\int_{\mathbb{R}^+\times SO(2)}\int_{\mathbb{R}^2}\overline{\mathscr{F}_q\big[\psi\big]\big(R_{-\theta}a\mathbf{w}'\big)}\mathscr{F}_q\big[\psi\big]\big(R_{-\theta}a\mathbf{w}'\big)d\mathbf{w}'\,\frac{da\,d\theta}{a}\right]\overline{g(\mathbf{x})}\,d\mathbf{x}\\
&= \int_{\mathbb{R}^2} f(\mathbf{x})\,C_\psi\,\overline{g(\mathbf{x})}\,d\mathbf{x}\\
&= C_\psi\left\langle f,g\right\rangle_{L^2(\mathbb{R}^2,\mathbb{H})},
\end{aligned}
$$

where C_ψ is given in (3.8). This completes the proof of Theorem 3.3. □

Remarks: (i). For $f = g$, Theorem 3.3 yields the energy preserving relation

$$\int_{\mathbb{R}^+\times\mathbb{R}^2\times SO(2)\times\mathbb{R}^2}\left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2\frac{da\,d\mathbf{b}\,d\theta d\mathbf{w}}{a^3} = C_\psi\left\|f\right\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2. \tag{3.10}$$

(ii). The operator $\mathscr{P}_\psi^{\mathbb{H}}$ is bounded and for $C_\psi = 1$, it becomes an isometry from $L^2(\mathbb{R}^2,\mathbb{H})$ to the space of transformations $L^2(\mathbb{R}^+\times\mathbb{R}^2\times SO(2)\times\mathbb{R}^2,\mathbb{H})$.

The next theorem guarantees the reconstruction of the input quaternion-valued signal f from the corresponding two-sided quaternion wave-packet transform.

Theorem 3.4 (Reconstruction Formula). *If $\psi\in L^2(\mathbb{R}^2,\mathbb{H})$ is admissible and $\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a,\mathbf{b},\theta,\mathbf{w}\big)$ is the two-sided quaternion wave-packet transform of an arbitrary function $f\in L^2(\mathbb{R}^2,\mathbb{H})$, then f can be reconstructed via*

$$f(\mathbf{x}) = \frac{1}{C_\psi}\int_{\mathscr{G}} e^{2\pi i x_1 w_1}\,\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)e^{2\pi j x_2 w_2}\,\psi\left(\frac{R_{-\theta}\big(\mathbf{x}-\mathbf{b}\big)}{a}\right)\frac{da\,d\mathbf{b}\,d\theta\,d\mathbf{w}}{a^4},\quad a.e. \tag{3.11}$$

Proof. According to Theorem 3.3, we can write

$$\begin{aligned} C_\psi\Big\langle f,\, g\Big\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} &= \Big\langle \mathscr{P}_\psi^{\mathbb{H}}\big[f\big],\, \mathscr{P}_\psi^{\mathbb{H}}\big[g\big]\Big\rangle_{L^2(\mathscr{G},\mathbb{H})} \\ &= \int_{\mathscr{G}} \mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big)\, \overline{\mathscr{P}_\psi^{\mathbb{H}}\big[g\big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big)}\, d\eta \\ &= \int_{\mathscr{G}} \mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big)\, \overline{\frac{1}{a}\int_{\mathbb{R}^2} e^{-2\pi i z_1 w_1}\, g(\mathbf{z})\overline{\psi\left(\frac{R_{-\theta}(\mathbf{z}-\mathbf{b})}{a}\right)} e^{-2\pi j z_2 w_2}\, d\mathbf{z}}\, d\eta \\ &= \int_{\mathbb{R}^2}\int_{\mathscr{G}} \mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big)\, e^{2\pi j x_2 w_2}\, \psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right) \overline{g(\mathbf{x})}\, e^{2\pi i x_1 w_1}\, \frac{d\eta}{a}\, d\mathbf{x} \\ &= \int_{\mathbb{R}^2}\int_{\mathscr{G}} e^{2\pi i x_1 w_1}\, \mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big)\, e^{2\pi j x_2 w_2}\, \psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right) \overline{g(\mathbf{x})}\, \frac{d\eta}{a}\, d\mathbf{x} \\ &= \int_{\mathbb{R}^2}\int_{\mathscr{G}} e^{2\pi i x_1 w_1}\, \mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big)\, e^{2\pi j x_2 w_2}\, \psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right) \frac{d\eta}{a}\cdot \overline{g(\mathbf{x})}\, d\mathbf{x} \\ &= \left\langle \int_{\mathscr{G}} e^{2\pi i x_1 w_1}\, \mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big)\, e^{2\pi j x_2 w_2}\, \psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right) \frac{d\eta}{a},\, g\right\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}. \end{aligned}$$

Since g is chosen arbitrarily from $L^2(\mathbb{R}^2,\mathbb{H})$, therefore, we obtain the desired result:

$$f(\mathbf{x}) = \frac{1}{C_\psi}\int_{\mathscr{G}} e^{2\pi i x_1 w_1}\, \mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big)\, e^{2\pi j x_2 w_2}\psi\left(\frac{R_{-\theta}\big(\mathbf{x}-\mathbf{b}\big)}{a}\right)\frac{da\, d\mathbf{b}\, d\theta\, d\mathbf{w}}{a^4}, \quad a.e.$$

This completes the proof of theorem. □

Theorem 3.5 (Reproducing Kernel Hilbert Space). *For a normalized admissible function $\psi \in L^2(\mathbb{R}^2,\mathbb{H})$, the range of the two-sided quaternion wave-packet transform (3.1) is a reproducing kernel Hilbert space in $L^2\left(\mathbb{R}^+\times\mathbb{R}^2\times SO(2)\times\mathbb{R}^2,\mathbb{H}\right)$ with kernel given by*

$$K_\psi\Big(a,\mathbf{b},\theta,\mathbf{w};a',\mathbf{b}',\theta',\mathbf{w}'\Big) = \frac{1}{aa'C_\psi}\left\langle e^{2\pi j x_2 w_2}\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right), e^{-2\pi i x_1(w_1-w_1')}e^{2\pi j x_2 w_2'}\psi\left(\frac{R_{-\theta'}\big(\mathbf{x}-\mathbf{b}'\big)}{a'}\right)\right\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{3.12}$$

Moreover, we have

$$\Big|K_\psi\Big(a,\mathbf{b},\theta,\mathbf{w};a',\mathbf{b}',\theta',\mathbf{w}'\Big)\Big| \le C_\psi^{-1}\,\big\|\psi\big\|_{L^2(\mathbb{R}^2,\mathbb{H})}, \quad \text{whenever } C_\psi > 0. \tag{3.13}$$

Proof. By invoking Definition 3.1 and the reconstruction formula (3.11), we obtain

$$\begin{aligned} &\mathscr{P}_\psi^{\mathbb{H}}[f]\Big(a',\mathbf{b}',\theta',\mathbf{w}'\Big) \\ &= \frac{1}{a'}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1'} f(\mathbf{x})\,\overline{\psi\left(\frac{R_{-\theta'}\big(\mathbf{x}-\mathbf{b}'\big)}{a'}\right)} e^{-2\pi j x_2 w_2'} d\mathbf{x} \\ &= \frac{1}{a'}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1'}\frac{1}{C_\psi}\int_{\mathscr{G}} e^{2\pi i x_1 w_1}\, \mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big)\, e^{2\pi j x_2 w_2}\, \psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)\frac{d\eta}{a}\, \overline{\psi\left(\frac{R_{-\theta'}\big(\mathbf{x}-\mathbf{b}'\big)}{a'}\right)} e^{-2\pi j x_2 w_2'}\, d\mathbf{x} \\ &= \frac{1}{a'C_\psi}\int_{\mathscr{G}}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1'}\, e^{2\pi i x_1 w_1}\, \mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\Big(a,\mathbf{b},\theta,\mathbf{w}\Big)\, e^{2\pi j x_2 w_2}\, \psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)\, \overline{\psi\left(\frac{R_{-\theta'}\big(\mathbf{x}-\mathbf{b}'\big)}{a'}\right)} e^{-2\pi j x_2 w_2'}\, \frac{d\mathbf{x}\, d\eta}{a} \end{aligned}$$

$$= \frac{1}{a'C_\psi}\int_{\mathscr{G}}\int_{\mathbb{R}^2} e^{2\pi i x_1(w_1-w_1')}\,\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\,e^{2\pi j x_2 w_2}\,\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)\overline{\psi\left(\frac{R_{-\theta'}\big(\mathbf{x}-\mathbf{b}'\big)}{a'}\right)}e^{-2\pi j x_2 w_2'}\,\frac{d\mathbf{x}\,d\eta}{a}$$

$$= \frac{1}{a'C_\psi}\int_{\mathscr{G}}\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\int_{\mathbb{R}^2} e^{2\pi j x_2 w_2}\,\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)\overline{\psi\left(\frac{R_{-\theta'}\big(\mathbf{x}-\mathbf{b}'\big)}{a'}\right)}e^{-2\pi j x_2 w_2'}\,e^{2\pi i x_1(w_1-w_1')}\,\frac{d\mathbf{x}\,d\eta}{a}$$

$$= \frac{1}{a'C_\psi}\int_{\mathscr{G}}\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\left\langle e^{2\pi j x_2 w_2}\,\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right),\; e^{-2\pi i x_1(w_1-w_1')}\,e^{2\pi j x_2 w_2'}\,\psi\left(\frac{R_{-\theta'}\big(\mathbf{x}-\mathbf{b}'\big)}{a'}\right)\right\rangle_{L^2(\mathbb{R}^2,\,\mathbb{H})}\frac{d\eta}{a}$$

$$= \int_{\mathscr{G}}\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\frac{1}{aa'C_\psi}\left\langle e^{2\pi j x_2 w_2}\,\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right),\; e^{-2\pi i x_1(w_1-w_1')}\,e^{2\pi j x_2 w_2'}\,\psi\left(\frac{R_{-\theta'}\big(\mathbf{x}-\mathbf{b}'\big)}{a'}\right)\right\rangle_{L^2(\mathbb{R}^2,\,\mathbb{H})} d\eta$$

$$= \int_{\mathscr{G}}\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)K_\psi\big(a,\mathbf{b},\theta,\mathbf{w};a',\mathbf{b}',\theta',\mathbf{w}'\big)\,d\eta.$$

Or equivalently,

$$K_\psi\big(a,\mathbf{b},\theta,\mathbf{w};a',\mathbf{b}',\theta',\mathbf{w}'\big)$$
$$= \frac{1}{aa'C_\psi}\left\langle e^{2\pi j x_2 w_2}\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right), e^{-2\pi i x_1(w_1-w_1')}e^{2\pi j x_2 w_2'}\psi\left(\frac{R_{-\theta'}\big(\mathbf{x}-\mathbf{b}'\big)}{a'}\right)\right\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}.$$

This completes the proof of first assertion.

Furthermore, we have

$$\Big|K_\psi\big(a,\mathbf{b},\theta,\mathbf{w};a',\mathbf{b}',\theta',\mathbf{w}'\big)\Big|$$
$$= \left|\frac{1}{aa'C_\psi}\left\langle e^{j x_2 w_2}\,\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right),\; e^{-2\pi i x_1(w_1-w_1')}\,e^{2\pi j x_2 w_2'}\,\psi\left(\frac{R_{-\theta'}\big(\mathbf{x}-\mathbf{b}'\big)}{a'}\right)\right\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}\right|$$
$$\le \frac{1}{\big|aa'C_\psi\big|}\int_{\mathbb{R}^2}\left|\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)\right|\left|\overline{\psi\left(\frac{R_{-\theta'}\big(\mathbf{x}-\mathbf{b}'\big)}{a'}\right)}\right|d\mathbf{x}$$
$$\le \frac{1}{\big|aa'C_\psi\big|}\left\|\psi\left(R_{-\theta}\left(\frac{\mathbf{x}}{a}-\frac{\mathbf{b}}{a}\right)\right)\right\|_{L^2(\mathbb{R}^2,\,\mathbb{H})}\left\|\overline{\psi\left(R_{-\theta'}\left(\frac{\mathbf{x}}{a'}-\frac{\mathbf{b}'}{a'}\right)\right)}\right\|_{L^2(\mathbb{R}^2,\mathbb{H})}$$

$$\le \frac{1}{\big|C_\psi\big|}\left\|\psi\left(R_{-\theta}\left(\mathbf{z}-\frac{\mathbf{b}}{a}\right)\right)\right\|_{L^2(\mathbb{R}^2,\,\mathbb{H})}\left\|\psi\left(R_{-\theta'}\left(\mathbf{z}'-\frac{\mathbf{b}'}{a'}\right)\right)\right\|_{L^2(\mathbb{R}^2,\,\mathbb{H})}$$
$$\le \frac{1}{\big|C_\psi\big|}\big\|\psi\big\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}$$
$$= C_\psi^{-1}\big\|\psi\big\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}, \quad \text{provided } C_\psi > 0.$$

This completes the proof of Theorem 3.5. □

4. Uncertainty Principles for the Quaternion wave-packet transform

The celebrated Heisenberg's uncertainty principle in harmonic analysis states that "a function can not be sharply localized in both the time and frequency domains". This principle plays a significant role in the modern signal analysis as it provides a lower bound for the optimal resolution of a signal in both time and frequency domains [14]. Since its inception, many ramifications of the uncertainty principle have appeared in literature, which resulted in the expansion of uncertainty principle from the classical Fourier domain to the fractional Fourier, linear canonical, special affine Fourier domains [3, 7, 9, 23, 41]. Motivated and inspired by the contemporary developments in the theory of uncertainty principles, our aim is to establish some new versions of the Heisenberg and the logarithmic-type uncertainty inequalities for the two-sided quaternion wave-packet transform. The results are obtained by using the machinery of two-side quaternion Fourier transforms and some fundamental inequalities of functional analysis. To facilitate the narrative, we need the following lemma.

Lemma 4.1. *Let $\psi \in L^2(\mathbb{R}^2, \mathbb{H})$ be an admissible quaternion-valued function, then for every $f \in L^2(\mathbb{R}^2, \mathbb{H})$, we have*

$$\int_{\mathscr{G}} \left|\xi\right|^2 \left|\mathscr{F}_q\Big[\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\Big](\xi)\right|_{\mathbb{H}}^2 d\eta \ = C_\psi \left|\xi\right|^2 \left\|\mathscr{F}_q[f](\xi)\right\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2. \tag{4.1}$$

Proof. Combining the inner product relations of two-sided quaternion Fourier (2.10) and wavelet-packet transforms (3.9), we obtain

$$C_\psi\Big\langle \mathscr{F}_q[f],\ \mathscr{F}_q[g]\Big\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \int_{\mathbb{R}^+\times\mathbb{R}^2\times SO(2)\times\mathbb{R}^2} \mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\,\overline{\mathscr{P}_\psi^{\mathbb{H}}[g]\big(a,\mathbf{b},\theta,\mathbf{w}\big)}\,\frac{da\,d\mathbf{b}\,d\theta\,d\mathbf{w}}{a^3}.$$

Identifying $\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)$ as a function of the translation parameter $\mathbf{b}$ and using (2.10), we have

$$C_\psi\Big\langle \mathscr{F}_q[f],\ \mathscr{F}_q[g]\Big\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \int_{\mathbb{R}^+\times\mathbb{R}^2\times SO(2)} \int_{\mathbb{R}^2} \mathscr{F}_q\left[\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\right](\xi)\,\overline{\mathscr{F}_q\left[\mathscr{P}_\psi^{\mathbb{H}}[g]\right](\xi)}\,d\xi\,\frac{da\,d\theta\,d\mathbf{w}}{a^3}.$$

Multiplying the above expression on both sides by $|\xi|^2$, we get

$$C_\psi\Big\langle \xi\,\mathscr{F}_q[f],\ \xi\,\mathscr{F}_q[g]\Big\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \int_{\mathbb{R}^+\times\mathbb{R}^2\times SO(2)} \int_{\mathbb{R}^2} \xi\,\mathscr{F}_q\left[\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\right](\xi)\cdot\overline{\xi\,\mathscr{F}_q\left[\mathscr{P}_\psi^{\mathbb{H}}[g]\right](\xi)}\,d\xi\,\frac{da\,d\theta\,d\mathbf{w}}{a^3}.$$

Finally, for $f = g$, we get the desired identity

$$\int_{\mathscr{G}} \left|\xi\right|^2 \left|\mathscr{F}_q\Big[\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\Big](\xi)\right|_{\mathbb{H}}^2 d\eta \ = C_\psi \left|\xi\right|^2 \left\|\mathscr{F}_q[f](\xi)\right\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2.$$

This completes the proof of Lemma 4.1. □

We are now ready to derive the Heisenberg-type inequalities for the proposed two-sided quaternion wave-packet transform (3.1).

Theorem 4.2. *Let $\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a,\mathbf{b},\theta,\mathbf{w}\big)$ be the two-sided quaternion wave-packet transform of any quaternion-valued function $f \in L^2(\mathbb{R}^2, \mathbb{H})$. Then, we have*

$$\left\{\int_{\mathscr{G}} \left|\mathbf{b}\right|^2 \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\eta\right\}^{1/2} \left\{\int_{\mathbb{R}^2} \left|\xi\right|^2 \left|\mathscr{F}_q\big[f\big](\xi)\right|_{\mathbb{H}}^2 d\xi\right\}^{1/2} \geq \frac{\sqrt{C_\psi}}{2}\left\|f\right\|_{L^2(\mathbb{R}^2,\,\mathbb{H})}^2. \tag{4.2}$$

Proof. The Heisenberg's inequality in the quaternion Fourier domain [3] is given by

$$\left\{\int_{\mathbb{R}^2} \left|\mathbf{b}\right|^2 \left|f(\mathbf{b})\right|_{\mathbb{H}}^2 d\mathbf{b}\right\}^{1/2} \left\{\int_{\mathbb{R}^2} \left|\xi\right|^2 \left|\mathcal{F}_q\left[f\right](\xi)\right|_{\mathbb{H}}^2 d\xi\right\}^{1/2} \geq \left\{\frac{1}{2}\int_{\mathbb{R}^2} \left|f(\mathbf{b})\right|_{\mathbb{H}}^2 d\mathbf{b}\right\}. \tag{4.3}$$

Replacing the quaternion-valued function f in (4.4) with $\mathscr{P}_\psi^{\mathbb{H}}[f](\cdot, \mathbf{b}, \cdot, \cdot)$ yields

$$\left\{\int_{\mathbb{R}^2} \left|\mathbf{b}\right|^2 \left|\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\mathbf{b}\right\}^{1/2} \left\{\int_{\mathbb{R}^2} \left|\xi\right|^2 \left|\mathcal{F}_q\left[\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right](\xi)\right|_{\mathbb{H}}^2 d\xi\right\}^{1/2} \geq \left\{\frac{1}{2}\int_{\mathbb{R}^2} \left|\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\mathbf{b}\right\}. \tag{4.4}$$

After integrating the inequality (4.3) with respect to measure $da\, d\theta\, d\mathbf{w}/a^3$, we obtain

$$\int_{\mathbb{R}^+\times SO(2)\times\mathbb{R}^2} \left\{\left\{\int_{\mathbb{R}^2} \left|\mathbf{b}\right|^2 \left|\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\mathbf{b}\right\}^{1/2} \left\{\int_{\mathbb{R}^2} \left|\xi\right|^2 \left|\mathcal{F}_q\left[\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right](\xi)\right|_{\mathbb{H}}^2 d\xi\right\}^{1/2}\right\} \frac{da\, d\theta\, d\mathbf{w}}{a^3} \geq \left\{\frac{1}{2}\int_{\mathbb{R}^+\times SO(2)\times\mathbb{R}^2} \int_{\mathbb{R}^2} \left|\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\mathbf{b}\right\} \frac{da\, d\theta\, d\mathbf{w}}{a^3}. \tag{4.5}$$

Thus, as a consequence of the quaternion Cauchy-Schwartz inequality (2.7), we may write

$$\left\{\int_{\mathbb{R}^+\times SO(2)\times\mathbb{R}^2} \int_{\mathbb{R}^2} \left|\mathbf{b}\right|^2 \left|\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\mathbf{b} \frac{da\, d\theta\, d\mathbf{w}}{a^3}\right\}^{1/2} \times \left\{\int_{\mathbb{R}^+\times SO(2)\times\mathbb{R}^2} \int_{\mathbb{R}^2} \left|\xi\right|^2 \left|\mathcal{F}_q\left[\mathscr{P}_\psi^{\mathbb{H}}[f]\right](\xi)\right|_{\mathbb{H}}^2 d\xi \frac{da\, d\theta\, d\mathbf{w}}{a^3}\right\}^{1/2} \geq \frac{1}{2}\int_{\mathbb{R}^+\times SO(2)\times\mathbb{R}^2} \int_{\mathbb{R}^2} \left|\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\mathbf{b} \frac{da\, d\theta\, d\mathbf{w}}{a^3}.$$

Applying the Lemma 4.1, the above expression can be simplified as

$$\left\{\int_{\mathbb{R}^+\times SO(2)\times\mathbb{R}^2} \int_{\mathbb{R}^2} \left|\mathbf{b}\right|^2 \left|\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\mathbf{b} \frac{da\, d\theta\, d\mathbf{w}}{a^3}\right\}^{1/2} \left\{C_\psi \int_{\mathbb{R}^2} \left|\xi\right|^2 \left|\mathscr{F}_q[f](\xi)\right|_{\mathbb{H}}^2 d\xi\right\}^{1/2} \geq \frac{1}{2}\int_{\mathbb{R}^+\times SO(2)\times\mathbb{R}^2} \int_{\mathbb{R}^2} \left|\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\mathbf{b} \frac{da\, d\theta\, d\mathbf{w}}{a^3}. \tag{4.6}$$

Finally, employing the energy persevering relation (3.10) in the R.H.S of (4.6), we obtain the desired result as

$$\left\{\int_{\mathscr{G}} \left|\mathbf{b}\right|^2 \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\eta\right\}^{1/2} \left\{\int_{\mathbb{R}^2} \left|\xi\right|^2 \left|\mathscr{F}_q\big[f\big](\xi)\right|_{\mathbb{H}}^2 d\xi\right\}^{1/2} \geq \frac{\sqrt{C_\psi}}{2} \left\|f\right\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2.$$

Following the idea of Cowling and Price [10], we shall derive a generalized inequality of Theorem 4.2 for $L^p(\mathbb{R}^2, \mathbb{H}), p \geq 1$ in the following theorem.

Theorem 4.3 (Generalised HUP). *Let $\psi \in L^2(\mathbb{R}^2, \mathbb{H})$ be an admissible quaternion-valued function with $\mathscr{F}_q[\psi]$ being real-valued. Then, for every $f \in L^2(\mathbb{R}^2, \mathbb{H})$, we have*

$$\left\{\int_{\mathscr{G}} |\mathbf{b}|^{2p} \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\eta\right\}^{1/p} \left\{\int_{\mathbb{R}^2} |\xi|^{2p} \left|\mathscr{F}_q\big[f\big](\xi)\right|_{\mathbb{H}}^2 d\xi\right\}^{1/p} \geq \frac{(C_\psi)^{1/p}}{4} \left\|f\right\|_{L^2(\mathbb{R}^2,\mathbb{H})}^{4/p}, \quad p \geq 1. \tag{4.7}$$

Proof. By virtue of Hölder's inequality, we can write

$$\begin{aligned}
&\int_{\mathscr{G}} |\mathbf{b}|^2 \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\eta \\
&= \int_{\mathscr{G}} \left\{|\mathbf{b}|^2 \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^{2/p}\right\} \left\{\left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^{\left(2-\frac{2}{p}\right)}\right\} d\eta \\
&\le \left\{\int_{\mathscr{G}} \left(|\mathbf{b}|^2 \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^{2/p}\right)^p d\eta\right\}^{1/p} \left\{\int_{\mathscr{G}} \left(\left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^{\left(2-\frac{2}{p}\right)}\right)^{\frac{p}{p-1}} d\eta\right\}^{\left(1-\frac{1}{p}\right)} \\
&= \left\{\int_{\mathscr{G}} |\mathbf{b}|^{2p} \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\eta\right\}^{1/p} \left\{\int_{\mathscr{G}} \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\eta\right\}^{\left(1-\frac{1}{p}\right)}.
\end{aligned}$$

Therefore, we have

$$\left\{\int_{\mathscr{G}} |\mathbf{b}|^{2p} \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\eta\right\}^{1/p} \ge \frac{\left\{\int_{\mathscr{G}} |\mathbf{b}|^2 \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\eta\right\}}{\left\{\int_{\mathscr{G}} \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\eta\right\}^{\left(1-\frac{1}{p}\right)}}. \tag{4.8}$$

In analogy with above and by virtue of orthogonality relation (3.9), we have

$$\left\{\int_{\mathbb{R}^2} |\xi|^{2p} \left|\mathscr{F}_q[f](\xi)\right|_{\mathbb{H}}^2 d\xi\right\}^{1/p} \ge \frac{\left\{\int_{\mathbb{R}^2} |\xi|^2 \left|\mathscr{F}_q[f](\xi)\right|_{\mathbb{H}}^2 d\xi\right\}}{\left\{\left\|f\right\|_{\mathbb{H}}^2\right\}^{\left(1-\frac{1}{p}\right)}}. \tag{4.9}$$

Multiplying the inequalities (4.8) and (4.9) and employing Theorem 4.2, we obtain

$$\begin{aligned}
&\left\{\int_{\mathscr{G}} |\mathbf{b}|^{2p} \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\eta\right\}^{1/p} \left\{\int_{\mathbb{R}^2} |\xi|^{2p} \left|\mathscr{F}_q[f](\xi)\right|_{\mathbb{H}}^2 d\xi\right\}^{1/p} \\
&\qquad \ge \frac{\int_{\mathscr{G}} |\mathbf{b}|^2 \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\eta \int_{\mathbb{R}^2} |\xi|^2 \left|\mathscr{F}_q[f](\xi)\right|_{\mathbb{H}}^2 d\xi}{\left\{\int_{\mathscr{G}} \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\eta\right\}^{\left\{1-\frac{1}{p}\right\}} \left\{\left\|f\right\|_{\mathbb{H}}^2\right\}^{\left(1-\frac{1}{p}\right)}} \\
&\qquad \ge \frac{1}{4} \frac{C_\psi \left\|f\right\|_{L^2(\mathbb{R}^2,\mathbb{H})}^4}{\left\{\int_{\mathscr{G}} \left|\mathscr{P}_\psi^{\mathbb{H}}\big[f\big]\big(a,\mathbf{b},\theta,\mathbf{w}\big)\right|_{\mathbb{H}}^2 d\eta\right\}^{\left\{1-\frac{1}{p}\right\}} \left\{\left\|f\right\|_{\mathbb{H}}^2\right\}^{\left(1-\frac{1}{p}\right)}} \\
&\qquad \ge \frac{1}{4} \frac{C_\psi \left\|f\right\|_{L^2(\mathbb{R}^2,\mathbb{H})}^4}{\left\{C_\psi \left\|f\right\|_{\mathbb{H}}^2\right\}^{\left(1-\frac{1}{p}\right)} \left\{\left\|f\right\|_{\mathbb{H}}^2\right\}^{\left(1-\frac{1}{p}\right)}} \\
&\qquad \ge \frac{(C_\psi)^{1/p}}{4} \left\|f\right\|_{L^2(\mathbb{R}^2,\mathbb{H})}^{4/p}.
\end{aligned}$$

The proof of Theorem 4.3 is complete. □

It is pertinent to that for $p = 1$, Theorem 4.3 boils down to the Theorem 4.2.

The rest of the section is devoted to establish an analogue of the logarithmic inequality for the two-sided quaternion wave-packet transform. To facilitate our intention, we start with the following definition.

Definition 4.4. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, the Schwartz space in $L^2(\mathbb{R}^2, \mathbb{H})$ is defined by

$$\mathscr{S}\left(\mathbb{R}^2, \mathbb{H}\right) = \left\{ f \in \mathbb{C}^\infty(\mathbb{R}^2, \mathbb{H}); \sup_{t\in\mathbb{R}^2} \left(1 + |t|^k\right) \left| \frac{\partial^{\alpha_1+\alpha_2}[f(t)]}{\partial_{t_1}^{\alpha_1} \partial_{t_2}^{\alpha_2}} \right| < \infty \right\}, \tag{4.10}$$

where $\mathbb{C}^\infty(\mathbb{R}^2, \mathbb{H})$, denote the space of all smooth functions from $\mathbb{R}^2$ to $\mathbb{H}$.

We now establish the logarithmic uncertainty principle for the two-sided quaternion wave-packet transform $\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a, \mathbf{b}, \theta, \mathbf{w}\big)$ as defined by (3.1).

Theorem 4.5. *Let $\psi \in \mathscr{S}\left(\mathbb{R}^2, \mathbb{H}\right)$ be an admissible quaternion and, suppose that $\mathscr{P}_\psi^{\mathbb{H}}[f] \in \mathscr{S}\left(\mathbb{R}^2, \mathbb{H}\right)$, then the two-sided quaternion wave-packet transform (3.1) satisfies the following logarithmic estimate of the uncertainty inequality:*

$$\int_{\mathscr{G}} \ln|\mathbf{b}| \left| \mathscr{P}_\psi^{\mathbb{H}}[f]\big(a, \mathbf{b}, \theta, \mathbf{w}\big) \right|_{\mathbb{H}}^2 d\eta + C_\psi \int_{\mathbb{R}^2} \ln|\xi| \left| \mathscr{F}_q\big[f\big](\xi) \right|_{\mathbb{H}}^2 d\xi \geq \left[\frac{\Gamma'(1/2)}{\Gamma(1/2)} - \ln\pi \right] \cdot C_\psi \left\| f \right\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2. \tag{4.11}$$

Proof. For any quaternion $f \in \mathscr{S}(\mathbb{R}^2, \mathbb{H})$, we have the following inequality [9]

$$\int_{\mathbb{R}^2} \ln|\mathbf{b}| \left| f(\mathbf{b}) \right|_{\mathbb{H}}^2 d\mathbf{b} + \int_{\mathbb{R}^2} \ln|\xi| \left| \mathscr{F}_q\big[f\big](\xi) \right|_{\mathbb{H}}^2 d\xi \geq \left[\frac{\Gamma'(1/2)}{\Gamma(1/2)} - \ln\pi \right] \int_{\mathbb{R}^2} \left| f(\mathbf{b}) \right|_{\mathbb{H}}^2 d\mathbf{b}.$$

By considering $\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a, \mathbf{b}, \theta, \mathbf{w}\big)$ as function of $\mathbf{b}$ and replacing f by $\mathscr{P}_\psi^{\mathbb{H}}[f]$ in the above inequality, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^2} \ln|\mathbf{b}| \left| \mathscr{P}_\psi^{\mathbb{H}}[f]\big(a, \mathbf{b}, \theta, \mathbf{w}\big) \right|_{\mathbb{H}}^2 d\mathbf{b} + \int_{\mathbb{R}^2} \ln|\xi| \left| \mathscr{F}_q\left[\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a, \mathbf{b}, \theta, \mathbf{w}\big) \right](\xi) \right|_{\mathbb{H}}^2 d\xi \\ &\qquad \geq \left[\frac{\Gamma'(1/2)}{\Gamma(1/2)} - \ln\pi \right] \int_{\mathbb{R}^2} \left| \mathscr{P}_\psi^{\mathbb{H}}[f]\big(a, \mathbf{b}, \theta, \mathbf{w}\big) \right|_{\mathbb{H}}^2 d\mathbf{b}. \end{aligned} \tag{4.12}$$

Integrating (4.12) under $da\, d\theta\, d\mathbf{w}/a^3$, and using the Fubini's theorem, we get

$$\begin{aligned} &\int_{\mathbb{R}^+\times SO(2)\times\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|\mathbf{b}| \left| \mathscr{P}_\psi^{\mathbb{H}}[f]\big(a, \mathbf{b}, \theta, \mathbf{w}\big) \right|_{\mathbb{H}}^2 d\mathbf{b} \frac{da\, d\theta\, d\mathbf{w}}{a^3} \\ &\qquad + \int_{\mathbb{R}^+\times SO(2)\times\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|\xi| \left| \mathscr{F}_q\left[\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a, \mathbf{b}, \theta, \mathbf{w}\big) \right](\xi) \right|_{\mathbb{H}}^2 d\xi \frac{da\, d\theta\, d\mathbf{w}}{a^3} \\ &\qquad \geq \left[\frac{\Gamma'(1/2)}{\Gamma(1/2)} - \ln\pi \right] \int_{\mathbb{R}^+\times SO(2)\times\mathbb{R}^2} \int_{\mathbb{R}^2} \left| \mathscr{P}_\psi^{\mathbb{H}}[f]\big(a, \mathbf{b}, \theta, \mathbf{w}\big) \right|_{\mathbb{H}}^2 d\mathbf{b} \frac{da\, d\theta\, d\mathbf{w}}{a^3}. \end{aligned}$$

As a consequence of Lemma 4.1, we obtain the desired inequality

$$\int_{\mathscr{G}} \ln|\mathbf{b}| \left| \mathscr{P}_\psi^{\mathbb{H}}[f]\big(a, \mathbf{b}, \theta, \mathbf{w}\big) \right|_{\mathbb{H}}^2 d\eta + C_\psi \int_{\mathbb{R}^2} \ln|\xi| \left| \mathscr{F}_q\big[f\big](\xi) \right|_{\mathbb{H}}^2 d\xi \left[\frac{\Gamma'(1/2)}{\Gamma(1/2)} - \ln\pi \right] \cdot C_\psi \left\| f \right\|_{L^2(\mathbb{R}^2,\mathbb{H})}^2.$$

This completes the proof of Theorem 4.5. □

5. Examples

In this section, we shall present some illustrative examples for the demonstration of the proposed two-sided quaternion wave-packet transform.

Example 5.1. Consider the two-dimensional function

$$\psi(\mathbf{x}) = \lambda^{-2} e^{\frac{-|\mathbf{x}|^2}{2\lambda^2}} - e^{\frac{-|\mathbf{x}|^2}{2}}, \quad 0 < \lambda < 1. \tag{5.1}$$

Then, we shall compute the two-sided quaternion wave-packet transform of the quaternion-valued signal $f(\mathbf{x}) = e^{-(ix_1+jx_2)},\ x_1, x_2 \in \mathbb{R}$, with respect to ψ for $R_{-\theta} = I$ as

$$\begin{aligned}
\mathscr{P}_\psi^{\mathbb{H}}[f]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big) &= \frac{1}{a}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1}\, f(\mathbf{x}) \overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)} e^{-2\pi j x_2 w_2}\, d\mathbf{x} \\
&= \frac{1}{a}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1}\, e^{-(ix_1+jx_2)} \left[\lambda^{-2}\, e^{\frac{-\left((x_1-b_1)^2+(x_2-b_2)^2\right)}{2a^2\lambda^2}} - e^{\frac{-\left((x_1-b_1)^2+(x_2-b_2)^2\right)}{2a^2}}\right] e^{-2\pi j x_2 w_2}\, d\mathbf{x} \\
&= \frac{1}{a\lambda^2}\, e^{\frac{-(b_1^2+b_2^2)}{2a^2\lambda^2}} \int_{\mathbb{R}} e^{\frac{-x_1^2-2x_1(-b_1+ia^2\lambda^2+2\pi i w_1 a^2\lambda^2)}{2a^2\lambda^2}}\, dx_1 \int_{\mathbb{R}} e^{\frac{-x_2^2-2x_2(-b_2+ja^2\lambda^2+2\pi j w_2 a^2\lambda^2)}{2a^2\lambda^2}}\, dx_2 \\
&\qquad - \frac{1}{a}\, e^{\frac{-(b_1^2+b_2^2)}{2a^2}} \int_{\mathbb{R}} e^{\frac{-x_1^2-2x_1(-b_1+ia^2+2\pi i w_1 a^2)}{2a^2}}\, dx_1 \int_{\mathbb{R}} e^{\frac{-x_2^2-2x_2(-b_2+ja^2+2\pi j w_2 a^2)}{2a^2}}\, dx_2 \\
&= \frac{1}{a\lambda^2}\, e^{\frac{-(b_1^2+b_2^2)}{2a^2\lambda^2}} \sqrt{2\pi a^2\lambda^2}\, e^{\frac{(-b_1+ia^2\lambda^2+2\pi i w_1 a^2\lambda^2)^2}{a^4\lambda^4}\times\frac{a^2\lambda^2}{2}} \sqrt{2\pi a^2\lambda^2}\, e^{\frac{(-b_2+ja^2\lambda^2+2\pi j w_2 a^2\lambda^2)^2}{2a^2\lambda^2}} \\
&\qquad - \frac{1}{a}\, e^{\frac{-(b_1^2+b_2^2)}{2a^2}} \sqrt{2\pi a^2}\, e^{\frac{(-b_1+ia^2+2\pi i w_1 a^2)^2}{2a^2}} \sqrt{2\pi a^2}\, e^{\frac{(-b_2+ja^2+2\pi j w_2 a^2)^2}{2a^2}} \\
&= 2\pi a\, e^{\frac{-(b_1^2+b_2^2)}{2a^2\lambda^2}}\, e^{\frac{(-b_1+ia^2\lambda^2+2\pi i w_1 a^2\lambda^2)^2}{2a^2\lambda^2}}\, e^{\frac{(-b_2+ja^2\lambda^2+2\pi j w_2 a^2\lambda^2)^2}{2a^2\lambda^2}} - 2\pi a\, e^{\frac{-(b_1^2+b_2^2)}{2a^2}}\, e^{\frac{(-b_1+ia^2+2\pi i w_1 a^2)^2}{2a^2}}\, e^{\frac{(-b_2+ja^2+2\pi j w_2 a^2)^2}{2a^2}} \\
&= 2\pi a\, e^{\frac{-(b_1^2+b_2^2)+(ia^2\lambda^2+2\pi i w_1 a^2\lambda^2-b_1)^2+(ja^2\lambda^2+2\pi j w_2 a^2\lambda^2-b_2)^2}{2a^2\lambda^2}} - 2\pi a\, e^{\frac{-(b_1^2+b_2^2)+(ia^2+2\pi i w_1 a^2-b_1)^2+(ja^2+2\pi j w_2 a^2-b_2)^2}{2a^2}}.
\end{aligned} \tag{5.2}$$

For computational convenience, we choose $a = 1, b_1 = b_2 = 1, \lambda = 0.5$, so that (5.2) yields

$$\begin{aligned}
\mathscr{P}_\psi^{\mathbb{H}}[f]\Big(1, 1, 0, \mathbf{w}\Big) &= 2\pi\, e^{\frac{-2+(i(0.5)^2+2\pi i w_1(0.5)^2-1)^2+(j(0.5)^2+2\pi j w_2(0.5)^2-1)^2}{2(0.5)^2}} - 2\pi e^{\frac{-2+(i+2\pi i w_1-1)^2+(j+2\pi j w_2-1)^2}{2}} \\
&= 2\pi\, e^{(-2-4\pi^2 w_1^2-4\pi^2 w_2^2-4\pi w_1-4\pi w_2)/8} \cdot e^{-i(1+2\pi w_1)} \cdot e^{-j(1+2\pi w_2)} \\
&\qquad - 2\pi\, e^{(-2-4\pi^2 w_1^2-4\pi^2 w_2^2-4\pi w_1-4\pi w_2)/2} \cdot e^{-i(1+2\pi w_1)} \cdot e^{-j(1+2\pi w_2)} \\
&= 2\pi \left[e^{(-2-4\pi^2 w_1^2-4\pi^2 w_2^2-4\pi w_1-4\pi w_2)/8} - e^{(-2-4\pi^2 w_1^2-4\pi^2 w_2^2-4\pi w_1-4\pi w_2)/2}\right] e^{-i(1+2\pi w_1)} \cdot e^{-j(1+2\pi w_2)} \\
&= 2\pi \left[e^{(-1-2\pi^2 w_1^2-2\pi^2 w_2^2-2\pi w_1-2\pi w_2)/4} - e^{(-1-2\pi^2 w_1^2-2\pi^2 w_2^2-2\pi w_1-2\pi w_2)}\right] \\
&\qquad \times \Big(\cos(1+2\pi w_1)\cos(1+2\pi w_2) - j\cos(1+2\pi w_1)\sin(1+2\pi w_2) \\
&\qquad\qquad - i\sin(1+2\pi w_1)\cos(1+2\pi w_2) + i\cdot j\sin(1+2\pi w_1)\sin(1+2\pi w_2)\Big).
\end{aligned} \tag{5.3}$$

The graphical representation of the given quaternion-valued signal $f(\mathbf{x}) = e^{-(ix_1+jx_2)},\ x_1, x_2 \in \mathbb{R}$ is presented in Fig. 1, whereas its two-sided quaternion wave-packet transform is depicted in Fig.2, for $a = 1$, $b_1 = b_2 = 1$, and $\lambda = 0.5$.

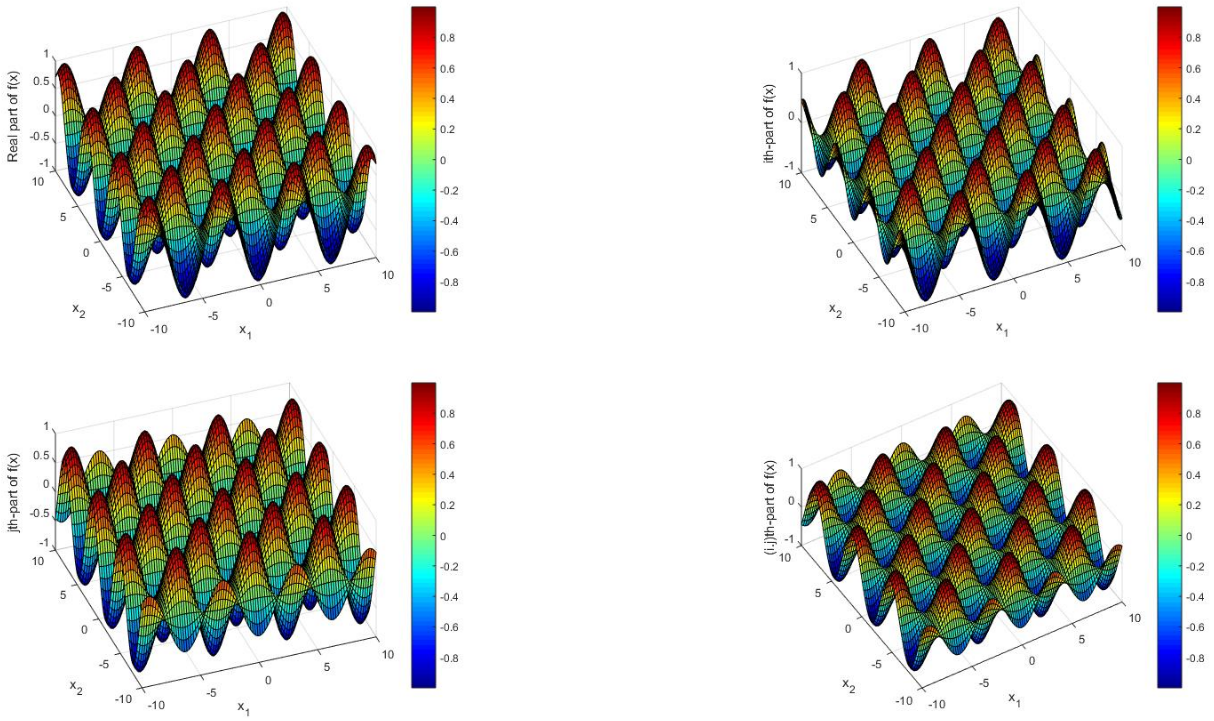

Figure 1: Real part (top left), i^{th}-imaginary part (top right), j^{th}-imaginary part(bottom left) and ij^{th}-imaginary part (bottom right) of the signal $f(x)$.

Example 5.2. Consider the two-dimensional Haar wavelet

$$\psi(\mathbf{x}) = \begin{cases} 1, & \text{if } 0 \le x_1, x_2 < 1/2 \\ -1, & \text{if } 1/2 \le x_1, x_2 < 1 \\ 0, & \text{otherwise}. \end{cases} \tag{5.4}$$

Then, the two-sided quaternion wave-packet transform of the signal $f(\mathbf{x}) = e^{-(x_1^2+x_2^2)}$, $x_1, x_2 \in \mathbb{R}$ with respect to ψ for $R_{-\theta} = I$ can be evaluated as

$$\begin{aligned}
\mathscr{P}_\psi^{\mathbb{H}}[f]\Big(a, \mathbf{b}, \theta, \mathbf{w}\Big) &= \frac{1}{a}\int_{\mathbb{R}^2} e^{-2\pi i x_1 w_1}\, f(\mathbf{x})\, \overline{\psi\left(\frac{R_{-\theta}(\mathbf{x}-\mathbf{b})}{a}\right)} e^{-2\pi j x_2 w_2}\, d\mathbf{x} \\
&= \frac{1}{a}\int_{b_1}^{\frac{a}{2}+b_1}\int_{b_2}^{\frac{a}{2}+b_2} e^{-2\pi i x_1 w_1} e^{-(x_1^2+x_2^2)} e^{-2\pi j x_2 w_2} dx_1 dx_2 \\
&\qquad - \frac{1}{a}\int_{\frac{a}{2}+b_1}^{a+b_1}\int_{\frac{a}{2}+b_2}^{a+b_2} e^{-2\pi i x_1 w_1} e^{-(x_1^2+x_2^2)} e^{-2\pi j x_2 w_2} dx_1 dx_2 \\
&= \frac{1}{a}\int_{b_1}^{\frac{a}{2}+b_1} e^{-(x_1^2+2\pi i x_1 w_1)} dx_1 \int_{b_2}^{\frac{a}{2}+b_2} e^{-(x_2^2+2\pi j x_2 w_2)} dx_2 \\
&\qquad - \frac{1}{a}\int_{\frac{a}{2}+b_1}^{a+b_1} e^{-(x_1^2+2\pi i x_1 w_1)} dx_1 \int_{\frac{a}{2}+b_2}^{a+b_2} e^{-(x_2^2+2\pi j x_2 w_2)} dx_2 \\
&= \frac{1}{a}\, e^{\frac{\pi^2(w_1^2+w_2^2)}{4}} \int_{b_1}^{\frac{a}{2}+b_1} e^{-(x_1+\pi i w_1)^2}\, dx_1 \cdot \int_{b_2}^{\frac{a}{2}+b_2} e^{-\left(x_2+\pi j w_2\right)^2}\, dx_2
\end{aligned}$$

$$-\frac{1}{a}\, e^{\frac{\pi^2(w_1^2+w_2^2)}{4}} \int_{\frac{a}{2}+b_1}^{a+b_1} e^{-(x_1+\pi i w_1)^2}\, dx_1 \cdot \int_{\frac{a}{2}+b_2}^{a+b_2} e^{-\left(x_2+\pi j w_2\right)^2}\, dx_2. \tag{5.5}$$

After substituting $z_1 = x_1 + \pi i w_1, z_2 = x_2 + \pi j w_2$, equation (5.5) becomes

$$\begin{aligned}
\mathscr{P}_\psi^{\mathbb{H}}[f]\big(a, \mathbf{b}, \theta, \mathbf{w}\big) &= \frac{e^{\pi^2(w_1^2+w_2^2)/4}}{a}\left\{\int_{b_1+\pi i w_1}^{\frac{a}{2}+b_1+\pi i w_1} e^{-z_1^2}\, dz_1 \cdot \int_{b_2+\pi j w_2}^{\frac{a}{2}+b_2+\pi j w_2} e^{-z_2^2}\, dz_2 \right.\\
&\qquad \left. - \int_{\frac{a}{2}+b_1+\pi i w_1}^{a+b_1+\pi i w_1} e^{-z_1^2}\, dz_1 \cdot \int_{\frac{a}{2}+b_2+\pi j w_2}^{a+b_2+\pi j w_j} e^{-z_2^2}\, dz_2 \right\}\\
&= \frac{1}{a}\, e^{\frac{\pi^2(w_1^2+w_2^2)}{4}} \left\{\left[-\operatorname{err}f\,(b_1+\pi i w_1) + \operatorname{err}f\left(\frac{a}{2}+b_1+\pi i w_1\right)\right]\right.\\
&\qquad \times\left[\operatorname{err}f\,(b_2+\pi j w_2) + \operatorname{err}f\left(\frac{a}{2}+b_2+\pi j w_2\right)\right]\\
&\qquad -\left[\operatorname{err}f\left(\frac{a}{2}+b_1+\pi i w_1\right) + \operatorname{err}f\,(a+b_1+\pi i w_1)\right]\\
&\qquad \left.\times\left[\operatorname{err}f\left(\frac{a}{2}+b_2+\pi j w_2\right) + \operatorname{err}f\,(a+b_2+\pi j w_2)\right]\right\},
\end{aligned} \tag{5.6}$$

where $\operatorname{err}f(z) = \int_0^z e^{-t^2}dt$.

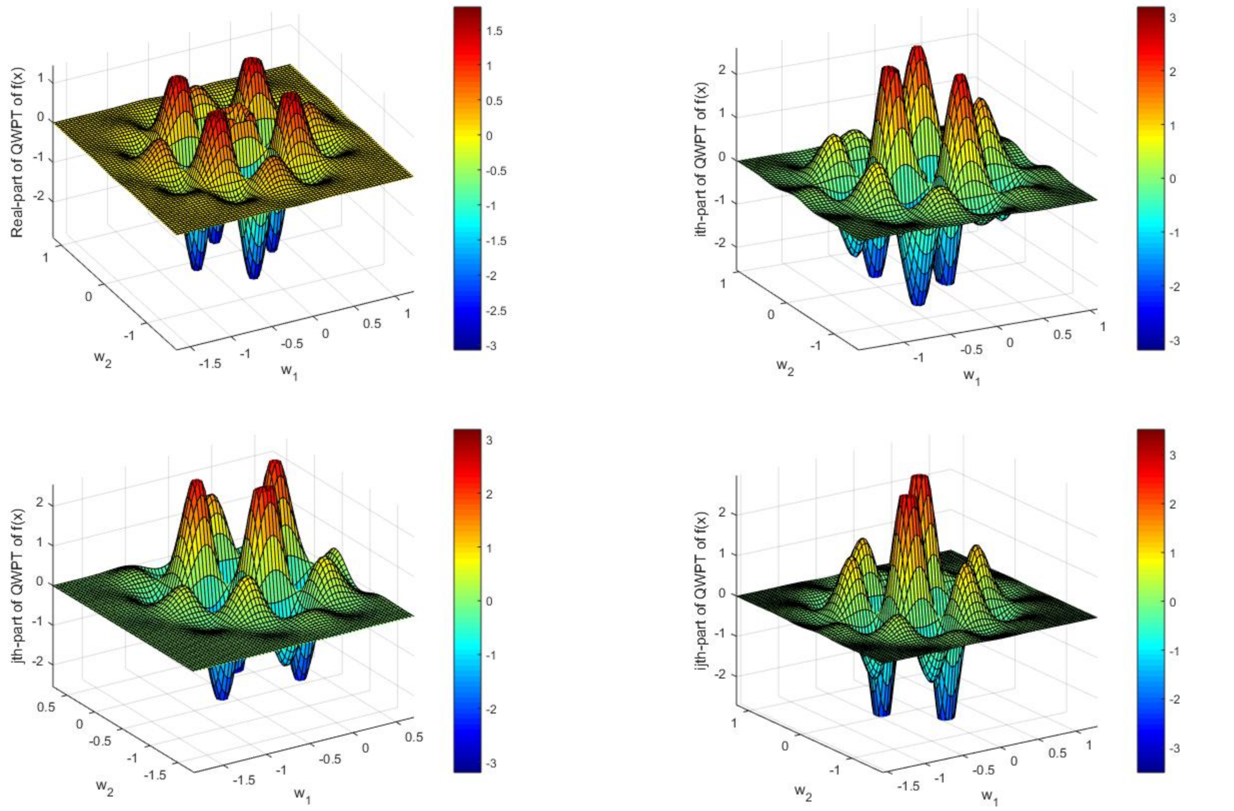

Figure 2: Two-sided quaternion wave-packet transform of $f(x)$, for $a = 1$, $b_1 = b_2 = 1$, and $\lambda = 0.5$.

6. Conclusion

In this article, we introduced a novel integral transform coined quaternion wave-packet transform which is capable of providing better time-frequency resolution over the high-frequency regions. Besides studying all the fundamental properties, we also illustrate the fundamental results via some lucid examples. Finally, we establish some analogues of the Heisenberg's and logarithmic uncertainty principles for the proposed two-sided quaternion wave-packet transform. It is hoped that quaternion wave-packet transform might be useful in three-dimensional colour images and video processing, aerospace engineering, oil exploration, crystallography and for the solution of other geometrical problems.

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Conflict of interests:

The authors declare that they have no conflict of interest.

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